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Subcritical transition and multistability in liquid metal magnetoconvection with sidewalls

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The motionless conducting state of liquid metal convection with an applied vertical magnetic field confined in a vessel with insulating side walls becomes linearly unstable to wall modes through a supercritical pitchfork bifurcation. Nevertheless, we show that the transition proceeds subcritically, with stable finite-amplitude solutions with different symmetries existing at parameter values beneath this linear stability threshold. Under increased thermal driving, the branch born from the linear instability becomes unstable and solutions are attracted to the most subcritical branch, which follows a quasiperiodic route to chaos. Thus, we show that the transition to turbulence is controlled by this subcritical branch and hence, turbulent solutions have no connection to the initial linear instability. This is further quantified by observing that the subcritical equilibrium solution sets the spatial symmetry of the turbulent mean flow and thus, organises large-scale structures in the turbulent regime.

Key words: Enter online.

1. Introduction

Liquid metal magnetoconvection (MC), where an electrically conducting fluid is subjected to a time-independent magnetic field and an imposed temperature gradient, is relevant to engineering applications such as liquid-metal cooling systems for nuclear fusion reactors (Davidson 1999), geo-astronomical systems such as flows in liquid-metal planetary cores (Jones 2011), and associated laboratory experiments (Stefani 2024). Despite its relevance, however, a number of uncertainties persist regarding the transition to turbulence in this system, especially in confined containers relevant to experiments and engineering applications. Here, we focus on the case where the uniform applied magnetic field is orientated in the vertical direction, directly opposing gravity. In contrast to magnetoconvection in a laterally periodic/infinite plane layer, where a linear instability of the motionless conducting state classically gives rise to domain-filling arrays of convection rolls (Chandrasekhar 1961; Busse & Clever 1982), wall-localised onset modes, named *wall modes*, are observed when sidewalls are introduced to the system with a sufficiently strong magnetic field. Wall modes

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observed in direct numerical simulations (DNS) (Liu *et al.* 2018; Akhmedagaev *et al.* 2020; Xu *et al.* 2023; McCormack *et al.* 2023; Wu *et al.* 2025) and experiments (Zürner *et al.* 2020; Xu *et al.* 2023) have been thought to originate from a linear instability of the conducting state, originating from the analytic-numerical hybrid approach of Houchens *et al.* (2002), and the asymptotic (large magnetic field limit) linear stability analysis of Busse (2008). However, recent work suggests discrepancies between linear stability theory, simulations and experiments in the measurement of the onset of convection (Xu *et al.* 2023), and short-time simulations displaying exponential growth from a perturbed conducting state were observed to have a different modal structure than converged wall mode equilibria at similar parameter values (McCormack *et al.* 2023). In this work, we show for the first time that multiple stable wall mode solutions exist at identical operating conditions, with fundamental consequences for the transition to turbulence in this flow.

In a general fluid dynamics setting, systems typically either follow a supercritical or subcritical transition to turbulence. A supercritical transition occurs when the laminar flow undergoes successive supercritical bifurcations, transitioning the flow to an increasingly spatio-temporally complex state. Each transition point is clearly identifiable in parameter space, and may be found using a linear stability analysis due to the local nature of the instabilities. Such behaviour has been identified in many flow configurations such as Taylor-Couette flow (TCF) (Taylor 1923; Gollub & Swinney 1975; Brandstater & Swinney 1987; Chossat & Iooss 1994), and Rayleigh–Bénard convection (RBC) (Rayleigh 1916; Malkus & Veronis 1958; Chandrasekhar 1961; Libchaber *et al.* 1982; Ecke *et al.* 1991; Bodenschatz *et al.* 2000; Borońska & Tuckerman 2010; Oteski *et al.* 2015). In other cases, however, the transition proceeds subcritically, where finite-amplitude disturbances trigger the transition despite a linearly stable laminar flow. In this case, non-trivial nonlinear solutions must be identified beneath the linear stability threshold to understand the transition. This leads to a different phenomenology, featuring hysteresis and large, unpredictable changes in the system response triggered at a range of parameter values depending on the supplied disturbance. In fluid dynamics and related fields, subcritical transitions and hysteretic effects occur in a wide range of systems. They are prominent concerning the transition to turbulence in wall-bounded shear flows of neutral (Kerswell 2005; Eckhardt *et al.* 2007; Eckhardt 2018; Avila *et al.* 2023; Hof 2023) and electrically conducting fluids (for comprehensive review see Zikanov *et al.* (2014)), in the transition to elasto-inertial and elastic turbulence for flows with straight streamlines (Morozov & van Saarloos 2007; Bonn *et al.* 2011; Pan *et al.* 2013; Lellep *et al.* 2024), and for types of (approximately) homogeneous turbulence (Linkmann & Morozov 2015). Finite-amplitude transitions between different turbulent states, such as those characterised by the presence or absence of a condensate, occur in continuum models describing certain regimes of active turbulence (Linkmann *et al.* 2019, 2020*b*), two-dimensional turbulence depending on the type of forcing (Linkmann *et al.* 2020*a*; Gallet 2024) and in quasi-two-dimensional turbulence occurring in rotating systems or thin fluid layers (Favier *et al.* 2019; de Wit *et al.* 2022*b,a*; Yokoyama & Takaoka 2017). Moreover, there are more complex scenarios, such as the transition to the ultimate regime of thermal convection (Roche 2020; Lohse & Shishkina 2023, 2024; Shishkina & Lohse 2024) which occurs through a subcritical transition from laminar to turbulent boundary layers, in presence of turbulent flow in the bulk.

Liquid metal MC (*i.e.* large magnetic diffusivity) transitions supercritically in a laterally periodic layer (Chandrasekhar 1961; Busse & Clever 1982; Proctor & Weiss 1982; Weiss & Proctor 2014), and the motionless conducting state undergoes a supercritical bifurcation when laterally confined between sidewalls (Houchens *et al.* 2002; Busse 2008; Bhattacharya *et al.* 2024). Despite this, we show that non-trivial nonlinear equilibrium solutions exist beneath the wall mode linear stability threshold, and that the transition proceeds on this subcritical

branch of solutions. We further observe that this has a fundamental impact on the organisation of large-scale structures in the turbulent regime. This implies that (a), the transition to turbulence and turbulent solutions in closed vessel experiments are fundamentally different to the laterally unbounded geo-astronomical flows they often intend to replicate in nature, and (b), we must change the way we think about the transition and the formation of large-scale structures in convective and potentially other non-linear pattern-forming systems subject to external forces, even when linear stability theory predicts a supercritical transition.

2. Formulation

An incompressible flow of viscous and electrically conducting fluid is driven by an imposed vertical temperature difference, δT , and a constant vertical magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$. The equations of motion under the quasistatic approximation (magnetic Reynolds $\text{Rm} = U\ell/\eta \ll 1$ and magnetic Prandtl number $\text{Pm} = \nu/\eta \ll 1$, where U and ℓ are characteristic velocity and length scales, ν is the kinematic viscosity and η is the magnetic diffusivity) and the Oberbeck-Boussinesq approximation are

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \sqrt{\frac{\text{Pr}}{\text{Ra}}} [\Delta \mathbf{u} + \text{Ha}^2 (\mathbf{j} \times \mathbf{e}_z)] + T \mathbf{e}_z, \quad (2.1a)$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \frac{1}{\sqrt{\text{Ra Pr}}} \Delta T, \quad (2.1b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{j} = -\nabla \phi + (\mathbf{u} \times \mathbf{e}_z), \quad \Delta \phi = \nabla \cdot (\mathbf{u} \times \mathbf{e}_z), \quad (2.1c)$$

where $\mathbf{u} = [u, v, w]^T$ is the velocity, T the temperature, p the kinematic pressure, \mathbf{j} the electric current density, and ϕ the electric field potential. Quantities have been non-dimensionalised using the height of the fluid layer H , the temperature difference $\delta T \equiv T_+ - T_- > 0$, where T_+ and T_- are the temperatures at the bottom and top plates respectively, the free-fall velocity $u_{ff} \equiv (\alpha g H \delta T)^{1/2}$, the free-fall time $t_{ff} \equiv H/u_{ff}$, the pressure ρu_{ff}^2 and the external magnetic field strength B_0 , where α is the thermal expansion coefficient and g the acceleration due to gravity. The control parameters are the Rayleigh number Ra , Prandtl number Pr , and Hartmann number Ha ,

$$\text{Ra} \equiv \frac{\alpha g \delta T H^3}{\kappa \nu}, \quad \text{Pr} \equiv \frac{\nu}{\kappa}, \quad \text{Ha} \equiv B_0 H \sqrt{\frac{\sigma}{\rho \nu}}, \quad (2.2)$$

where κ is the thermal diffusivity, σ the electrical conductivity, and ρ the mass density. We apply no-slip boundary conditions (BCs) for the velocity at all boundaries $\mathbf{u} = 0$, and adiabatic BC at the side walls, $\partial T / \partial \mathbf{n} = 0$, where \mathbf{n} is the vector orthogonal to the surface. All solid boundaries are considered electrically insulating $\partial \phi / \partial \mathbf{n} = 0$. The spatial domain $\Omega \subset \mathbb{R}^3$ is a cube with width W and length L , i.e. $H = W = L$, and is resolved on non-uniform grids using the MC extension of *GOLDFISH* (Kooij *et al.* 2018; Reiter *et al.* 2021, 2022; McCormack *et al.* 2023; Teimurazov *et al.* 2024), which uses a third-order Runge-Kutta time integration scheme and a fourth-order finite-volume discretisation. The coordinates of the grid points, discretizing the interval $[0, 1]$, are defined as $x_k = [1 - (2\tilde{x}_k / (\tilde{x}_1 - \tilde{x}_n))] / 2$ for $k = 1, \dots, n$, where \tilde{x}_k are generalized Chebyshev nodes $\tilde{x}_k = \cos((2k + 2k_{clip} - 1)\pi / (2n + 4k_{clip}))$ with uniformity parameter $k_{clip} \in \mathbb{N}$. In our DNS, we use at least 220^2 points in the cross-plane direction with $k_{clip} = 10$, and 350 points in the vertical direction with $k_{clip} = 1$ in the vertical direction. With this grid, spatial flow fluctuations are resolved to less than 2 Kolmogorov microscales for all simulations used in the stability analysis, with no less than 9 points in the Hartmann boundary layer which forms on the top/bottom boundaries of the domain. Here, the Hartmann layer thickness is defined as $\delta_\nu \approx 1/\text{Ha}$ which uses the prefactor measured by

Teimurazov *et al.* (2024). This layer first forms near onset in the boundary layer underneath the wall modes and thus, resolving this layer is critical to correctly analysing the stability of the solutions. Simulations with $Ra \geq 10^8$ have been resolved on a finer $250^2 \times 400$ grid to explore the long-time behaviour of the system, the results of which have been validated with shorter runs using a $350^2 \times 600$ grid. We set $Pr = 0.025$ for liquid metals such as Gallium-Indium-Tin, and fix the Hartmann number at $Ha = 500$ or $Ha = 1000$, leaving the Rayleigh number Ra as our bifurcation parameter.

3. Multistability of nonlinear wall mode equilibria

3.1. Linear stability and the linear onset branch

Although a strong magnetic field can completely suppress convection, the motionless conducting state becomes linearly unstable with sufficient thermal driving in an infinitely wide and long domain, with the critical Rayleigh number $Ra_c \sim Ha^2$ (Chandrasekhar 1961) (shown by the open black star in Fig. 1(a)). However, including insulating sidewalls results in a different linear instability giving rise to wall modes beneath the onset of bulk convection (Houchens *et al.* 2002; Busse 2008; Bhattacharya *et al.* 2024), even in large-aspect ratio domains (also with finite sidewall conductivity). The critical Rayleigh number for a sidewall in a semi-infinite domain with free-slip top and bottom boundaries is $Ra_c \sim Ha^{3/2}$ for large Ha (Busse 2008) (shown by the filled black star in Fig. 1(a)). Since no stability theory has been performed for the domain and boundary conditions we consider, or any other fully confined domain, we first perform a linear stability analysis for the conducting state. The linear stability threshold is obtained by perturbing the conducting state with random noise of magnitude $O(10^{-10})$ and scanning Ra to find an approximate threshold. Two equilibria are converged on each side of the threshold with $|\partial_t E_k| = O(10^{-14})$ where $E_k = \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \mathbf{u}) \, dx$, ensuring agreement between the fastest growing/slowest decaying modes. This modal structure (similar to Fig. 1(b) (LB) at $Ra = 8 \times 10^5$) is used as the spatial eigenmode to bound the linear stability threshold $Ra_{c,L}$ by bisection. For $Ha = 500$ we have $6.89 \times 10^5 < Ra_{c,L} < 6.90 \times 10^5$ and at $Ha = 1000$ we have $1.83 \times 10^6 < Ra_{c,L} < 1.84 \times 10^6$. Locally, based on these two points, $Ra_{c,L} \sim Ha^{1.412 \pm 0.005}$, where the error corresponds to the uncertainty in $Ra_{c,L}$ at each Ha . Although this is a local measurement of the scaling at finite Ha , it is close to the asymptotic result for a single sidewall obtained by Busse (2008) where $Ra_c \sim Ha^{3/2}$ at leading order. The stable equilibria born from the linear instability (linear onset branch (LB) in the bifurcation diagram in Fig. 1(a)) exhibit a discrete 4-fold rotational symmetry in the vertical velocity field (Fig. 1(b) (LB)), and is thus invariant under the rotation $r : rw(x, y, z) = \mathcal{R}_{\pi/2} w(x, y, z)$, where \mathcal{R}_{θ} represents a rotation θ about \mathbf{e}_z . The solution additionally has reflection symmetries in the x - y plane (about the x and y axes, and about the two diagonals). Thus, these symmetries may be described by a group isomorphic to the dihedral group D_4 , generated by r , the identity id and the reflection s about the $x = y$ diagonal, which we will denote $D_4^+ = \{id, r, r^2, r^3, s, sr, sr^2, sr^3\}$. These solutions have thin convective wall modes with a two-layer structure near the sidewalls, with convection almost completely suppressed in the bulk. Each sidewall features two counter-rotating rolls, which lead to the spatially periodic pattern of up/down flows shown in the vertical velocity isosurfaces in Fig. 1(b) (LB). A secondary solution can be obtained by changing the direction of the rolls and flipping the vertical coordinate $\mathbf{z} : \mathbf{z}w(x, y, z) = -w(x, y, -z)$, and thus, the bifurcation is a supercritical pitchfork, which has broken the \mathbb{Z}_2 Boussinesq symmetry of the $D_4 \times \mathbb{Z}_2 = D_4^+ \times \{id, \mathbf{z}\}$ conducting state. Increasing Ra results in protrusions growing from the wall modes into the bulk of the flow, similar to other observed solutions (Liu *et al.* 2018; Xu *et al.* 2023; McCormack *et al.* 2023), and the equilibria eventually lose stability

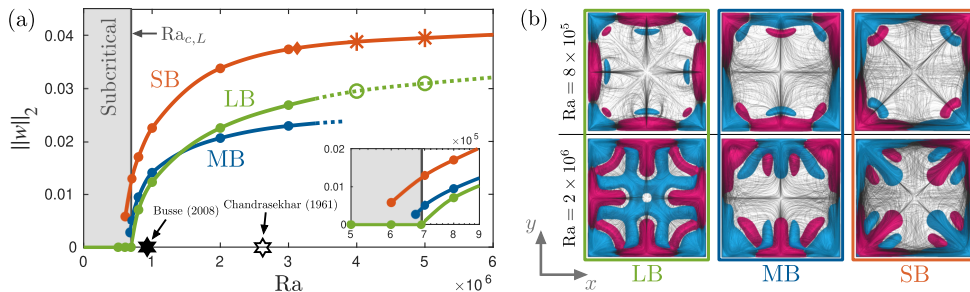


Figure 1: (a) Bifurcation diagram at $Ha = 500$ showing stable/unstable equilibria (filled/open markers), limit cycles (diamonds), invariant tori (asterisks) on the linear onset branch (LB) stemming from the linear instability at $Ra_{c,L}$, the mixed symmetry branch (MB) and the subcritical branch (SB). The mean of time-dependent solutions is shown. The grey area represents $Ra < Ra_{c,L}$. (b) Upflow (pink) and downflow (blue) vertical velocity isosurfaces and streamlines (black) of solutions on the various branches shown from the top view at $Ha = 500$ at $Ra = 8 \times 10^5$ (top) and $Ra = 2 \times 10^6$ (bottom) with $w = \pm 0.005$ and ± 0.01 , respectively.

between $3 \times 10^6 < Ra < 4 \times 10^6$. Since solutions on this branch are only weakly unstable up to $Ra \approx 10^7$, we have estimated the continuation of the unstable branch in Fig. 1(a).

3.2. Subcritical branch

Having determined the threshold for a linear instability, we now assess the nonlinear stability of the conducting state and search for subcritical solutions. We note that the solutions on LB have a different spatial symmetry than other wall mode solutions in the same domain (McCormack *et al.* 2023). Thus, to try to construct a different branch of solutions, we initialise from a solution obtained far from the threshold of the linear instability ($Ra = 5 \times 10^7$) at the given Ha , which displays chaotic dynamics, and then slowly reduce Ra until we encounter equilibria. The obtained solutions again feature wall-mode structures but indeed have a different symmetry in the vertical velocity field (the same as in McCormack *et al.* (2023)), now invariant under the rotation $\mathbf{z}r : (\mathbf{z}r)w(x, y, z) = -\mathcal{R}_{\pi/2}w(x, y, -z)$ which features a reflection in sign and vertical coordinate (Fig. 1(b) (SB)). This state still exhibits reflection symmetries about the diagonals in the $x - y$ plane, but also along the x and y axes, this time accompanied by a reflection in sign and vertical coordinate. Thus, the symmetry group is again isomorphic to D_4 but now with the different generators, $r_- = (\mathbf{z}r)$, id , and s , which we will denote $D_4^- = \{\text{id}, r_-, r_-^2, r_-^3, s, sr_-, sr_-^2, sr_-^3\}$. Importantly, we have found that this solution branch extends beneath the linear stability threshold $Ra_{c,L}$ (Fig. 1(a)) to as low as $Ra = 6 \times 10^5$ ($\approx 0.87Ra_c$) for $Ha = 500$, and $Ra = 1.7 \times 10^6$ ($\approx 0.93Ra_c$) for $Ha = 1000$, and is thus subcritical. These simulations required lengths of approximately 800 or 700 free-fall times (6.5 or 3.4 diffusion times) respectively to assess stability. Solutions at lower Ra evolve extremely slowly, and we have not been able to determine whether they are stable. The simplest explanation from the given data is that this subcritical branch (SB) stems from a saddle-node bifurcation, although we have been unable to determine where the unstable branch from this bifurcation reconnects with the other branches due to computational cost. We note that tests conducted using an under-resolved $60^2 \times 80$ grid did not obtain the correct qualitative transition scenario exhibited by the properly resolved simulations we present here, and thus, resolving the thin boundary layers on all walls is imperative to assessing the stability of the solutions. Importantly, the subcritical branch at both magnetic field strengths extends quite close to the conducting state, suggesting that experimentally small disturbances with the right modal structure could initiate the transition subcritically. Although the magnitude of the subcriticality ($Ra/Ra_{c,L} \approx 0.9$ in both cases) is quite small, the clear difference in the symmetries of the solutions is promising and should be exploited to measure this

phenomenon in experiments. Additionally, the difference between the lowest Ra subcritical solution found and $Ra_{c,L}$ grows with increased magnetic field strength by a factor of about 1.5 from $Ha = 500$ to $Ha = 1000$, suggesting that the gap in Ra between the point of linear instability and the point of the expected saddle-node bifurcation grows with increased magnetic field strength as approximately $Ha^{3/2}$. Thus, the subcriticality becomes increasingly important at higher Ha.

3.3. Mixed symmetry branch

A third branch of equilibrium solutions denoted the mixed symmetry branch (MB) has also been identified, first at $Ha = 1000$ by trying a number of different initial conditions at parameter values where stable equilibria exist. This branch has then been continued to $Ha = 500$ (Fig. 1(a)). The solutions have a symmetry that is mixed between the two other solutions, featuring one/two roll combinations on adjacent walls with matched symmetry on opposing walls (Fig. 1(b) (MB)). The vertical velocity field is invariant under the 180° rotation $rr_- : (rr_-)w(x, y, z) = -\mathcal{R}_\pi w(x, y, -z)$, a reflection in y , and a reflection in x accompanied by a reflection in sign and vertical coordinate. Thus, the symmetry group is isomorphic to the Klein 4-group $K_4 = \{\text{id}, rr_-, sr^3, sr_-\}$. These stable solutions are interesting as they suggest that narrow/wide rolls characteristic of the linear onset/subcritical branches respectively may coexist. This is relevant to larger aspect ratio domains where recent simulations from Wu *et al.* (2025) show near-onset solutions that have various combinations of wide/narrow rolls. This suggests that the results are unlikely to be specific to this geometry or aspect ratio.

3.4. Amplitude equations from symmetry

A model amplitude system which captures the main features of the bifurcation diagram (Fig. 1(a)), may be deduced from symmetries of the system. Associated with each of the three equilibrium solutions found, a secondary solution can be obtained by reversing the direction of the rolls and flipping in the vertical direction $z \mapsto -z$. As Ra is increased, LB first bifurcates from the conducting state through a *supercritical* pitchfork bifurcation. We then suppose that MB and SB subsequently bifurcate through *subcritical* pitchfork bifurcations as Ra is increased further. Close to onset, when the reduced Rayleigh number $R \sim (Ra - Ra_{c,L})/Ra_{c,L} \ll 1$, we assume that only these three modes are relevant to the dynamics of the system, which can thus be described by three amplitudes $A_n(t)$ corresponding to the LB, SB, and MB ($n = 1, 2, 3$) equilibria respectively

$$\Psi(x, y, z, t) = \sum_{n=1}^3 \Psi_n(x, y, z, t) + \text{s.m.} = \sum_{n=1}^3 A_n(t) \psi_n(x, y, z) + \text{s.m.}, \quad (3.1)$$

where Ψ is the state vector of the system and s.m. are stable modes which will be neglected. Since for each solution $\Psi_n(x, y, z, t)$, there exists a distinct antisymmetric solution $\mathfrak{z}\Psi_n(x, y, z, t) = -\Psi_n(x, y, -z, t)$, the equation that governs the evolution of the amplitudes must respect this symmetry. Thus, writing a general nonlinear evolution equation for the amplitudes as $\partial_t \mathbf{A} = \mathcal{N}(\mathbf{A})$, where $\mathbf{A} = [A_1, A_2, A_3]^T$, we note that \mathcal{N} must be equivariant with respect to the reflection $\varpi \mathbf{A} \equiv -\mathbf{A}$, meaning the commutative property $(\varpi \mathcal{N})(\mathbf{A}) = (\mathcal{N} \varpi)(\mathbf{A})$ must hold (Crawford & Knobloch 1991). This implies that \mathcal{N} is odd $-\mathcal{N}(\mathbf{A}) = \mathcal{N}(-\mathbf{A})$, and thus, the evolution equation must have the form $\partial_t \mathbf{A} = \mathcal{F}(A_k^2, R) \mathbf{A}$, for $k = \{1, 2, 3\}$. We note that these equations are also equivariant with respect to the additional symmetries of the different states. The information about these symmetries are contained in the eigenfunctions ψ_n and do not change the amplitudes themselves. We then Taylor expand \mathcal{F} , for $R \ll 1$, and truncate the system to the lowest order that qualitatively

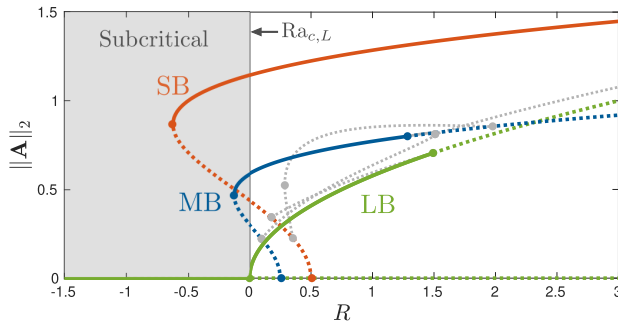


Figure 2: Bifurcation diagram of the amplitude equations 3.2, showing the norm of the amplitudes $\|A\|_2$ as a function of the reduced Rayleigh number R for the various stable/unstable equilibria denoted by solid/dotted lines. Single mode solutions corresponding to the LB, MB and SB states are shown in green, blue and orange respectively. Mixed mode solutions are shown in grey. Markers show the bifurcation points.

captures the desired dynamics, obtaining

$$\partial_t A_1 = [R - b_1 A_2^2 - c_1 A_3^2 - d_1 A_1^2] A_1, \quad (LB) \quad (3.2a)$$

$$\partial_t A_2 = [(R - a_2) - b_2 A_1^2 - c_2 A_3^2 + d_2 A_2^2 - e_2 A_2^4] A_2, \quad (SB) \quad (3.2b)$$

$$\partial_t A_3 = [(R - a_3) - b_3 A_1^2 - c_3 A_2^2 + d_3 A_3^2 - e_3 A_3^4] A_3, \quad (MB) \quad (3.2c)$$

where $a_n, b_n, c_n, d_n, e_n > 0$ are undetermined real coefficients. The position of the subcritical pitchfork bifurcations of the SB and MB branches are set by the coefficients a_2 and a_3 . The position of the saddle-node bifurcations of the SB and MB branches occur at $R_{sn,n} = a_n - d_n^2/4e_n$ at amplitudes $A_n = \pm\sqrt{d_n/2e_n}$ for $n = \{2, 3\}$. The coupling coefficients b_i, c_i determine the bifurcation points, stability and domains of existence of the mixed-mode solutions. For illustration, we have set $a_2 = 0.5, a_3 = 0.25, b_1 = 1.5, e_2 = b_2 = c_n = 2, d_2 = 3, d_3 = 3.5, b_3 = 6, e_3 = 8$. The bifurcation diagram for this choice of parameters is shown in figure 2, exhibiting good qualitative agreement with the bifurcation diagram of the full system in figure 1(a). Thus, although additional complexity may exist in the full system, the behaviour we observe in the DNS can be rationalised by a simplified system, derived from assumptions about the origin of the SB and MB branches, and a consideration of the symmetries of the problem.

A natural question that arises is whether this behaviour is also expected in a cylindrical geometry, often used in experiments, where the system has a continuous rotational symmetry i.e. $O(2) \times \mathbb{Z}_2$ instead of the discrete $D_4 \times \mathbb{Z}_2$ as before. We describe the competition between N unstable modes of azimuthal wavenumber n in cylindrical coordinates by expanding the state vector as

$$\Psi(r, \theta, z, t) = \Re \left[\sum_{n=1}^N a_n(t) e^{in\theta} \psi_n(r, z) \right] + \text{s.m.}, \quad (3.3)$$

where $\Re[\cdot]$ takes the real part of the now complex eigenfunction decomposition. Again, the equation evolving the amplitudes a_n must be equivariant with respect to the symmetries of the system, and thus, must commute with rotations $\theta \rightarrow \theta + \theta_0$ and the reflection. Thus, $\partial_t a_n = \mathcal{F}(|a_k|^2, R) a_n$, for $k \in \{1, \dots, N\}$. Since the conducting state now bifurcates through a circle pitchfork bifurcation, meaning the solutions are neutrally stable with respect to azimuthal rotations, we write the complex amplitudes in polar form $a_n = A_n e^{i\varphi_n}$, noting that due to this neutrality, $\partial_t \varphi_n = 0$. Again, Taylor expanding \mathcal{N} , for $R \ll 1$, we arrive at a system for the real amplitudes A_n which is equivalent to Eq. (3.2) for N equations. This means that the continuous symmetry of a cylinder does not preclude the possibility of having

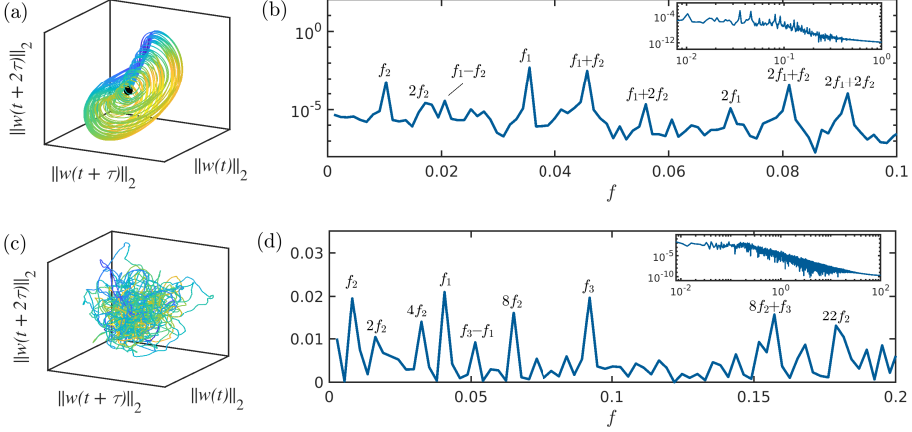


Figure 3: Phase portrait at $Ha = 500$ of the (a) invariant 2-torus at $Ra = 4 \times 10^6$ and (c) chaotic solution at $Ra = 10^8$ constructed using time-delay embedding with $\tau \approx 4$. The colour map corresponds to $\|w(t+3\tau)\|_2$. The unstable equilibrium point (LB) is shown in black. (b,d) Corresponding power spectral density of $\|w(t)\|_2$ respectively as a function of the frequency f . Fundamental frequencies are labelled f_i . Zoomed out spectra are shown in the insets.

the same behaviour we have observed in the cube. Whether the same behaviour occurs as in the cylinder depends on the sign of the coefficients as described before, which must be determined to make qualitative statements about the behaviour of the system. However, we observe that the continuous symmetry of the cylinder does not affect the final structure of the equations we obtain compared to the cube. Furthermore, any finite subgroup of $O(2)$ is isomorphic to a dihedral D_n or cyclic \mathbb{Z}_n group and thus, the solutions bifurcating from the conducting state through these circle pitchforks will generically have D_n or \mathbb{Z}_n symmetries. This suggests that similar symmetry breaking phenomena observed in the cube, may plausibly occur in the cylinder.

4. Subcritical transition to turbulence

Although the presence of subcritical solutions is of fundamental interest, the dynamical role that these solutions play in the transition to chaos must be investigated. Thus, we assess the route to chaos for all three branches at $Ha = 500$. As previously mentioned, the equilibria on the linear onset branch (LB) become unstable between $3 \times 10^6 < Ra \leq 4 \times 10^6$, and undergo long transients of approximately 700 free-fall times before ultimately being attracted to solutions on the subcritical branch. The mixed symmetry branch (MB) similarly becomes unstable with increased Ra . The subcritical branch first undergoes a Hopf bifurcation as Ra is increased between $3 \times 10^6 < Ra < 3.1 \times 10^6$ which breaks the s reflection symmetry, producing a limit cycle featuring a periodic flapping of four r_- rotationally symmetric wall mode extrusions in the horizontal plane. This is equivalent to the secondary bifurcation previously observed at $Ha = 1000$ (McCormack *et al.* 2023). A secondary Hopf bifurcation (Neimark-Sacker bifurcation on the Poincaré section of the limit cycle) subsequently occurs upon increased Ra , forming an invariant 2-torus. The invariant 2-torus is characterised by two fundamental frequencies ($f_1 \approx 0.03542$ and $f_2 \approx 0.01028$ at $Ra = 4 \times 10^6$) which are incommensurable (*i.e.* not rationally related), and thus, the solution is not periodic and densely fills \mathbb{T}^2 . This property is confirmed by constructing an arbitrary time series using f_1 and f_2 in a Fourier expansion, the phase portrait of which is seen to become increasingly dense as time is made arbitrarily large. The power density spectrum of the solution at $Ra = 4 \times 10^6$

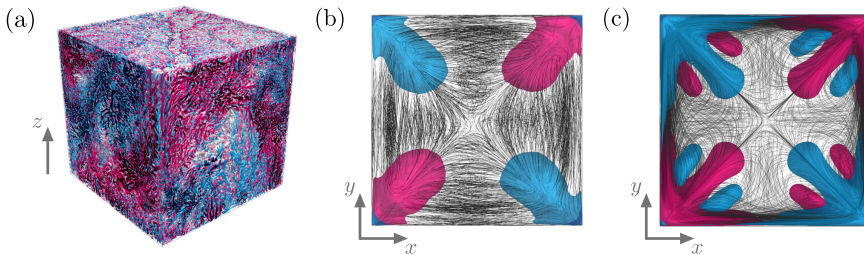


Figure 4: (a) Instantaneous Q-criterion isosurfaces ($Q=3$) coloured by the vertical vorticity for the flow at $Ha = 500$, $Ra = 10^9$ and (b) the corresponding mean flow ($w = \pm 0.1$ isosurfaces (pink/blue)). (c) Equilibrium solution on the subcritical branch (SB) at $Ra = 2 \times 10^6$ ($w = \pm 0.01$ isosurfaces).

is shown in Fig. 3(b) with additional peaks appearing as harmonics of the fundamental frequencies. The phase portrait of the solution is shown in Fig. 3(a). The solution continues to feature flapping wall modes, but the additional frequency modulates the protrusions as they flap, moving their tips back and forth from each other, maintaining the $\mathbb{Z}_4 = \{id, r_-\}$ symmetry despite the additional wall mode modulation. Increasing the Rayleigh number to $Ra = 5 \times 10^6$ leads to a \mathbb{Z}_4 to $\mathbb{Z}_2 = \{id, r^2\}$ symmetry break. Here, the solution still features four wall modes flapping back and forth as before, but as the wall mode dynamics become more vigorous, extending further into the bulk, the solution now has a preferred diagonal where two opposing wall modes dominate over the others. The solution becomes chaotic by $Ra = 10^7$ and is accompanied by a large increase in fast timescale activity, revealing a broadband spectrum. Clear trends are more easily observed at $Ra = 10^8$, whose power density spectrum is shown in Fig. 3(d) and phase portrait is constructed in Fig. 3(c). The solution features three incommensurate frequencies and a wide broadband spectrum, which can be compared to the spectrum of the 2-torus at $Ra = 4 \times 10^6$ in Fig. 3(b). Thus, the system undergoes a quasiperiodic route to chaos where a finite number of Hopf bifurcations lead to the formation of a chaotic attractor. This is in the spirit of the Ruelle-Takens-Newhouse theorem (Ruelle & Takens 1971; Newhouse *et al.* 1978; Eckmann 1981), which predicts more generally that three-tori are likely to be structurally unstable. However, more detailed analysis is required to determine precise details of the route in the present system, due to the possibility of phase-locking (tori-periodic orbit transitions) (Arnol'd 1961; Reichardt & Nori 1999), periodic-chaotic-periodic transitions (Knobloch *et al.* 1986) or other effects. Exploring the behaviour of the system at finer increments of increased nonlinearity (Ra), as has been done in other fluid systems (Knobloch *et al.* 1986; Oteski *et al.* 2015; Launay *et al.* 2019; Wang *et al.* 2025), may well reveal additional complexity.

At $Ha = 1000$, the wall modes display smaller length scales and are increasingly confined near the sidewall, resulting in a different transition pathway. The subcritical branch undergoes a Hopf bifurcation (similar to the $Ha = 500$ case), but then displays complex homoclinic behaviour reminiscent of Shilnikov dynamics (McCormack *et al.* 2023). Importantly, the transition again occurs on the subcritical branch and has no connection to the initial linear instability.

4.1. Consequences of the subcritical transition

At all values of Ha studied, the route to chaos originates through a sequence of bifurcations on the subcritical branch. Often in supercritical transitions, the equilibrium solution born from the first linear instability in the system imprints the turbulent flow, organising large-scale features (*e.g.* Taylor vortices in turbulent TCF (Taylor 1923; Grossmann *et al.* 2016) or convection rolls in turbulent RBC (Chandrasekhar 1961; Busse 1978; Ahlers *et al.* 2009)). Although the subcriticality is reasonably small in this system at the considered Ha , it has

a significant effect on the flow in the turbulent regime. At $Ra = 10^9$ ($\approx 1000 Ra_{c,L}$), the flow field exhibits clear multiscale behaviour with small-scale vortices being observed in an instantaneous snapshot of the flow in Fig. 4(a). However, unlike the supercritical cases, this turbulent flow has been formed through bifurcations on the subcritical branch and thus, the turbulent flow retains properties of the subcritical equilibria and not the linear instability. This is clearly observed in the mean flow of the solution which retains the same D_4^- symmetry as the subcritical equilibria, and not the D_4^+ symmetry of the solutions born from the linear instability. This means that understanding the subcritical nature of the transition is imperative to understanding properties of turbulent MC, such as the globally organising flow structures.

5. Conclusions

The addition of insulating sidewalls changes the phenomenology of the transition to turbulence in liquid metal magnetoconvection, with stable solutions existing beneath the linear stability threshold. This occurs despite the conducting state undergoing a supercritical bifurcation, and the transition proceeding supercritically in a laterally periodic layer. Although the magnitude of the subcriticality is relatively small at the studied magnetic field strengths, our results suggest the transition becomes increasingly subcritical at higher magnetic field strengths, and is thus, centrally important in high Ha regimes currently inaccessible to simulations and experiments. Multiple stable equilibrium solutions exist near onset, which are distinguished by their spatial symmetries and have different transport properties. The qualitative features of the near onset behaviour can be rationalised by low-dimensional amplitude equations, deduced from symmetries of the system. We have shown that the solutions born from the linear instability (LB), and the mixed symmetry branch (MB) quickly become unstable, and solutions are attracted to the subcritical branch (SB), which is the primary branch involved in the route to chaos. Thus, we show that both the transition and the formation of global flow structures in turbulent regimes can only be understood through the subcriticality, and have no connection to the linear instability, despite linear stability theory predicting a supercritical transition. This must be considered when laboratory experiments with sidewalls are compared to laterally unbounded geophysical or astrophysical flows. Since wall modes are observed in a variety of container geometries and with finite wall conductivity, both numerically and experimentally, we anticipate the subcriticality is insensitive to these properties. Wall modes are also well known to form in rotating convection with sidewalls (Goldstein *et al.* 1993; Herrmann & Busse 1993; Ning & Ecke 1993; Kuo & Cross 1993; Goldstein *et al.* 1994; Liu & Ecke 1999; Liao *et al.* 2006; Kunnen *et al.* 2011; Horn & Schmid 2017; Favier & Knobloch 2020; de Wit *et al.* 2020; Zhang *et al.* 2021; Wedi *et al.* 2022; de Wit *et al.* 2023; Zhang *et al.* 2024; Ravichandran & Wettlaufer 2024; Vasil *et al.* 2025), and are robust to varied sidewall geometry (Favier & Knobloch 2020). Furthermore, wall localised modes are known to form in a variety of simple pattern forming systems on bounded domains, such as the cubic-quintic Swift–Hohenberg equation, which are expected to have wide applicability (Verschueren *et al.* 2021). Thus, one might anticipate that similar results to those obtained here may extend to a wider class of systems, such as rotating convection or rotating MC. Further open questions in this direction relate to obtaining a detailed understanding of these effects for domains with continuous symmetries (*e.g.* for MC in a cylinder), and for larger aspect ratio domains.

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