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A Geometric Approach to Lower Bounds for the Maximum of Gaussian Random Processes

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Abstract—The paper proposes an approach to lower bounds for the expectation of the maximum of a Gaussian process based on classical results from the geometry of convex bodies.

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In what follows, by $\{g_k\}_{k\in\Lambda}$ we denote a tuple of independent Gaussian random variables with $\mathbb{E}g_k = 0$ and $\mathbb{E}g_k^2 = 1$, $k \in \Lambda$, indexed by elements of some finite set Λ and by $\{\varepsilon_k\}_{k\in\Lambda}$, a tuple of independent Bernoulli variables ($\mathbb{P}\{\varepsilon_k = 1\} = \mathbb{P}\{\varepsilon_k = -1\} = 1/2$). If $\Lambda = \{1, 2, \dots, n\}$, then by G_n we denote a random vector $\{g_k\}_{k=1}^n$.

The paper considers lower bounds for the expectation of the supremum of random processes, primarily, Gaussian ones. Such bounds have important applications outside probability theory, in particular, in the theory of orthogonal series. In this regard, we cite an interesting result by Rider.

Theorem A [1]. *Let $\Lambda \subset \mathbb{Z}$ be a set such that for some absolute constant $c_1 > 0$ and any coefficients $\{a_k\}_{k\in\Lambda}$ one has the relation*

$$\mathbb{E}\biggl(\biggl\|\sum_{k\in\Lambda} \varepsilon_k a_k e^{ikx}\biggr\|_{L^\infty}\biggr) \ge c_1 \sum_{k\in\Lambda} |a_k|.$$

Then Λ is a Sidon set; *i.e.,*

$$\biggl\|\sum_{k\in\Lambda} a_k e^{ikx}\biggr\|_{L^\infty} \ge c_2 \sum_{k\in\Lambda} |a_k|, \qquad c_2 > 0,$$

for any coefficients $\{a_k\}$. (*Here and below, $c_1, c_2, \dots$ denote various absolute constants.*)

For results related to Theorem A and pertaining to series in general jointly bounded orthonormal systems, see [2] and [3].

The first sharp-in-order lower bounds for random processes associated with orthogonal series were obtained by Salem and Zygmund in 1954.

Theorem B [4]. *One has the estimate*

$$\mathbb{E}\biggl(\biggl\|\sum_{k=1}^{N} \varepsilon_k e^{ikx}\biggr\|_{L^\infty}\biggr) \ge c_3 (N\log N)^{1/2}, \qquad c_3 > 0, \qquad N = 1, 2, \dots .$$

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The paper [4] used the so-called second moment method based on the comparison of the L^1- and L^2-norms of some random functions. This method applies to a wide class of independent random variables (see, in particular, Theorem D below). Another lower bound method was developed for Gaussian processes on the basis of the classical Slepian lemma (see, e.g., [5, p. 213]). The following assertion is one of the most important results obtained by this method.

Theorem C (Sudakov [6]). *Let*

$$V = \{v_j\}_{j=1}^N \subset \mathbb{R}^n \tag{1}$$

be a tuple of vectors such that for some $a > 0$ one has

$$\|v_j - v_{j'}\|_{\ell_2^n} \geq a, \qquad 1 \leq j, j' \leq N, \quad j \neq j'. \tag{2}$$

Then

$$\mathbb{E}\Big(\max_{1\leq j\leq N} |\langle G_n, v_j\rangle|\Big) \geq c_3 a (\log N)^{1/2}, \tag{3}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product on the corresponding Euclidean space.

Using results of the type of Theorem C, Marcus and Pisier [7] established a criterion (a condition on the sequence $\{a_k\}$ of coefficients) for the almost sure continuity of the sum of random series of the form

$$\sum_{k\in\mathbb{Z}} \varepsilon_k a_k e^{ikx} \quad \text{or} \quad \sum_{k\in\mathbb{Z}} g_k a_k e^{ikx}.$$

For an arbitrary tuple (1) of vectors, an order-sharp estimate of the left-hand side of (3) was obtained by Talagrand (see [8]). Just as the previous results in this direction, this estimate is stated in terms of entropy characteristics of the set V. In a number of important cases, estimating the ε-entropy of the set V is a very complicated problem, which prevents one from bringing the estimate "to a number." This is partly why this method has produced no results concerning estimates of the uniform norm of random series in general orthonormal systems. The following result was obtained in [9], [10] (see also [11]) using the second moment method and the central limit theorem for two-dimensional vectors with a sharp estimate of the remainder term.

Theorem D. *Let $\{\xi_k\}_{k=1}^n$ be a tuple of independent random variables with $\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 = 1$, and $\mathbb{E}|\xi_k| \geq 1/M$, $1 \leq k \leq n$, let $\Phi = \{\varphi_k\}_{k=1}^n$ be an orthonormal function system defined on a measure space (X,μ), $\mu(X) = 1$, and assume that*

$$\|\varphi_k\|_{L^{2+\delta}(X,\mu)} \leq M, \quad 1 \leq k \leq n, \qquad \text{where} \quad \delta > 0.$$

Then for any coefficients $\{a_k\}_{k=1}^n$ one has

$$\mathbb{E}\bigg(\bigg\|\sum_{k=1}^n \xi_k a_k \varphi_k\bigg\|_{L^\infty(X)}\bigg) \geq C_{M,\delta}\bigg(\sum_{k=1}^n a_k^2\bigg)^{1/2}\bigg(1 + \log\bigg[\frac{(\sum_{k=1}^n a_k^2)^2}{\sum_{k=1}^n a_k^4}\bigg]\bigg)^{1/2}.$$

In the case of independent Gaussian variables, there exists a lower bound that holds for any orthonormal system.

Theorem 1. *There exists an absolute constant $c_4 > 0$ such that for each $n = 1, 2, \dots$ and any orthonormal system $\Phi = \{\varphi_k\}_{k=1}^n \subset L^2(X,\mu)$, $\mu(X) = 1$, one has the inequality*

$$\mathbb{E}\bigg(\sup_{x\in X}\bigg|\sum_{k=1}^n g_k \varphi_k(x)\bigg|\bigg) \geq c_4 (n\log n)^{1/2}.$$

Proof. Let $W_\Phi = \{v_x, x \in X\} \subset R^n$, where $v_x = \{\varphi_k(x)\}_{k=1}^n$.

Lemma 1. *For $r = 0, 1, \dots, n$, one has the following lower bound for the Kolmogorov width of the set W_Φ in the space ℓ_2^n:*

$$d_r(W_\Phi, \ell_2^n) \geq (n - r)^{1/2}.$$

We omit the proof of the lemma, because it is completely similar to that of the classical lower bound for the width $d_r(B_1^n, \ell_2^n)$ (see, e.g., [12, p. 139]). Lemma 1 readily implies the following assertion.

Lemma 2. *For any orthonormal system* $\Phi = \{\varphi_k\}_{k=1}^n$, $n > 3$, *there exists a tuple of points* (“*points of the width*”) $\{x_1, \dots, x_{[n/2]}\} \subset X$ *such that*

$$\rho_{\ell_2^n}(v_{x_p}, \operatorname{span}\{v_{x_q}, 1 \le q \le p-1\}) \ge \left(\frac{n}{2}\right)^{1/2}, \qquad p = 2, \dots, [n/2],$$

where $\rho_{\ell_2^n}(v, L)$ *is the Euclidean distance from a point* $v \in \mathbb{R}^n$ *to a set* $L \subset \mathbb{R}^n$ *and* $\operatorname{span}\{v_\alpha\}$ *is the linear span of the set* $\{v_\alpha\}$.

To complete the proof of Theorem 1, let us consider the set of points $\{x_q\}_{q=1}^{[n/2]}$ constructed in Lemma 2 and show that

$$\mathbb{E}\big(\max_q |\langle G_n, v_{x_q}\rangle|\big) \ge c_4 (n \log n)^{1/2}. \tag{4}$$

By construction, for $q = 2, \dots, [n/2]$ we have

$$\begin{gathered} v_{x_q} = u_q + z_q, \qquad u_q \perp L_{q-1} \equiv \operatorname{span}\{v_{x_\nu}, 1 \le \nu \le q-1\}, \\ \|u_q\|_{\ell_2^n} \ge \left(\frac{n}{2}\right)^{1/2}, \qquad z_q \in L_{q-1}. \end{gathered} \tag{5}$$

Further, of course, one can obtain the desired statement with the help of Theorem C, but it is much easier to use the fact that jointly Gaussian uncorrelated variables are independent. Therefore, for $q = 2, \dots, [n/2]$ and for any $a > 0$, the probability

$$\alpha(q, A) \equiv \mathbb{P}\Big\{\max_{1 \le \nu \le q} |\langle G_n, v_{x_\nu}\rangle| \le A\Big\}$$

satisfies the relation

$$\alpha(q, A) \le \alpha(q-1, A) \cdot \mathbb{P}\{|\langle G_n, u_q\rangle| \le A\}. \tag{6}$$

If $A = \gamma(n \log n)^{1/2}$, where $\gamma > 0$ is a sufficiently small absolute constant, then from (6) and the lower bound (5) for $\|u_q\|_{\ell_2^n}$ we obtain

$$\alpha([n/2]), A) \le \left(1 - \frac{1}{n^{1/3}}\right)^{[n/2]-1} \le c_5 \exp(-c_6 n^{2/3}).$$

The proof of Theorem 1 is complete. □

Below is a proof of Theorem C based on the following classical Urysohn inequality, published in *Matematicheskii Sbornik* exactly one hundred years ago.

Theorem E [13]. *For any convex body* K *and the unit Euclidean ball* B *in* $\mathbb{R}^n$, *one has the inequality*

$$\left(\frac{\operatorname{Vol} K}{\operatorname{Vol} B}\right)^{1/n} \le \frac{w(K)}{w(B)},$$

where $\operatorname{Vol} K$ *and* $w(K)$ *are the volume and the mean width, respectively, of the body* K.

The relationship between the Urysohn inequality and estimates of the ϵ-entropy of sets in the space ℓ_2^n was established by Milman [14] (with the participation of A. Pajor, see [14]), but, to the best of the author’s knowledge, this relationship has not been used to prove results like Theorem C. The approach based on the Urysohn inequality has the advantage that it potentially allows one to avoid estimating the ε-entropy and replace it with geometric characteristics of the set V in (1), which admit a simpler estimate (see, e.g., Theorem 2 below).

Thus, for a given set (1), let K be the convex hull of $V \cup -V$. Since the random vector $G_n/\|G_n\|_{\ell_2^n}$ is uniformly distributed on the sphere S^{n-1}, we can readily verify that

$$\mathbb{E}\Big(\sup_{1\le j\le N} |\langle G_n, v_j\rangle|\Big) = \gamma_n w(K), \tag{7}$$

where $0 < c_6 n^{1/2} \le \gamma_n \le c_7 n^{1/2}$.

First, consider the case in which the number of vectors in V is exponentially large compared with n; i.e., $\log N = \delta \cdot n$, $\delta \ge \delta_0 > 0$, where δ_0 is an arbitrary fixed absolute constant. Following [14], we apply the Urysohn inequality to the Minkowski sum of the bodies K and $(a/2)B$. By the conditions of Theorem C, we have

$$\operatorname{Vol}\left(K + \frac{a}{2}B\right) \ge N\left(\frac{a}{2}\right)^n \operatorname{Vol} B;$$

therefore,

$$w\left(K + \frac{a}{2}B\right) \ge w(B)N^{1/n}\frac{a}{2} = N^{1/n}\cdot a.$$

However,

$$w\left(K + \frac{a}{2}B\right) = w(K) + w\left(\frac{a}{2}B\right) = w(K) + a.$$

As a result, we obtain

$$w(K) \ge (N^{1/n} - 1)a = (e^{\log N/n} - 1)a \ge C_{\delta_0}\frac{\log N}{n}a \ge C'_{\delta_0}a. \tag{8}$$

Now consider the general case of Theorem C, where we can always assume that $n \le N$. We need a standard estimate (see, e.g., [15]) for the distribution of the length of the orthogonal projection of a random vector on the sphere S^{n-1} onto a given r-dimensional subspace: for some positive c_8, c_9, and c_{10} and for $r = 1, 2, \dots, n$, one has

$$\mu_{n-1}\left\{z = \{z_\nu\}_{\nu=1}^n \in S^{n-1} \colon \left(\sum_{\nu=1}^r z_\nu^2\right)^{1/2} \notin \left[c_8\sqrt{\frac{r}{n}},\, c_9\sqrt{\frac{r}{n}}\right]\right\} \le 4\exp(-c_{10}r). \tag{9}$$

Set $r = [c_{11}\log N]$, where the constant c_{11} is large enough that

$$4N^3\exp(-c_{10}r) < 1,$$

and fix a constant δ_0 large enough that the condition $\log N \le \delta_0 n$ implies the inequality $r < n$. In what follows, we assume that $\log N < \delta_0 n$, because the assertion of Theorem C for the case in which $\log N \ge \delta_0 n$ has been verified above.

Just as in the proof of the classical Johnson–Lindenstrauss lemma (which also relies on estimates like (9)), consider the orthogonal projection π_L of V onto a random r-dimensional subspace $L \subset \mathbb{R}^n$ (uniformly distributed with respect to the Haar measure). The estimate (9) and the choice of r guarantee that for a random subspace L and for each pair (j, j'), $1 \le j, j' \le N$, $j \ne j'$, one has

$$\|\pi_L v_j - \pi_L v_{j'}\|_{\ell_2^n} \ge ac_8\sqrt{\frac{r}{n}}. \tag{10}$$

Moreover, the Haar measure of those subspaces for which (10) is violated for at least one pair (j, j') does not exceed N^{-1}.

For a random subspace L, let $w_L(K)$ be the mean width of the body K along the directions in L. Since $\log N \ge r/c_{11}$, we can use the already established estimate (8) and conclude that, for a majority (in the Haar measure) of r-dimensional subspaces L,

$$w_L(K) \ge c_{12}a\left(\frac{r}{n}\right)^{1/2} \ge c_{13}a\sqrt{\frac{\log N}{n}}. \tag{11}$$

Integrating inequality (11) over all r-dimensional subspaces, we arrive at the assertion of Theorem C. More precisely, we use the following equality, which holds for any continuous function f on the sphere S^{n-1}:

$$\int_{S^{n-1}} f\,d\mu_{n-1} = \int_{O^n} \int_{S(TL_r^\circ)} f\,d\mu_{r-1}\,d\nu_H,$$

where O^n is the group of orthogonal transformations of the space $\mathbb{R}^n$ with the Haar measure ν_H, $T \in O^n$, $L_r^\circ = \{a_1, a_2, \dots, a_r, 0, \dots, 0\} \subset \mathbb{R}^n$, and $S(TL_r^\circ)$ is the unit sphere in the subspace $T(L_r^\circ)$.

The above proof of Theorem C permits a significant weakening of the entropy condition (2). In particular, the following result was actually established above (for simplicity, we restrict ourselves to the case in which N is much larger than n).

Theorem 2. *Let* $\log N \ge \delta_0 n > 0$*, and let the tuple* (1) *satisfy*

$$\frac{N}{3}\,\mathrm{Vol}\{B_{a/2}(0)\} \ge \sum_{\substack{(j,j')\\ j\ne j}} \mathrm{Vol}\{B_{a/2}(v_j) \cap B_{a/2}(v_{j'})\}, \tag{12}$$

where $B_a(v)$ *is the Euclidean ball of radius* a *centered at* v*. Then*

$$\mathbb{E}\Big(\max_{1\le j\le N} |\langle G_n, v_j\rangle|\Big) \ge c_{\delta_0} a (\log N)^{1/2}.$$

Note that the right-hand side of (12) admits an effective estimate.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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