Post AdS/CFT

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Abstract

The Hamiltonian governing the gravitational interaction of N relativistic particles in a four-dimensional anti-de Sitter background is derived to leading order in Newton's constant. The resulting pairwise interactions, combined with the confining nature of motion in anti-de Sitter spacetime, are expected to lead to classical chaos. In the context of the AdS/CFT correspondence, the emergence of a chaotic classical limit on the gravity side has important implications for the dual three-dimensional conformal field theory, including that the spectrum of conformal primary operators at strong coupling should exhibit level repulsion in line with the Wigner surmise.

I. INTRODUCTION

The study of the gravitational interaction of N particles in asymptotically flat spacetime has a long history, beginning with the post-Newtonian approach of [1], where a derivative/momentum expansion is made. Later a post-Minkowskian expansion was developed where one expands in powers of Newton's constant, but retains all orders in momentum [2, 3]. Such an expansion is, for instance, relevant for deriving gravitational wave emission from coalescing binary black hole systems. In the present work we consider the gravitational interaction of N particles in an asymptotically anti-de Sitter (AdS) spacetime, hence we formulate a post-AdS expansion.

Our motivation is rather different from the asymptotically flat case, as our ultimate goal is to explore features of the anti-de Sitter spacetime/conformal field theory (AdS/CFT) correspondence that are relevant to the black hole information problem. In earlier work [4], we framed the formation of a typical small AdS black hole in four-dimensional anti-de Sitter spacetime as the gradual coalescence a cloud of collapsing particles, that is well described when the particles are well-separated using the holographic reconstruction methods of HKLL [5–8]. A small AdS black hole has a mass that is smaller than the characteristic AdS mass scale and a finite lifetime due to black hole evaporation. Much of the literature on AdS black holes instead considers the limit, where the black hole mass is large compared to the AdS scale and a high temperature limit of the canonical ensemble matches well with the large mass limit of the microcanonical ensemble, but large AdS black holes essentially behave as stable massive remnants in the context of the black hole information problem.

Within the setup of [4], the gravitational collapse to a small AdS black hole is understood as eigenstate thermalization on the space of small black hole states. This assumes that the evaluation of semi-classical observables involves averaging over a dense population of energy eigenstates within a narrow range of energies determined by the finite black hole lifetime. The microscopic counting of states for the initial cloud of particles agrees with the Bekenstein-Hawking entropy (up to numerical factors of order 1) [9, 10] so there is indeed a large number of CFT states (of order e^S) that correspond to small AdS black holes. However a puzzle emerges because perturbative excitations in AdS have energies (or equivalently conformal dimensions) that are quantized in units of the inverse AdS radius of curvature. If the pattern of the perturbative level spacing persists in the interacting theory, these states would have enormous degeneracies, and the conditions for eigenstate thermalization [11], would not be satisfied. In order for eigenstate thermalization to hold, the spectrum of quantum energy eigenstates must for the most part be non-degenerate, leading to a quasi-continuum of energy levels. While this assumption receives indirect support in the literature for states that collapse to black holes [12] there is surprisingly little direct evidence that this is the case. In the present paper we consider the classical limit of a *typical* scattering state, relaxing the condition of black hole formation, and ask whether such a state exhibits classical chaos.

Due to the negative spacetime curvature of AdS, massive particles will be confined to the interior region, and interaction energies will always be finite. As such, one has infinite "dwell" time and even relatively simple pairwise interactions are expected to lead to classical chaos in systems with more than just a few particles (see below for further discussion). The case of massless particles is more delicate. Null geodesics reach null infinity in AdS at finite affine parameter, indicating that massless particles can reach null infinity and that additional boundary conditions must be provided. Here we have in mind simple energy conserving Dirichlet boundary conditions, with a flat metric on the conformal boundary. Massless particles will then reflect off the boundary at infinity in a geodesic approximation. While their interactions vanish at infinity, the time average of the interaction energy is finite and non-vanishing. Again this is expected to lead to classical chaos.[1]

At this level of approximation, the Lyapunov time associated with chaos appears to be very long, of order R^3_{AdS}/l^2_{pl}, with particles needing many AdS crossing times to randomize their momenta by factors of order one. At higher orders in the post-AdS approximation, higher order interactions will appear which will only decrease the estimate of the Lyapunov time. However, in galactic N-body simulations [13] in the asymptotically flat case, the Lyapunov time for multiparticle states tends to be dominated not by averages of long-range interactions of many particles, but by close approach of pairs of particles. We expect to see a similar phenomenon in asymptotically AdS spacetime. In particular, for particles that collapse to black holes, we expect a thermalization time on a timescale set by the local physics of the black hole, perhaps of order $M \log M$.

[1] We note that the geodesic approximation fails before massless radiation reaches null infinity due to the infinite gravitational redshift in AdS spacetime.. A more complete treatment, would involve solving the massless field equations and consider the reflection of wave packets from the asymptotic region. We expect the time average of the interaction energy to be non-vanishing and finite in this case as well.

Having established that the classical limit of the bulk dual of a holographic conformal field theory is chaotic, one may then follow the general logic of the Wigner surmise [14, 15], that the full quantum system will exhibit level repulsion. We note that this holds for any finite value of the Lyapunov time, so establishing an upper bound is sufficient to make the argument. Hence, for N-particle scattering, we may expect a quasi-continuum of states with level spacing of order $e^{-S(E)}$ where S is the microcanonical entropy of the conformal field theory at energy E. As the microcanonical entropy at a given fixed energy is finite in the three (or higher) dimensional conformal field theories appearing in the AdS/CFT correspondence, our argument yields a sharp prediction for the spectrum of conformal field theories in dimensions three and higher with holographic duals, that generic primary operators will experience level-repulsion and develop a near-continuous spectrum.

It is interesting to compare to the situation in asymptotically flat spacetime. In this case, the dwell time of typical incoming clouds of particles will be finite, and one expects ordinary perturbative scattering, perhaps governed by underlying integrability, rather than chaotic dynamics. Moreover, in the atypical sector of the space of states that contains black holes, they may be dressed by infinitely degenerate soft hair in the classical limit [16]. Having a negative cosmological constant avoids both of these issues. The soft hair is no longer present, as the spectrum of excitations becomes discrete. Likewise in AdS one expects integrability will only apply to certain special limits of amplitudes where the dwell time can be made finite. Conversely, in AdS one can expect eigenstate thermalization to work for typical states (of energies larger than the Planck mass) whereas in asymptotically flat spacetime it may only work for very special families of states. This suggests that AdS/CFT in the limit of vanishing cosmological constant provides a regularized description of dynamical black hole formation in quantum gravity, while that dynamics may be much harder to access by considering the asymptotically flat case directly.

In the following we briefly review the Hamiltonian approach to gravity applied to an anti-de Sitter background, and then derive the Hamiltonian for N-particles interacting at leading order in Newton's constant and to all orders in momentum. We end with some brief conclusions.

II. BULK DYNAMICS

Following the Hamiltonian formulation of [17–20] and its generalization to anti-de Sitter spacetime in [21], we begin with the gravitational action with no matter sources,

$$S_{grav} = \frac{1}{16\pi G_N}\int d^4x\, g^{1/2}\left(\pi^{ij}\partial_0 g_{ij} + N\left(R^{(3)} - 2\Lambda + \frac{1}{2}\pi^2 - g_{ik}g_{jl}\pi^{ij}\pi^{kl}\right) + 2N_i\pi^{ij}_{\ \ |j}\right), \tag{1}$$

where G_N is Newton's constant, $R^{(3)}$ is the three-dimensional Ricci scalar, Λ is the cosmological constant and $\pi^{ij}_{\ \ |j}$ is the three-dimensional divergence of the conjugate momentum on a spatial hypersurface. We also have the definitions[2]

$$N = (-g^{00})^{-1/2}, \qquad N_i = g_{0i}\,, \tag{2}$$

$$g = \det(g_{ij})\,, \qquad \pi = g_{ij}\pi^{ij}\,, \tag{3}$$

$$\pi^{ij} = N\left(\Gamma^0_{kl} - g_{kl}g^{mn}\Gamma^0_{mn}\right)g^{ik}g^{jl}\,. \tag{4}$$

With matter present, the constraint equations become

$$g^{1/2}\left(R - 2\Lambda + \frac{1}{2}\pi^2 - g_{ik}g_{jl}\pi^{ij}\pi^{kl}\right) = -16\pi G_N T^0_{\ 0}\,, \tag{5}$$

$$g^{1/2}\pi^{ij}_{\ \ |j} = 8\pi G_N T^i_{\ 0}\,, \tag{6}$$

where $T^0_{\ 0}$ and $T^i_{\ 0}$ are the energy and momentum densities, respectively. For point particle matter we have

$$T^0_{\ 0} = -\sum_A \left(g^{ij}p_{Ai}p_{Aj} + m_A^2\right)^{1/2}\delta(\mathbf{x} - \mathbf{x}_A)\,, \tag{7}$$

$$T^i_{\ 0} = -\sum_A g^{ij}p_{Aj}\delta(\mathbf{x} - \mathbf{x}_A)\,, \tag{8}$$

where A is summed over the different point particle sources (see for example [20]). The three-dimensional Dirac delta functions are defined so that $\int d^3x f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_A) = f(\mathbf{x}_A)$ for a smooth scalar function f on a spatial hypersurface.

Our strategy will be to solve the Hamiltonian constraint (5) and the momentum constraints (6) order-by-order in G_N and express the leading order Hamiltonian for N interacting particles in terms of the particle positions and momenta along with the propagating

[2] Our conventions differ from [21] by a factor of $g^{1/2}$ in the definition of the conjugate momentum variable. Here π^{ij} is a tensor rather than a tensor density as in [21].

degrees of freedom of the gravitational field. We adapt the gauge conditions of [2, 22] for four-dimensional asymptotically flat spacetime to an asymptotically AdS background with $\Lambda = -3$ as follows,

$$g_{ij} = \frac{e^{4\phi}}{z^2}\delta_{ij} + h_{ij}^{TT}\,, \tag{9}$$

$$\delta_{ij}\pi^{ij} = 0\,. \tag{10}$$

Here h_{ij}^{TT} is a transverse-traceless metric perturbation, $\delta^{ij}h_{ij}^{TT} = 0$, $\delta^{ij}\nabla_i h_{jk}^{TT} = 0$, where the covariant derivative ∇_i is with respect to the background hypersurface metric $g_{ij}^{(0)} = \delta_{ij}/z^2$. The trace part of the metric involves $\phi = \phi_1 + \phi_2 + \ldots$, where the subscript denotes the order of the expansion in G_N.

The conjugate momentum π^{ij} may be decomposed as follows [23],

$$\pi^{ij} = \pi_{TT}^{ij} + \nabla^i \pi_T^j + \nabla^j \pi_T^i + \left(\nabla^i\nabla^j - \frac{1}{3}g^{(0)\,ij}\nabla^2\right)\pi_L\,, \tag{11}$$

into a transverse-traceless symmetric tensor satisfying $\nabla_j \pi_{TT}^{ij} = 0$ and $\delta_{ij}\pi_{TT}^{ij} = 0$, a transverse vector satisfying $\nabla_j \pi_T^j = 0$, and a scalar mode π_L. In general, such a decomposition also includes a trace part but this vanishes due to the gauge condition (10).

Inserting the coordinate conditions (9), (10) and the decomposition (11) on the left-hand side of the Hamiltonian constraint (5) and working to second order in perturbations, we find after some algebra that

$$\begin{aligned} g^{1/2}\,(R-2\Lambda) = \sqrt{g^{(0)}}\Big(&24(\phi_1 + 4\phi_1^2 + \phi_2) + 8(\nabla^2\phi_1 + (\nabla\phi_1)^2 + 2\phi_1\nabla^2\phi_1 + \nabla^2\phi_2) \\ &+ 4h_{ij}^{TT}\nabla^i\nabla^j\phi_1 + h_{TT}^{ij}h_{ij}^{TT} + \frac{3}{4}\nabla^i h_{TT}^{jk}\nabla_i h_{jk}^{TT} - \frac{1}{2}\nabla^i h_{TT}^{jk}\nabla_k h_{ij}^{TT} + h_{TT}^{ij}\nabla^2 h_{ij}^{TT}\Big)\,, \end{aligned} \tag{12}$$

and

$$g^{1/2}\left(\frac{1}{2}\pi^2 - g_{ik}g_{jl}\pi^{ij}\pi^{kl}\right) = \sqrt{g^{(0)}}\left(-\pi_{TT}^{ij}\pi_{ij}^{TT} - 2\pi_{TT}^{ij}\tilde{\pi}_{ij} - \tilde{\pi}^{ij}\tilde{\pi}_{ij}\right)\,, \tag{13}$$

where indices are raised and lowered using the background hypersurface metric and we have introduced the shorthand notation

$$\tilde{\pi}_{ij} = \nabla_i \pi_j^T + \nabla_j \pi_i^T + \left(\nabla_i\nabla_j - \frac{1}{3}g_{ij}^{(0)}\nabla^2\right)\pi_L\,. \tag{14}$$

For the purposes of the present paper it is sufficient to expand the left-hand side of the momentum constraint (6) to first order in perturbations, giving

$$g^{1/2}\pi^{ij}_{\ \ |j} = \sqrt{g^{(0)}}g^{(0)ij}\left[\left(\nabla^2-2\right)\pi^T_j + \frac{2}{3}\nabla_j\left(\nabla^2-3\right)\pi^L\right], \tag{15}$$

where we have used the decomposition (11).

III. FIRST ORDER SOLUTION

Our goal is to compute the Hamiltonian at linear order in G_N but to arbitrary order in momenta. Our first task is to compute the order G_N contributions to ϕ and π^{ij}. We will assume our initial state does not contain any transverse-traceless gravitational modes at order G_N^0, but that these will subsequently be generated due to radiative couplings. The general techniques we employ in this section were developed in [24, 25] and later adapted to anti-de Sitter spacetime in [23], where various Euclidean AdS Green functions are computed.

A. Scalar perturbation

We begin by solving the Hamiltonian constraint at first order in G_N, which reduces to the following equation for the leading order perturbation of the metric trace,

$$\left(\nabla^2-3\right)\phi_1(\mathbf{x}) = -2\pi G_N\sum_A\left(z_A^2\delta^{ij}p_{Ai}p_{Aj}+m_A^2\right)^{1/2} z_A^3\,\delta(\mathbf{x}-\mathbf{x}_A)\,. \tag{16}$$

The solution is a sum over the point particle sources,

$$\phi_1(\mathbf{x}) = \sum_A \phi_{1A}(\mathbf{x}) = 2\pi G_N\sum_A\left(z_A^2\delta^{ij}p_{Ai}p_{Aj}+m_A^2\right)^{1/2} G(\mathbf{x},\mathbf{x_A})\,, \tag{17}$$

where the defining equation for the scalar Green function is

$$\left(\nabla^2-3\right)G(\mathbf{x},\mathbf{x}') = -\frac{1}{\sqrt{g^{(0)}}}\delta(\mathbf{x}-\mathbf{x}')\,. \tag{18}$$

AdS symmetry implies the Green function can be expressed as a function of the geodesic distance $\lambda(\mathbf{x},\mathbf{x}')$ between its arguments, but for our purposes it is more convenient to work with the so-called chordal distance,

$$u(\mathbf{x},\mathbf{x}') = \frac{1}{2zz'}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right), \tag{19}$$

which is related to the geodesic distance by $u+1=\cosh\lambda$. The chordal distance and its derivatives satisfy a number of useful identities that are listed in [23]. The ones that enter into our considerations are included in Appendix A below for easy reference.

Inserting $G(\mathbf{x},\mathbf{x}') = G(u)$ into (18) and using the identities (A1) and (A2), the equation for the scalar Green function reduces to

$$u(u+2)G''(u) + 3(u+1)G'(u) - 3G(u) = -z'^3\delta(\mathbf{x}-\mathbf{x}')\,. \tag{20}$$

The corresponding homogeneous differential equation has the following general solution,

$$G(u) = c_1(u+1) + c_2\frac{2u(u+2)+1}{\sqrt{u(u+2)}}\,. \tag{21}$$

Setting $c_1 = -2c_2$ ensures maximal fall-off as $u \to \infty$. If we further choose $c_1 = \frac{1}{4\pi}$ we have in fact obtained a correctly normalized solution to the full inhomogeneous problem in (20).[3] The resulting scalar Green function is given by

$$G(u) = \frac{1}{4\pi\sqrt{u(u+2)}}\left(\sqrt{u(u+2} - u - 1\right)^2, \tag{22}$$

and the leading order perturbation of the metric trace is given by the sum $\phi_1(\mathbf{x}) = \sum_A \phi_{1A}(\mathbf{x})$, where

$$\phi_{1A}(\mathbf{x}) = \frac{G_N}{2}\left(z_A^2\delta^{ij}p_{Ai}p_{Aj} + m_A^2\right)^{1/2} u_A^{-1/2}(u_A+2)^{-1/2}\left(\sqrt{u_A(u_A+2)} - u_A - 1\right)^2, \tag{23}$$

with $u_A = u(\mathbf{x},\mathbf{x_A})$. We note that the metric trace perturbation falls off as

$$\phi_1 \sim z^3 \tag{24}$$

near the boundary at $z \to 0$. This will be important when we consider the boundary Hamiltonian in Section (IV).

B. Conjugate momentum perturbation

Next we need the first order solution for the momentum constraints,

$$\left(\nabla^2 - 2\right)\pi_i^T(\mathbf{x}) + \frac{2}{3}\nabla_i\left(\nabla^2 - 3\right)\pi^L(\mathbf{x}) = -8\pi G_N \sum_A \frac{p_{Ai}}{\sqrt{g^{(0)}}}\delta(\mathbf{x}-\mathbf{x}_A)\,. \tag{25}$$

To solve this, we act with a transverse vector projector on the source term to give the equation for the transverse vector Green function,

$$\left(\nabla^2 - 2\right)G_{ij'}^T(\mathbf{x},\mathbf{x}') = -\frac{g_{ij'}^{(0)}}{\sqrt{g^{(0)}}}\delta(\mathbf{x}-\mathbf{x}') + \nabla_i\frac{1}{\nabla^2}\nabla_{j'}\,, \tag{26}$$

[3] This is easily verified by multiplying both sides of 20 by a test function in $\mathbf{x}$ and integrating over a small spherical volume centered on $\mathbf{x}'$.

which will generate the solution for π_i^T. The transverse vector Green function is a bivector, with the unprimed index associated to position $\mathbf{x}$ and the primed index associated to position $\mathbf{x}'$. The parallel transporter, denoted by $g^{(0)}_{ij'}$, maps the unit vectors tangent to the geodesic connecting $\mathbf{x}$ and $\mathbf{x}'$ into each other via the relation

$$t_i = -g^{(0)\,j'}_{\;\;i}\, t_{j'}\,, \tag{27}$$

where $t_i = \partial_i u/\sqrt{u(u+2)}$ and $t_{j'} = \partial_{j'} u/\sqrt{u(u+2)}$. The parallel transporter can be expressed in terms of derivatives of u as follows [23],[4]

$$g^{(0)}_{ij'} = -\partial_i \partial_{j'} u + \frac{\partial_i u \partial_{j'} u}{u+2}\,, \tag{28}$$

as can be verified by insertion into (27). This expression will be useful momentarily, when we solve for $G^T_{ij'}(\mathbf{x},\mathbf{x}')$.

To generate the solution for π_L we instead project onto the longitudinal component of the source. This yields the Green function equation

$$\frac{2}{3}\nabla_i \left(\nabla^2 - 3\right) G^L_{j'} = -\nabla_i \frac{1}{\nabla^2}\nabla_{j'}\,. \tag{29}$$

Let us consider (29) and (26) in turn.

1. *Longitudinal component*

To proceed we need to solve for the inverse Laplacian $1/\nabla^2$ via

$$\nabla^2 \Delta_0(\mathbf{x},\mathbf{x}') = -\frac{1}{\sqrt{g^{(0)}}}\delta(\mathbf{x}-\mathbf{x}')\,. \tag{30}$$

Inserting $\Delta_0(\mathbf{x},\mathbf{x}') = \Delta_0(u)$ leads to the following ordinary differential equation,

$$u(u+2)\Delta_0''(u) + 3(u{+}1)\Delta_0'(u) = -z'^3\delta(\mathbf{x}-\mathbf{x}')\,. \tag{31}$$

The general solution to the homogenous problem is

$$\Delta_0(u) = c_1 + c_2\frac{u+1}{\sqrt{u(u+2)}}\,. \tag{32}$$

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[4] We note that the expression for $g^{(0)}_{ij'}$ in Table 2 of [23] contains a typo.

The coefficients $c_1$and c_2 are again determined by requiring maximal fall-off at infinity and the correct normalization in the $u \to 0$ limit to match the $\delta(\mathbf{x}-\mathbf{x}')$ on the right hand side of (30),

$$\Delta_0(u) = \frac{1}{4\pi}\left(\frac{u+1}{\sqrt{u(u+2)}} - 1\right) . \tag{33}$$

This can now be inserted on the right hand side of the equation for the longitudinal Green function,

$$\frac{2}{3}\left(\nabla^2 - 3\right) G^L_{j'} = -\frac{1}{\nabla^2}\nabla_{j'} = \left(\nabla_{j'}\frac{1}{\nabla^2}\right) = -\left(\partial_{j'}u\right)\Delta_0'(u) . \tag{34}$$

We look for a solution of the form

$$G^L_{j'}(\mathbf{x},\mathbf{x}') = \left(\partial_{j'}u\right) a(u) , \tag{35}$$

with $a(u)$ to be determined. Inserting this ansatz on the left hand side of (34), and using the identities (A1) - (A4) in Appendix (A), leads to an inhomogeneous ordinary differential equation for $a(u)$,

$$u(u+2)a''(u) + 5(u+1)a'(u) = \frac{3}{8\pi}\frac{1}{u^{3/2}(u+2)^{3/2}} , \tag{36}$$

whose general solution is given by

$$a(u) = c_1\frac{(u+1)(2u^2+4u-1)}{3u^{3/2}(u+2)^{3/2}} + c_2 - \frac{2u^2+6u+3}{8\pi\sqrt{u}(u+2)^{3/2}} . \tag{37}$$

To ensure maximal falloff at large u we get the condition $\frac{2}{3}c_1 + c_2 - \frac{1}{4\pi} = 0$. As $u \to 0$ the solution goes like

$$a(u) = -\frac{c_1}{3\left(2u\right)^{3/2}} + O\left(\frac{1}{\sqrt{u}}\right) , \tag{38}$$

so to avoid a δ function in (29) we need $c_1 = 0$. Thus $c_2 = \frac{1}{4\pi}$.

Bringing everything together, we obtain the following expression for the longitudinal component of the momentum perturbation at first order,

$$\pi^L(\mathbf{x}) = 2G_N\sum_A z_A^2\delta^{i'j'}p_{Ai'}\partial_{j'}u_A\left(1 - \frac{2u_A^2+6u_A+3}{2\sqrt{u_A}(u_A+2)^{3/2}}\right) . \tag{39}$$

2. Transverse component

Next we want to solve equation (26) for the transverse Green function. Inserting the expression (28) for the parallel transporter in terms of derivatives of u leads to the differential equation

$$\left(\nabla^2 - 2\right) G^T_{ij'}(\mathbf{x}, \mathbf{x}') = (\partial_i \partial_{j'} u)\, z'^3 \delta(\mathbf{x} - \mathbf{x}') + (\partial_i u)\,(\partial_{j'} u)\,\Delta_0''(u) + (\partial_i \partial_{j'} u)\,\Delta_0'(u)\,, \tag{40}$$

where $\Delta_0(u)$ is given by (33). The $\partial_i u \partial_{j'} u$ term in $g_{ij'}$ vanishes in the coincident limit, so does not appear when multiplied by the delta function. Following [23] we decompose $G_{ij'}$ into two independent tensor structures,

$$G^T_{ij'}(\mathbf{x}, \mathbf{x}') = (\nabla_i \nabla_{j'} u)\ A(u) + (\nabla_i u \nabla_{j'} u)\ B(u)\,, \tag{41}$$

and proceed to solve for the unknown functions $A(u)$ and $B(u)$. Using the identities in Appendix A for various derivatives of the chordal distance u we end up with a pair of coupled ordinary differential equations,

$$u(u+2)A'' + 3(u+1)A' - A + 2(u+1)B = z'^3 \delta(\mathbf{x} - \mathbf{x}') + \Delta_0'(u)\,, \tag{42}$$

$$u(u+2)B'' + 7(u+1)B' + 2A' + 2B = \Delta_0''(u)\,. \tag{43}$$

The transverse gauge condition

$$g^{(0)ab} \nabla_a G^T_{bj'} = 0\,, \tag{44}$$

implies

$$3A + (u+1)A' + 4(u+1)B + u(u+2)B' = 0\,, \tag{45}$$

which can be re-expressed as

$$u(u+2)C'(u) + 2(u+1)C(u) = 2A(u)\,, \tag{46}$$

where we have defined the auxiliary function $C(u)$ as the following linear combination of the transverse vector propagator functions,

$$C(u) = (u+1)A(u) + u(u+2)B(u)\,. \tag{47}$$

A suitably chosen linear combination of (42) and (43) yields the following ordinary differential equation for $C(u)$ alone,

$$u\,(u+2)\,C''(u) + 5\,(u+1)\,C'(u) = z'^3 \delta(\mathbf{x} - \mathbf{x}') + (u+1)\,\Delta_0'(u) + u\,(u+2)\,\Delta_0''(u)\,. \tag{48}$$

The right hand side may be further simplified using (31) and (33) to give

$$u\,(u+2)\,C''(u) + 5\,(u+1)\,C'(u) = \frac{u+1}{2\pi u^{3/2}(u+2)^{3/2}}\,. \tag{49}$$

We can solve for $C(u)$ using the same strategy as before. The homogeneous problem is in fact identical to the one we encountered for the function $a(u)$ when solving for the longitudinal Green function but the particular solution to the inhomogeneous problem is different,

$$C(u) = c_1 \frac{(u+1)(2u^2+4u-1)}{3u^{3/2}(u+2)^{3/2}} + c_2 - \frac{u+1}{4\pi\sqrt{u(u+2)}} . \tag{50}$$

Fall-off as $u \to \infty$ requires $\frac{2}{3}c_1 + c_2 = \frac{1}{4\pi}$ while at small u we have

$$C(u) = -\frac{1}{3(2u)^{3/2}} c_1 + O\left(\frac{1}{\sqrt{u}}\right) , \tag{51}$$

which fixes $c_1 = 0$, hence $c_2 = \frac{1}{4\pi}$.

The next step is to solve for the transverse vector propagator functions $A(u)$ and $B(u)$ by inserting the solution for $C(u)$ into (46) and (47),

$$A(u) = -\frac{1}{8\pi\sqrt{u(u+2)}} \left(u+1-\sqrt{u(u+2)}\right)^2 , \tag{52}$$

$$B(u) = -\frac{1}{4\pi} \left(1 - \frac{(1+u)\left(u(u+2)-\frac{1}{2}\right)}{\left(u(u+2)\right)^{3/2}}\right) . \tag{53}$$

Finally, the full expression for π_i^T is obtained by summing over contributions from the different point particle sources,

$$\pi_i^T(\mathbf{x}) = 8\pi G_N \sum_A z_A^2 \delta^{k'j'} p_{Ak'} \left(\partial_i \partial_{j'} u_A A(u_A) + \partial_i u_A \partial_{j'} u_A B(u_A)\right) . \tag{54}$$

IV. HAMILTONIAN

As is well-known, the Hamiltonian and momentum constraints reduce the bulk contribution over a spacelike hypersurface to a total derivative term, that may be expressed as a boundary contribution. Brown and York [26] showed that if one wishes to impose Dirichlet boundary conditions at infinity, the term takes the form of the trace of the extrinsic curvature of the boundary embedded in the spacelike hypersurface. In asymptotically anti-de Sitter spacetimes, this term can lead to extra divergent terms which may be cancelled by the addition of boundary counterterms [27, 28].

We begin by computing the trace of the extrinsic curvature of the 2-boundary B embedded in the constant time hypersurface Σ, using the perturbed metric (9) and assembling that into the boundary Hamiltonian

$$H = \frac{1}{8\pi G_N} \int_B \sqrt{\det\sigma}\, N(k-k_0) = \frac{1}{2\pi G_N} \int d^2x \frac{1}{z^3} \left(\phi + z\partial_z\phi + \cdots\right) , \tag{55}$$

where the subtraction term with $k_0 = -2$ ensures that the Hamiltonian vanishes for an unperturbed AdS background and the ... on the right hand side denotes higher order terms in the perturbation expansion. Noting the rapid fall-off of the trace perturbation of the metric (24), we may discard the higher order contributions $\mathcal{O}(\phi^2)$ near the boundary, and need only retain the $\phi + z\partial_z\phi$ contribution. This may be converted into a bulk integral

$$\begin{aligned} H &= \frac{1}{2\pi G_N}\int d^3x \partial_z \left(\frac{1}{z^3}\left(\phi + z\partial_z\phi\right)\right) \\ &= -\frac{1}{2\pi G_N}\int d^3x \sqrt{\det g^{(0)}} N^{(0)} \left(-3\phi + \nabla^2\phi\right) \,, \end{aligned} \tag{56}$$

with $g^{(0)}_{ij} = \delta_{ij}/z^2$ and $N^{(0)} = 1/z$. We may then compute the right hand side using the Hamiltonian constraint (5), evaluated at second order. We note (55) expresses the Hamiltonian as a boundary term, which can be matched directly with the conformal field theory time evolution operator acting on a set of boundary operator insertions. On the other hand, one also has a bulk interpretation of the same Hamiltonian in (56). In each case, the canonical variables will be a set of particle positions and momenta, along with the radiation degrees of freedom of the gravitational field π^{ij}_{TT} and h^{TT}_{ij}.

The resulting Hamiltonian can be expressed as a sum of three terms,

$$H = H_{rad} + H_A + H_{AB} \,, \tag{57}$$

where H_{rad} is the Hamiltonian of the radiation degrees of freedom, H_A is a set of terms dependent on the positions and momenta of single particles (and their couplings to the radiation terms) while H_{AB} denotes terms involving pairwise couplings to other fields. At higher order in G_N many-body interactions will appear.

The term quadratic in the radiation fields is

$$\begin{aligned} H_{rad} = &-\frac{1}{16\pi G_N}\int d^3x \sqrt{g^{(0)}} N^{(0)} \Big(g^{(0)ij} g^{(0)kl} h^{TT}_{ik} h^{TT}_{jl} - g^{(0)}_{ij} g^{(0)}_{kl} \pi^{ik}_{TT}\pi^{jl}_{TT} \\ &+ g^{(0)ij} g^{(0)kl} g^{(0)mn} \left(\frac{3}{4}\nabla_i h^{TT}_{km}\nabla_j h^{TT}_{ln} - \frac{1}{2}\nabla_i h^{TT}_{km}\nabla_n h^{TT}_{jl} + h^{TT}_{km}\nabla_i\nabla_j h^{TT}_{ln}\right)\Big) \,, \end{aligned} \tag{58}$$

while the term dependent on a single particle position/momentum takes the form

$$\begin{aligned} H_A = &\sum_A \left(\frac{\bar{m}_A}{z_A} - \frac{1}{2z_A\bar{m}_A} p^i_A p^j_A h^{TT}_{ij}(\mathbf{x}_A)\right) \\ &- \frac{1}{8\pi G_N}\int d^3x \sqrt{g^{(0)}} N^{(0)} \left(2\nabla^i\nabla^j\phi_1 h^{TT}_{ij} - \tilde{\pi}_{ij}\pi^{ij}_{TT}\right) \,, \end{aligned} \tag{59}$$

where

$$\bar{m}_A = \sqrt{g^{ij}_{(0)} p_{Ai} p_{Aj} + m_A^2}\,, \tag{60}$$

and we are using the shorthand notation from (14).

The expression for ϕ_1 has been given in the previous section (23), expressed as a sum over A, and $\nabla_b \nabla_d \phi$ may be straightforwardly computed using that formula. Likewise the first order evaluation of π_a^T and π^L of the previous section determines the integrand for the last term. The terms inside the integral in H_A reduce to total derivatives in flat spacetime, but survive in anti-de Sitter due to the nontrivial lapse function.

The final term to be assembled is H_{AB}. Here we run into the issue that terms arising from sources where $B = A$ give rise to divergent terms. In the flat spacetime case [2] this type of term is dealt with via Hadamard's method of partie finie [29]. Instead we note these terms are divergent and independent of the particle position. Therefore they may be removed by position independent counterterms in the Hamiltonian via renormalization. We proceed with the understanding that only the finite $B \neq A$ terms are to be considered. With that in mind,

$$\begin{aligned} H_{AB} = &\sum_{A,B\neq A} \frac{2}{z_A \bar{m}_A} \phi_{1B}(\mathbf{x}_A) g^{(0)ij}_{(0)}(x_A) p_{Ai} p_{Aj} \\ &- \frac{1}{16\pi G_N} \int d^3x \sqrt{g^{(0)}} N^{(0)} \left(96\phi_1^2 - 16\phi_1 \nabla^2 \phi_1 - 8(\nabla\phi_1)^2 - \tilde{\pi}^{ij}\tilde{\pi}_{ij}\right) \,. \end{aligned} \tag{61}$$

As before, the integrand in this expression is given explicitly in terms of the expression (23) for ϕ_1 and the first order evaluation of of π_a^T and π^L of the previous section.

This concludes the derivation of the Hamiltonian at linear order in G_N in the post-anti-de Sitter approximation, where the dynamical variables are the set of particle momenta and positions (x_A, p_A) and the transverse-traceless radiation variables (h_{ij}, π^{ij}_{TT}). The answer contains integrals that resemble one-loop integrals in quantum field theory in anti-de Sitter spacetime. It seems likely some or all of these integrals can be obtained in closed form, however we leave that issue for future work. We have checked that the integrals fall off as the separation of the particles increases, and vanish in the limit that a particle approaches the boundary. In the flat spacetime case, the corresponding integrals can be explicitly performed and are tabulated in appendix 3 of [30].

It should be noted the conformally flat metric we have chosen does not provide a set of global coordinates on AdS. Instead one must glue together a sequence of such coordinates patches to cover the (universal cover of) AdS. To fully explore this, one must then provide matching conditions for the particle momenta and positions as they transition from one coordinate patch to another. We will not attempt a detailed construction here.

V. CONCLUSIONS

In this work we have obtained the Hamiltonian for N-particles of arbitrary masses undergoing mutual gravitational interactions, at leading order in G_N and to all orders in momenta, akin to the so-called post-Minkowskian approximation to general relativity in asymptotically flat spacetime [2]. At this order, a pairwise interaction is present, in addition to couplings to gravitational waves. In AdS spacetime, generic particles remain at finite separation on average, so interactions will make finite contributions to time-averaged observables. This is rather different from a generic scattering process in asymptotically flat spacetime where the particles scatter off each other and move off to infinity. However, for special initial conditions in asymptotically flat spacetime that correspond to mutually bound orbits, the persistent pairwise interactions lead to chaotic behavior [13].

This provides strong evidence that there is a hard upper bound on the Lyapunov time governing typical scattering states in a conformal field theory dual to gravity in an asymptotically anti-de Sitter background. We expect this upper bound to change qualitatively once one goes to higher orders, due to the 3-body and beyond interactions that then begin to appear. However this will only serve to lower the bound. As argued in the introduction, this in turn provides strong evidence that the spectrum of primary operators in the CFT will exhibit level repulsion, in line with the Wigner surmise.

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Appendix A: Useful identities

In this appendix we have collected together some identities satisfied by the chordal distance variable (19) and its derivatives that are referred to in the main text.

$$g^{(0)ij}\,(\nabla_i u)\,(\nabla_j u) = u(u+2)\,, \tag{A1}$$

$$\nabla_i \nabla_j u = g^{(0)}_{ij}(u+1)\,, \tag{A2}$$

$$g^{(0)ij}\,(\nabla_i u)\,(\nabla_j \nabla_{j'} u) = (u+1)\nabla_{j'} u\,, \tag{A3}$$

$$\nabla_i \nabla_j \nabla_{j'} u = g^{(0)}_{ij} \nabla_{j'} u\,, \tag{A4}$$

$$\nabla^2\,(\nabla_i \nabla_{j'} u) = \nabla_i \nabla_{j'} u\,, \tag{A5}$$

$$\nabla^2\,(\nabla_i u \nabla_{j'} u) = 4\nabla_i u \nabla_{j'} u + 2(u+1)\nabla_i \nabla_{j'} u\,, \tag{A6}$$

$$g^{(0)ab}\,(\nabla_b u)\,(\nabla_a \nabla_i \nabla_{j'} u) = \nabla_i u \nabla_{j'} u\,, \tag{A7}$$

$$g^{(0)ab} \nabla_a\,(\nabla_i u \nabla_{j'} u)\,\nabla_b u = 2(1+u)\nabla_i u \nabla_{j'} u\,. \tag{A8}$$

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