The Development of Fuzzy Algebra And Its Applications: A Chronological Perspective

Ogiugo, Mike E.
Department of Mathematics, Yaba College of Technology, Nigeria
E-mail: mike.ogiugo@yabatech.edu.ng

Abstract: This paper explores the historical development of fuzzy algebra, an extension of classical algebra that incorporates the principles of fuzzy set theory to address uncertainty and partial membership. Originating from Lotfi A. Zadeh's introduction of fuzzy set theory in 1965, fuzzy algebra has evolved through significant contributions from researchers worldwide. This work highlights key contributions, mathematical formalism, and theoretical advancements that have shaped this field over the decades

1. Introduction

Fuzzy algebra is a branch of mathematics that extends classical algebraic structures (groups, rings, fields, lattices) through the lens of fuzzy set theory, enabling the modelling of uncertainty and imprecision. Classical algebra deals with well-defined operations and crisp set memberships. However, many real-world problems involve uncertainty, prompting the need for a more flexible algebraic framework.

In 1965, Lotfi A. Zadeh introduced the concept of a fuzzy set. Zadeh had his training in electrical engineering, including the initial system theory as a student in Tehran, Iran. Following his immigration to the USA in 1942, Zadeh continued his studies at the Massachusetts Institute of Technology (MIT) in Cambridge, Massachusetts. He moved to New York in 1946, where he was awarded a Ph.D by Columbia University in 1949. Since 1958, he has been a Professor of Electrical Engineering at the University of California at Berkeley. During his early years at Berkeley, Zadeh worked on various problems emerging from system theory, including adaptive and time-varying systems, optimal control, and system identification. In the early 1960s, he began to question the adequacy of conventional mathematics for dealing with highly complex systems, as exemplified by the following quotation from one of his papers on systems theory (Zadeh, 1962):

> There are some who feel that this gap reflects the fundamental inadequacy of conventional mathematics, the mathematics of precisely-defined points, functions, sets, probability measures, etc. For coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities that are not describable in terms of probability distributions. Indeed, the need for such mathematics is becoming increasingly apparent even in the realm of inanimate systems, for in most practical cases, the a priori data and criteria by which the performance of a man-made system is judged are far from being precisely specified or having accurately known probability distributions.

Zadeh begins his paper with a clear and convincing motivation for his groundbreaking concept (Zadeh 1965a):

> Clearly, the "class of all real numbers which are much greater than 1," or" the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction. The purpose of this note is to explore preliminarily some of the basic properties and implications of a concept which may be of use in dealing with "classes" of the type cited above. The concept in question is that of a fuzzy set, that is, a "class" with a continuum of grades of membership function... The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability. . . . Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership

The key idea in Zadeh's paper of 1965 is, of course, the concept of a fuzzy set, which is a generalisation of the classical idea of a set. Intuitively, a classical set is any collection of definite and distinct objects conceived as a whole. Objects that are included in a set are usually called its members. Each classical set must satisfy two requirements. First, members of each set must be distinguishable from one another; and second, any given object either is or is not a member of the set. The proposition "a is a member of A" for any given object "a" and any given set A is either true or false. We say that each classical set has a sharp boundary which separates objects that are members of the set from those that are not its members. Fuzzy sets differ from classical sets by rejecting the second requirement. As a consequence, their boundaries are not necessarily sharp. Zadeh explained that these concepts relate to situations in which the source of imprecision is not a random variable or a stochastic process but rather a class or classes which do not possess sharply defined boundaries.

2. Introduction to Fuzzy Set

Zadeh's introduction of fuzzy set theory provided a mathematical model for handling vagueness, leading to the development of fuzzy algebra. Zadeh's formulation of fuzzy set theory marked a paradigm shift in mathematics, where binary logic was no longer sufficient to model real-world vagueness

A fuzzy set A on a universe X is defined by a membership function:

$$\mu_A : X \to [0, 1],$$

where $\mu_A(x)$ represents the degree to which x belongs to A.

Zadeh also recognised, somewhat indirectly, that each standard fuzzy set is associated with a special family of classical sets. These classical sets are now commonly known as level-cuts,

$$\mu_{A_\alpha=\{x\in A|\mu(x)\geq\alpha\}, \quad \alpha\in[0,1].}$$

Zadeh devotes almost one-third of his 1965 paper to discussing operations on fuzzy sets. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets in a universal set U, with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$. The fuzzy union and intersection are defined as:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad \forall x \in U$$
$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \quad \forall x \in U$$
$$\mu_{\tilde{A}}(x) = 1 - \mu_{A}(x), \quad \forall x \in U$$

Zadeh's motivation was practical, that the traditional logic and mathematics were insufficient to model imprecise linguistic and cognitive information. This call to broaden formal frameworks laid the groundwork for fuzzy extensions in many mathematical disciplines, including algebra. The concept of fuzziness deals with imprecise data through what he termed fuzzy sets.

A practical application of the fuzzy-based model for evidence-based clinical decision support systems (Navin & Mukesh Krishnan , 2024)

Problem: Diagnosing diabetes based on "high" blood sugar levels, where thresholds are not absolute.

Fuzzy Approach: Define fuzzy sets for blood glucose (mg/dL):

Normal:

$$\mu_{normal}(x) = \begin{cases} 1 & \text{if } x \leq 90 \\ \frac{110-x}{20} & \text{if } 90 < x < 110 \\ 0 & \text{if } x \geq 110 \end{cases}$$

Prediabetic:

$$\mu_{prediabetic}(x) = \begin{cases} 0 & \text{if } x \leq 100 \\ \frac{x-100}{20} & \text{if } 100 < x < 140 \\ 1 & \text{if } 140 \leq x \leq 200 \end{cases}$$

Diabetic:

$$\mu_{diabetic}(x) = \begin{cases} 0 & \text{if } x \leq 140 \\ \frac{x-140}{60} & \text{if } 140 < x < 200 \\ 1 & \text{if } x \geq 200 \end{cases}$$

Application: A patient with 130 mg/dL has:

$\mu_{prediabetic}(130) = 0.75$

$\mu_{diabetic}(130) = 0.0$

$\mu_{normal}(130) = 0.0$

The system concludes "likely prediabetic" and recommends further tests.

3. The Fuzziness of Algebraic Structures: Early Development of Fuzzy Algebra (1970s– 2000s)

The notion of fuzziness has captured the interest of many researchers worldwide because it can potentially revolutionise the nature of research in the area of man-machine systems and humanistic processes. Fuzzy algebra extends classical algebraic structures such as groups, rings, and vector spaces by incorporating fuzzy membership. It allows for the degree of membership of elements in algebraic sets, thereby enabling the modelling of real-world phenomena with inherent uncertainty.

Around the same time, Goguen (1967) proposed L-fuzzy sets, extending membership values to complete lattices rather than just [0,1]. This generalisation laid the theoretical basis for L-fuzzy groups and rings, opening a more algebraically flexible framework. By the early 1970s, mathematicians began exploring how algebraic structures could be fuzzified. The key idea was to generalise classical algebraic axioms (e.g., associativity, commutativity) using fuzzy membership functions. The first significant application of fuzzy set theory in algebra appeared in 1971 when A.

Rosenfeld introduced the notion of fuzzy subgroups.
A fuzzy subset μ of a group $(G, \cdot)$ is a fuzzy subgroup if:
1. Closure under Operation:

$$\mu(x \cdot y) \geq \min(\mu(x), \mu(y)) \quad \forall x, y \in G.$$

2. Closure under Inverses:

$$\mu(x^{-1}) \geq \mu(x) \quad \forall x \in G.$$

Remarks:

Non-Uniqueness of Inverses: In classical group theory, inverses are unique. While fuzzy group theory, if $\mu(x) = \mu(x^{-1})$, multiple elements may satisfy the inverse condition to the same degree.

Weaker Closure: Classical closure requires exact membership, while fuzzy closure allows approximate membership (e.g., $\mu(x \cdot y) \geq 0.5$ even if $\mu(x) = 0.6$ and $\mu(y) = 0.5$).

Rosenfeld's fuzzy subgroup axioms preserve associativity:
For all $x, y, z \in G$,
$\mu((x \cdot y) \cdot z) \geq \min(\mu(x \cdot y);$
$\mu(z)) \geq \min(\mu(x), \mu(y), \mu(z))$
Similarly, $\mu(x \cdot (y \cdot z)) \geq \min(\mu(x), \mu(y), \mu(z))$.

This condition ensures that the structure of the group is preserved under fuzziness. This work laid the foundation for further studies on normality, homomorphisms, and quotient structures in fuzzy group theory. Considering a finite group G, the number of subgroups of G is finite, while the number of level subgroups of G appears infinite. By definition, every level subgroup is indeed a subgroup of G; not all these level subgroups are distinct. It is revealed that μ is a fuzzy subgroup of G on the condition that its level subgroups are subgroups of G. In 1981, Das in his paper studied Zadeh's notion of level subsets to define level subgroups of fuzzy subgroups. Fuzzy subgroups have been characterised by using their level subgroups; hence, it has become one of the significant tools used in the study of fuzzy group theory.

Suresh and Bhattacharya (1971) focused on characterising fuzzy subsets that preserve algebraic operations, particularly under group homomorphism. Anthony and Sherwood (1979) introduced the notion of fuzzy subgroups under t-norms, generalising Rosenfeld's min-operation.

The introduction of fuzzy equivalence relations and fuzzy algebraic structures further enriched the field of fuzzy group theory. The cardinality of the number of all fuzzy subsets of even a singleton set is infinite. Without some sort of equivalence relation on the set of fuzzy subsets, the number of such fuzzy subsets is unmanageable. Therefore, Researchers impose an equivalence relation based on the equality of sets on the collection of all fuzzy subsets of a given set. When comparing various notions of fuzzy equivalence relation in the literature, and discovered that each definition of fuzzy equivalence relation determines the number of distinct fuzzy subgroups of a finite group. The previous works have shown that even with the same groups, as in the case of the symmetric groups and alternating groups (see Ogiugo and EniOluwafe, 2019; Ogiugo and Amit, 2020), different results for the number of distinct fuzzy subgroups are obtained depending on the definition of equivalence relations used.

A fuzzy relation $\tilde{R}$ between two sets X and Y is defined by a membership function $\mu_{\tilde{R}} : X \times Y \to [0,1]$, representing the degree of relation between elements of X and Y.

A fuzzy relation $\tilde{R}$ on a set X is an equivalence relation if it satisfies:
1. Reflexivity: $\mu_{\tilde{R}}(x,x) = 1, \quad \forall x \in X$
2. Symmetry: $\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x), \quad \forall x, y \in X$
3. Transitivity: $\mu_{\tilde{R}}(x,z) \geq \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{R}}(y,z)\}, \quad \forall x, y, z \in X$

In 1982, Liu's paper extended fuzzy concepts to ring theory. Let R be a ring and $\mu : R \to [0,1]$ a fuzzy subset. Then μ is a fuzzy ideal of R if for all $x, y \in R$:
1. $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(rx) \geq \mu(x)$, $\mu(xr) \geq \mu(x)$, for all $r \in R$

This forms a fuzzy ideal because it preserves addition and multiplication under min. These axioms extend the classical definition of an ideal into the fuzzy context while preserving the ring's structural properties. Kumar (1983) introduced fuzzy prime ideals, extending classical ideal theory. Mordeson & Malik (1986) explored fuzzy field extensions, paving the way for fuzzy Galois theory. Fuzzy fields were later explored by Kumar (1987), who introduced membership-based definitions for invertibility and algebraic closure.

Atanassov (1986) introduced intuitionistic fuzzy sets (IFS), where an element has both a membership $\mu(x)$ and non-membership $\nu(x)$, with $\mu(x) + \nu(x) \leq 1$. This inspired Intuitionistic fuzzy groups and Interval-valued fuzzy rings

These decades mark a phase of experimentation and generalisation, where researchers explored how far fuzzy set concepts could penetrate algebraic theories without compromising their structural integrity. The study of fuzzy normal subgroups, where $\mu(xgx^{-1}) = \mu(g)$, added depth to group-theoretic analysis. The lattice structure of fuzzy subgroups and ideals was investigated, leading to the development of fuzzy algebraic lattices. The study of fuzzy Lie algebras was initiated in the paper of Kim and Lee (1998). For fuzzy Lie algebras, where a fuzzy subset μ of a Lie algebra $\mathfrak{g}$ satisfies:

$$\mu([x,y]) \geq \min(\mu(x), \mu(y)) \quad \forall x, y \in \mathfrak{g}.$$

This decade has seen an exponential growth in the development of fuzzy mathematics.

3. Contemporary Research and Applications (2010s–Present)

The systematic study of fuzzy algebraic structures is one of the main areas of current fuzzy algebra. For instance, Jun and Kang (2011) expanded the framework to fuzzy ideals in BCI- and BCK-algebras, revealing novel algebraic invariants, while Bhakat and Das (2014) investigated their homomorphic images and provided modifications to the idea of fuzzy normal subgroups. Additionally, the addition of interval-valued, bipolar fuzzy, and intuitionistic fuzzy sets to algebraic systems has produced richer models that more accurately represent dual viewpoints and partial truths.

Applications have evolved in tandem with theory. Fuzzy groups and semigroups, in particular, are now utilised to represent systems with inherent uncertainties in network security, pattern recognition, and cryptography. In 2019, Sahu and Rajaraman, in their paper, used fuzzy automata based on fuzzy semigroups to enhance secure key exchange protocols. According to Zimmermann (2010), fuzzy rings and lattices are used in artificial intelligence to support reasoning mechanisms in fuzzy logic controllers and decision-support systems.

Additionally, fuzzy logic has been combined with algebraic methods utilising soft sets to tackle difficult decision-making issues, particularly in the fields of engineering and medical diagnostics (Alcantud and Mathew , 2017). Fuzzy soft groups, soft fuzzy lattices, and other hybrid structures are currently the focus of a thriving field of study that aims to manage various degrees of uncertainty. Classifying data into fuzzy groups for clustering in Pattern Recognition (Bezdek, 1999). Fuzzy algebra has found applications in decision-making, optimisation, and artificial intelligence.

4. Conclusion

Fuzzy algebra has developed into a vibrant, multidisciplinary field, to sum up. Since the 2010s, it has evolved to strike a balance between advancing abstract theory and meeting real-world demands in unpredictable situations. In fuzzy group theory, many researchers have worked on the classification of fuzzy subgroups of finite abelian and non-abelian groups. The interesting story is that the work on the classification of fuzzy subgroups of both groups is ongoing. There is potential for significant applications and new theoretical understandings with more research in this field. Such as :

1. Fuzzy Algebraic Machine Learning: Hybrid models combining fuzzy algebra with deep learning (e.g., fuzzy neural networks for interpretability),
2. Fuzzy Topological Data Analysis: using fuzzy algebra to analyse high-dimensional datasets with noise,
3. Quantum Fuzzy Algebra: extending fuzzy concepts to quantum computing for uncertainty-aware algorithms, and computational intelligence.

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