Spinning up the spool:
Massive spinning fields in 3d quantum gravity

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Abstract

We show how to incorporate massive spinning fields into the Euclidean path integral of three-dimensional quantum gravity via its Chern-Simons formulation. The coupling of the spinning fields to gravity is captured by a Wilson spool, a collection of Wilson loops winding around closed paths of the geometry, and generalizes the proposal of [1,2]. We present a robust derivation of the Wilson spool by providing a new group-theoretic perspective of the quasinormal mode method for one-loop determinants. We test our proposal on Euclidean BTZ and S^3 backgrounds. We also evaluate explicitly the quantum corrections to the path integral on S^3, and report on how G_N and the mass are renormalized to leading order in perturbation theory.

Contents

1 Introduction

Low-dimensional gravity is an exciting arena to explore and test the gravitational path integral. In two and three spacetime dimensions, there is no propagating graviton and all of the effective degrees of freedom are long-range. A prime example of this phenomenon is the rewriting of pure Einstein gravity with a cosmological constant (of either sign) as a Chern-Simons gauge theory [3,4] which is the quintessential example of a topological field theory in three-dimensions. A full leveraging of this fact allows the exact evaluation of the gravitational path integral either about a saddle point [5–9] or as a non-perturbative sum over saddles [10–13]. While Chern-Simons gravity is not a UV-complete model of quantum gravity [12–14], its all-loop exactness provides strong tests for potential microscopic models in the spirit of e.g. [8,15–19].

One feature that is expected of a UV-complete model of quantum gravity is that it includes matter, in particular massive fields that couple to gravity. The manifest topological invariance that makes Chern-Simons theory so powerful as a description of the gravitational path integral also presents a challenge to incorporating matter. On a practical level, this is simple to illustrate: the action of a massive field theory minimally coupled

to a geometry involves both inverse metrics and metric determinants. The rewriting of these terms as Chern-Simons connections is highly non-linear and indicates that integrating out of the massive field will result in a non-local effective action. However, we can take inspiration from the general philosophy that the low-energy avatar of the worldline of a massive degree of freedom is a line-operator of the effective gauge theory [20].

This philosophy was made precise in [1] for massive scalar fields minimally coupled to gravity with a positive cosmological constant. The key result was the expression of the one-loop determinant of a massive scalar field coupled to a background metric, $g_{\mu\nu}$, as a gauge-invariant object of the Chern-Simons connections, $A_{L/R}$. That is,

$$Z_{\text{scalar}}[g_{\mu\nu}] = \exp \frac{1}{4} \mathbb{W}[A_L, A_R] \ . \tag{1.1}$$

The object $\mathbb{W}[A_L, A_R]$, coined the Wilson spool, is a collection of Wilson loop operators wrapped many times around cycles of the base geometry. The equality in (1.1) is expected to apply to three-dimensional gravity of either sign of cosmological constant, and this was explicitly shown for Euclidean black holes in Anti-de Sitter (i.e., Euclidean BTZ) and Euclidean de Sitter (i.e., the three-sphere S^3) in [1, 2]. It has also been upheld on $T\bar{T}$ deformations of AdS$_3$ [21]. The importance of (1.1) is not only conceptual, it is practical: it was additionally shown in [1] that certain "exact methods" in Chern-Simons theory (such as Abelianisation [22–24] and supersymmetric localization [25, 26]) extend to three-dimensional de Sitter (dS$_3$) Chern-Simons gravity with the Wilson spool inserted into the path integral. This allows a precise and efficient calculation of the quantum gravitational corrections to Z_{scalar} at any order of perturbation theory of Newton's constant, G_N.

An important element excluded from the dictionary established in [1] is the description of massive fields with spin. The representation theory of dS$_3$ spacetime admits a series of representations describing single-particle states with both mass and integer spin [27,28]. It is important to know how the field theory of those particles fits into quantum gravity about the dS$_3$ background. This consideration is more than pedestrian: we expect the massive states of a UV description of quantum gravity to carry spin and it is sensible to organize these states into the representation theory of the low energy effective theory. In [2] a Wilson spool describing massive spinning fields on Euclidean BTZ was conjectured and shown to reproduce the correct one-loop determinants as $G_N \to 0$, however, no principled derivation was provided there. More importantly, it was shown that a naïve generalization of the Wilson spool does not correctly reproduce the physics of spinning fields on S^3.

In this paper we address this mismatch by revisiting the derivation of $\mathbb{W}[A_L, A_R]$. The original derivation in [1] to obtain (1.1) used some of the conditions behind the quasinormal mode method to evaluate one-loop determinants developed in [29]. Here, we give a robust derivation where we cast all aspects of the method in terms of conditions that rely solely on group-theoretic concepts. These conditions apply to gravity with either sign of cosmological constant, Λ. We show in explicit examples how from the group-theoretic picture one can write the one-loop determinant in terms of (A_L, A_R) and representations of the algebra. This then gives the Wilson spool, which describes the minimal coupling of massive spinning fields to gravity.

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To be specific, we state our main result, the generalization of (1.1) for massive spinning fields, as the following. Consider the local path integral,[1] $Z_{\Delta,\mathsf{s}}$, of a spin-s field $\Phi_{\mu_1\mu_2\dots\mu_s}$ with mass

$$\frac{m^2}{\Lambda} = (\Delta + \mathsf{s} - 2)(\mathsf{s} - \Delta) \ , \tag{1.2}$$

minimally coupled to a metric geometry, (M_3, g_{M_3}), where M_3 is topologically either Euclidean BTZ or Euclidean dS$_3$. Then, we propose

$$\log Z_{\Delta,\mathsf{s}}[g_{M_3}] = \frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \ , \tag{1.3}$$

where

$$\mathbb{W}_{j_L,j_R}[A_L, A_R] = \frac{i}{2}\int_{\mathcal{C}}\frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\left(1 + \mathsf{s}^2\sin^2\left(\frac{\alpha}{2}\right)\right)\times$$
$$\sum_{\mathsf{R}_L\otimes\mathsf{R}_R}\mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_\gamma A_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_\gamma A_R}\right) \ . \tag{1.4}$$

The details of $\mathbb{W}_{j_L,j_R}$ will be made explicit as we continue, however let us briefly summarize the parts appearing in (1.4). The Chern-Simons connections, $A_{L/R}$, are related to the metric, g_{M_3}, in (1.3) through the usual Chern-Simons gravity dictionary and they are integrated over a non-trivial cycle, γ, of the base geometry. The representations, $\mathsf{R}_{L/R}$, appearing in the Wilson loops are summed over a set determined by the mass and spin, (Δ, s), of (1.3) and labelled by weights (j_L, j_R). Lastly the parameter α is integrated along a contour, $\mathcal{C}$, determined by a regularization scheme appropriate for the sign of cosmological constant. The ultimate effect of the α integral is to implement a "winding" of the Wilson loop operators around γ; this occurs through the summing the residues of the poles of its measure (as well as any of representation traces themselves). We coin the above object, (1.4), the *spinning Wilson spool*.

In the remainder of this paper we will establish the results upholding (1.3) and (1.4) as well as the necessary details to utilize (1.4). Much of this analysis will be done in the context of a positive cosmological constant, where the difference between the scalar and spinning Wilson spools are most apparent. In Sec. 2 we will establish the representation theory necessary for describing massive spin-s particles in terms of the Euclidean isometry group $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. In Sec. 3 we will utilize this representation theory to build the spinning Wilson spool, (1.4), when the background is classically S^3. This follows from a re-organization of the quasinormal mode method for one-loop determinants [29] in terms of representation theory following a set of broad guiding principles, which we establish in Sec. 3.1 and apply to Chern-Simons gravity in Sec. 3.2. In Sec. 4 we propose the spinning Wilson spool as an operator, allowing us to insert it off-shell into the Chern-Simons path integral. The full power of this is manifest in the context of a dS$_3$ background where we show that the Chern-Simons path integral with a spool insertion reduces to an ordinary integral that can be evaluated to any order in G_N perturbation theory. We illustrate

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[1]Including any additional Stückelberg fields to fix its invariances and associated ghosts.

how this can be interpreted as either a renormalization of the particle mass or Newton's constant, depending on choice of renormalization condition. Lastly in Sec. 5, to establish the scope of the results in Sec. 3, we will also illustrate how (1.4) also holds when the background is Euclidean BTZ *mutatis mutandis*. We conclude the paper in Sec. 6 with potential uses for the spinning Wilson spool as well as a discussion of open problems. Details on the dictionary relating Chern-Simons theory and gravity as well as details on the spin-s quasinormal modes on S^3 are found in App. A and App. B, respectively.

2 Non-standard spinning representations of $\mathfrak{su}(2)$

In this section, we will describe single-particle states of massive spin-s fields living on dS$_3$, with de Sitter radius ℓ_{dS}, as representations of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. The guiding principle of the construction is to mimic the unitary representations of the Lorentzian dS$_3$ isometry group, $\mathfrak{so}(1,3)$. In [1] this was done for cases with $s = 0$, and here, following the same arguments and conventions, we extend that construction to include spin.

Unitary representations of $\mathfrak{so}(1,3)$ are labeled by a conformal dimension Δ (the eigenvalue of the dilatation operator D) and a spin s (the eigenvalue of the spin operator iM) and come in two series:[2]

$$
\begin{aligned}
\Delta &= 1 + \nu , & \nu &\in (-1,1) , & s &= 0 , \\
\Delta &= 1 - i\mu , & \mu &\in \mathbb{R} , & s &\in \mathbb{Z} .
\end{aligned} \tag{2.1}
$$

The first line is the *complementary series* describing light scalars with masses $\ell_{\mathrm{dS}}^2 m^2 = 1 - \nu^2$ while the second line is the *spinning principal series* describing spinning particles with masses $\ell_{\mathrm{dS}}^2 m^2 = (s-1)^2 + \mu^2$.

To connect this to $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, let us introduce some basic aspects of $\mathfrak{su}(2)$. The algebra is generated by L_3 and $L_\pm$, where

$$
[L_3, L_\pm] = \pm L_\pm , \qquad [L_+, L_-] = 2L_3 . \tag{2.2}
$$

The Casimir of the representation is $c_2^{\mathfrak{su}(2)} = \frac{1}{2}(L_+L_- + L_-L_+) + L_3^2$. Representations of $\mathfrak{su}(2)$ will be characterized by j, the eigenvalue of L_3. The $\mathfrak{so}(1,3)$ algebra shares a complexification with $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, such that the dilatation and spin generators can be identified with the Cartan elements of $\mathfrak{su}(2)$ as

$$
D = -L_3 - \bar{L}_3 , \qquad M = iL_3 - i\bar{L}_3 , \tag{2.3}
$$

respectively. Similar relations follow for the remaining generators, which can be found in [1]. In (2.3) we have distinguished the generators of $\mathfrak{su}(2)_R$ from those of $\mathfrak{su}(2)_L$ by an overbar. Within this complexification, the quadratic Casimir of $\mathfrak{so}(1,3)$ is equal to the sum of the $\mathfrak{su}(2)$ Casimirs:

$$
c_2^{\mathfrak{so}(1,3)} = -2c_2^{\mathfrak{su}(2)_L} - 2c_2^{\mathfrak{su}(2)_R} . \tag{2.4}
$$

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[2]See [1] for a basic introduction on $\mathfrak{so}(1,3)$ representation theory and for our notation here. See [27, 28, 30–32] for a more thorough treatment.

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Equation (2.3) suggests that we need to look for $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ representations with

$$\Delta = -j_R - j_L \ , \qquad\qquad \mathsf{s} = j_R - j_L \ , \qquad\qquad (2.5)$$

where j_L is the eigenvalue of L_3 in $\mathfrak{su}(2)_L$, and similarly for j_R.

The relations (2.5), together with (2.1), indicate to us that we will need to construct representations with continuous and complex eigenvalues (j_L, j_R). These do not fall into the standard finite-dimensional representation theory of $\mathfrak{su}(2)$. However in [1, 33] it was shown how to construct an alternative inner product in $\mathfrak{su}(2)$ (or equivalently, an alternative notion of Hermitian conjugation) allowing for highest-weight representations with a continuous and complex weight while preserving norm-positivity. Such representations were coined *non-standard representations* in the latter reference. These representations are built from a highest weight state, $|j,0\rangle$, satisfying

$$L_3|j,0\rangle = j|j,0\rangle \ , \qquad\qquad L_+|j,0\rangle = 0 \ , \qquad\qquad (2.6)$$

and $j \in \mathbb{C}$ is the weight of the representation. Acting with lowering operators defines

$$|j,p\rangle = N_{j,p} \left(L_-\right)^p |j,0\rangle \ , \qquad\qquad (2.7)$$

for some normalizations $N_{j,p}$ that we will determine shortly.

The key ingredient distinguishing the non-standard representations is a map, $\mathcal{S}$, between highest weight representations

$$\mathcal{S}|j,p\rangle = |j^*,p\rangle \ , \qquad\qquad (2.8)$$

where j^* is the complex conjugate of j. This map is utilized in Hermitian conjugation in the following way:

$$L_3^\dagger := \mathcal{S}^{-1} L_3 \mathcal{S} \ , \qquad L_\pm^\dagger := -\mathcal{S}^{-1} L_\mp \mathcal{S} \ . \qquad\qquad (2.9)$$

Note that $j \overset{\mathcal{S}}{\to} j^*$ is consistent with the action of $L_3^\dagger$. In [1], norm-positive inner products were constructed for non-standard representations with $j = -\frac{1}{2}(1+\nu)$ or $j = -\frac{1}{2}(1-i\mu)$ (and $\mu, \nu \in \mathbb{R}$) which relate to the scalar complementary and principal series with $\mathsf{s} = 0$. Here we will relax this condition and show that a norm-positive inner product can be constructed for *any* complex j.

We will first determine the normalizations $N_{j,p}$. We note that

$$\langle j, p-1|L_+|j,p\rangle = \frac{N_{j,p}}{N_{j,p-1}} p(2j+1-p)\langle j, p-1|j,p-1\rangle \ , \qquad\qquad (2.10)$$

which can be seen from replacing $L_+ L_-$ with $c_2^{\mathfrak{su}(2)} + L_3 - L_3^2$ in the matrix element. Alternatively, utilizing (2.9) and (2.8), this same matrix element is

$$\langle j, p-1|L_+|j,p\rangle = \langle j, p|L_+^\dagger|j, p-1\rangle^* = -\left(\frac{N_{j^*,p-1}}{N_{j^*,p}}\right)^* \langle j, p|j,p\rangle \ . \qquad\qquad (2.11)$$

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If we assume that we can set $\langle j, p-1 | j, p-1 \rangle = 1$ and also $\langle j, p | j, p \rangle = 1$, then (2.10) and (2.11) imply

$$\left(\frac{N_{j^*,p-1}}{N_{j^*,p}} \right)^* = p(p-2j-1) \left(\frac{N_{j,p}}{N_{j,p-1}} \right) . \qquad (2.12)$$

To solve this constraint, we write

$$\alpha_{j,p} = \mathrm{Arg}\,(N_{j,p}) \; , \qquad \phi_{j,p} = \mathrm{Arg}\,(p - 2j - 1) \; , \qquad (2.13)$$

leading to recurrence relations

$$\frac{|N_{j^*,p-1}|}{|N_{j^*,p}|} = p|p - 2j - 1| \frac{|N_{j,p}|}{|N_{j,p-1}|} \; ,$$
$$\alpha_{j^*,p} - \alpha_{j^*,p-1} = \alpha_{j,p} - \alpha_{j,p-1} + \phi_{j,p} \mod(2\pi) \; . \qquad (2.14)$$

There is a lot of freedom in solving these recurrence relations. We will choose a particular solution by fixing

$$|N_{j,p}| = |N_{j^*,p}| \; , \qquad \alpha_{j,0} = \alpha_{j^*,0} = 0 \; , \qquad \alpha_{j,p} = -\alpha_{j^*,p} \; \forall \, p \; . \qquad (2.15)$$

This sets

$$\frac{|N_{j,p-1}|}{|N_{j,p}|} = \frac{|N_{j^*,p-1}|}{|N_{j^*,p}|} = \sqrt{p|p - 2j - 1|} \qquad (2.16)$$

and[3]

$$\alpha_{j,p} = -\frac{1}{2} \sum_{p'=1}^{p} \phi_{j,p'} \; . \qquad (2.17)$$

For concreteness we summarize the generator actions on this representation as

$$L_3 | j, p \rangle = (j - p) | j, p \rangle \; ,$$
$$L_- | j, p \rangle = e^{i\phi_{j,p}/2} \sqrt{(p+1)|p - 2j|} \, | j, p+1 \rangle \; ,$$
$$L_+ | j, p \rangle = -e^{i\phi_{j,p}/2} \sqrt{p|p - 2j - 1|} \, | j, p-1 \rangle \; , \qquad (2.18)$$

with the action on $| j^*, p \rangle$ obtained by simply replacing $j \to j^*$ in the above formulas. We can now show norm-positivity by induction starting with $\langle j, 0 | j, 0 \rangle = 1$. Let us investigate the first descendant state:

$$\begin{aligned}
|L_- | j, 0 \rangle|^2 &= -\langle j, 0 | \mathcal{S}^{-1} L_+ \mathcal{S} L_- | j, 0 \rangle \\
&= e^{i\phi_{j,1}/2 + i\phi_{j^*,1}/2} \sqrt{|2j|} \sqrt{|2j|} \langle j, 0 | j, 0 \rangle \\
&= |2j| > 0 \; .
\end{aligned} \qquad (2.19)$$

---

[3]Note that $\phi_{j^*,p} = -\phi_{j,p}$.

Similarly norm-positivity of $|j,p\rangle$ induces norm-positivity of $|j,p+1\rangle$[4]:

$$
\begin{aligned}
|L_-|j,p\rangle|^2 &= -\langle j,p|\mathcal{S}^{-1}L_+\mathcal{S}L_-|j,p\rangle \\
&= e^{i\phi_{j,p+1}/2+i\phi_{j^*,p+1}/2}\sqrt{(p+1)|p-2j|}\sqrt{(p+1)|p-2j|}\langle j,p|j,p\rangle \\
&= (p+1)|p-2j|\langle j,p|j,p\rangle > 0 \ .
\end{aligned}
\tag{2.20}
$$

This establishes norm-positivity for all states in the representation. We additionally note that these representations have well-defined characters

$$
\chi_j(z) = \mathrm{Tr}_j\left(e^{i2\pi z L_3}\right) = \sum_{p=0}^{\infty} e^{i2\pi z(j-p)} = \frac{e^{i\pi(2j+1)z}}{2i\sin(\pi z)} \ .
\tag{2.21}
$$

In [1] the restrictions of the highest-weights, $j = -\frac{1}{2}(1+\nu)$ or $j = -\frac{1}{2}(1-i\mu)$, followed from Hermiticity of the $\mathfrak{su}(2)$ Casimirs, separately. In this paper we will take a more Lorentzian perspective, and impose reality of the $\mathfrak{so}(1,3)$ Casimir at the level of (2.4); doing so ties the $\mathfrak{su}(2)_{L/R}$ representations together and the highest weights take the generic form

$$
j_L = -\frac{1}{2}(1+\mathsf{s}-i\mu) \ , \qquad j_R = -\frac{1}{2}(1-\mathsf{s}-i\mu) \ .
\tag{2.22}
$$

Let us briefly show this now. Assuming that j_L and j_R are generically complex the condition for reality of the $\mathfrak{so}(1,3)$ Casimir is that

$$
\mathrm{Im}\left(j_L(j_L+1)\right) = -\mathrm{Im}\left(j_R(j_R+1)\right) \ ,
\tag{2.23}
$$

or equivalently

$$
\mathrm{Im}(j_L)\left(1+2\mathrm{Re}(j_L)\right) = -\mathrm{Im}(j_R)\left(1+2\mathrm{Re}(j_R)\right) \ .
\tag{2.24}
$$

If both sides are simultaneously zero then we must have either $j \in \mathbb{R}$ or $j \in -\frac{1}{2}+i\mathbb{R}$ (for either j_L or j_R) which lead to the complimentary and principal-type representations discussed in [1]. More generally there will be a family of highest weight solutions satisfying (2.24). However, if we further insist[5] that $j_L - j_R \in \mathbb{R}$ then it must be the case that

$$
\mathrm{Im}(j_L) = \mathrm{Im}(j_R) \ , \qquad\qquad \mathrm{Re}(j_L) + \mathrm{Re}(j_R) = -1 \ ,
\tag{2.25}
$$

which lead to the highest-weights appropriate for the spinning principal series, (2.22). We will refer to such representations as the *spinning principal-type representations*.

As noted already in [1], our non-standard characters can be massaged into the suggestive Lorentzian form via

$$
\chi_{j_L}(z_L)\chi_{j_R}(z_R) + \chi_{j_L^*}(z_L)\chi_{j_R^*}(z_R) = \frac{w^{\mathsf{s}}q^{\Delta} + w^{-\mathsf{s}}q^{\bar{\Delta}}}{(1-w^{-1}q)(1-wq)} \ ,
\tag{2.26}
$$

---

[4]We are assuming j is a generic complex number and ignoring the potential for possible null states when $j \in \frac{1}{2}\mathbb{N}_0$. Of course, in these cases the representation simply terminates and we recover the standard, finite-dimensional, representations of $\mathfrak{su}(2)$.

[5]At this point we will impose this by hand; we will see later that in the quantum theory only representations with $j_L - j_R \in \mathbb{Z}$ contribute to the matter one-loop determinant.

where $q = e^{-i\pi(z_L + z_R)}$ and $w = e^{i\pi(z_L - z_R)}$ and we have identified (2.5) as well as $\bar{\Delta} = 2 - \Delta$. This matches the Harish-Chandra character, $\text{Tr}\left(q^D w^{iM}\right)$, for the $\mathfrak{so}(1,3)$ spinning principal series [31].

3   Spinning spool on S^3

In this section, we present the construction of the Wilson spool for massive spin-s fields on S^3. That is, we will derive an expression for the one-loop determinant of these fields on S^3 in terms of the representations of $\mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R$ constructed in the previous section.

To start, let us describe the path integral of a single massive spin-s field, with no self-interactions. The local partition function for this theory contains a symmetric spin-s tensor, $\Phi_{\mu_1 \mu_2 \dots \mu_s}$, as well as a tower of Stückelberg fields which enforce that $\Phi_{\mu_1 \mu_2 \dots \mu_s}$ is transverse and traceless [34]:

$$\nabla^\nu \Phi_{\nu \mu_2 \dots \mu_s} = \Phi^\nu{}_{\nu \mu_3 \dots \mu_s} = 0 \ . \tag{3.1}$$

As emphasized in [8], on a compact manifold the path integral over symmetric, transverse, traceless (STT) tensors leads to non-local divergences which cannot be canceled by local counterterms. This path integral must be compensated by the path integral over the Stückelberg fields and ghosts which leave behind a finite product from integrating over normalizable zero modes. To that end, we write

$$Z_{\Delta,s} = Z_{\text{zero}} Z_{\text{STT}} \ , \tag{3.2}$$

where

$$Z_{\text{STT}} = \int [\mathcal{D}\Phi_{\mu_1 \mu_2 \dots \mu_s}]_{\text{STT}} e^{-\frac{1}{2}\int \Phi\left(-\nabla^2_{(s)} + \ell^2_{\text{dS}} \bar{m}^2_s\right)\Phi} \ . \tag{3.3}$$

Above $\nabla^2_{(s)}$ is the Laplace-Beltrami operator

$$\left[\nabla^2_{(s)}\Phi\right]_{\mu_1 \mu_2 \dots \mu_s} = \nabla_\nu \nabla^\nu \Phi_{\mu_1 \mu_2 \dots \mu_s} \ , \tag{3.4}$$

and $\bar{m}^2_s$ is an "effective mass"

$$\ell^2_{\text{dS}} \bar{m}^2_s = \ell^2_{\text{dS}} m^2 + 3s - s^2 \ , \tag{3.5}$$

where we recall that m^2 is the standard mass parameter in dS$_3$ [8], and related to the representation theory of Sec. 2 via

$$\ell^2_{\text{dS}} m^2 = (\Delta + s - 2)(s - \Delta) \ . \tag{3.6}$$

The zero mode contribution in (3.2) follows from counting conformal Killing tensor modes on S^3 and is given by [8][6]

$$Z_{\text{zero}} = \left[(\Delta - 1)(\bar{\Delta} - 1)\right]^{\frac{s^2}{2}} \prod_{n=0}^{s-2} \left[(\Delta + n)(\bar{\Delta} + n)\right]^{-(n+1+s)(n+1-s)} \ , \tag{3.7}$$

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[6]This is true for $s \geq 2$ while for $s = 0, 1$ the product over n is replaced by 1.

where we recall that $\bar{\Delta} = 2 - \Delta$.

In the following, we will show that

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{4}\mathbb{W}_{j_L, j_R} , \tag{3.8}$$

for fields on S^3. That is, we will express $Z_{\Delta,\mathsf{s}}$ as a function of the Chern-Simons connections which we will explicitly construct utilizing the non-standard representation theory of Sec. 2. While our construction takes place on a fixed classical background, we will see that $\mathbb{W}_{j_L, j_R}$ is an integral over gauge-invariant Wilson loop operators and naturally generalizes into an off-shell operator that can be inserted into the Chern-Simons path integral, which we will discuss in Sec 4.

The roadmap to derive (3.8) is as follows. We will first focus on Z_{STT}. This determinant can be evaluated via the method of quasinormal modes pioneered by Denef, Hartnoll, and Sachdev (DHS) [29]. We will adapt this method such that each component has a group theoretic interpretation: we will show how defining properties of quasinormal modes can be translated to conditions on the representation of the fields. This follows [1], however, for spinning fields, we will take particular care with the role of global conditions (i.e., Euclidean solutions are regular and single-valued) in isolating physical contributions to the quasinormal mode product. The additional contribution of Z_{zero}, which is not part of the quasinormal mode product but crucial for maintaining locality of $Z_{\Delta,\mathsf{s}}$, will permit a Schwinger parameterization of $\log Z_{\Delta,\mathsf{s}}$, regularized by an $i\varepsilon$ prescription. This will organize the quasinormal mode sum into an integral over representation traces of the background holonomies. From this follows our main result.

3.1   A group theory perspective on S^3 quasinormal modes

As the first step in our construction, we will recast the functional determinant

$$Z_{\mathrm{STT}} = \det\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\mathrm{dS}}\bar{m}^2_{\mathsf{s}}\right)^{-\frac{1}{2}} \tag{3.9}$$

in $\mathfrak{su}(2)$ representation theoretic language. We recall that the DHS method instructs us to treat Z^2_{STT} as meromorphic function of Δ. Then, up to a holomorphic function, Z^2_{STT} is equal to the product containing the same zeros and poles. Here Z_{STT} only has poles on states satisfying $(-\nabla^2_{(\mathsf{s})} + \bar{m}^2_{\mathsf{s}})\Phi_{\mu_1\mu_2...\mu_\mathsf{s}} = 0$. These are precisely the spin-s quasinormal modes. In App. B, we explicitly compute these modes and their product to obtain Z_{STT} directly. Here we will reinterpret these modes in terms of $\mathfrak{su}(2)$ representation theory to obtain an expression natural to the Chern-Simons theory formulation of gravity.

We note that the isometry algebra of the three-sphere is generated by two mutually commuting sets of $\mathfrak{su}(2)$ vector fields $\{\zeta_a\}$ and $\{\bar{\zeta}_b\}$ which are the infinitesimal left and right group actions acting on $S^3 \simeq SU(2)$. On spin-s STT tensors the Casimirs of their Lie derivatives, $\{\mathcal{L}_{\zeta_a}\}$ and $\{\mathcal{L}_{\bar{\zeta}_b}\}$, act as the Laplace-Beltrami operator [35]:

$$-2\delta^{ab}\left(\mathcal{L}_{\zeta_a}\mathcal{L}_{\zeta_b} + \mathcal{L}_{\bar{\zeta}_a}\mathcal{L}_{\bar{\zeta}_b}\right)\Phi_{\mu_1\mu_2...\mu_\mathsf{s}} = \left[\nabla^2_{(\mathsf{s})} - \mathsf{s}(\mathsf{s}+1)\right]\Phi_{\mu_1\mu_2...\mu_\mathsf{s}} . \tag{3.10}$$

Hence we can write (3.9) suggestively as

$$Z_{\text{STT}} = \det\left(2c_2^{\mathfrak{su}(2)_L} + 2c_2^{\mathfrak{su}(2)_R} + \Delta(2-\Delta) - \mathsf{s}^2\right)^{-\frac{1}{2}} . \tag{3.11}$$

Following the DHS methodology, we then expect Z_{STT}^2 to have pole contributions from states in $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ representations satisfying

$$\left(-2c_2^{\mathfrak{su}(2)_L} - 2c_2^{\mathfrak{su}(2)_R}\right)|\psi\rangle = \left[\Delta(2-\Delta) - \mathsf{s}^2\right]|\psi\rangle . \tag{3.12}$$

This is precisely the condition satisfied by the non-standard representations constructed in Sec. 2 with highest weights $(j_L, j_R) = (-\frac{\Delta+\mathsf{s}}{2}, -\frac{\Delta-\mathsf{s}}{2})$. We are interested in the poles in Z_{STT}^2 arising from weights of the representations $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ as we continue Δ in the complex plane.[7] In principle we should consider all representations that satisfy (3.12), so for a given (Δ, s), we also encounter poles associated to highest weight representations arrived at by sending $\Delta \to \bar{\Delta} = 2 - \Delta$ as well as $\mathsf{s} \to -\mathsf{s}$.[8] If we define $(j_L, j_R) = (-\frac{\Delta+\mathsf{s}}{2}, -\frac{\Delta-\mathsf{s}}{2})$ then we will denote

$$(\bar{j}_L, \bar{j}_R) = \left(-\frac{\bar{\Delta}+\mathsf{s}}{2}, -\frac{\bar{\Delta}-\mathsf{s}}{2}\right) , \tag{3.13}$$

while $\mathsf{s} \to -\mathsf{s}$ is equivalent to $j_L \leftrightarrow j_R$. We will then have pole contributions from any of the representations appearing in

$$\mathcal{R}_{\Delta,\mathsf{s}} = \left\{\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}, \mathsf{R}_{\bar{j}_L} \otimes \mathsf{R}_{\bar{j}_R}, \mathsf{R}_{j_R} \otimes \mathsf{R}_{j_L}, \mathsf{R}_{\bar{j}_R} \otimes \mathsf{R}_{\bar{j}_L}\right\} . \tag{3.14}$$

We make a special note that for scalar representations ($j_L = j_R = j$) it is sufficient to consider the smaller set

$$\mathcal{R}_{\Delta,\text{scalar}} = \left\{\mathsf{R}_j \otimes \mathsf{R}_j, \mathsf{R}_{\bar{j}} \otimes \mathsf{R}_{\bar{j}}\right\} \tag{3.15}$$

as in [1].

The "mass-shell condition" (3.12) is only a necessary condition to contribute a physical pole to Z_{STT}^2. Functional determinants come with boundary and regularity conditions on their functional domain and we must impose these on weight spaces satisfying (3.12) to obtain a physical answer. We will state these up-front in an $\mathfrak{su}(2)$ natural language as the following:

Condition I. Single-valued solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must return to itself under parallel transport around any closed cycle in the Euclidean manifold.

Condition II. Globally regular solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must be globally regular on the Euclidean manifold. When the base space is homogeneous this means $\mathsf{R}_L \otimes \mathsf{R}_R$ lifts from a representation of the *isometry algebra* to a representation of the *isometry group*.

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[7]It will be important as we progress to take special care of the cases when Δ is such that $j_{L/R} \in \frac{1}{2}\mathbb{N}$; in these cases the weight spaces terminate discontinuously to finite-dimensional representations.

[8]This is consistent with the Lorentzian picture: a spin-s field is built out of $\mathfrak{so}(1,3)$ representations labelled by both $(\Delta, \pm\mathsf{s})$ while the (Δ, s) representation is isomorphic to that labelled by $(\bar{\Delta}, -\mathsf{s})$ through the $\mathfrak{so}(1,3)$ shadow map [32].

Let us first expand upon **Condition I** for spin-s fields on S^3. A field Φ living in a representation $\mathsf{R}_L \otimes \mathsf{R}_R$ is parallel transported around a cycle, γ, through the background connections:

$$\Phi_f = \mathsf{R}_L \left[\mathcal{P} \exp \left(\oint_\gamma a_L \right) \right] \Phi_i \, \mathsf{R}_R \left[\mathcal{P} \exp \left(-\oint_\gamma a_R \right) \right] . \tag{3.16}$$

When $a_{L/R}$ are flat this conjugation is trivial. However for the backgrounds appropriate for describing the S^3 metric geometry, the background connections take non-trivial holonomies

$$\mathcal{P} \exp \left(\oint_\gamma a_L \right) = u_L^{-1} \, e^{i2\pi L_3 \mathsf{h}_L^{(\gamma)}} \, u_L^{-1} \, , \qquad \mathcal{P} \exp \left(\oint_\gamma a_R \right) = u_R^{-1} \, e^{i2\pi \bar{L}_3 \mathsf{h}_R^{(\gamma)}} \, u_R^{-1} \, , \tag{3.17}$$

when γ wraps one of two lines on the base S^3 [1]. These lines are Hopf-linked and Wick-rotate to the coordinate positions of the static patch origin and horizon. They yield respective holonomies

$$\gamma_{\text{orig.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1, 1) \, , \qquad \gamma_{\text{hor.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1, -1) \, . \tag{3.18}$$

More details can be found in App. A.1. The salient point is that a single-valued field will satisfy

$$\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R \in \mathbb{Z} \, , \tag{3.19}$$

for each of the two sets of holonomies in (3.18) and for all weights (λ_L, λ_R) in the representation $\mathsf{R}_L \otimes \mathsf{R}_R$.

From cycles wrapping the origin weights must satisfy

$$\lambda_L - \lambda_R \in \mathbb{Z} \tag{3.20}$$

to contribute a pole to Z_{STT}^2. Weights of a highest-weight representation $\mathsf{R}_{j_{L/R}}$ necessarily take the form

$$\lambda_{L/R} = j_{L/R} - p_{L/R} \, , \qquad p_{L/R} \in \mathbb{N}_0 \, , \tag{3.21}$$

simply via the structure of the $\mathfrak{su}(2)$ algebra. Single-valuedness around the static patch origin then requires

$$j_L - j_R \in \mathbb{Z} \qquad \Leftrightarrow \qquad \mathsf{s} \in \mathbb{Z} \, . \tag{3.22}$$

This condition is the same for all other representations in $\mathcal{R}_{\Delta,\mathsf{s}}$. We pause here to note that while the representation theory in Sec. 2 only relies on $j_L - j_R \in \mathbb{R}$, we now see that only fields with quantized spin can contribute physical poles to Z_{STT}^2. We will thus fix $\mathsf{s} \in \mathbb{Z}$ and consider the analytic structure of Z_{STT}^2 as a function of Δ. This analytic structure is constrained by single-valuedness around the static patch horizon, which requires

$$\lambda_L + \lambda_R \in \mathbb{Z} \, . \tag{3.23}$$

We will return to this condition shortly.

We now address **Condition II**, that configurations contributing to Z_{STT}^2 are globally regular. Without loss of generality we will state this for $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R} \in \mathcal{R}_{\Delta,\mathsf{s}}$. For the

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S^3 background in question the isometry group acts transitively. Thus regularity at a point guarantees global regularity as long as the isometry group, $SU(2)_L \times SU(2)_R$, acts faithfully on the field in question: that is, $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ lifts to a representation of the isometry group. The Peter-Weyl theorem states that these must be finite-dimensional representations of $\mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R$ (see, for example, [36]), where such representations have weights (3.21) satisfying

$$\lambda_{L/R} = j_{L/R} - p_{L/R} , \qquad 0 \leq p_{L/R} \leq 2j_{L/R} , \qquad j_{L/R} \in \frac{1}{2}\mathbb{N}_0 . \tag{3.24}$$

To be clear about interpretation: the DHS method instructs us to consider the structure of Z_{STT}^2 as Δ continues to the complex plane. The mass-shell condition, (3.12), then instructs us to consider representations with generically complex highest weights, $j_{L/R} \in \mathbb{C}$. Such representations are non-standard and infinite-dimensional. However **Condition II** simply tells us that the poles of Z_{STT}^2 are located at $\Delta \in \mathbb{Z}_{\leq -\mathsf{s}}$ and the orders of these poles are correctly counted not by weights of an infinite dimensional representation but instead by (3.24). In this counting we notice that weights of finite dimensional representations of $SU(2)$ are centered about zero and so for any weight satisfying $\lambda_L + \lambda_R = N > 0$ there is a corresponding weight with $\lambda_L + \lambda_R = -N$. Thus for the purposes of counting the number of weights contributing to a particular pole, we can restate (3.23) as

$$\lambda_L + \lambda_R = |N| , \qquad N \in \mathbb{Z} \tag{3.25}$$

as a necessary condition for incorporating **Condition II**.

We pause to note that for minimally coupled scalar fields (3.25) is also sufficient to imply **Condition II**. Thus the scalar one-loop determinant can be written as

$$Z_{\mathrm{scalar}} = \prod_{\substack{(\lambda_L, \lambda_R) \\ \in \mathsf{R}_j \otimes \mathsf{R}_j}} \prod_{N \in \mathbb{Z}} \left(|N| - \lambda_L - \lambda_R\right)^{-1/2} \times \prod_{\substack{(\bar{\lambda}_L, \bar{\lambda}_R) \\ \in \mathsf{R}_{\bar{j}} \otimes \mathsf{R}_{\bar{j}}}} \prod_{\bar{N} \in \mathbb{Z}} \left(|\bar{N}| - \bar{\lambda}_L - \bar{\lambda}_R\right)^{-1/2} , \quad (3.26)$$

where we have written explicitly the product over the two representations appearing in $\mathcal{R}_{\Delta, \mathrm{scalar}}$. From here the expression of Z_{scalar} as a Wilson spool follows the procedure in [1].

For massive spin-s fields, (3.25) is no longer sufficient and we must impose additional constraints to reproduce Z_{STT}^2. For a given $|N|$ in (3.25), **Condition II** additionally implies the weights $\lambda_{L/R} = j_{L/R} - p_{L/R}$ must satisfy

$$p_L \geq -|N| - \mathsf{s} , \qquad p_R \geq -|N| + \mathsf{s} . \tag{3.27}$$

While the first of these is always satisfied (for positive s) the second is an additional constraint on counting the order of the poles appearing in Z_{STT}^2 and is only non-trivial when $|N| \leq \mathsf{s}$. For these $2\mathsf{s} + 1$ cases we observe that $\tilde{p}_R = p_R + |N| - \mathsf{s} \geq 0$ and (3.25) is equivalently written

$$j_L + j_R - p_L - \tilde{p}_R = \mathsf{s} , \qquad p_L, \tilde{p}_R \geq 0 . \tag{3.28}$$

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We can thus treat this as a condition on the weights $(\lambda_L, \tilde{\lambda}_R) = (j_L - p_L, j_R - \tilde{p}_R)$ of highest-weight representations $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ and for each pole arising from this condition being satisfied, it arises $2\mathsf{s} + 1$ times.

Applying this same procedure to all representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$ we arrive at

$$Z_{\text{STT}} = \prod_{\mathcal{R}_{\Delta,\mathsf{s}}} \prod_{(\lambda_L,\lambda_R)} \left((\mathsf{s} - \lambda_L - \lambda_R)^{-\frac{2\mathsf{s}+1}{2}} \prod_{\substack{N \in \mathbb{Z}, \\ |N| > \mathsf{s}}} (|N| - \lambda_L - \lambda_R)^{-\frac{1}{2}} \right) , \qquad (3.29)$$

where the first product is understood to take the product over all pairs $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}_{\Delta,\mathsf{s}}$ and the second product is taken over all weights $(\lambda_L, \lambda_R) \in \mathsf{R}_L \otimes \mathsf{R}_R$ of a particular pair in $\mathcal{R}_{\Delta,\mathsf{s}}$. Where it does not cause confusion we will maintain this shorthand (in both products and sums) for compactness of notation.

As mentioned at the beginning to this section, the local spin-s partition function, $Z_{\Delta,\mathsf{s}}$, includes, in addition to this quasinormal product, the product from integrating over normalizable zero modes, (3.7):

$$Z_{\Delta,\mathsf{s}} = Z_{\text{zero}} Z_{\text{STT}} . \qquad (3.30)$$

In the next section we will show how this combination, with the expression of Z_{STT} as a product over representation weights, (3.29), will lead to the Wilson spool.

3.2   Constructing the spool

The procedure to cast $\log Z_{\Delta,\mathsf{s}}$ as an integral over Wilson loop operators starts by rearranging (3.7) and (3.29). We first make use of the Schwinger parameterization of the logarithm

$$\log M = -\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} e^{-\alpha M} , \qquad (3.31)$$

with a regularization of the divergence at $\alpha \to 0$ that we will leave unspecified for now. We will address this regularization through a suitable $i\varepsilon$ prescription below. Applying (3.31) to (3.29), we first see that the sum over weights in $\log Z_{\text{STT}}$ can then be organized into representation traces

$$\sum_{(\lambda_L,\lambda_R)} e^{\alpha(\lambda_L + \lambda_R)} = \text{Tr}_{\mathsf{R}_L} \left(e^{\alpha L_3} \right) \text{Tr}_{\mathsf{R}_R} \left(e^{\alpha \bar{L}_3} \right) , \qquad (3.32)$$

which are the characters of the non-standard representation in (2.21). This then leads to

$$\log(Z_{\text{STT}}) = \frac{1}{2} \int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} \left(\sum_{\substack{N \in \mathbb{Z} \\ |N| > \mathsf{s}}} e^{-|N|\alpha} + (2\mathsf{s}+1)e^{-\mathsf{s}\alpha} \right) \times$$

$$\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \text{Tr}_{\mathsf{R}_L} \left(e^{\alpha L_3} \right) \text{Tr}_{\mathsf{R}_R} \left(e^{\alpha \bar{L}_3} \right) . \qquad (3.33)$$

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Similarly, we can introduce a Schwinger parameter to $\log Z_{\text{zero}}$, where now (3.7) reads

$$\log Z_{\text{zero}} = \int_\times^\infty \frac{d\alpha}{\alpha} \left(\sum_{n=0}^{\mathsf{s}-2} ((n+1)^2 - \mathsf{s}^2) e^{(j_L + j_R - n)\alpha} - \frac{\mathsf{s}^2}{2} e^{(j_L + j_R + 1)\alpha} \right) + (j_{L/R} \to \bar{j}_{L/R})$$

$$= \frac{1}{2} \int_\times^\infty \frac{d\alpha}{\alpha} (e^\alpha - 1)^2 e^{-2\alpha} \left(\sum_{n=0}^{\mathsf{s}-2} ((n+1)^2 - \mathsf{s}^2) e^{-n\alpha} - \frac{\mathsf{s}^2}{2} e^\alpha \right) \times$$

$$\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \text{Tr}_{\mathsf{R}_L} \left(e^{\alpha L_3} \right) \text{Tr}_{\mathsf{R}_R} \left(e^{\alpha \bar{L}_3} \right) . \tag{3.34}$$

In the first line, we used $\Delta = -j_L - j_R$, and in the second, the characters (2.21) to cast this as a trace. The zero mode contribution (3.34) combines nicely with $\log(Z_{\text{STT}})$ to give

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{2} \int_\times^\infty \frac{d\alpha}{\alpha} \left(\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} - \mathsf{s}^2 \sinh(\alpha) \right) \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \text{Tr}_{\mathsf{R}_L} \left(e^{\alpha L_3} \right) \text{Tr}_{\mathsf{R}_R} \left(e^{\alpha \bar{L}_3} \right) , \tag{3.35}$$

where we used

$$\sum_{n \in \mathbb{Z}} e^{-|n|\alpha} = \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} . \tag{3.36}$$

At this point we use the even parity of the integrand to regulate the divergence through the following $i\varepsilon$ prescription:

$$\int_\times^\infty \frac{d\alpha}{\alpha} f(\alpha) := \lim_{\varepsilon \to 0} \frac{1}{4} \sum_\pm \int_{-\infty}^\infty \frac{d\alpha}{\alpha \pm i\varepsilon} f(\alpha \pm i\varepsilon) . \tag{3.37}$$

This is a choice of regularization scheme for the one-loop determinant. Finally, under a change of integration variables $\alpha \to -i\alpha$ we can write the partition function as

$$\log Z_{\Delta,\mathsf{s}} = \frac{i}{8} \int_\mathcal{C} \frac{d\alpha}{\alpha} \left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2 \cos\left(\frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) \right) \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \text{Tr}_{\mathsf{R}_L} \left(e^{i\alpha L_3} \right) \text{Tr}_{\mathsf{R}_R} \left(e^{i\alpha \bar{L}_3} \right) , \tag{3.38}$$

where the contour $\mathcal{C}$ runs upwards along the imaginary α axis to the left and right of the divergence at the origin, as depicted in Fig. 1.

As a final step we now rewrite the holonomies inside the traces to restore the background connections, arriving at (3.8) with $\mathbb{W}_{j_L,j_R}$ the spinning Wilson spool:

$$\mathbb{W}_{j_L,j_R}[a_L, a_R] := \frac{i}{2} \int_\mathcal{C} \frac{d\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \left(1 + 2\mathsf{s}^2 \sin^2\left(\frac{\alpha}{2}\right) \right) \times$$

$$\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \text{Tr}_{\mathsf{R}_L} \left(\mathcal{P} e^{\frac{\alpha}{2\pi} \oint_\gamma a_L} \right) \text{Tr}_{\mathsf{R}_R} \left(\mathcal{P} e^{-\frac{\alpha}{2\pi} \oint_\gamma a_R} \right) , \tag{3.39}$$

where above $\gamma = \gamma_{\text{hor.}}$ is a cycle wrapping the singular point corresponding to the horizon.

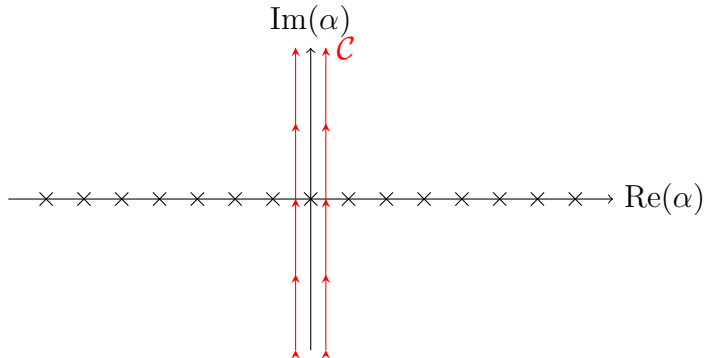At this point let us make several comments:

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Figure 1: The integration contour regulating the $\alpha \to 0$ divergence.

- The spinning Wilson spool takes a form similar to that of the scalar spool found in [1,2]; importantly the "operator pieces" of the expression (3.39) have been organized into gauge-invariant Wilson loop operators. The only modification the spinning spool brings is in the integration measure.

- The modification to the integration measure, proportional to s^2, will have the effect of lowering the degree of each pole at $\alpha \in 2\pi\mathbb{Z}$ by two. As we will shortly see, this effect reproduces the "edge partition function" of [8].

- Mathematically the holonomies corresponding to $\gamma_{\text{hor.}}$ appear in the one-loop determinant because they are sensitive to Δ on which Z_{STT} is treated as an meromorphic function. The physics behind this is clear: we are reproducing a one-loop determinant of massive fields. In the worldline quantum mechanics framework this corresponds to averaging over worldlines of a massive particle in the static patch. Such wordlines are timelike and rotate to a contour gauge equivalent to $\gamma_{\text{hor.}}$.

3.3 Testing the Wilson spool

We now uphold equations (3.8) and (3.39) by verifying that it indeed reproduces the correct path integral of a massive spinning field on S^3. There are several ways how to evaluate this path integral, and here we will focus on two approaches to use as a comparison. The first is to implement the DHS method traditionally: in App. B we evaluate this path integral by explicitly listing the quasinormal modes and applying DHS. The second approach we can compare to are the expressions found in [8], which cast the results in terms of $\mathfrak{so}(1,3)$ characters.

Since we are not turning on gravity ($G_N \to 0$), we evaluate the Wilson loop operators in (3.39) as characters with the appropriate holonomies in (3.18). Using the form of our non-standard representation character, (2.21), we then write

$$\log Z_{\Delta,\mathsf{s}} = -\frac{i}{8} \int_{\mathcal{C}} \frac{d\alpha}{\alpha} \left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2 \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \right) e^{i\alpha(1-\Delta)} , \qquad (3.40)$$

where we note that the sum over representations in $\mathcal{R}_{\Delta,\mathsf{s}}$ is already neatly packaged into our two contours. We recognize the first term in the parentheses of (3.40) as twice the

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on-shell scalar Wilson spool found in [1]. We evaluate both terms by deforming the α contours to run above and below the real α axis to pick up the residues at the poles lying at $2\pi\mathbb{Z}_{\neq 0}$. This deformation is depicted in Fig. 2.
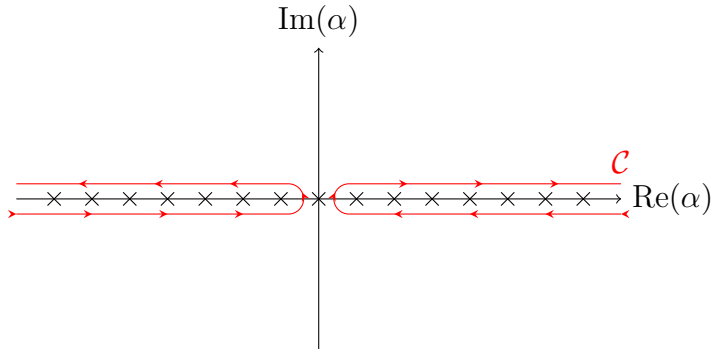

Figure 2: We deform the α integration contour to wrap the poles lying along the real axis.

Summing the towers of poles and expressing the $\mathfrak{su}(2)_{L/R}$ highest-weight labels in terms of μ and s as in (2.22), we write this as

$$
\begin{aligned}
\log Z_{\Delta,\mathsf{s}} &= \sum_{\pm}\left(-\frac{1}{4\pi^2}\mathrm{Li}_3\left(e^{\mp 2\pi\mu}\right) \mp \frac{\mu}{2\pi}\mathrm{Li}_2\left(e^{\mp 2\pi\mu}\right) - \frac{\mu^2+\mathsf{s}^2}{2}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right)\right) \\
&= 2\log Z_{\text{scalar}} - \frac{\mathsf{s}^2}{2}\sum_{\pm}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right)\ ,
\end{aligned}
\tag{3.41}
$$

where

$$
\mathrm{Li}_p(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^p}
\tag{3.42}
$$

is the polylogarithm. As mentioned above, $\log Z_{\text{scalar}}$ is the one-loop determinant of a massive scalar field on S^3 with a mass set by $\ell_{\text{dS}}^2 m^2 = \Delta(2-\Delta) = 1 + \mu^2$. Up to an overall phase, unfixed by the quasinormal mode method, (3.41) matches the path integral of a massive spin-s field on S^3 via DHS in App. B and the results reported in [8].

4 Coupling quantum matter to dS$_3$ quantum gravity

Having constructed the spinning Wilson spool on S^3 in Sec. 3, in this section we will promote it to an off-shell object with the aim of incorporating quantum effects from gravity around S^3. More concretely, we posit the following: let A_L and A_R be $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ connections respectively, yielding a non-degenerate dreibein $e^a = -i\left(A_L^a - A_R^a\right)$ with an associated metric geometry (M_3, g_{M_3}) that is topologically equivalent to the round S^3. Then the partition function of massive spin-s fields minimally coupled to g_{M_3} is

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determined by the spinning Wilson spool given by (3.39) with $a_{L/R}$ replaced with $A_{L/R}$:

$$\log Z_{\Delta,\mathsf{s}}[g_{M_3}] = \frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \ . \tag{4.1}$$

For scalar fields it was argued in [1] that the manipulations leading to the Wilson spool remain valid for off-shell geometries. This argumentation relied on the expression of the Laplacian $-\nabla^2_{g_{M_3}}$ as a Casimir of local $\mathfrak{su}(2)_{L/R}$ action (see App. D of [1]). While this remains true when acting on spin-s fields (via the transverse-traceless condition), various manipulations leading to (3.39) make the leap to (4.1) less rigorous than for scalar fields. This includes both manipulations relying on the integer nature of the holonomies, $\mathsf{h}_{L/R}$, as well as the explicit form of Z_{zero}. Barring a detailed analysis of normalizable zero modes for the ghost and Stückelberg fields appearing in $Z_{\Delta,\mathsf{s}}$ on generic three-geometries, at this point we take (4.1) *as a proposal*. This proposal is upheld on the grounds that it utilizes and generalizes naturally the gauge-invariant observables in Chern-Simons theory (namely its Wilson loops); these observables appear in a form that reduces to twice[9] the scalar path integral when $\mathsf{s} \to 0$ and, as we will see in the following section, continues to work for negative cosmological constant.

With this proposal in hand, the Wilson spool gives us a concrete route for calculating finite G_N effects[10] to $\log Z_{\Delta,\mathsf{s}}$. That is we can consider

$$\begin{aligned}
\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\text{grav}} &:= \int [\mathcal{D}g_{M_3}]_{S^3} \, e^{-I_{\text{EH}}[g_{M_3}]} \log Z_{\Delta,\mathsf{s}}[g_{M_3}] \\
&\equiv \int [\mathcal{D}A_L \mathcal{D}A_R]_{S^3} \, e^{iS[A_L, A_R]} \left(\frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \right) \ ,
\end{aligned} \tag{4.2}$$

where the second line follows from the rewriting of the gravitational variables and action (in a first-order formalism) as two Chern-Simons path integrals.[11] Additionally, in the first line, $[\mathcal{D}g_{M_3}]_{S^3}$ indicates that we integrate over metric geometries topologically equivalent to S^3, and in the second line, $[\mathcal{D}A_L \mathcal{D}A_R]_{S^3}$ indicates we perform the Chern-Simons path integral on the base S^3 topology.

The power of the second line of (4.2) lies in the breadth of techniques for evaluating Wilson loop observables for $SU(2)$ Chern-Simons theories on the three-sphere [22–26, 37, 38]. In [1] it was shown how to adapt two such techniques, Abelianisation [22–24] and localization through a $\mathcal{N} = 2$ supersymmetric extension [25, 26], for the evaluation of Wilson loop expectation values in light of the features unique to Chern-Simons gravity: non-trivial background connections, complex levels, and non-standard representations appearing in Wilson loop operators. While *prima facie* these are two very different techniques, they both lead to the expression of a Wilson loop expectation

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[9]This stems simply from the fact that a STT field contains roughly two scalars corresponding to the polarizations with s and $-\mathsf{s}$ and has nothing to do with the ambiguities of a compact space and possible zero modes. This counting is obviously not continuous as $\mathsf{s} \to 0$.

[10]To be clear on scope: here we mean to all orders in G_N perturbation theory about the S^3 saddle. We do not consider topology change or other non-perturbative effects here.

[11]In App. A.1 we review dictionary between Chern-Simons theory and dS$_3$ gravity.

value as a deformation of its on-shell (i.e. flat) background value integrated over a single modulus:

$$\int [\mathcal{D}A] e^{ik\, S_{\text{CS}}[A]} \text{Tr}_{\mathsf{R}}\, \mathcal{P} e^{\oint_\gamma A} = e^{ir S_{\text{CS}}[a]} \int_{\mathbb{R}} \mathrm{d}\sigma\, \sin^2(\pi\sigma) e^{ir\frac{\pi}{2}\sigma^2} \chi_{\mathsf{R}}(\sigma + \mathsf{h}) \ , \qquad (4.3)$$

where $r = k + 2$ is the renormalized level, a is the on-shell background connection, h is its holonomy about the contour γ, and $\chi_{\mathsf{R}}(z) = \text{Tr}_{\mathsf{R}}\, e^{i2\pi z\, L_3}$ is the $\mathfrak{su}(2)$ representation character. Equation (4.3) holds for complex levels and non-standard $\mathfrak{su}(2)$ representations, including those found in Sec. 2. Applying this to (4.2), we can write

$$\big\langle \log Z_{\Delta,\mathsf{s}} \big\rangle_{\text{grav}} = \frac{i}{8} e^{ir_L S_{\text{CS}}[a_L] + ir_R S_{\text{CS}}[a_R]} \int \mathrm{d}\sigma_L \mathrm{d}\sigma_R\, e^{ir_L \frac{\pi}{2}\sigma_L^2 + ir_R \frac{\pi}{2}\sigma_R^2} \sin^2(\pi\sigma_L)\sin^2(\pi\sigma_R) \times$$

$$\sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \left(1 + 2\mathsf{s}^2 \sin^2\left(\frac{\alpha}{2}\right)\right) \times$$

$$\chi_{\mathsf{R}_L}\left(\frac{\alpha}{2\pi}(1 + \sigma_L)\right) \chi_{\mathsf{R}_R}\left(\frac{\alpha}{2\pi}(1 - \sigma_R)\right) \ , \quad (4.4)$$

where we used the off-shell version of the spool in (3.39). Upon writing

$$r_L = \hat{\delta} + is \ , \qquad r_R = \hat{\delta} - is \ , \qquad s = \frac{\ell_{\text{dS}}}{4G_N} \ , \qquad \hat{\delta} \in \mathbb{Z} \ , \qquad (4.5)$$

per the Chern-Simons gravity dictionary outlined in App. A.1, we posit that (4.4) is *exact* (that is, holding to all orders) in G_N perturbation theory about the S^3 saddle.

While this claim has power in principle, in practice (4.4) is complicated as an integral. By rescaling $\sigma_{L/R} \to r_{L/R}^{-1/2}\sigma_{L/R}$, we can proceed systematically in $\ell_{\text{dS}}^{-1}G_N$ perturbation theory which amounts to a Taylor expansion in $\sigma_{L/R}$ of non-Gaussian pieces of the integrand in (4.4). At any order in this expansion the Gaussian integrals over $\sigma_{L/R}$ can be performed. This leaves the contour integral over α which can be deformed to pick up its poles (which remain at $2\pi\mathbb{Z}_{\neq 0}$ at each order of perturbation theory) in a similar spirit as our computation in Sec. 3.3. This procedure completely mirrors that outlined for the scalar Wilson spool in [1] and can be efficiently implemented on a computer algebra system.

To illustrate this concretely, we evaluate $\big\langle \log Z_{\Delta,\mathsf{s}} \big\rangle_{\text{grav}}$ normalized by the gravitational path integral to the first non-zero order of $\ell_{\text{dS}}^{-1}G_N$ perturbation theory. The gravitational path integral on S^3 has a close form given by

$$\mathcal{Z}_{\text{grav}} = e^{ir_L S_{\text{CS}}[a_L] + ir_R S_{\text{CS}}[a_R]} \int \mathrm{d}\sigma_L \mathrm{d}\sigma_R\, e^{ir_L \frac{\pi}{2}\sigma_L^2 + ir_R \frac{\pi}{2}\sigma_R^2} \sin^2(\pi\sigma_L)\sin^2(\pi\sigma_R)$$

$$= e^{2\pi s} \left(i e^{-i2\pi \frac{\hat{\delta}}{\hat{\delta}^2 + s^2}}\right) \frac{2}{\sqrt{\hat{\delta}^2 + s^2}} \left|\sin\left(\frac{\pi}{\hat{\delta} + is}\right)\right|^2 \ , \qquad (4.6)$$

as outlined in [1] (see also [8, 13, 39]). By implementing the rescaling $\sigma_{L/R} \to r_{L/R}^{-1/2}\sigma_{L/R}$ in (4.4), to leading order in the coupling we find

$$\frac{\big\langle \log Z_{\Delta,\mathsf{s}} \big\rangle_{\text{grav}}}{\mathcal{Z}_{\text{grav}}} = \log Z_{\Delta,\mathsf{s}}[S^3] + \left(\frac{G_N}{\ell_{\text{dS}}}\right)^2 [\log Z_{\Delta,\mathsf{s}}]_{(2)} + \dots \ , \qquad (4.7)$$

where the dots correspond to subleading corrections in $\ell_{\rm dS}^{-1} G_N$, and the first non-trivial correction is

$$[\log Z_{\Delta,\mathsf{s}}]_{(2)} = \sum_{\pm} \sum_{i=0}^{3} z_i^{(2)}[\pm\mu,\mathsf{s}]\mathrm{Li}_{-i}\left(e^{\mp 2\pi\mu}\right) \ , \tag{4.8}$$

with

$$z_0^{(2)}[\mu,\mathsf{s}] = -\left(\frac{16\pi}{3} - \frac{8}{\pi} - 8i\hat{\delta}\right)\mu^3 - \left(16\pi - \frac{144}{\pi} - 24i\hat{\delta} + 16\pi\mathsf{s}^2 - \frac{216}{\pi}\mathsf{s}^2 - 24i\hat{\delta}\mathsf{s}^2\right)\mu \ ,$$

$$z_1^{(2)}[\mu,\mathsf{s}] = \left(\frac{16\pi^2}{3} + 12 - i8\pi\hat{\delta}\right)\mu^4 + \left(16\pi^2 - 276 - i24\pi\hat{\delta} - 432\mathsf{s}^2\right)\mu^2$$
$$- (16\pi^2 - 372 - i24\pi\hat{\delta})\mathsf{s}^4 + (16\pi^2 - 588 - i24\pi\hat{\delta})\mathsf{s}^2 - 48 \ ,$$

$$z_2^{(2)}[\mu,\mathsf{s}] = -\frac{24\pi}{5}\mu^5 + \left(104\pi + 96\pi\mathsf{s}^2\right)\mu^3 + \left(96\pi - 168\pi\mathsf{s}^4 + 792\pi\mathsf{s}^2\right)\mu \ ,$$

$$z_3^{(2)}[\mu,\mathsf{s}] = -\left(8\pi^2 - 16\pi^2\mathsf{s}^2\right)\mu^4 - \left(32\pi^2 + 40\pi^2\mathsf{s}^4 + 160\pi^2\mathsf{s}^2\right)\mu^2$$
$$+ 8\pi^2\mathsf{s}^6 + 24\pi^2\mathsf{s}^4 - 32\pi^2\mathsf{s}^2 \ . \tag{4.9}$$

We note that taking $\mathsf{s} \to 0$ doubles the corrections to the scalar one-loop determinant [1] as expected. At leading order in a large μ expansion (which amounts to a large mass expansion while holding s fixed)

$$[\log Z_{\Delta,\mathsf{s}}]_{(2)} = -\frac{48\pi\,\mu^5}{5}e^{-2\pi\mu} + \left(24 - \frac{16\pi^2}{3} + 32\pi^2\mathsf{s}^2 - i16\pi\hat{\delta}\right)\mu^4\,e^{-2\pi\mu} + \dots \ , \tag{4.10}$$

where we have kept to next-to-leading order in the large mass expansion where the first contribution from spin appears.

It is natural at this stage to interpret this as a renormalization of the mass[12] of the spin-s field. Writing an expansion for the renormalized mass

$$\mu_R = \mu + \left(\frac{G_N}{\ell_{\rm dS}}\right)^2 \delta_\mu^{(2)} + \dots \ , \tag{4.11}$$

we note to $O(G_N^2 \ell_{\rm dS}^{-2})$

$$\log Z_{\Delta,\mathsf{s}} = \log Z_{\Delta_R,\mathsf{s}} - \pi\frac{\cosh(\pi\mu_R)}{\sinh(\pi\mu_R)}(\mu_R^2 + \mathsf{s}^2)\left(\frac{G_N^2}{\ell_{\rm dS}^2}\right)\delta_\mu^{(2)} + \dots \ , \tag{4.12}$$

where $\Delta_R = 1 - i\mu_R$. Interpreting the corrections due to quantum gravity as a mass renormalization then sets as a renormalization condition

$$\frac{\left\langle \log Z_{\Delta,\mathsf{s}}\right\rangle_{\rm grav}}{\mathcal{Z}_{\rm grav}} \overset{!}{=} \log Z_{\Delta_R,\mathsf{s}} \ , \tag{4.13}$$

---

[12]Importantly the spin of the field does not get renormalized.

with G_N held fixed. This then determines the renormalized mass as

$$\delta_\mu^{(2)} = \frac{1}{\pi} \frac{\tanh(\pi \mu_R)}{\mu_R^2 + \mathsf{s}^2} \left[\log Z_{\Delta,\mathsf{s}}\right]_{(2)}$$
$$= -\frac{48}{5} \mu_R^3 e^{-2\pi\mu_R} + \left(\frac{24}{\pi} - \frac{16\pi}{3} + 32\pi\mathsf{s}^2 - i16\hat{\delta}\right) \mu_R^2 e^{-2\pi\mu_R} + \dots , \qquad (4.14)$$

where in the second line we have written the leading and next to leading terms in a large mass expansion. Note that the magnitude of the leading term is s independent and consistent[13] with the leading mass renormalization for scalar fields [1].

We note that this is not the only renormalization condition that one may choose to set. We could instead to renormalize Newton's constant, $G_N \to G_{N,R}$, while holding the mass of the spinning field fixed. To illustrate this, we will set $\hat{\delta} = 0$ and consider the following renormalization condition:

$$\frac{\int [\mathcal{D}g] e^{-I_{\mathrm{EH}}[g]} Z_{\Delta,\mathsf{s}}[g]}{Z_{\Delta,\mathsf{s}}[g_{S^3}]} \Bigg|_{G_N} = \mathcal{Z}_{\mathrm{grav}} \left(1 + \frac{\left\langle \log Z_{\Delta,\mathsf{s}} \right\rangle_{\mathrm{grav}}}{\mathcal{Z}_{\mathrm{grav}}} - \log Z_{\Delta,\mathsf{s}}[g_{S^3}] + \dots \right)\Bigg|_{G_N}$$
$$\overset{!}{=} \mathcal{Z}_{\mathrm{grav}}\Big|_{G_{N,R}} , \qquad (4.16)$$

or equivalently

$$\frac{\mathcal{Z}_{\mathrm{grav}}\big|_{G_{N,R}}}{\mathcal{Z}_{\mathrm{grav}}\big|_{G_N}} = 1 + \left(\frac{G_{N,R}}{\ell_{\mathrm{dS}}}\right)^2 \left[\log Z_{\Delta,\mathsf{s}}\right]_{(2)} + \dots . \qquad (4.17)$$

Note we have normalized by $Z_{\Delta,\mathsf{s}}[g_{S^3}]$ in (4.16) as this leading term decouples from metric fluctuations. It would be responsible for a renormalization of the cosmological constant, but does not mediate the gravitational self-interactions relevant for renormalizing the coupling, G_N. Writing

$$G_N = G_{N,R} \left(1 + \delta_{G_N}^{(2)} \left(\frac{G_{N,R}}{\ell_{\mathrm{dS}}}\right)^2 + \dots \right) , \qquad (4.18)$$

we find from expanding (4.6) and comparing to the right-hand side of (4.17)

$$\delta_{G_N}^{(2)} = -\frac{1}{3} \left[\log Z_{\Delta,\mathsf{s}}\right]_{(2)}$$
$$= \frac{16\pi}{5} \mu^5 e^{-2\pi\mu} - \left(8 - \frac{16\pi^2}{9} + \frac{32\pi^2}{3}\mathsf{s}^2\right) \mu^4 e^{-2\pi\mu} + \dots , \qquad (4.19)$$

---

[13] In [1] the renormalization condition was taken (tacitly) as

$$\frac{\left\langle \log Z_{\Delta_R,\mathrm{scalar}} \right\rangle_{\mathrm{grav}}}{\mathcal{Z}_{\mathrm{grav}}} \overset{!}{=} \log Z_{\Delta,\mathrm{scalar}} , \qquad (4.15)$$

which leads to an overall minus sign with respect to (4.14).

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where again we have written the leading and next-to-leading terms in a large mass expansion.

It is worth noting that the results of this section, both the renormalization of the field mass, (4.14), and the renormalization of G_N, (4.17), are novel. To our knowledge, corresponding computations have not been carried out in the metric formulation of dS_3.[14] To this end, the results of this section provide concrete and testable predictions of the Chern-Simons formulation of gravity.

5 Spinning spool on AdS_3

To illustrate the general utility of the spinning Wilson spool constructed in Sec. 3.2, as well as bolster the **Conditions I & II** that lead to it in Sec. 3.1, we can repeat this construction in an AdS_3 background. An expression for the one-loop determinant of massive spinning fields on a BTZ black hole as a classical Wilson spool was conjectured in [2] based on an extension of the result for a massive scalar field. In this section we show how to derive the spinning spool on AdS_3 from the principles set out in Sec. 3.1 and demonstrate that the final expression accords with the general result (1.4).

We wish to compute

$$Z_{\Delta,\mathsf{s}} = \det\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}}\right)^{-\frac{1}{2}} \tag{5.1}$$

with $\nabla^2_{(\mathsf{s})}$ being the Laplace-Beltrami operator acting on spin-s STT tensors. The background geometry entering in (5.1) will be the rotating BTZ black hole [41, 42] with AdS_3 radius ℓ_{AdS}; properties of this geometry are described in App. A.2. The effective mass $\ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}}$ is related to the standard mass, m^2, via [43]

$$\ell^2_{\mathrm{AdS}}\bar{m}^2_{\mathsf{s}} = \ell^2_{\mathrm{AdS}}m^2 + \mathsf{s}(\mathsf{s}-3) . \tag{5.2}$$

We recall that the standard mass is related to the conformal dimension of a dual primary through [44]

$$\ell^2_{\mathrm{AdS}}m^2 = (\Delta + \mathsf{s} - 2)(\Delta - \mathsf{s}) . \tag{5.3}$$

$Z_{\Delta,\mathsf{s}}$ is completely captured by this one-loop determinant over STT tensors: in contrast to the previous section, there is no additional zero mode product as AdS_3 is non-compact.

The isometry group of Lorentzian AdS_3 is $SO(2,2)$ with an algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$ which we will take to be generated by $\{L_0, L_\pm\}$ and $\{\bar{L}_0, \bar{L}_\pm\}$, respectively.[15] As done for dS_3, it is useful to briefly establish our conventions regarding the description of single-particle states living in AdS_3 as representations of $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$. Lowest-weight representations $\mathsf{R}^{\mathrm{LW}}_j$ are defined by a lowest-weight state, $|j,0\rangle_{\mathrm{LW}}$, which is annihilated by L_- and labelled by its L_0 eigenvalue:

$$L_-|j,0\rangle_{\mathrm{LW}} = 0 , \qquad L_0|j,0\rangle_{\mathrm{LW}} = j|j,0\rangle_{\mathrm{LW}} . \tag{5.4}$$

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[14]In the absence of a cosmological constant, a three-loop computation involving gravitons was done in [40]. It seems feasible that one could apply that approach to dS_3 and verify (4.14) and (4.17).

[15]Our conventions for the $\mathfrak{sl}(2,\mathbb{R})$ algebra follow [45, 46].

All other states in $\mathsf{R}_j^{\mathrm{LW}}$ are generated by the action of L_+ acting on $|j,0\rangle_{\mathrm{LW}}$. This representation has a character

$$\chi_{j,\mathrm{LW}}(z) = \mathrm{Tr}_{\mathsf{R}_j^{\mathrm{LW}}}\left(e^{i2\pi z L_0}\right) = \frac{e^{i\pi z(2j-1)}}{2\sinh(-i\pi z)} \ . \tag{5.5}$$

Highest-weight representations $\mathsf{R}_j^{\mathrm{HW}}$ are defined by a highest-weight state, $|j,0\rangle_{\mathrm{HW}}$, which is instead annihilated by L_+. We will take by convention its L_0 eigenvalue to be $-j$:

$$L_+|j,0\rangle_{\mathrm{HW}} = 0 \ , \qquad\qquad L_0|j,0\rangle_{\mathrm{HW}} = -j|j,0\rangle_{\mathrm{HW}} \ . \tag{5.6}$$

All other states of $\mathsf{R}_j^{\mathrm{HW}}$ are generated by the action of L_- on $|j,0\rangle_{\mathrm{HW}}$. This representation has a character

$$\chi_{j,\mathrm{HW}}(z) = \mathrm{Tr}_{\mathsf{R}_j^{\mathrm{HW}}}\left(e^{i2\pi z L_0}\right) = \frac{e^{-i\pi z(2j-1)}}{2\sinh(i\pi z)} \ . \tag{5.7}$$

Our conventions have been chosen so that the Casimir of both highest and lowest-weight representations is given by

$$c_2^{\mathfrak{sl}(2,\mathbb{R})}|j,p\rangle_{\mathrm{LW/HW}} = j(j-1)|j,p\rangle_{\mathrm{LW/HW}} \ , \qquad\qquad \forall \ |j,p\rangle_{\mathrm{LW/HW}} \in \mathsf{R}_j^{\mathrm{LW/HW}} \ . \tag{5.8}$$

Returning to the functional determinant (5.1), we now want to implement the steps in Sec. 3.1: the "mass-shell condition" and **Conditions I & II**. To start, we write the Laplace-Beltrami operator in terms of the Casimir of the isometries of AdS$_3$ [44]:

$$2c_2^{\mathfrak{sl}(2,\mathbb{R})_L} + 2c_2^{\mathfrak{sl}(2,\mathbb{R})_R} = \nabla_{(\mathsf{s})}^2 + \mathsf{s}(\mathsf{s}+1) \ . \tag{5.9}$$

In this language, a pole contributing to $Z_{\Delta,\mathsf{s}}^2$ corresponds to a state $|\psi\rangle$ in a representation of $\mathfrak{sl}(2,\mathbb{R})_L \oplus \mathfrak{sl}(2,\mathbb{R})_R$ satisfying

$$\left(2c_2^{\mathfrak{sl}(2,\mathbb{R})_L} + 2c_2^{\mathfrak{sl}(2,\mathbb{R})_R}\right)|\psi\rangle = \left(\Delta(\Delta-2) + \mathsf{s}^2\right)|\psi\rangle \ . \tag{5.10}$$

This is the "mass-shell condition" and it is satisfied for pairs of highest and lowest-weight representations labeled by

$$j_L = \frac{\Delta \pm \mathsf{s}}{2} \ , \qquad j_R = \frac{\Delta \mp \mathsf{s}}{2} \ . \tag{5.11}$$

Note that representations labeled by (5.11) with Δ replaced by $\bar\Delta = 2-\Delta$ share the same Casimir. However, unlike in dS$_3$, we are forced to choose either Δ or $\bar\Delta$ due to Dirichlet boundary conditions tacitly imposed on solutions contributing to (5.1).[16] For what follows we will assume, without loss of generality, that Δ corresponds to a normalizable massive spin-s solution.[17] Fixing $j_{L/R}$ as in (5.11) with the upper sign, we can then have pole contributions from any representation appearing in

$$\mathcal{R}_{\Delta,\mathsf{s}} = \mathcal{R}_{\Delta,\mathsf{s}}^{\mathrm{HW}} \cup \mathcal{R}_{\Delta,\mathsf{s}}^{\mathrm{LW}} = \left\{\mathsf{R}_{j_L}^{\mathrm{HW}} \otimes \mathsf{R}_{j_R}^{\mathrm{HW}}, \mathsf{R}_{j_R}^{\mathrm{HW}} \otimes \mathsf{R}_{j_L}^{\mathrm{HW}}\right\} \cup \left\{\mathsf{R}_{j_L}^{\mathrm{LW}} \otimes \mathsf{R}_{j_R}^{\mathrm{LW}}, \mathsf{R}_{j_R}^{\mathrm{LW}} \otimes \mathsf{R}_{j_L}^{\mathrm{LW}}\right\} \ . \tag{5.12}$$

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[16]Dirichlet boundary conditions also exclude $\mathfrak{sl}(2,\mathbb{R})$ representations that are neither highest nor lowest-weight from contributing to the one-loop determinant.

[17]In the special cases where Δ and $\bar\Delta$ both correspond to normalizable solutions, we simply choose Δ, again without loss of generality.

For scalars, $\mathsf{s} = 0$, we have the reduced set

$$\mathcal{R}_{\Delta,\text{scalar}} = \mathcal{R}_{\Delta,\text{scalar}}^{\text{HW}} \cup \mathcal{R}_{\Delta,\text{scalar}}^{\text{LW}} = \left\{\mathsf{R}_j^{\text{HW}} \otimes \mathsf{R}_j^{\text{HW}}\right\} \cup \left\{\mathsf{R}_j^{\text{LW}} \otimes \mathsf{R}_j^{\text{LW}}\right\} . \qquad (5.13)$$

We now apply the group theoretic conditions of Sec. 3.1 to the representations contributing to (5.1). **Conditions I & II** as they are stated in Sec. 3.1 can be readily applied to the one-loop determinant on BTZ, and they respectively imply:

Condition I. Single-valued solutions. In Euclidean signature, BTZ is a solid torus that has two cycles that characterize its global properties. Requiring solutions to be single-valued around the contractible thermal cycle, γ_{th}, of the BTZ geometry requires $\mathsf{s} \in \mathbb{Z}$. Requiring solutions to be single-valued around the non-contractible spatial cycle, γ_{sp}, of the BTZ geometry requires a weight $(\lambda_L, \lambda_R) \in \mathsf{R}_L \otimes \mathsf{R}_R$ to satisfy

$$\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R \in \mathbb{Z} , \qquad (5.14)$$

where $\mathsf{h}_{L,R}$ are holonomies around the non-contractible cycle

$$\mathsf{h}_L = -\frac{1}{\tau} , \qquad \mathsf{h}_R = -\frac{1}{\bar{\tau}} , \qquad (5.15)$$

and τ is the modular parameter defining the geometry. See App. A.2 for details.

Condition II. Globally regular solutions. A representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}_{\Delta,\mathsf{s}}$ is required to lift to a group representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. In contrast to Sec. 3.1, in the present case **Condition II** is trivial since *every* representation of $\mathfrak{sl}(2,\mathbb{R})$ lifts to a (not necessarily unitary) representation of the universal cover of $SL(2,\mathbb{R})$ [47].

Thus we fix $\mathsf{s} \in \mathbb{Z}$ and the one-loop determinant (5.1) is then the product of poles in the complex Δ plane given by

$$Z_{\Delta,\mathsf{s}} = \prod_{\mathcal{R}_{\Delta,\mathsf{s}}} \prod_{(\lambda_L,\lambda_R)} \prod_{N\in\mathbb{Z}} (|N| - \lambda_L \mathsf{h}_L + \lambda_R \mathsf{h}_R)^{-1/4} (|N| + \lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R)^{-1/4} . \qquad (5.16)$$

Following (3.31), we next implement a Schwinger parameterization of the logarithm of this expression which reads

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{1}{4} \int_\times^\infty \frac{d\alpha}{\alpha} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \sum_{(\lambda_L,\lambda_R)} \left(e^{\alpha(\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R)} + e^{-\alpha(\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R)}\right) , \qquad (5.17)$$

where we have performed the sum over N as in (3.36). We will regulate the $\alpha \to 0$ divergence of the above expression by combining the two terms in the bracket into a single contour integral regulated about the origin by an $i\varepsilon$ prescription. To ensure convergence of the representation traces, a separate $i\varepsilon$ prescription must be given to the highest and lowest-weight representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$:

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{1}{4} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \int_{-\infty\pm i\varepsilon}^{\infty\pm i\varepsilon} \frac{d\alpha}{\alpha} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} \sum_{(\lambda_L,\lambda_R)} e^{\alpha(\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R)} . \qquad (5.18)$$

with the (+/-) sign applying to representations in $\mathcal{R}_{\Delta,\mathsf{s}}^{\text{LW/HW}}$, respectively. Recognizing the sum over weights as a representation trace and redefining $\alpha \to -i\alpha$ we then can write this suggestively as

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{i}{4} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}^{\text{LW}}} \int_{\mathcal{C}_+} \frac{d\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \text{Tr}_{\mathsf{R}_L} \left(\mathcal{P} e^{\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_L}\right) \text{Tr}_{\mathsf{R}_R} \left(\mathcal{P} e^{-\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_R}\right)$$
$$+ \frac{i}{4} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}^{\text{HW}}} \int_{\mathcal{C}_-} \frac{d\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \text{Tr}_{\mathsf{R}_L} \left(\mathcal{P} e^{\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_L}\right) \text{Tr}_{\mathsf{R}_R} \left(\mathcal{P} e^{-\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_R}\right) , \quad (5.19)$$

where the contours $\mathcal{C}_\pm$ are depicted in Fig. 3 and applied separately to the lowest and highest-weight representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$. We have written $\log Z_{\Delta,\mathsf{s}}$ in this form
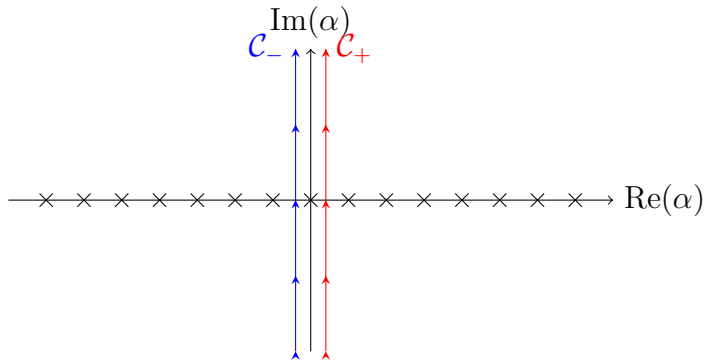

Figure 3: The integration contours relevant for the Wilson spool in AdS$_3$ gravity. Highest-weight representations are integrated against $\mathcal{C}_-$ (in blue) while lowest-weight representations are integrated against $\mathcal{C}_+$ (in red). The black crosses depict the poles of the integration measure, $\frac{1}{\alpha} \frac{\cos \alpha/2}{\sin \alpha/2}$. Not depicted are the poles of the representation traces which lie along the imaginary α axis.

to draw comparison to the dS$_3$ Wilson spool in Sec. 3.2. In the construction from that section, all representations are integrated along both $\mathcal{C}_\pm$. This is natural: as was argued in Sec. 3.1, the poles of the one-loop determinant lie on states of finite-dimensional $\mathfrak{su}(2)$ representations which are simultaneously highest and lowest-weight. In the present case, $\mathcal{C}_\pm$ appear distinctly because highest and lowest-weight representations are distinct for $\mathfrak{sl}(2,\mathbb{R})$.

With this comparison noted, we can express $\log Z_{\Delta,\mathsf{s}}$ purely in terms of lowest-weight representations, which are more standard in the AdS/CFT dictionary, by returning to (5.18) and recalling that every weight of a highest-weight representation is the negative of a weight in a corresponding lowest-weight representation. This allows to write $\log Z_{\Delta,\mathsf{s}}$ in the form conjectured in the supplemental material of [2]

$$\log(Z_{\Delta,\mathsf{s}}) = \frac{i}{4} \sum_{\mathcal{R}_{\Delta,\mathsf{s}}^{\text{LW}}} \int_{2\mathcal{C}_+} \frac{d\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \text{Tr}_{\mathsf{R}_L} \left(\mathcal{P} e^{\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_L}\right) \text{Tr}_{\mathsf{R}_R} \left(\mathcal{P} e^{-\frac{\alpha}{2\pi} \oint_{\gamma_{\text{sp}}} a_R}\right). \quad (5.20)$$

providing a principled derivation of that result. It was shown there that utilizing the holonomies of the background connections corresponding to a spinning BTZ black hole (see App. A.2 for the explicit form), the right-hand side of (5.20) evaluates to the correct one-loop determinant of massive spin-s fields on that background:

$$\log Z_{\Delta,\mathsf{s}} = -\sum_{\pm} \sum_{l,\bar{l}=0}^{\infty} \log\left(1 - q^{\frac{\Delta\pm\mathsf{s}}{2}+l}\bar{q}^{\frac{\Delta\mp\mathsf{s}}{2}+\bar{l}}\right) , \qquad q = e^{i\frac{2\pi}{\tau}} , \qquad \bar{q} = e^{-i\frac{2\pi}{\bar{\tau}}} . \qquad (5.21)$$

This agrees with the one-loop determinant of BTZ for spinning fields as reported in, for example, [43, 48–50].

We finish this section by reconciling the spinning Wilson spool in AdS$_3$ gravity with our main result (1.4), namely the lack of the s^2 correction to the α integration measure. This lack stems from two places in our derivation: (i) the absence of normalizable zero-modes and (ii) the triviality of **Condition II** which leaves the product over the weight spaces unrestricted. However it is interesting to note that, on-shell, $\text{Tr}_{\mathsf{R}_{L/R}}\mathcal{P}e^{\pm\frac{\alpha}{2\pi}\oint_{\gamma_{\mathrm{sp}}} a_{L/R}}$ only have poles along the imaginary α axis [2]. Thus this correction term is *completely regular* along the real α axis and integrates to zero. We may then include it into the definition of the spinning Wilson spool and state succinctly, for both signs of the cosmological constant,

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{4}\mathbb{W}_{j_L,j_R}[a_L, a_R] , \qquad (5.22)$$

with $\mathbb{W}_{j_L,j_R}$ appearing in (1.4) and the representations $\mathsf{R}_{j_{L,R}}$, the integration contour, and the background-connections $a_{L,R}$ chosen appropriately.

6   Discussion

In this paper we extended the Wilson spool constructions of [1,2] to incorporate one-loop determinants of massive spinning fields in both Euclidean dS$_3$ and AdS$_3$ backgrounds. This construction was based upon arranging the quasinormal mode spectra of the respective backgrounds into the representation theory of the isometry algebra. Along the way we codified two important principles for evaluating one-loop determinants in a representation-theoretic framework. These conditions lead naturally to a spinning Wilson spool expression for the local path integral, $\log Z_{\Delta,\mathsf{s}}$, of a massive spinning field, (1.4). This expression mimics the scalar Wilson spool: it is an integral over gauge invariant Wilson loop operators with the integral providing a mechanism for "wrapping" the loops around cycles of the base manifold. The spinning Wilson spool only departs from the scalar expression in its integration measure which cleanly captures the "edge contributions" of [8]. We posited an off-shell expression for the spinning Wilson spool which allows its insertion into the Chern-Simons path integral. In the context of a Euclidean dS$_3$ background, exact methods in Chern-Simons theory provide efficient methods for evaluating quantum gravity corrections to $\log Z_{\Delta,\mathsf{s}}$. We discussed two renormalization conditions in which these quantum corrections can be interpreted as renormalizing either the particle mass

or Newton's constant. This provides concrete predictions for testing the correspondence between Chern-Simons theory and three-dimensional quantum gravity.

There are multiple comments in order about our results, as well as future directions to which the spinning Wilson spool may prove useful. We briefly discuss these below.

Group theoretic perspective on one-loop determinants

One important result of Sec. 3.1 is the restating of the quasinormal mode method in a manner directly utilizing aspects of the representation theory and isometries of the background. At the core of that section are **Conditions I & II** which precisely isolate the representations and the states of a representation that contribute poles to massive one-loop determinants. We regard these conditions to be a significant new asset to the evaluation of one-loop determinants. For one, **Conditions I & II** will play an important role in establishing the Wilson spool for matter on Euclidean saddles beyond S^3, which is ongoing work [51]. For instance, our preliminary findings indicate that these two conditions correctly predict the placement and the degeneracies of the poles of scalar one-loop determinants on Lens spaces. This gives us strong credence that a Wilson spool built from **Conditions I & II** will accurately describe the physics of matter on those backgrounds.

More broadly, we have taken care to state the content of these conditions in a manner not specific to spacetime dimension or sign of cosmological constant. Even outside the context of 3d quantum gravity, it is our expectation that **Conditions I & II** will provide a powerful asset to organizing quasinormal modes and evaluating one-loop determinants for any manifold which possesses a transitive group action, regardless of dimension.

On the role of finite dimensional $\mathfrak{su}(2)$ representations

In Sec. 2 we took great care to build $\mathfrak{su}(2)$ representations corresponding to massive spinning particle states; these representations are infinite dimensional and non-standard. The reader might then find it surprising that finite dimensional representations played such important role in Sec. 3.1 (particularly in the implementation of **Condition II**). It is worth disentangling the roles of these two separate series of representations.

We regard the mass-shell condition, (3.12), as always providing a link between representation theory and the particle mass. For physical values of the mass, these representations are generically non-standard. The implementation of **Condition II** is a separate statement about the analytic structure of Z_{STT}^2, namely it encounters poles on finite dimensional representations of $\mathfrak{su}(2)$. These however do not (necessarily) lie on physical values of the mass. In evaluating $\mathbb{W}_{j_L,j_R}$ our goal is not to land on a pole; our goal is to evaluate a one-loop determinant for a physical field. As such the representations appearing in (1.4) should be appropriate for a physical value of Δ and are, again, generically non-standard. Obtaining the right hand side of (3.41) is a non-trivial verification of this fact.

We can further illustrate this distinction by investigating what happens if we insert finite dimensional representations $j_{L/R} \in \frac{1}{2}\mathbb{Z}$ directly into the Wilson spool (say, for the classical S^3 background). For the sake of this illustration we will take a non-spinning case

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$j_L = j_R = j$:

$$\mathbb{W}_j\big|_{\text{finite dim rep}} = \frac{i}{2} \int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha} \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} \frac{\sin^2\left((2j+1)\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)} \ , \tag{6.1}$$

where we have inserted directly the finite dimensional $\mathfrak{su}(2)$ characters $\chi_j(z) = \frac{\sin(\pi(2j+1)z)}{\sin(\pi z)}$ and utilized the holonomies around $\gamma_{\text{hor.}}$, (3.18). However it is clear what will happen if we wrap $\mathcal{C}$ as in Fig. 2: the poles at $\alpha \in 2\pi\mathbb{Z}_{\neq 0}$ are now only first order and now without any exponential damping. We thus find

$$\mathbb{W}_j\big|_{\text{finite dim rep}} = \sum_{n=1}^{\infty} \frac{(2j+1)^2}{n} = (2j+1)^2\zeta(1) = \infty \ , \tag{6.2}$$

where ζ is the Riemann zeta function. This illustrates the importance of utilizing the non-standard representations (corresponding to physical masses) in $\mathbb{W}$.

Massive versus massless spinning fields

It is worth commenting on the important distinction between describing massive spinning fields (the focus of this work) versus massless (higher) spin fields in the Chern-Simons formalism. Our comments are very much motivated by viewing Chern-Simons gravity as an effective field theory. This effective field theory provides a description of the physics below some mass gap; indeed we can expect that massive degrees of freedom can be "integrated" out, leaving behind an effective response on the remaining low-energy degrees of freedom. This is precisely what the spinning Wilson spool encapsulates: the response of massive degrees of freedom directly in variables natural to the Chern-Simons path integral.

We contrast this with massless degrees of freedom which cannot be integrated out. Instead their presence must alter the low-energy effective field theory. This is an indication that the one-loop determinants of massless higher-spin fields do not have a Wilson spool description. Instead, we already know they are described by modifying the effective field theory to a theory of *higher-spin gravity*. For example, one simple way of describing massless higher-spin fields up to spin-N is by replacing $SL(2,\mathbb{R}) \to SL(N,\mathbb{R})$ in asymptotically AdS$_3$ spacetimes [52–54] and $SU(2) \to SU(N)$ in asymptotically Euclidean dS$_3$ spacetimes [8, 55].

These above comments lead naturally to a line of inquiry: "How does one couple massive matter to a theory of higher-spin gravity?" The relations between Wilson lines and particle worldlines in higher-spin gravity have been explored, e.g., in [56, 57]. However we believe the Wilson spool (generalized to, e.g., $SU(N)$ or $SL(N,\mathbb{R})$) will be an invaluable tool for a more complete answer to this important open question.

Generalized symmetries in 3d gravity

Let us offer a final, speculative, remark on the insights that the Wilson spool can possibly lend to our understanding of gravity as an effective field theory. A modern perspective

on organizing low-energy physics is the extension of the Landau paradigm of broken symmetries to higher-form and non-invertible (what we will collectively call, generalized) symmetries [58, 59]. Attempts to categorize gravity as a low-energy phase through this lens include [60–62]. An important conjecture along these lines is the wide-held belief that all global symmetries are either gauged or explicitly broken in a UV theory of quantum gravity [63] and these statements extend to generalized symmetries.

Chern-Simons theory provides a natural area for exploring these discussions applied to three-dimensional gravity: the Wilson line operators of Chern-Simons theory both generate and are mutually charged under a generalized symmetry.[18] It naturally follows that the Wilson spool is also charged under this generalized symmetry and its non-zero expectation value is a direct sign of the explicit breaking of this symmetry through the inclusion of matter. While the spectrum of operators in Chern-Simons theory might be too large, we note the special role non-standard representation theory plays in this statement: as noted above, the Wilson spool associated to a finite-dimensional $\mathfrak{su}(2)$ representation diverges even classically, (6.2). It would be very interesting to make more precise the relation between the Wilson spool and the generalized symmetries of Chern-Simons theory as well as what these relations lend to the question, "What is 3d gravity?"[19]

Acknowledgements

A   Chern-Simons gravity dictionary

A.1   Euclidean dS$_3$

Three-dimensional de Sitter spacetime is a maximally symmetric spacetime with two-sphere spatial slices expanding into future and past infinity. Due to this expansion, inertial

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[18]We are being imprecise about the distinction between topological operators and topological defects here. Because we often have in mind Euclidean signature, we will call an operator anything that can be inserted into the path integral to obtain an expectation value.

[19]We thank Nabil Iqbal for extended discussions in this direction.

observers have access to a finite portion of spacetime called the static patch. A coordinate patch covering one-half of this patch is given by (t, ρ, φ) with $\rho \in [0, \frac{\pi}{2})$, $t \in (-\infty, \infty)$, and $\varphi \in [0, 2\pi)$ with φ a periodic coordinate. The observer's origin and causal horizon lie at $\rho = 0$ and $\rho = \frac{\pi}{2}$, respectively. The corresponding metric of the static patch is

$$\frac{\mathrm{d}s^2}{\ell_{\mathrm{dS}}^2} = -\cos^2 \rho \, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2 \rho \, \mathrm{d}\varphi^2 \ . \tag{A.1}$$

Above, the "de Sitter radius," ℓ_{dS}, sets the length scale for this maximally symmetric spacetime. For the purpose of moving between the following Chern-Simons description and the metric description, as well for computing one-loop determinants, it will be useful to go to Euclidean signature through the Wick-rotation, $t = -i\tau$. Under this rotation the static-patch rotates to a three-sphere in "torus coordinates"

$$\frac{\mathrm{d}s^2}{\ell_{\mathrm{dS}}^2} = \cos^2 \rho \, \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sin^2 \rho \, \mathrm{d}\varphi^2 \ . \tag{A.2}$$

Absence of a conical singularity at the horizon, $\rho = \frac{\pi}{2}$, sets $\tau \sim \tau + 2\pi$. The isometry group of this Euclidean space is $SU(2)_L \times SU(2)_R/\mathbb{Z}_2$ with the L/R denoting left/right group action. We will denote the generators of these two actions as $\{L_a\}_{a=1,2,3}$ and $\{\bar{L}_a\}_{a=1,2,3}$, respectively.

In accordance with the split structure of this isometry group, we will describe Euclidean dS$_3$ gravity with a pair of $SU(2)$ Chern-Simons theories

$$S = k_L \, S_{\mathrm{CS}}[A_L] + k_R \, S_{\mathrm{CS}}[A_R] \ , \tag{A.3}$$

with

$$S_{\mathrm{CS}}[A] = \frac{1}{4\pi} \mathrm{Tr} \int \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \ , \tag{A.4}$$

and the trace taken in the fundamental representation. The basic ingredients of the dictionary between this Chern-Simons description and the more familiar "metric description" is given by relating the gauge connections, $A_{L/R}$, to a dreibein, e^a, and (dual) spin-connection, $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$, via

$$A_L = i \left(\omega^a + \frac{1}{\ell_{\mathrm{dS}}} e^a \right) L_a \ , \qquad A_R = i \left(\omega^a - \frac{1}{\ell_{\mathrm{dS}}} e^a \right) \bar{L}_a \ . \tag{A.5}$$

The levels, $k_{L/R}$, are written as

$$k_L = \delta + is \ , \qquad k_R = \delta - is \ . \tag{A.6}$$

We can rewrite the action (A.3) as

$$iS = -I_{\mathrm{EH}} - i\delta I_{\mathrm{GCS}} \ , \tag{A.7}$$

with

$$I_{\mathrm{EH}} = -\frac{s}{4\pi\ell_{\mathrm{dS}}} \int \epsilon_{abc} e^a \wedge \left(R^{bc} - \frac{1}{3\ell_{\mathrm{dS}}^2} e^b \wedge e^c \right) \tag{A.8}$$

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the Euclidean Einstein-Hilbert term in the first-order formalism[20] with positive cosmological constant, $\Lambda = \ell_{\mathrm{dS}}^{-2}$. Thus $s \equiv \frac{\ell_{\mathrm{dS}}}{4G_N}$ is the (inverse) gravitational coupling; in this paper we will find it convenient to keep it written as s and keep in mind the classical limit is $s \to \infty$. I_{GCS} is the gravitational Chern-Simons term

$$I_{\mathrm{GCS}} = \frac{1}{2\pi}\mathrm{Tr}\int\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) + \frac{1}{2\pi\ell_{\mathrm{dS}}^2}\mathrm{Tr}\int e \wedge T \,, \qquad (A.9)$$

with T the torsion two-form. Under quantization the levels will undergo a finite renormalization $k_{L/R} \to r_{L/R} \equiv k_{L/R} + 2$ [64] amounting to a renormalization of the I_{GCS} coupling, $\delta \to \hat{\delta} = \delta + 2$. For the rest of this paper we will always work directly with the renormalized levels.

Appropriate classical flat background connections, $a_{L/R}$, describing the S^3 are given by

$$\begin{aligned} a_L &= iL_1\,\mathrm{d}\rho + i\left(\sin\rho L_2 - \cos\rho L_3\right)(\mathrm{d}\varphi - \mathrm{d}\tau) \,, \\ a_R &= -i\bar{L}_1\,\mathrm{d}\rho - i(\sin\rho\bar{L}_2 + \cos\rho\bar{L}_3)(\mathrm{d}\varphi + \mathrm{d}\tau) \,. \end{aligned} \qquad (A.10)$$

An important aspect of the above connections is that they each possess ring singularities at $\rho = 0$ and $\rho = \frac{\pi}{2}$ where the φ and τ coordinates degenerate, respectively. These Wick-rotate to the worldline at the static patch origin and to the causal horizon bifurcation surface, respectively. There is potential for holonomy around these singularities which we will write generically as

$$\mathcal{P}\exp\oint_\gamma a_L = g_\rho^{-1}e^{i2\pi L_3 \mathsf{h}_L}g_\rho \,, \qquad \mathcal{P}\exp\oint_\gamma a_R = \bar{g}_\rho\, e^{i2\pi\bar{L}_3\mathsf{h}_R}\bar{g}_\rho^{-1} \,, \qquad (A.11)$$

for periodic group elements, $g_\rho = e^{iL_1\rho}$ and $\bar{g}_\rho = e^{i\bar{L}_1\rho}$, and holonomies, $\mathsf{h}_{L/R}$. Given the solution (A.10), it is easy to deduce that for cycles, γ_{orig}, wrapping the static-patch origin at $\rho = 0$

$$\gamma_{\mathrm{orig}}: \qquad \mathsf{h}_L = 1 \,, \qquad \mathsf{h}_R = 1 \,, \qquad (A.12)$$

while for cycles, γ_{hor}, wrapping the causal horizon at $\rho = \frac{\pi}{2}$,

$$\gamma_{\mathrm{hor}}: \qquad \mathsf{h}_L = 1 \,, \qquad \mathsf{h}_R = -1 \,. \qquad (A.13)$$

Lastly we point out that these singularities give delta function sources of curvature[21] that is important for reproducing the on-shell action

$$ir_L S_{\mathrm{CS}}[a_L] + ir_R S_{\mathrm{CS}}[a_R] = \frac{\pi\ell_{\mathrm{dS}}}{2G_N} \,, \qquad (A.14)$$

which is the tree-level de Sitter entropy.

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[20]Here R^{ab} is the Riemann two-form.

[21]Importantly the metric geometry remains smooth everywhere.

A.2 Lorentzian AdS$_3$

For Anti-de Sitter space, we will start directly in Lorentzian signature where the relevant isometry group exhibits a similar split, $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. Accordingly gravity is described by two Chern-Simons theories, this time with opposite levels:

$$S = k S_{\text{CS}}[A_L] - k S_{\text{CS}}[A_R] = S_{\text{EH}} , \qquad k = \frac{\ell_{\text{AdS}}}{4 G_N} , \tag{A.15}$$

where S_{EH} is the Lorentzian Einstein-Hilbert action and ℓ_{AdS} is the AdS radius. This matching is facilitated by writing the connections as

$$A_L = \left(\omega^a + \frac{1}{\ell_{\text{AdS}}} e^a \right) L_a , \qquad A_R = \left(\omega^a - \frac{1}{\ell_{\text{AdS}}} e^a \right) \bar{L}_a , \tag{A.16}$$

where now $\{L_a\}_{a=0,\pm}$ and $\{\bar{L}_a\}_{a=0,\pm}$ generate the Lie algebras $\mathfrak{sl}(2,\mathbb{R})_L$ and $\mathfrak{sl}(2,\mathbb{R})_R$, respectively. We will follow the conventions set in [45, 46] for the $\mathfrak{sl}(2,\mathbb{R})$ algebra.

The general asymptotically AdS$_3$ vacuum solutions are Bañados geometries [65] described by background connections

$$a_L = L_0 \, \mathrm{d}\eta + \left(e^\eta L_+ - e^{-\eta} \frac{2\pi\mathcal{L}}{k} L_- \right) \mathrm{d}x^+ ,$$

$$a_R = - \bar{L}_0 \, \mathrm{d}\eta - \left(e^\eta \bar{L}_- - e^{-\eta} \frac{2\pi\bar{\mathcal{L}}}{k} \bar{L}_+ \right) \mathrm{d}x^- , \tag{A.17}$$

where $\mathcal{L}(x^+)$ and $\bar{\mathcal{L}}(x^-)$ are arbitrary functions. Here η is a radial coordinate with the conformal boudary at $\eta \to \infty$ and $x^\pm = t \pm \phi$ are the lightcone coordinates of the boundary cylinder. A special case of interest in this paper are the rotating BTZ black hole geometries described by

$$\mathcal{L} = \frac{M \ell_{\text{AdS}} + J}{4\pi} , \qquad \bar{\mathcal{L}} = \frac{M \ell_{\text{AdS}} - J}{4\pi} , \tag{A.18}$$

where M and J are the mass and spin of the black hole, respectively [41, 42].

In Euclidean signature, $t \to it_E$, smoothness of the horizon requires that Euclidean time is compact and so the geometry is the interior of a complex torus

$$(w, \bar{w}) \sim (w + 2\pi m + 2\pi n \tau, \bar{w} + 2\pi m + 2\pi \bar{\tau}) , \qquad m, n \in \mathbb{Z} , \tag{A.19}$$

where $w = \phi + it_E$ and $\bar{w} = \phi - it_E$ and with modular parameter

$$\tau = \frac{i}{2}\sqrt{\frac{k}{2\pi\mathcal{L}}} , \qquad \bar{\tau} = -\frac{i}{2}\sqrt{\frac{k}{2\pi\bar{\mathcal{L}}}} . \tag{A.20}$$

Around the two cycles of this torus the connections have holonomy

$$\mathcal{P} \exp \oint_\gamma a_L = u_L^{-1} \, e^{i2\pi L_0 \mathsf{h}_L} \, u_L , \qquad \mathcal{P} \exp \oint_\gamma a_R = u_R \, e^{i2\pi \bar{L}_0 \mathsf{h}_R} \, u_R^{-1} , \tag{A.21}$$

where $u_{L/R}$ are periodic along both cycles (only depending on η). For the contractible, thermal, cycle $\gamma_{\text{th}} : (w, \bar{w}) \to (w + \tau, \bar{w} + \bar{\tau})$ the connections have holonomy

$$\gamma_{\text{th}} : \qquad \mathsf{h}_L = -1 \ , \qquad \mathsf{h}_R = -1 \ , \tag{A.22}$$

while for the non-contractible spatial cycle, $\gamma_{\text{sp}} : (w, \bar{w}) \to (w + 2\pi, \bar{w} + 2\pi)$, they have holonomy

$$\gamma_{\text{sp}} : \qquad \mathsf{h}_L = -\frac{1}{\tau} \ , \qquad \mathsf{h}_R = -\frac{1}{\bar{\tau}} \ . \tag{A.23}$$

B   DHS for massive spinning fields on S^3

In this appendix, we will review DHS [29] as it was originally intended: first, determine the quasinormal modes from the equations of motion of a massive spinning field, Wick-rotating them carefully to Euclidean space, and then evaluate the one-loop as a product of the poles.

Special care is needed when implementing DHS for massive spinning fields. The quasinormal modes of massive fields on de Sitter are found directly from the equations of motion of the field in the manner of [66]. However, as shown in [67], for spin $\mathsf{s} > 0$ some care is required in Wick-rotating de Sitter quasinormal modes into regular modes on S^3, that is symmetric, transverse, traceless spherical harmonics. In particular, imposing the naïve fall-off condition on the frequency ω is not sufficient. We will review carefully and explicitly this regularity condition.

We will then construct the one-loop determinant from the quasinormal mode frequencies of a spin-s field on S^3 using the DHS method and regularise it via zeta function regularisation, as in the case for a scalar field in [29]. Identities of special functions are then utilised to rewrite these expressions in the form of polylogarithms to make explicit the comparison to the one-loop determinants derived in [8] and Sec. 3.3.

The metric for the de Sitter static patch in "torus coordinates" is given by (A.1). For the sake of finding the quasinormal modes of this background, we will also make use of the radial coordinate

$$r = \sin \rho \ . \tag{B.1}$$

B.1   Real massive scalar field

To introduce notation and the basic ingredients needed in subsequent sections, we will start by deriving the quasinormal modes of a real scalar field in dS$_3$ and then review the derivation of the one-loop determinant as done originally in DHS [29]. This will set up the procedure for massive spinning fields in the next subsections.

Quasinormal modes

First, we construct the solutions to the equations of motion for a scalar field Φ on dS$_3$

$$\nabla^2 \Phi = \ell_{\text{dS}}^2 m^2 \Phi \ , \tag{B.2}$$

where ∇^2 is the Laplacian on (A.1) and as in the main text $\ell_{\mathrm{dS}}^2 m^2 = \Delta(2 - \Delta)$. Suitable boundary conditions must also be imposed on the solutions to ensure that they Wick-rotate to a regular solution on S^3. These are:

- The solution must be single-valued in φ.

- The solution must be single-valued in imaginary time.

- The solution must be regular at the origin of the static patch ($r = 0$).

- The solution must be regular and purely in-going or out-going at the horizon ($r = 1$).

Proceeding via separation of variables we encounter a second order ODE for r, resulting in two solutions. Imposing regularity at the origin removes one of these solutions leaving us with the modes

$$\Phi = e^{-i\omega t + il\varphi} \left(r^2\right)^{\left|\frac{l}{2}\right|} \left(1 - r^2\right)^{-\frac{i\omega}{2}} {}_2F_1\left(a, b; c; r^2\right) , \tag{B.3}$$

where

$$a = \frac{\Delta + |l| - i\omega}{2} , \qquad b = \frac{\bar{\Delta} + |l| - i\omega}{2} , \qquad c = 1 + |l| , \tag{B.4}$$

and $l, i\omega \in \mathbb{Z}$ are constants constrained to the integers by the single-valuedness of Φ around φ and imaginary time respectively.

To consider regularity at the horizon we will expand the solution to lowest order as $\frac{\varepsilon^2}{2} = 1 - r^2$ approaches zero. Taking care as we are in a non-generic case for the hypergeometric function when $c - a - b = i\omega \in \mathbb{Z}$, the connection formulae for the hypergeometric function about r^2 and $1 - r^2$ [68] imply that a purely ingoing or outgoing solution will require at least one of $a, b, (c - a), (c - b) \in \mathbb{Z}_{\leq 0}$. Noting that $a \to b$ simply takes $\Delta \to \bar{\Delta}$ which has no effect on the form of the mode Φ, we need consider only two (not necessarily distinct) cases:

Case 1: $a = -n \in \mathbb{Z}_{\leq 0} \implies \Delta = -2n - |l| + i\omega$.

Case 2: $c - b = -n \in \mathbb{Z}_{\leq 0} \implies \Delta = -2n - |l| - i\omega$.

In both cases, the hypergeometric function takes the form of a Jacobi polynomial in terms of Pochhammer symbols (for the rising factorial). For **Case 1**, $a = -n$ gives

$$
{}_2F_1\left(-n, b; c; 1 - \frac{\varepsilon^2}{2}\right) = \sum_{d=0}^{n} \left(\frac{\varepsilon^2}{2}\right)^d \frac{(b)_d (c - b)_{n-d}}{(c)_n} , \tag{B.5}
$$

and for **Case 2** we make use of the identity ${}_2F_1\left(a, b; c; u\right) = (1-u)^{c-a-b} {}_2F_1\left(c - b, c - a; c; u\right)$. Substituting for a, b, c, we then find that to lowest order in ε the following:

Case 1 : $\Delta = -2n - |l| + i\omega$, $\qquad n \in \mathbb{Z}_{\leq 0}, i\omega \in \mathbb{Z}$,

$$
\Phi \sim \begin{cases} e^{-i\omega t} \varepsilon^{+i\omega} & \text{(anti-quasinormal)} \quad 0 \leq i\omega \leq n , \\ e^{-i\omega t} \varepsilon^{-i\omega} & \text{(quasinormal)} \qquad \text{else} , \end{cases} \tag{B.6}
$$

Case 2 : $\Delta = -2n - |l| - i\omega$, $\qquad n \in \mathbb{Z}_{\leq 0}, i\omega \in \mathbb{Z}$,

$$\Phi \sim \begin{cases} e^{-i\omega t}\varepsilon^{-i\omega} & \text{(quasinormal)} & 0 \leq -i\omega \leq n \ , \\ e^{-i\omega t}\varepsilon^{+i\omega} & \text{(anti-quasinormal)} & \text{else .} \end{cases} \tag{B.7}$$

For quasinormal modes regularity demands $i\omega \leq 0$, for anti-quasinormal modes regularity demands $i\omega \geq 0$. Note that for $i\omega = 0$ the modes are equivalent.

For $i\omega \geq 0$ **Case 2** is a subset of **Case 1** and the possible regular anti-quasinormal modes are

$$\Delta = -2\tilde{n} - |l| - i\omega \ , \qquad n - i\omega \equiv \tilde{n} \in \mathbb{N}_0 \ . \tag{B.8}$$

For $i\omega < 0$ **Case 1** is a subset of **Case 2** and the possible regular quasinormal modes are

$$\Delta = -2\tilde{n} - |l| + i\omega \ , \qquad n + i\omega \equiv \tilde{n} \in \mathbb{N}_0 \ . \tag{B.9}$$

These two expressions can be combined by allowing $i\omega \equiv k \in \mathbb{Z}$ to range over all integers (and relabelling $\tilde{n} \to n$ for convenience). The Euclidean regular (anti-)quasinormal modes take the values

$$\Delta = -2n - |l| - |k| \ . \tag{B.10}$$

Note that the modes with $\Delta \leftrightarrow \bar{\Delta}$ are equivalent due to the ambiguity in defining Δ from the mass. This was seen above by the invariance of Φ to the exchange, and we have hence counted these modes only once. In using the DHS method we consider the one-loop determinant as a meromorphic function of Δ. Since the determinant also cannot distinguish between $\Delta \leftrightarrow \bar{\Delta}$ there must be a pole at both Δ and $\bar{\Delta}$. The poles in the one-loop determinant thus appear at

$$\Delta, \bar{\Delta} = -2n - |l| - |k| \ . \tag{B.11}$$

It can also be confirmed that the modes we have obtained are (expectedly) the scalar spherical harmonics on S^3. More simply the degeneracy of each value of $\Delta = -p, p \in \mathbb{Z}_{\geq 0}$ can be counted as $(p+1)^2$ matching the degeneracy of the spherical harmonics fitting into representations of $\mathfrak{so}(4)$.

One-loop determinant

Following [29] the one-loop determinant can be constructed from the location of the quasinormal frequencies, treating the functional determinant as a meromorphic function of Δ. For convenience in this section the notation of [29] is also followed such that $\Delta \equiv \Delta_+$, and $\bar{\Delta} \equiv \Delta_-$.

Up to a holomorphic factor, which in this case is trivial, the partition function is simply a product of poles at the (anti-)quasinormal mode frequencies:

$$Z_{\text{scalar}} = \prod_{\substack{\pm, n \in \mathbb{N} \\ l, k \in \mathbb{Z}}} (|k| + 2n + |l| + \Delta_\pm)^{-\frac{1}{2}} \ . \tag{B.12}$$

Noting that $|k| + |l| + 2n = p$ has degeneracy $(p+1)^2$, we can rewrite (B.12) as

$$\log(Z_{\text{scalar}}) = -\frac{1}{2} \sum_{\pm, p \in \mathbb{N}} (p+1)^2 \log(p + \Delta_\pm)$$
$$= \frac{1}{2} \sum_{\pm} \zeta'(-2, 1 \pm i\mu) \mp 2i\mu\zeta'(-1, 1 \pm i\mu) - \mu^2\zeta'(0, 1 \pm i\mu) , \tag{B.13}$$

where $\Delta_\pm = 1 \pm i\mu$ for the principle series and ζ is the Hurwitz zeta function. This is the formula derived in [29] and it will be convenient to write in terms of polylogarithms. The two key the identities are

$$\zeta'(-2, \pm i\mu) \mp 2i\mu\zeta'(-1, \pm i\mu) - \mu^2\zeta'(0, \pm i\mu)$$
$$= \zeta'(-2, 1 \pm i\mu) \mp 2i\mu\zeta'(-1, 1 \pm i\mu) - \mu^2\zeta'(0, 1 \pm i\mu) , \tag{B.14}$$

and[22]

$$\zeta'(-n, x) + (-1)^n \zeta'(-n, 1 - x) = -\pi i \frac{B_{n+1}(x)}{n+1} + e^{\frac{\pi i n}{2}} \frac{n!}{(2\pi)^n} \text{Li}_{n+1}(e^{-2\pi i x}) , \tag{B.15}$$

where B_n are the Bernoulli polynomials. Substituting these into (B.13) we find

$$\log(Z_{\text{scalar}}) = \sum_{\pm} \left(-\frac{\mu^2}{4} \text{Li}_1(e^{\pm 2\pi\mu}) \pm \frac{\mu}{4\pi} \text{Li}_2(e^{\pm 2\pi\mu}) - \frac{1}{8\pi^2} \text{Li}_3(e^{\pm 2\pi\mu}) \right) , \tag{B.16}$$

in agreement with [8].

B.2 Massive spin-1 field

Next we work out in detail the quasinormal modes of a massive spin-1 field on dS$_3$, and then its one-loop determinant via DHS. Many results follow from the analysis of a scalar field. However, it is important to highlight the subtleties of the regularity condition in this case, and how it reproduces the appearance of the "edge" partition function [67].

Quasinormal modes

The equation of motion is

$$\left(\nabla^2_{(1)} - \ell^2_{\text{dS}} m^2 - 2 \right) \Phi_\mu = 0 , \tag{B.17}$$

accompanied by a transversality condition

$$\nabla^\mu \Phi_\mu = 0 . \tag{B.18}$$

---

[22]The identity (B.15) can be derived by adapting the proof from [69] to the region of the complex plane where $x \in i\mathbb{R}$.

These equations of motion are solved in [50] for an AdS_3 background using a first-order formalism, we present here the corresponding results in dS_3.[23]

As in the scalar case we discard one set of solutions to the radial part of the PDE as non-regular at the origin, and impose $l, i\omega \in \mathbb{Z}$ to ensure single-valuedness. The resulting expressions are

$$\Phi_L = e_L e^{-i\omega t + il\varphi} \left(r^2\right)^{\left|\frac{l}{2}\right|} \left(1 - r^2\right)^{-\frac{i\omega}{2}} {}_2F_1 \left(\frac{\bar{\Delta} + 1 + |l| - i\omega}{2}, \frac{\Delta - 1 + |l| - i\omega}{2}; 1 + |l|; r^2\right) ,$$

$$\Phi_R = -e_R e^{-i\omega t + il\varphi} \left(r^2\right)^{\left|\frac{l}{2}\right|} \left(1 - r^2\right)^{-\frac{i\omega}{2}} {}_2F_1 \left(\frac{\bar{\Delta} - 1 + |l| - i\omega}{2}, \frac{\Delta + 1 + |l| - i\omega}{2}; 1 + |l|; r^2\right) ,$$

$$\Phi_r = -\frac{i}{(\Delta - 1)} \frac{1}{r(1 - r^2)} \left[(\omega + il)\Phi_L + (\omega - il)\Phi_R\right] ,$$

$$\text{(B.19)}$$

where $\Phi_{L/R}$ are related to the components of the vector field via

$$i\Phi_\varphi = \Phi_L + \Phi_R ,$$
$$\Phi_t = \Phi_L - \Phi_R .$$
$$\text{(B.20)}$$

In (B.19) $e_{L/R}$ are constants of integration and describe the polarisation vector. These constants are further constrained by (B.17), (B.18) to satisfy one of two conditions depending on the sign of l.

$$(-\omega + il + i(\Delta - 1))e_R = (-\omega - il + i(\Delta - 1))e_L , \qquad l \leq 0 ,$$
$$(-\omega + il - i(\Delta - 1))e_R = (-\omega - il - i(\Delta - 1))e_L , \qquad l \geq 0 .$$
$$\text{(B.21)}$$

Next, we look for those modes which Wick-rotate into regular Euclidean modes on S^3. In order for a mode to be single-valued and regular a necessary condition is for Φ_L, Φ_R themselves to be single-valued and regular at the horizon. Applying the results from the scalar case (B.10) to (B.19) with $\Delta \to \Delta \pm 1$ implies that regular modes potentially occur at

$$\Phi_L : \qquad \Delta = 1 - 2N_L - |l| - |k| \tag{B.22}$$
$$\text{or } \bar{\Delta} = 1 - 2(N_L' + 1) - |l| - |k|$$
$$\Phi_R : \qquad \Delta = 1 - 2(N_R' + 1) - |l| - |k| \tag{B.23}$$
$$\text{or } \bar{\Delta} = 1 - 2N_R - |l| - |k| .$$

Unlike in the scalar case there is no symmetry of $\Delta \to \bar{\Delta}$ in Φ_μ and so the two cases result in distinct physical modes.

---

[23]One method to derive these solutions is the analytic continuation of $\ell_{\text{dS}} \to i\ell_{\text{AdS}}$ at which point the results of [50] can be used directly.

For Φ_μ to have a mode at some value of Δ or $\bar{\Delta}$ it is necessary for both Φ_L and Φ_R to support a mode. We will suppose that k, l are fixed and consider two[24] families of solutions:

- A mode with $N_{L/R} \geq 1$ in one of the two components and $N'_{R/L} = N_{L/R} - 1 \geq 0$ in the other. We denote these the 'general modes' valid for all $N' \geq 0$:

values	ranges
$\Delta = -1 - 2N' - |l| - |k|$	$k, l \in \mathbb{Z}, N' \in \mathbb{Z}_{\geq 0}$
$\bar{\Delta} = -1 - 2N' - |l| - |k|$	$k, l \in \mathbb{Z}, N' \in \mathbb{Z}_{\geq 0}$

- If $N_{L/R} = 0$ in one component it may be possible to still have a consistent mode if the other component is identically 0. For example setting $N_L = 0$ in (B.22) it may be permissible from (B.21) to set $e_R = 0$. A unique list (accounting for overlaps where $k, l = 0$) of these 'special modes' is

values	ranges
	$k < 0, l \geq 0$
$\Delta = 1 - |l| - |k|$	$k = 0, l \in \mathbb{Z}$
	$k > 0, l \leq 0$
	$k < 0, l \leq 0$
$\bar{\Delta} = 1 - |l| - |k|$	$k = 0, l \in \mathbb{Z} \setminus \{0\}$
	$k > 0, l \geq 0$

Since the regularity of Φ_L, Φ_R is only a necessary condition it is still important to check the regularity of the Euclidean norm $|\Phi|^2 = \Phi_\mu \Phi^\mu$. This will determine regularity in a coordinate-independent manner. Consider first regularity as $r \to 0$ at the origin of the static patch. The only cases which can cause divergences are those for which $l = 0$. In this case, to lowest order in r we find

$$|\Phi|^2 \sim \frac{1}{r^2} \left(\frac{|\omega|^2}{(\Delta - 1)^2} + 1 \right) |e_L - e_R|^2 . \tag{B.24}$$

The first bracket is always non-zero due to the reality of Δ and so regularity implies $e_L = e_R$. For $l = 0$ general modes this is always true from (B.21). For the special modes one of $e_{L/R}$ is zero and the other is non-zero and so these modes are irregular.

As in the scalar case, looking at the horizon we can expand in small ε for $r^2 = 1 - \frac{\varepsilon^2}{2}$. Again using the expansion (B.5) the only potentially problematic case is $k = 0$, in which case this expansion gives to leading order

$$|\Phi|^2 \sim \frac{1}{\varepsilon^2} \left(1 + \frac{l^2}{(\Delta - 1)^2} \right) \left| e_L \frac{(c_L - b_L)_{n_L}}{(c_L)_{n_L}} + e_R \frac{(c_R - b_R)_{n_R}}{(c_R)_{n_R}} \right|^2 , \tag{B.25}$$

---

[24]There is also a single exceptional case where $\Delta = \bar{\Delta} = 1$ and we can choose a mode in each component by taking $N_L = N_R = l = k = 0$. This also automatically satisfies the constraints of (B.21). It can be checked using the regularity conditions below that this solution cannot be regular at both the origin and horizon.

where again the constants $a = -n, b, c$ are defined as coefficients in the hypergeometric function for each component. The general modes with $k = 0$ satisfy

$$\Delta = -1 - 2N' - |l| \qquad \text{or} \qquad \bar{\Delta} = -1 - 2N' - |l| \ . \tag{B.26}$$

Inserting the first of these into the coefficients of the hypergeometric function and noting that from (B.21) it is possible to choose

$$
\begin{aligned}
e_L &= +|l| - \Delta + 1 = 2(N' + 1 + |l|) \ , \\
e_R &= -|l| - \Delta + 1 = 2(N' + 1) \ ,
\end{aligned}
\tag{B.27}
$$

it therefore follows[25] that

$$
\begin{aligned}
&e_L \frac{(c_L - b_L)_{n_L}}{(c_L)_{n_L}} + e_R \frac{(c_R - b_R)_{n_R}}{(c_R)_{n_R}} \\
&= 2(N' + 1 + |l|) \frac{(-N' - 1)_{N'+1}}{(1 + |l|)_{N'+1}} + 2(N' + 1) \frac{(-N')_{N'}}{(1 + |l|)_{N'}} = 0 \ .
\end{aligned}
\tag{B.28}
$$

The same process works in the $\bar{\Delta}$ case. All the general modes therefore are regular.

In the special cases with $k = 0$ and $\Delta, \bar{\Delta} = 1 - |l|$ either $\Phi_L = 0, \Phi_R \neq 0, n_R = 0$ or $\Phi_R = 0, \Phi_L \neq 0, n_L = 0$. Noting that $(x)_0 \equiv 1$ the leading order term does not vanish and the special modes with $k = 0$ are not regular. Combining the results of regularity at the origin and horizon the final list of regular Euclidean modes is

values	ranges
$\Delta = -1 - 2N' - |l| - |k|$	$k, l \in \mathbb{Z}, N' \in \mathbb{Z}_{\geq 0}$
$\bar{\Delta} = -1 - 2N' - |l| - |k|$	$k, l \in \mathbb{Z}, N' \in \mathbb{Z}_{\geq 0}$
$\Delta = 1 - |l| - |k|$	$k < 0, l > 0$
	$k > 0, l < 0$
$\bar{\Delta} = 1 - |l| - |k|$	$k < 0, l < 0$
	$k > 0, l > 0$

It can be confirmed that the degeneracy of the modes at $\Delta = -p, p \geq 1$ is $p(p + 2)$. This aligns with the expected degeneracies of transverse spherical vector harmonics on S^3, which fit into representations of $\mathfrak{so}(4)$.

As in the scalar case we note that while each of these modes is distinct, the one-loop determinant has an ambiguity under $\Delta \leftrightarrow \bar{\Delta}$ and so we must include a pole for each. Since the (anti-)quasinormal modes listed above take the same form for Δ and $\bar{\Delta}$ this effectively doubles the degeneracy of the poles in the one-loop determinant relative to the modes listed in the table.

---

[25] The identities $(-n)_n = (-1)^n n!$ and $(1 + |l|)_{N'+1} = (1 + |l|)_{N'}(1 + |l| + N')$ are useful here.

One-loop determinant

We recall from Sec. 3 that for spinning fields we must account for both a set of zero modes (3.7), due to the compact nature of S^3, as well as the functional determinant of a symmetric, transverse, traceless field (3.2). The DHS method can only reproduce this second part of the calculation. Repeating the procedure in the scalar case above for the spin-1 field, the relevant functional determinant for a transverse vector field is

$$Z_{\text{STT}} = \prod_{\substack{\pm, n \in \mathbb{N} \\ l, k \in \mathbb{Z}}} (|k| + 2n + |l| + \Delta_\pm + 1)^{-1} \prod_{\substack{\pm, \{l>0, k<0\} \\ \cup \{l<0, k>0\}}} (|k| + |l| + \Delta_\pm - 1)^{-1} \tag{B.29}$$

and hence

$$\begin{aligned}
\log(Z_{\text{STT}}) &= -\sum_{\substack{\pm, n \in \mathbb{N} \\ l, k \in \mathbb{Z}}} \log\left(|k| + 2n + |l| + \Delta_\pm + 1\right) - \sum_{\substack{\pm, \{l>0, k<0\} \\ \cup \{l<0, k>0\}}} \log\left(|k| + |l| + \Delta_\pm - 1\right) \\
&= -\sum_{\pm, p=0}^{\infty} \log(\Delta_\pm + 1 + p)(p+1)^2 - \sum_{\pm, p=1}^{\infty} \log(\Delta_\pm + p)(2p) \\
&= -\sum_{\pm, p=0}^{\infty} \log(\Delta_\pm + p)(p+1)^2 + \sum_{\pm, p=0}^{\infty} \log(\Delta_\pm + p) \ .
\end{aligned} \tag{B.30}$$

We now identify the first of these sums as exactly $2\log(Z_{\text{scalar}})$ in (B.12), and for the second term it is convenient to define

$$\begin{aligned}
\log(Z_{\text{edge}}) &\equiv -\sum_{\pm, p=0}^{\infty} \log(\Delta_\pm + p) \\
&= \sum_{\pm} \zeta'(0, \Delta_\pm) \ .
\end{aligned} \tag{B.31}$$

We refer to this part of the partition function by the term 'edge' in reference to [8], where this term is viewed as an 'edge character integral'. Similarly to before the expansion of the zeta function implies that

$$\zeta'(0, \pm i\mu) = \zeta'(0, 1 \pm i\mu) - \log(\pm i\mu) \ , \tag{B.32}$$

and hence

$$\begin{aligned}
\log(Z_{\text{edge}}) &= \frac{1}{2} \sum_{\pm} \zeta'(0, 1 \pm i\mu) + \zeta'(0, \pm i\mu) + \log(\pm i\mu) \\
&= \frac{1}{2} \sum_{\pm} -\pi i B_1(\pm i\mu) + \text{Li}_1(e^{\pm 2\pi\mu}) + \log(\pm i\mu) \ .
\end{aligned} \tag{B.33}$$

Using the definition of the Bernoulli polynomial and Li_1 and inserting this expression for $\log(Z_{\text{edge}})$ into the main calculation, we arrive at

$$\log(Z_{\text{STT}}) = 2\log(Z_{\text{scalar}}) - \left(-\pi\mu - \log\left(1 - e^{-2\pi\mu}\right)\right) - \frac{1}{2}\log\left(\mu^2\right) \ . \tag{B.34}$$

Finally, we need to account for the zero modes. Following [8], the contribution is given by (3.7), and with this the full one-loop determinant is

$$Z_{\Delta,\mathsf{s}=1} = Z_{\text{STT}} Z_{\text{zero}} = Z_{\text{STT}} \prod_{\pm} (\Delta_{\pm} - 1)^{\frac{1}{2}} \,, \tag{B.35}$$

and so

$$\log(Z_{\Delta,\mathsf{s}=1}) = 2\log(Z_{\text{scalar}}) - \left(-\pi\mu - \log\left(1 - e^{-2\pi\mu}\right)\right) \,. \tag{B.36}$$

B.3 Massive spin-s fields

It is now clear how the story should continue for massive spin-s fields, when $\mathsf{s} \geq 2$. We solve the equations of motion for a spin-s field

$$\nabla^2 \Phi_{\mu_1\ldots\mu_\mathsf{s}} = \ell_{\text{dS}}^2 \bar{m}_\mathsf{s}^2 \Phi_{\mu_1\ldots\mu_\mathsf{s}} \,, \tag{B.37}$$

where Φ is symmetric and subject to the usual constraints

$$\nabla^{\mu_1} \Phi_{\mu_1\ldots\mu_\mathsf{s}} = 0, \qquad \Phi^{\nu}{}_{\nu\mu_3\ldots\mu_\mathsf{s}} = 0 \,. \tag{B.38}$$

Applying (anti-)quasinormal mode boundary conditions we can extract the (anti-)quasinormal mode frequencies. A careful treatment of these quasinormal modes will show that (as expected) the quasinormal modes which Wick-rotate to regular Euclidean modes on S^3 are the symmetric, transverse, traceless spherical harmonics. These lie in representations of $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ labelled by highest weights $(\frac{n}{2} + \mathsf{s}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n}{2} + \mathsf{s})$.

The corresponding degeneracies for $\Delta = -p, p \geq -\mathsf{s}$ are $(\mathsf{s} + 1 + p)(1 - \mathsf{s} + p)$ and so

$$\begin{aligned}
\log(Z_{\text{STT}}) &= -\sum_{\pm,p=\mathsf{s}}^{\infty} \log(\Delta_{\pm} + p)(p + 1 + \mathsf{s})(p + 1 - \mathsf{s}) \\
&= -\sum_{\pm,p=0}^{\infty} \log(\Delta_{\pm} + p)(p+1)^2 + \mathsf{s}^2 \sum_{\pm,p=0}^{\infty} \log(\Delta_{\pm} + p) \\
&\quad + \sum_{\pm,p=0}^{\mathsf{s}-2} \log(\Delta_{\pm} + p)(p + 1 + \mathsf{s})(p + 1 - \mathsf{s}) \,.
\end{aligned} \tag{B.39}$$

The first of these sums reproduces again $2\log(Z_{\text{scalar}})$ in (B.13) and the second sum is proportional to $\log(Z_{\text{edge}})$ as defined in (B.31). We can directly insert our previous expression for this.

$$\begin{aligned}
\log(Z_{\text{STT}}) =& 2\log(Z_{\text{scalar}}) - \mathsf{s}^2 \left(-\pi\mu - \log\left(1 - e^{-2\pi\mu}\right)\right) - \frac{\mathsf{s}^2}{2} \log(\mu^2) \\
& + \sum_{\pm,p=0}^{\mathsf{s}-2} \log(\Delta_{\pm} + p)(p + 1 + \mathsf{s})(p + 1 - \mathsf{s}) \,.
\end{aligned} \tag{B.40}$$

40

As in the spin-1 case, to compute the full local partition function we also need to account for a set of zero modes (3.7). Including these, we obtain the final result

$$\log(Z_{\Delta,\mathsf{s}}) = \log(Z_{\text{STT}}) + \log(Z_{\text{zero}}) = 2\log(Z_{\text{scalar}}) - \mathsf{s}^2 \left(-\pi\mu - \log\left(1 - e^{-2\pi\mu}\right)\right) \ . \quad \text{(B.41)}$$

This formula is again in agreement with the calculation in [8], and it matches with the Wilson spool in (3.41) (up to overall phase).

References

[1] A. Castro, I. Coman, J. R. Fliss, and C. Zukowski, "Keeping matter in the loop in dS$_3$ quantum gravity," *JHEP* **07** (2023) 120, `arXiv:2302.12281 [hep-th]`.

[2] A. Castro, I. Coman, J. R. Fliss, and C. Zukowski, "Coupling Fields to 3D Quantum Gravity via Chern-Simons Theory," *Phys. Rev. Lett.* **131** (Oct, 2023) 171602, `arXiv:2304.02668 [hep-th]`.

[3] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys. Lett.* **B180** (1986) 89.

[4] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl. Phys. B* **311** (1988) 46.

[5] M. Banados, T. Brotz, and M. E. Ortiz, "Quantum three-dimensional de Sitter space," *Phys. Rev. D* **59** (1999) 046002, `arXiv:hep-th/9807216`.

[6] T. R. Govindarajan, R. K. Kaul, and V. Suneeta, "Quantum gravity on dS(3)," *Class. Quant. Grav.* **19** (2002) 4195–4205, `arXiv:hep-th/0203219`.

[7] J. Cotler and K. Jensen, "A theory of reparameterizations for AdS$_3$ gravity," *JHEP* **02** (2019) 079, `arXiv:1808.03263 [hep-th]`.

[8] D. Anninos, F. Denef, Y. T. A. Law, and Z. Sun, "Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions," *JHEP* **01** (2022) 088, `arXiv:2009.12464 [hep-th]`.

[9] S. Collier, L. Eberhardt, and M. Zhang, "Solving 3d gravity with Virasoro TQFT," *SciPost Phys.* **15** no. 4, (2023) 151, `arXiv:2304.13650 [hep-th]`.

[10] S. Carlip, "The Sum over topologies in three-dimensional Euclidean quantum gravity," *Class. Quant. Grav.* **10** (1993) 207–218, `arXiv:hep-th/9206103`.

[11] E. Guadagnini and P. Tomassini, "Sum over the geometries of three manifolds," *Phys. Lett. B* **336** (1994) 330–336.

[12] E. Witten, "Three-Dimensional Gravity Revisited," `arXiv:0706.3359 [hep-th]`.

[13] A. Castro, N. Lashkari, and A. Maloney, "A de Sitter Farey Tail," *Phys. Rev. D* **83** (2011) 124027, `arXiv:1103.4620 [hep-th]`.

[14] A. Maloney and E. Witten, "Quantum Gravity Partition Functions in Three Dimensions," *JHEP* **02** (2010) 029, `arXiv:0712.0155 [hep-th]`.

[15] S. Banerjee, R. K. Gupta, I. Mandal, and A. Sen, "Logarithmic Corrections to N=4 and N=8 Black Hole Entropy: A One Loop Test of Quantum Gravity," *JHEP* **11** (2011) 143, arXiv:1106.0080 [hep-th].

[16] A. Sen, "Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions," *JHEP* **04** (2013) 156, arXiv:1205.0971 [hep-th].

[17] S. Bhattacharyya, A. Grassi, M. Marino, and A. Sen, "A One-Loop Test of Quantum Supergravity," *Class. Quant. Grav.* **31** (2014) 015012, arXiv:1210.6057 [hep-th].

[18] A. Sen, "Logarithmic Corrections to N=2 Black Hole Entropy: An Infrared Window into the Microstates," *Gen. Rel. Grav.* **44** no. 5, (2012) 1207–1266, arXiv:1108.3842 [hep-th].

[19] S. Giombi and I. R. Klebanov, "One Loop Tests of Higher Spin AdS/CFT," *JHEP* **12** (2013) 068, arXiv:1308.2337 [hep-th].

[20] H. Ooguri and C. Vafa, "Knot invariants and topological strings," *Nucl. Phys. B* **577** (2000) 419–438, arXiv:hep-th/9912123.

[21] M. He, "One-loop partition functions in $T\overline{T}$-deformed AdS$_3$," *JHEP* **05** (2024) 067, arXiv:2401.09879 [hep-th].

[22] M. Blau and G. Thompson, "Derivation of the Verlinde formula from Chern-Simons theory and the G/G model," *Nucl. Phys. B* **408** (1993) 345–390, arXiv:hep-th/9305010.

[23] M. Blau and G. Thompson, "Chern-Simons theory on S1-bundles: Abelianisation and q-deformed Yang-Mills theory," *JHEP* **05** (2006) 003, arXiv:hep-th/0601068.

[24] M. Blau and G. Thompson, "Chern-Simons Theory on Seifert 3-Manifolds," *JHEP* **09** (2013) 033, arXiv:1306.3381 [hep-th].

[25] A. Kapustin, B. Willett, and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," *JHEP* **03** (2010) 089, arXiv:0909.4559 [hep-th].

[26] M. Marino, "Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories," *J. Phys. A* **44** (2011) 463001, arXiv:1104.0783 [hep-th].

[27] E. Joung, J. Mourad, and R. Parentani, "Group theoretical approach to quantum fields in de Sitter space. I. The Principle series," *JHEP* **08** (2006) 082, arXiv:hep-th/0606119.

[28] Z. Sun, "A note on the representations of $SO(1, d+1)$," arXiv:2111.04591 [hep-th].

[29] F. Denef, S. A. Hartnoll, and S. Sachdev, "Black hole determinants and quasinormal modes," *Class. Quant. Grav.* **27** (2010) 125001, arXiv:0908.2657 [hep-th].

[30] E. Joung, J. Mourad, and R. Parentani, "Group theoretical approach to quantum fields in de Sitter space. II. The complementary and discrete series," *JHEP* **09** (2007) 030, `arXiv:0707.2907 [hep-th]`.

[31] T. Basile, X. Bekaert, and N. Boulanger, "Mixed-symmetry fields in de Sitter space: a group theoretical glance," *JHEP* **05** (2017) 081, `arXiv:1612.08166 [hep-th]`.

[32] J. Penedones, K. Salehi Vaziri, and Z. Sun, "Hilbert space of Quantum Field Theory in de Sitter spacetime," `arXiv:2301.04146 [hep-th]`.

[33] A. Castro, P. Sabella-Garnier, and C. Zukowski, "Gravitational Wilson Lines in 3D de Sitter," *JHEP* **07** (2020) 202, `arXiv:2001.09998 [hep-th]`.

[34] Y. M. Zinoviev, "On massive high spin particles in (A)dS," `arXiv:hep-th/0108192`.

[35] M. A. Rubin and C. R. Ordóñez, "Eigenvalues and degeneracies for n-dimensional tensor spherical harmonics," *Journal of Mathematical Physics* **25** no. 10, (10, 1984) 2888–2894.

[36] R. Howe and E. C. Tan, *"Non-abelian harmonic analysis: applications of SL(2, R)"*. Universitext. 1st ed., 1992.

[37] C. Beasley and E. Witten, "Non-Abelian localization for Chern-Simons theory," *J. Diff. Geom.* **70** (2005) 183–323, `arXiv:hep-th/0503126`.

[38] C. Beasley, "Localization for Wilson Loops in Chern-Simons Theory," *Adv. Theor. Math. Phys.* **17** (2013) 1–240, `arXiv:0911.2687 [hep-th]`.

[39] Y. Hikida, T. Nishioka, T. Takayanagi, and Y. Taki, "Holography in de Sitter Space via Chern-Simons Gauge Theory," *Phys. Rev. Lett.* **129** no. 4, (2022) 041601, `arXiv:2110.03197 [hep-th]`.

[40] M. Leston, A. Goya, G. Pérez-Nadal, M. Passaglia, and G. Giribet, "3D Quantum Gravity Partition Function at Three Loops," *Phys. Rev. Lett.* **131** no. 18, (2023) 181601, `arXiv:2307.03830 [hep-th]`.

[41] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, `arXiv:hep-th/9204099`.

[42] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the (2+1) black hole," *Phys. Rev. D* **48** (1993) 1506–1525, `arXiv:gr-qc/9302012`. [Erratum: Phys.Rev.D 88, 069902 (2013)].

[43] S. Datta and J. R. David, "Higher Spin Quasinormal Modes and One-Loop Determinants in the BTZ black Hole," *JHEP* **03** (2012) 079, `arXiv:1112.4619 [hep-th]`.

[44] C. Sleight, "Metric-like Methods in Higher Spin Holography," *PoS* **Modave2016** (2017) 003, `arXiv:1701.08360 [hep-th]`.

[45] M. Gutperle and P. Kraus, "Higher Spin Black Holes," *JHEP* **05** (2011) 022, `arXiv:1103.4304 [hep-th]`.

[46] A. Castro, N. Iqbal, and E. Llabrés, "Wilson lines and Ishibashi states in AdS$_3$/CFT$_2$," *JHEP* **09** (2018) 066, `arXiv:1805.05398 [hep-th]`.

[47] A. Kitaev, "Notes on $\widetilde{sl}(2, \mathbb{R})$ representations," `arXiv:1711.08169 [hep-th]`.

[48] S. Giombi, A. Maloney, and X. Yin, "One-loop Partition Functions of 3D Gravity," *JHEP* **08** (2008) 007, `arXiv:0804.1773 [hep-th]`.

[49] J. R. David, M. R. Gaberdiel, and R. Gopakumar, "The Heat Kernel on AdS(3) and its Applications," *JHEP* **04** (2010) 125, `arXiv:0911.5085 [hep-th]`.

[50] A. Castro, C. Keeler, and P. Szepietowski, "Tweaking one-loop determinants in AdS$_3$," *JHEP* **10** (2017) 070, `arXiv:1707.06245 [hep-th]`.

[51] R. Bourne, A. Castro, J. R. Fliss, and A. Law (*work in progress*).

[52] M. P. Blencowe, "A Consistent Interacting Massless Higher Spin Field Theory in $D = (2+1)$," *Class. Quant. Grav.* **6** (1989) 443.

[53] E. Bergshoeff, M. P. Blencowe, and K. S. Stelle, "Area Preserving Diffeomorphisms and Higher Spin Algebra," *Commun. Math. Phys.* **128** (1990) 213.

[54] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, "Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields," *JHEP* **11** (2010) 007, `arXiv:1008.4744 [hep-th]`.

[55] E. Joung and K. Mkrtchyan, "Notes on higher-spin algebras: minimal representations and structure constants," *JHEP* **05** (2014) 103, `arXiv:1401.7977 [hep-th]`.

[56] J. de Boer and J. I. Jottar, "Entanglement Entropy and Higher Spin Holography in AdS$_3$," *JHEP* **04** (2014) 089, `arXiv:1306.4347 [hep-th]`.

[57] M. Ammon, A. Castro, and N. Iqbal, "Wilson Lines and Entanglement Entropy in Higher Spin Gravity," *JHEP* **10** (2013) 110, `arXiv:1306.4338 [hep-th]`.

[58] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," *JHEP* **02** (2015) 172, `arXiv:1412.5148 [hep-th]`.

[59] J. McGreevy, "Generalized Symmetries in Condensed Matter," *Ann. Rev. Condensed Matter Phys.* **14** (2023) 57–82, `arXiv:2204.03045 [cond-mat.str-el]`.

[60] V. Benedetti, H. Casini, and J. M. Magan, "Generalized symmetries of the graviton," *JHEP* **05** (2022) 045, `arXiv:2111.12089 [hep-th]`.

[61] K. Hinterbichler, D. M. Hofman, A. Joyce, and G. Mathys, "Gravity as a gapless phase and biform symmetries," *JHEP* **02** (2023) 151, `arXiv:2205.12272 [hep-th]`.

[62] F. Benini, C. Copetti, and L. Di Pietro, "Factorization and global symmetries in holography," *SciPost Phys.* **14** no. 2, (2023) 019, `arXiv:2203.09537 [hep-th]`.

[63] D. Harlow and H. Ooguri, "Symmetries in quantum field theory and quantum gravity," *Commun. Math. Phys.* **383** no. 3, (2021) 1669–1804, `arXiv:1810.05338 [hep-th]`.

[64] E. Witten, "Quantization of Chern-Simons Gauge Theory with Complex Gauge Group," *Commun. Math. Phys.* **137** (1991) 29–66.

[65] M. Banados, "Three-dimensional quantum geometry and black holes," *AIP Conf. Proc.* **484** no. 1, (1999) 147–169, `arXiv:hep-th/9901148`.

[66] A. López-Ortega, "Quasinormal modes of D-dimensional de Sitter spacetime," *General Relativity and Gravitation* **38** no. 11, (Sept., 2006) 1565–1591, `arXiv:gr-qc/0605027`.

[67] M. Grewal, Y. T. A. Law, and K. Parmentier, "Black hole horizon edge partition functions," *JHEP* **06** (2023) 025, `arXiv:2211.16644 [hep-th]`.

[68] Z. X. Wang and D. R. Guo, *Special Functions*. World Scientific, 1989.

[69] V. Adamchik, "A class of logarithmic integrals," in *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, ISSAC '97, p. 1–8. Association for Computing Machinery, New York, NY, USA, 1997. `https://doi.org/10.1145/258726.258736`.