On the removal of the barotropic condition in helicity studies of the compressible Euler and ideal compressible MHD equations

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Abstract

The helicity is a topological conserved quantity of the Euler equations which imposes significant constraints on the dynamics of vortex lines. In the compressible setting the conservation law only holds under the assumption that the pressure is barotropic. We show that by introducing a new definition of helicity density $h_\rho = (\rho \boldsymbol{u}) \cdot \operatorname{curl}(\rho \boldsymbol{u})$ this assumption on the pressure can be removed, although $\int_V h_\rho dV$ is no longer conserved. However, we show for the non-barotropic compressible Euler equations that the new helicity density h_ρ obeys an entropy-type relation (in the sense of hyperbolic conservation laws) whose flux $\boldsymbol{J}_\rho$ contains all the pressure terms and whose source involves the potential vorticity $q = \boldsymbol{\omega} \cdot \nabla \rho$. Therefore the rate of change of $\int_V h_\rho dV$ no longer depends on the pressure and is easier to analyse, as it only depends on the potential vorticity and kinetic energy as well as $\operatorname{div} \boldsymbol{u}$. This result also carries over to the inhomogeneous incompressible Euler equations for which the potential vorticity q is a material constant. Therefore q is bounded by its initial value $q_0 = q(\boldsymbol{x}, 0)$, which enables us to define an inverse resolution length scale λ_H^{-1} whose upper bound is found to be proportional to $\|q_0\|_\infty^{2/7}$. In a similar manner, we also introduce a new cross-helicity density for the ideal non-barotropic magnetohydrodynamic (MHD) equations.

Keywords: Helicity, topological fluid dynamics, barotropic approximation, potential vorticity, compressible Euler equations, inhomogeneous incompressible Euler equations, compressible MHD equations

Mathematics Subject Classification: 76N99 (primary), 76W05, 76B99 (secondary)

1 Introduction

It was first shown by Helmholtz (1858) that for an ideal, barotropic fluid with conservative body forces, vortex lines are transported by the flow. Thomson (1868) (later Lord Kelvin) then recognised that any knots and linkages in these lines are conserved. Almost a century later, Moreau (1961) showed that the helicity $\mathcal{H}$ is conserved, a result that was later proved

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independently by Moffatt (1969), who went further in recognizing its relationship with the magnetic helicity invariant of Woltjer (1958a), and with the cross-helicity invariant $\mathcal{H}_c$ of the ideal magnetohydrodynamic (MHD) equations (Woltjer 1958b).

In a fluid flow with velocity vector $\boldsymbol{u}$ and vorticity vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ the helicity is defined to be a volume integral of the form $\mathcal{H} = \int_V \boldsymbol{u} \cdot \boldsymbol{\omega}\, dV$. In this context, Moffatt (1969) showed explicitly that it is also a measure of the degree of vortex line linkages in a localised disturbance. This reference (Moffatt 1969) has been the foundation for a significant body of work on helicity applied to knots and linkages in both ideal fluids and MHD: see also Pouquet *et al* (1976), Berger and Field (1984), Moffatt (1978, 1985, 1990) and Moffatt and Ricca (1992). Moreover, it was shown in Enciso *et al* (2016) that the helicity is the only integral invariant for a general volume-preserving flow. We refer to the paper by Moffatt (2014) for an overview of results and a wide range of references in the subject. The effect of helicity dynamics on the energy cascade, as well as the energy dissipation rate, has been investigated in Biferale *et al* (2012); Capocci *et al* (2023); Linkmann (2018) (see references therein), see also Biferale and Titi (2013). In fact, tracking the helicity density has now become a standard diagnostic tool in the study of vortical structures in large scale numerical simulations of both incompressible Euler and Navier-Stokes flows: see Kerr (2018, 2023). For more general recent references in incompressible Navier-Stokes turbulence, see Iyer *et al* (2019), Yeung and Ravikumar (2020) and Buaria *et al* (2022).

In the setting of ideal compressible flows, a (nearly) universal feature in the study of helicity dynamics has been the assumption of barotropicity of the fluid: that is, the pressure P is taken to be solely a function of the fluid density ρ, so iso-surfaces of pressure and density are parallel. As has been described in Thorpe *et al* (2003), the results of Helmholtz and Kelvin on vortex lines and structures were introduced to the meteorological and oceanographic communities by Bjerknes (1898) (in which rotational effects were included) through his circulation theorem[1]. The essential role played by barotropicity had been pointed out earlier by Silberstein (1896). In meteorological reality, the validity of the barotropic assumption is somewhat limited. For instance, regions of barotropicity in the atmosphere have a uniform temperature distribution and are distinguished by the absence of fronts, so the barotropic assumption is generally restricted to the tropics (Rogachevskii 2021; Roulstone and Norbury 2013; Vallis 2019), whereas in the mid-latitudes the atmosphere is generally baroclinic. For instance, the Unified Model run by the UK Meteorological Office assumes that the pressure is a function of both density and temperature (Davies *et al* 2005). The barotropic assumption has greater validity, however, in the study of stellar interiors or of the interstellar medium (Yokoi 2013). One common class of barotropic models used in astrophysics are polytropic fluids, where it is assumed that $P \propto \rho^\gamma$ for $\gamma \geq 1$.

In the non-barotropic case, a generalised helicity (which is a conserved quantity) has been introduced in Mobbs (1981): see also Gaffet (1985) and Salmon (1988). Similarly, a generalised cross helicity can be introduced for the non-barotropic MHD equations (Yahalom 2017). As noted in Webb (2018) (see page 183), these generalised helicities are nonlocal quantities as they depend on a nonlocal variable, namely the Lagrangian time integral of the temperature. The goal of this paper is to introduce a generalised helicity which solely depends on *local* variables, even though it might not be fully conserved. Due to its local

[1]This is also called the Poincaré-Bjerknes circulation theorem (Poincaré 1893).

nature, the dynamics of this new generalised helicity is easier to interpret.

In particular, the aim of this paper is to explore the circumstances under which the barotropic condition can be removed and to study the consequences of its removal. In order to explain briefly how Moreau (1961) and Moffatt (1969) used the barotropic assumption, let us consider an ideal compressible fluid with velocity vector $\boldsymbol{u}$, density ρ, pressure P and vorticity $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$. In their standard form, without the inclusion of the (specific) internal energy (see §2.2), the barotropic compressible Euler equations take the form

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}P\,, \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\mathrm{div}\,\boldsymbol{u} = 0\,, \tag{1.1}$$

where $\mathrm{D}/\mathrm{D}t = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the material derivative and P is solely a function of ρ. The equation for the vorticity is

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} + \boldsymbol{\omega}\mathrm{div}\,\boldsymbol{u} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\,\boldsymbol{u} - \boldsymbol{\nabla}\left(\rho^{-1}\right)\times\boldsymbol{\nabla}P\,. \tag{1.2}$$

We note that the cross product vanishes as $\boldsymbol{\nabla}P$ and $\boldsymbol{\nabla}(\rho^{-1})$ are both parallel to $\boldsymbol{\nabla}\rho$, due to the barotropic assumption. Therefore the evolution of the helicity density can be written as an entropy-type relation (in the sense of hyperbolic conservation laws) of the form

$$\partial_t h + \mathrm{div}\,\boldsymbol{J}_\pi = 0\,, \tag{1.3}$$

where $h = \boldsymbol{u}\cdot\boldsymbol{\omega}$, $\Pi(\rho) = \int_0^\rho \eta^{-1}P'(\eta)d\eta$ and

$$\boldsymbol{J}_\pi = h\boldsymbol{u} + \boldsymbol{\omega}\left(\Pi - \tfrac{1}{2}|\boldsymbol{u}|^2\right)\,. \tag{1.4}$$

Thus, with suitable boundary conditions on the domain such as periodic boundary conditions, it follows that $\mathcal{H} = \int_V \boldsymbol{u}\cdot\boldsymbol{\omega}\,dV$ is conserved. In fact, Moffatt (1969) showed that the time derivative of the helicity integrated over a moving domain $\mathcal{V}(t)$ transported by the fluid is equal to a perfect divergence. Therefore its evolution on such a moving domain is only determined by boundary terms.

The above calculation illustrates the point that if the barotropic assumption is to be dropped then the helicity $\mathcal{H} = \int_V h\,dV$ is no longer conserved, because in equation (1.3) the term $\mathrm{div}\,(\Pi\boldsymbol{\omega})$ is replaced by $\rho^{-1}\mathrm{div}(\boldsymbol{\omega}P)$ and the additional term $\boldsymbol{u}\cdot(\boldsymbol{\nabla}(\rho^{-1})\times\boldsymbol{\nabla}P)$ appears. Therefore a change in the definition of the helicity $\mathcal{H}$ is required. It is hardly surprising that without the barotropic assumption, the property that vortex lines are transported with the flow will be lost. We introduce a new definition of helicity density h_ρ and helicity H as

$$h_\rho = (\rho\boldsymbol{u})\cdot\mathrm{curl}\,(\rho\boldsymbol{u}) = (\rho\boldsymbol{u})\cdot(\rho\boldsymbol{\omega}) \qquad \text{with} \qquad H = \int_V h_\rho\,dV\,. \tag{1.5}$$

In the context of helicity dynamics without the barotropic assumption, the challenge is to estimate the growth or decay of a topological quantity without explicitly imposing any assumptions on P. In §2 of this paper we will show that h_ρ obeys an entropy-type law

$$\partial_t h_\rho + \mathrm{div}\boldsymbol{J}_\rho = \sigma_\rho\,. \tag{1.6}$$

The pressure P appears only in $\boldsymbol{J}_\rho$ and thus disappears under integration over a periodic domain. In the case of the inhomogeneous incompressible Euler equations, a potential physical interpretation of these formal results lies in using the helicity H and the constant total energy E_0 to define an inverse length scale. In analogy with the Kolmogorov length scale, in §2.1 we define this inverse length scale based on the average rate of change of the helicity as follows

$$[\lambda_H]^{-1} = \left(\frac{\left\langle\left|\frac{\mathrm{d}H}{\mathrm{d}t}\right|\right\rangle}{\varrho_0^{1/2}E_0^{3/2}}\right)^{2/7}, \qquad \varrho_0 = L^{-3}\int_V \rho dV\,. \tag{1.7}$$

The brackets $\langle\cdot\rangle := \frac{1}{T}\int_0^T \cdot\, dt$ denote a finite-time average, ϱ_0 denotes the domain averaged density and the energy is defined by

$$E_0 = \int_V \mathcal{E}_0 dV \qquad \text{and} \qquad \mathcal{E}_0 = \tfrac{1}{2}\rho|\boldsymbol{u}|^2\,. \tag{1.8}$$

The length scale λ_H could be interpreted as the smallest length scale on which there are significant variations of H and hence significant topological variations. The source term σ_ρ is twice the product of the potential vorticity $q = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\,\rho$ and the energy density. It is shown in §2.1 that this leads to the estimate

$$[\lambda_H]^{-1} \leq \left(\frac{4}{E_0\rho_0}\right)^{1/7} \|q_0\|_\infty^{2/7}\,, \tag{1.9}$$

where $\|q_0\|_\infty$ is the spatial maximum norm of the (initial) potential vorticity, which is a material constant under the dynamics. We emphasise that for the inhomogeneous incompressible Euler equations the pressure solves an elliptic problem and the barotropic assumption is not relevant in this context, as the pressure also depends on velocity derivatives. As the canonical helicity is not a conserved quantity for these equations, the modified helicity H provides a new topological quantity whose dynamics is easy to interpret and its growth is bounded.

A similar calculation for the fully compressible Euler equations (including the specific internal energy) is explained in §2.2 with the complication that σ_ρ has an extra term involving $\operatorname{div}\boldsymbol{u}$ that weights the helicity density h_ρ. Finally in §3 we introduce a non-barotropic cross-helicity density $h_c = \rho\boldsymbol{u}\cdot\boldsymbol{B}$ for the ideal compressible MHD equations.

2 The evolution of h_ρ for both the inhomogeneous incompressible and the compressible $3D$ Euler equations

It is known that strong solutions of the $3D$ incompressible Euler equations can develop singularities in a finite time (Elgindi 2021): see also Drivas and Elgindi (2023) for a recent survey and references therein. In the compressible case it is also known that smooth solutions blow up in finite time (Sideris 1985). Once singularities or shocks develop, solutions could be too irregular to perform the manipulations needed to obtain the results in this paper: for example, the vorticity $\boldsymbol{\omega}$ might no longer be a well-defined pointwise quantity. Thus it should be understood that the results in the following sections are only valid for time intervals when sufficiently smooth solutions exist for both the Euler and ideal MHD equations. Sufficient regularity conditions for solutions of the incompressible Euler equations to conserve the helicity, can be found in Cheskidov *et al* (2008) and Boutros and Titi (2024) and references therein.

2.1 Results for the inhomogeneous incompressible $3D$ Euler equations

We recall that the $3D$ inhomogeneous incompressible Euler equations are given by

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P\,, \qquad \mathrm{div}\,\boldsymbol{u} = 0\,, \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0\,. \tag{2.1}$$

In this model the density ρ is allowed to vary but $\boldsymbol{u}$ remains divergence-free. The divergence-free condition implies that P must satisfy an elliptic equation that involves derivatives of the velocity

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla} P\right) = -(\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}) : (\boldsymbol{u}\otimes\boldsymbol{u})\,, \tag{2.2}$$

and so an imposition of barotropicity is invalid. The vorticity satisfies the following equation

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\left(\rho^{-1}\right)\times\boldsymbol{\nabla} P\,. \tag{2.3}$$

We note that the canonical helicity $\mathcal{H}$ is not conserved by solutions of the inhomogeneous incompressible Euler equations, as the integral $\int_V \boldsymbol{u}\cdot\left[\boldsymbol{\nabla}\left(\rho^{-1}\right)\times\boldsymbol{\nabla} P\right] dV$ generally does not vanish. The new helicity H introduced in (1.5) is also not a conserved quantity, but we will find that its growth is bounded and its evolution depends only on local quantities. First we need to recall two basic identities.

Firstly, it is not difficult to show that ρ, $\boldsymbol{u}$ and P obey the conservation law

$$\partial_t\mathcal{E}_0 + \mathrm{div}\left\{(\mathcal{E}_0 + P)\,\boldsymbol{u}\right\} = 0\,, \tag{2.4}$$

where $\mathcal{E}_0 = \frac{1}{2}\rho|\boldsymbol{u}|^2$ is the energy density. Therefore on a periodic domain the energy $E_0 = \int_V \mathcal{E}_0\, dV$ is conserved.

Secondly, we recall Ertel's theorem (Ertel 1942) for the potential vorticity $q = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\rho$. Using the evolution equation for $\boldsymbol{\omega}$ in (2.3), it can be shown that q satisfies

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left(\frac{\mathrm{D}\rho}{\mathrm{D}t}\right) - \left[\boldsymbol{\nabla}(\rho^{-1})\times\boldsymbol{\nabla} P\right]\cdot\boldsymbol{\nabla}\rho = 0\,, \tag{2.5}$$

where terms of the type $(\boldsymbol{\nabla}\rho)\cdot(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u})$ cancel, in tandem with the vanishing of the pressure term. It therefore follows that q is a material constant.

The material derivatives of $\rho\boldsymbol{u}$ and $\rho\boldsymbol{\omega}$ are easily found, thereby giving

$$\frac{\mathrm{D}h_\rho}{\mathrm{D}t} \;=\; \rho^2\boldsymbol{u}\cdot(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}) - \left\{\rho\,\boldsymbol{\omega}\cdot\boldsymbol{\nabla} P - \boldsymbol{u}\cdot(\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla} P)\right\}\,. \tag{2.6}$$

We note that the terms involving the pressure in (2.6) form a perfect divergence

$$\begin{aligned}\rho\,\boldsymbol{\omega}\cdot\boldsymbol{\nabla} P - \boldsymbol{u}\cdot[\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla} P] &= \boldsymbol{\nabla} P\cdot[\rho\,\boldsymbol{\omega} + \boldsymbol{\nabla}\rho\times\boldsymbol{u}]\\ &= \boldsymbol{\nabla} P\cdot\mathrm{curl}\,(\rho\boldsymbol{u})\end{aligned}$$

$$= \quad \mathrm{div}\left\{P\mathrm{curl}\,(\rho\boldsymbol{u})\right\}\,. \tag{2.7}$$

Equation (2.6) then becomes

$$\begin{aligned}\partial_t h_\rho + \mathrm{div}\left\{h_\rho \boldsymbol{u} + P\mathrm{curl}\,(\rho\boldsymbol{u})\right\} &= \rho^2\boldsymbol{\omega}\cdot\boldsymbol{\nabla}(\tfrac{1}{2}|\boldsymbol{u}|^2)\\ &= \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{2}\rho^2|\boldsymbol{u}|^2\right) - |\boldsymbol{u}|^2\boldsymbol{\omega}\cdot\boldsymbol{\nabla}(\tfrac{1}{2}\rho^2)\\ &= \mathrm{div}\left\{\tfrac{1}{2}\boldsymbol{\omega}\rho^2|\boldsymbol{u}|^2\right\} - q\left(\rho|\boldsymbol{u}|^2\right)\,,\end{aligned} \tag{2.8}$$

which can be written as an entropy-type relation

$$\partial_t h_\rho + \mathrm{div}\boldsymbol{J}_\rho = \sigma_\rho\,, \tag{2.9}$$

with the flux vector $\boldsymbol{J}_\rho$ and the scalar source term σ_ρ defined as

$$\boldsymbol{J}_\rho = h_\rho\boldsymbol{u} + P\mathrm{curl}\,(\rho\boldsymbol{u}) - \tfrac{1}{2}\boldsymbol{\omega}\rho^2|\boldsymbol{u}|^2\,, \qquad \sigma_\rho = -q\rho|\boldsymbol{u}|^2\,. \tag{2.10}$$

One can therefore infer that the sign of the potential vorticity impacts whether h_ρ increases or decreases. After integration of equation (2.9) over a periodic domain, the term $\mathrm{div}\boldsymbol{J}_\rho$ disappears and one finds

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2\int_V q\mathcal{E}_0\,. \tag{2.11}$$

The property that q is a material constant means that the growth or decay of H is bounded. In fact, equation (2.5) implies that $\|q(\cdot,t)\|_\infty \le \|q_0\|_\infty$. This immediately implies the bound

$$\left|\frac{\mathrm{d}H}{\mathrm{d}t}\right| \le 2\|q_0\|_\infty E_0\,, \tag{2.12}$$

where $\|q\|_\infty$ is the maximum spatial norm of the potential vorticity. In turn, this bound constrains the globally averaged alignment between $\boldsymbol{u}$ and $\boldsymbol{\omega}$ and hence the topological dynamics.

Next we notice that the evolution of H induces a length scale λ_H of the following form

$$[\lambda_H]^{-1} = \left(\frac{\left\langle\left|\frac{\mathrm{d}H}{\mathrm{d}t}\right|\right\rangle}{\varrho_0^{1/2}E_0^{3/2}}\right)^{2/7}\,, \tag{2.13}$$

where ϱ_0 is the average density $\varrho_0 = L^{-3}\int_V \rho\, dV$. The quantity λ_H can be interpreted as a resolution length scale: for example the smallest length scale on which there is significant topological dynamics. We can now use equation (2.12) to find the following upper bound on this inverse length scale

$$\lambda_H^{-1} \le \left(\frac{4}{E_0\rho_0}\right)^{1/7}\|q_0\|_\infty^{2/7}\,. \tag{2.14}$$

As has been said before, λ_H could be viewed as a cutoff length scale for the helicity dynamics.

2.2 Results for the fully compressible $3D$ Euler equations

The fully compressible Euler equations, including the specific internal energy e (per unit mass), require an equation for e in addition to those given in (1.1). They are given by

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}P\,, \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathrm{div}\,\boldsymbol{u} = 0\,, \qquad \rho\frac{\mathrm{D}e}{\mathrm{D}t} = -P\mathrm{div}\,\boldsymbol{u}\,. \tag{2.15}$$

Before considering the helicity density h_ρ, let us consider the well-known formula for the full energy density

$$\mathcal{E} = \rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + e\right)\,. \tag{2.16}$$

It is not difficult to show that $\mathcal{E}$ satisfies the exact continuity equation (2.4) (with $\mathcal{E}_0$ replaced by $\mathcal{E}$). Thus we find that the total energy $E = \int_V \mathcal{E}dV$ is constant for any equation of state. For the system (2.15) to be fully determined, an equation of state for P in terms of e and ρ is required. Our results however are independent of the choice of equation of state and therefore we do not fix a choice. We note that similar results as described below can also be obtained for different formulations of the compressible Euler equations involving temperature or entropy dynamics. This is because our results only rely on the form of the density and velocity equations. In addition, we note that the potential vorticity q is not a material constant for these equations. However, it was shown in Gibbon and Holm (2012) that q satisfies

$$\partial_t q + \mathrm{div}\,(q\boldsymbol{u}) + \mathrm{div}\,\left[\boldsymbol{\omega}\rho\mathrm{div}\,\boldsymbol{u}\right] = 0\,. \tag{2.17}$$

One can deduce from equation (2.15) that $\rho\boldsymbol{u}$ and $\rho\boldsymbol{\omega}$ evolve according to

$$\frac{\mathrm{D}(\rho\boldsymbol{u})}{\mathrm{D}t} + (\rho\boldsymbol{u})\mathrm{div}\,\boldsymbol{u} = -\boldsymbol{\nabla}P\,, \tag{2.18}$$

$$\frac{\mathrm{D}(\rho\boldsymbol{\omega})}{\mathrm{D}t} + 2(\rho\boldsymbol{\omega})\mathrm{div}\,\boldsymbol{u} = \rho\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \rho^{-1}\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}P\,. \tag{2.19}$$

Following the grouping of the pressure terms as in (2.7), from (2.18) and (2.19) we deduce that

$$\partial_t h_\rho + 2h_\rho\mathrm{div}\,\boldsymbol{u} + \mathrm{div}\left\{h_\rho\boldsymbol{u} + P\mathrm{curl}\,(\rho\boldsymbol{u}) - \tfrac{1}{2}\boldsymbol{\omega}\rho^2|\boldsymbol{u}|^2\right\} = -q\rho|\boldsymbol{u}|^2\,. \tag{2.20}$$

With $\boldsymbol{J}_\rho$ and σ_ρ defined in (2.10) and $\tilde{\sigma}_\rho = \sigma_\rho - 2h_\rho\,\mathrm{div}\,\boldsymbol{u}$, then we find a similar entropy-type relation to (2.9)

$$\partial_t h_\rho + \mathrm{div}\boldsymbol{J}_\rho = \tilde{\sigma}_\rho\,. \tag{2.21}$$

(2.21) integrates to

$$\frac{\mathrm{d}H}{\mathrm{d}t} + 2\int_V h_\rho\,\mathrm{div}\,\boldsymbol{u}\,dV = -2\int_V q\mathcal{E}_0 dV\,. \tag{2.22}$$

From this equation one can observe that the helicity H increases in regions of compression, while it decreases in regions of dilatation. Moreover, one can deduce an inequality of the following form

$$\left|\frac{\mathrm{d}H}{\mathrm{d}t}\right| \le 2\|q(\cdot,t)\|_\infty E_0 + 2\|h_\rho(\cdot,t)\|_\infty\int_V|\mathrm{div}\,\boldsymbol{u}(\cdot,t)|\,dV\,, \tag{2.23}$$

where we observe that bounds of this type are particularly useful in the perturbative regime of slightly compressible flows where the last term on the right hand side could be small.

3 Cross-helicity in ideal compressible MHD

Let us consider the $3D$ compressible ideal MHD equations. These are composed of

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = (\mathrm{curl}\,\boldsymbol{B})\times\boldsymbol{B} - \boldsymbol{\nabla}\,P \qquad \text{and} \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\mathrm{div}\,\boldsymbol{u} = 0\,, \tag{3.1}$$

together with the induction equation for the magnetic field $\boldsymbol{B}$ (as well as the divergence-free condition $\mathrm{div}\,\boldsymbol{B} = 0$) and the equation for the specific internal energy e

$$\partial_t\boldsymbol{B} = \mathrm{curl}\,(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\nabla}\,\boldsymbol{u} - \boldsymbol{B}\mathrm{div}\,\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{B}\,, \qquad \rho\frac{\mathrm{D}e}{\mathrm{D}t} = -P\mathrm{div}\,\boldsymbol{u}\,. \tag{3.2}$$

The energy density $\mathcal{E}_B$ is an extension of (2.16) to include $\boldsymbol{B}$

$$\mathcal{E}_B = \tfrac{1}{2}|\boldsymbol{B}|^2 + \mathcal{E}\,, \tag{3.3}$$

where $\mathcal{E} = \rho\,(\frac{1}{2}|\boldsymbol{u}|^2 + e)$ is the fluid energy density. Then we find that

$$\partial_t\mathcal{E}_B + \mathrm{div}\left\{\boldsymbol{u}\,(\mathcal{E}_B + P) + \tfrac{1}{2}|\boldsymbol{B}|^2\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{B}\right\} = 0\,, \tag{3.4}$$

from which it follows that the full energy

$$E_{0,B} = \int_V \left[\tfrac{1}{2}|\boldsymbol{B}|^2 + \rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + e\right)\right]\,dV \tag{3.5}$$

is conserved.

The canonical cross-helicity $\mathcal{H}_c = \int_V \boldsymbol{u}\cdot\boldsymbol{B}\,dV$ is a pseudo-scalar (Berger and Field 1984; Moffatt 1978; Pouquet *et al* 1976). In parallel with (1.5), we introduce the following generalised cross-helicity

$$H_c = \int_V \rho\boldsymbol{u}\cdot\boldsymbol{B}\,dV\,, \tag{3.6}$$

with the cross-helicity density defined as $h_c = \rho\boldsymbol{u}\cdot\boldsymbol{B}$. The MHD equivalent of the potential vorticity is $q_c = \boldsymbol{B}\cdot\boldsymbol{\nabla}\,\rho$. One can check that the magnetic potential vorticity satisfies

$$\partial_t q_c + \mathrm{div}\;(\boldsymbol{u}q_c) + \mathrm{div}\;(\rho\mathrm{div}\,\boldsymbol{u}\boldsymbol{B}) = 0\,. \tag{3.7}$$

The equation for $\rho\boldsymbol{u}$ is

$$\frac{\mathrm{D}(\rho\boldsymbol{u})}{\mathrm{D}t} + \rho\boldsymbol{u}\mathrm{div}\,\boldsymbol{u} = (\mathrm{curl}\boldsymbol{B})\times\boldsymbol{B} - \boldsymbol{\nabla}\,P\,, \tag{3.8}$$

from which we deduce that

$$\begin{aligned}\partial_t h_c &= -\boldsymbol{B}\cdot\{\boldsymbol{B}\times\mathrm{curl}\boldsymbol{B} + \boldsymbol{\nabla}\,P + \rho\boldsymbol{u}\mathrm{div}\,\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}(\rho\boldsymbol{u})\}\\ &+ \rho\boldsymbol{u}\cdot\{\boldsymbol{B}\cdot\boldsymbol{\nabla}\,\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{\nabla}\,\boldsymbol{B} - \boldsymbol{B}\mathrm{div}\,\boldsymbol{u}\} \qquad (3.9)\\ &= -2h_c\mathrm{div}\,\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{\nabla}\,h_c - \boldsymbol{B}\cdot\boldsymbol{\nabla}\,P + \rho\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right)\,. \qquad (3.10)\end{aligned}$$

After re-arrangement we find

$$\partial_t h_c + h_c\mathrm{div}\,\boldsymbol{u} + \mathrm{div}\boldsymbol{J}_c \;=\; -\tfrac{1}{2}q_c|\boldsymbol{u}|^2\,. \tag{3.11}$$

	$\mathcal{E}$	h	$\boldsymbol{J}$	σ
Baro-Euler	$\mathcal{E}_0$	$h = \boldsymbol{u}\cdot\boldsymbol{\omega}$	$\boldsymbol{J}_\pi = h\boldsymbol{u} + \boldsymbol{\omega}\left(\Pi - \frac{1}{2}\lvert\boldsymbol{u}\rvert^2\right)$	0
II-Euler	$\mathcal{E}_0$	$h_\rho = \rho\boldsymbol{u}\cdot\rho\boldsymbol{\omega}$	$\boldsymbol{J}_\rho = h_\rho\boldsymbol{u} + P\mathrm{curl}\,(\rho\boldsymbol{u}) - \frac{1}{2}\boldsymbol{\omega}\rho^2\lvert\boldsymbol{u}\rvert^2$	$\sigma_\rho = -q\rho\lvert\boldsymbol{u}\rvert^2$
Comp-Euler	$\mathcal{E}$	$h_\rho = \rho\boldsymbol{u}\cdot\rho\boldsymbol{\omega}$	$\boldsymbol{J}_\rho = h_\rho\boldsymbol{u} + P\mathrm{curl}\,(\rho\boldsymbol{u}) - \frac{1}{2}\boldsymbol{\omega}\rho^2\lvert\boldsymbol{u}\rvert^2$	$\tilde{\sigma}_\rho = \sigma_\rho - 2h_\rho\mathrm{div}\,\boldsymbol{u}$
MHD	$\mathcal{E} + \frac{1}{2}\lvert\boldsymbol{B}\rvert^2$	$h_c = \rho\boldsymbol{u}\cdot\boldsymbol{B}$	$\boldsymbol{J}_c = h_c\boldsymbol{u} + \boldsymbol{B}\left(P - \frac{1}{2}\rho\lvert\boldsymbol{u}\rvert^2\right)$	$\sigma_c = -\frac{1}{2}q_c\lvert\boldsymbol{u}\rvert^2 - h_c\mathrm{div}\,\boldsymbol{u}$

Table 1: The entries in the table represent the entropy-type relations $\partial_t h + \mathrm{div}\,\boldsymbol{J} = \sigma$ for the four different cases, which are the barotropic compressible Euler equations (Baro-Euler), the inhomogeneous incompressible Euler equations (II-Euler), the fully compressible Euler equations (Comp-Euler) and the ideal compressible MHD equations. Note that $\mathcal{E}_0 = \frac{1}{2}\rho|\boldsymbol{u}|^2$ and $\mathcal{E} = \rho\left(\frac{1}{2}|\boldsymbol{u}|^2 + e\right)$.

The equivalent of (2.10) is

$$\boldsymbol{J}_c = h_c\boldsymbol{u} + P\boldsymbol{B} - \tfrac{1}{2}\rho\boldsymbol{B}|\boldsymbol{u}|^2\,, \tag{3.12}$$

so with the definition $\sigma_c = -\frac{1}{2}q_c|\boldsymbol{u}|^2 - h_c\mathrm{div}\,\boldsymbol{u}$ we find

$$\partial_t h_c + \mathrm{div}\boldsymbol{J}_c = \sigma_c\,. \tag{3.13}$$

This integrates to

$$\frac{\mathrm{d}H_c}{\mathrm{d}t} + \int_V h_c\mathrm{div}\boldsymbol{u}\,dV = -\tfrac{1}{2}\int_V q_c|\boldsymbol{u}|^2 dV\,. \tag{3.14}$$

As before, we note that the sign of the potential vorticity impacts the evolution of the cross helicity H_c. Moreover, in regions of expansion H_c is decreasing, while it is increasing in regions of compression. Equation (3.13) is the equivalent of (2.21). On the left-hand side, $\mathrm{div}\,\boldsymbol{u}$ weights the cross-helicity density. In the case of the inhomogeneous incompressible MHD equations, one finds a similar relation to (2.9) and deduces that q_c is a material constant.

4 Comments and conclusion

Before making some comments, let us summarise what we have found so far. Until now, most results on helicity dynamics for compressible flows have required the barotropic approximation. The main thread in this paper has been the investigation of a different definition of helicity density for compressible flows which takes the form $h_\rho = (\rho\boldsymbol{u})\cdot(\rho\boldsymbol{\omega})$ for which the barotropic condition is no longer required. The entries in Table 1 show how the four cases are related. The inclusion of the ρ^2-term is crucial, as the results do not hold for other powers of ρ in the helicity density. Moreover, we repeat that the evolution of H (and H_c) can only be analysed provided there exist time intervals on which sufficiently smooth solutions exist for either the compressible Euler equations (including the internal energy), the equations of compressible ideal MHD or the inhomogeneous incompressible Euler equations respectively.

The inclusion of the specific internal energy per unit mass e, while it does not appear explicitly in the calculations for the dynamics of h_ρ, impacts not only the $\mathcal{E}_0$-term through its appearance in the total energy equation (2.16), but also the source term σ_ρ defined in (2.10). For the compressible Euler equations as given in equation (2.15) to be fully determined, an equation of state for specifying the pressure P is required. However, our results in this paper are independent of the choice of such an equation of state, which is why the dependence of the pressure on the density, temperature and entropy has been left unspecified. A typical non-barotropic choice might be the use of the ideal gas law, where P would be a function of both density and temperature (Davies *et al* 2005).

In the case of the inhomogeneous incompressible Euler equations, the new definition of h_ρ and its behaviour leads us to introduce a resolution length scale λ_H in (1.7). This new length scale λ_H is bounded from below in equation (2.14) and it suggests a typical length scale on which helicity (and hence topological) variations occur. The two main features of the dynamics are the predominance of the potential vorticity q which remains bounded in L^∞, and in the fully compressible case the (average) sign of $\operatorname{div} \boldsymbol{u}$.

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References

Berger, M. and Field, G. 1984 The topological properties of magnetic helicity, *J. Fluid Mech.*, **147**, pp. 133–148.

Biferale, L., Musacchio, S. and Toschi, F. 2012 Inverse energy cascade in three-dimensional isotropic turbulence, *Phys. Rev. Lett.*, **108**, pp. 164501.

Biferale, L. and Titi, E. S. 2013 On the global regularity of a helical-decimated version of the 3D Navier-Stokes equations, *J. Stat. Phys.*, **151**, pp. 1089-1098.

Bjerknes, V. 1898 Über einen hydrodynamischen Fundamentalsatz und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres, *Kongl. Sven. Vetensk. Akad. Handlingar*, **31**, pp. 1–35.

Boutros, D. W. and Titi, E. S. 2024 On the conservation of helicity by weak solutions of the 3D Euler and inviscid MHD equations, *arXiv:2410.00813.*

Buaria, D., Pumir, P. and Bodenschatz, E. 2022 Generation of intense dissipation in high Reynolds number turbulence, *Phil. Trans. R. Soc. A* **380**, 20210088.

Capocci, D., Johnson, P. L., Oughton, S., Biferale, L. and Linkmann, M. 2023 New exact Betchov-like relation for the helicity flux in homogeneous turbulence, *J. Fluid Mech.*, **963**, pp. R1.

Cheskidov, C., Constantin, P., Friedlander, S. and Shvydkoy, R. 2008 Energy conservation and Onsager's conjecture for the Euler equations, *Nonlinearity*, **21**, pp. 1233–1252.

Davies, T., Cullen, M. J. P., Malcolm, A. J., Mawson, M. H., Staniforth, A., White, A. A., and Wood, N. 2005 A new dynamical core for the Met Office's global and regional modelling of the atmosphere, *Quart. J. Royal Met. Soc.*, **131**, pp. 1759–1782.

Drivas, T. D. and Elgindi, T. M. 2023 Singularity formation in the incompressible Euler equation in finite and infinite time, *EMS Surv. Math. Sci.*, **10**, pp. 1–100.

Elgindi, T. M. 2021 Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$, *Ann. Math.*, **194**, pp. 647–727.

Enciso, A., Peralta-Salas, D. and de Lizaur, F. T. 2016 Helicity is the only integral invariant of volume-preserving transformations, *Proc. Natl. Acad. Sci. USA*, **113**, pp. 2035-2040.

Ertel, H. 1942 Ein neuer hydrodynamischer Wirbelsatz, *Meteorol. Z.*, **59**, pp. 271–281.

Gaffet, B. 1985 On generalized vorticity-conservation laws, *J. Fluid Mech.*, **156**, pp. 141–149.

Gibbon, J. D. and Holm, D. D. 2012 Quasiconservation laws for compressible 3D Navier-Stokes flow, *Phys. Rev. E*, **86**, 047301.

Helmholtz, H. 1858 Über Integrale der Hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, *J. Reine Angew. Math.*, **55**, pp. 25–55.

Iyer, K. P., Sreenivasan, K. R. and Yeung, P. K. 2019 Circulation in High Reynolds Number Isotropic Turbulence is a Bifractal, *Phys. Rev. X*, **9**, 041006.

Kerr, R. M. 2018 Topology of interacting coiled vortex rings, *J. Fluid Mech.*, **854**, pp. R2.

Kerr, R. M. 2023 Sensitivity of trefoil vortex knot reconnection to the initial vorticity profile, *Phys. Rev. Fluids*, **8**, 074701.

Linkmann, M. 2018 Effects of helicity on dissipation in homogeneous box turbulence, *J. Fluid Mech.*, **856**, pp. 79–102.

Mobbs, S. 1981 Some vorticity theorems and conservation laws for non-barotropic fluids, *J . Fluid Mech.*, **108**, pp. 475–483.

Moffatt, H. K. 1969 The degree of knottedness of tangled vortex lines, *J. Fluid Mech.*, **35**, pp. 117–129.

Moffatt, H. K. 1978 *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press.

Moffatt, H. K. 1985 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology, Part 1. Fundamentals, *J. Fluid Mech.*, **159**, pp. 359–378.

Moffatt, H. K. 1990 The energy spectrum of knots and links, *Nature*, **347**, pp. 367–369.

Moffatt, H. K. and Ricca, R. 1992 Helicity and the Călugăreanu Invariant, *Proc. R. Soc. Lond. A.*, **439**, pp. 411–429.

Moffatt, H. K. 2014 Helicity and singular structures in fluid dynamics, *Proc. Natl. Acad. Sci. USA*, **111**, pp. 3663–3670.

Moreau, J. J. 1961 Constantes d'un îlot tourbillonnaire en fluid parfait barotrope, *C. R . Acad, Sci. Paris*, **252**, pp. 2810–2812.

Poincaré, H. 1893 Théorie des tourbillons : leçons professées pendant le deuxième semestre 1891-1892 (Vol. 11), *Gauthier-Villars, Article 158.*

Pouquet, A., Frisch, U. and Léorat, J. 1976 Strong MHD helical turbulence and the nonlinear dynamo effect, *J. Fluid Mech.*, **77**, pp. 321–354.

Rogachevskii, I. 2021 *Introduction to Turbulent Transport of Particles, Temperature and Magnetic Fields*, Cambridge University Press.

Roulstone, I. and Norbury, J. 2013 *Invisible in the Storm : The Role of Mathematics in Understanding Weather*, Princeton University Press.

Salmon, R. 1988 Hamiltonian fluid mechanics, *Annu. Rev. Fluid Mech.*, **20**, pp. 225–256.

Sideris, T. 1985 Formation of singularities in three-dimensional compressible fluids, *Comm. Math. Phys.*, **101**, pp. 475–485.

Silberstein, L. 1896 O tworzeniu się wirów w płynie doskonałym (On the creation of eddies in an ideal fluid), *Proc. Cracow Acad. Sci.*, **31**, pp. 325–335. Also published as: Über die Entstehung von Wirbelbewegungen in einer reibungslosen Flüssigkeit, *Bull. Int. l'Acad. Sci. Cracovie*, pp. 280–290.

Thomson, W. 1868 On vortex motion, *Trans. Roy. Soc. Edin.*, **25**, pp. 217–260.

Thorpe, A. J., Volkert, H. and Ziemianski, M. J. 2003 The Bjerknes' circulation theorem: a historical perspective, *Bull. Amer. Met. Soc.*, April, pp. 471–480.

Vallis, G. K. 2019 *Essentials of Atmospheric and Oceanic Dynamics*, Cambridge University Press.

Webb, G. 2018 *Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws*, Springer.

Woltjer, L. 1958 A theorem on force-free magnetic fields. *Proc. Natl. Acad. Sci. USA*, **44**, pp. 489–491.

Woltjer, L. 1958 On hydromagnetic equilibrium. *Proc. Natl. Acad. Sci. USA*, **44**, pp. 833–841.

Yahalom, A. 2017 A conserved local cross-helicity for non-barotropic MHD, *Geo. Astrophys. Fluid Dyn.* **111:2**, pp. 131–137.

Yeung, P. K. and Ravikumar, K. 2020 Advancing understanding of turbulence through extreme-scale computation: Intermittency and simulations at large problem sizes, *Phys. Rev. Fluids*, **5**, 110517.

Yokoi, Y. 2013 Cross-helicity and related dynamo, *Geo. Astrophys. Fluid Dyn.*, **107**, pp. 114–184.