

Screening and localization in the nonlinear Anderson problem

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We study the spreading dynamics of an initially localized wave packet in 1D nonlinear Schrödinger lattices with random potential. It is shown that adding small dielectric coupling of the wave field to surrounding random medium results in asymptotic localization of the nonlinear field. The nonlinear localization length depends on dielectric loss of the medium at low temperatures and the value of nonlinearity parameter. The model predicts a possibility of self-induced localization when the “medium” to which the wave field is dielectrically coupled is the wave function itself.

Anderson localization is the absence of diffusion of waves in random lattices [1]. It is a generic wave phenomenon, which applies to any kind of wave (acoustic, matter, light, etc.) in a disordered environment. Anderson localization occurs because strong disorder induces multiple scattering paths along which the wave field interferes with itself, the result being that the envelope wave function decays exponentially fast with length. The phenomenon was initially discussed for the dynamics of noninteracting electrons in a semiconductor [1, 2] and at present is fairly well understood [3, 4].

Can nonlinearity or interactions destroy Anderson localization? The question—to which no unambiguous answer has been provided—challenges the classic theories of localization [3], which assume that the lattice is static and the absence of interactions. Over the years, two nonequivalent approaches to the problem and respectively two different lines of inquiry have emerged.

One—dating back to Fleishman and Anderson [5]—pursued the interaction path, with the major milestone settled in by Altshuler *et al.* [6]. This line of inquiry has evolved over time into a vast area of research commonly referred to as many-body localization, or MBL (Ref. [7] for review; references therein). The result of interest here is that electron-electron interactions alone are unable to cause the relaxation and establish the thermal equilibrium, that is one needs a thermal bath (coupling to phonons) to destroy the localized state [7].

The other line of inquiry—initiated by Pikovsky and Shepelyansky [8] and Flach *et al.* [9], and which is the main interest of the present study—followed the nonlinearity path, with most attention paid to the spreading dynamics of an initially localized wave packet of finite norm in 1D nonlinear Schrödinger lattices with random potential. The process is described by the discrete Anderson nonlinear Schrödinger equation (DANSE)

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta |\psi_n|^2 \psi_n, \quad (1)$$

where $\psi_n = \psi(n, t)$ is complex wave function which depends on time t and the position coordinate n ,

$$\hat{H}_L \psi_n = \varepsilon_n \psi_n + V(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \quad (2)$$

is the Anderson Hamiltonian in tight-binding approximation [1], V is hopping matrix element, $\beta > 0$ characterizes nonlinearity, on-site energies ε_n are randomly distributed with zero mean across a finite energy range, and the probability distribution is normalized to $\sum_n |\psi_n|^2 = 1$.

The various aspects of dynamical model (1) are discussed in Refs. [10–21]. The results can be summarized by a critical value $\beta = \beta_c$, above which the nonlinear field can spread indefinitely along the lattice, and below which it is Anderson localized similar to the linear field.

The relation between DANSE (1) and MBL is not at all trivial. In fact, DANSE (1) is a single-particle problem, which finds its origins in the conundrum of quantum localization of dynamical chaos [22, 23]. That said, DANSE (1) captures much of the observed phenomenology of the Bose-Einstein condensates in random potential (e.g., Refs. [8, 12, 24, 25]). This, together with its relative mathematical simplicity, makes DANSE (1) an attractive and powerful tool at describing Anderson localization in nonlinear lattices with disorder [12, 14].

It is the purpose of the present Letter to demonstrate that DANSE (1) leads to asymptotic localization of the nonlinear field for *any* $\beta > 0$ as soon as a small dielectric coupling to ambient medium is added. This coupling of the system and the environment introduces a nonlinear localization scale, which depends exclusively on dielectric loss of the material at low temperatures and the β value.

Our idea is as follows: By its formulation, DANSE (1) refers to a wave process in a medium—not in vacuum. The effect of the medium is included via the random potential in Eq. (2), yet there can be a nonzero dielectric response from the same random medium which is omitted in (1). We argue and confirm through the direct calculation that this response is such as to diminish (screen) the actual strength of nonlinear interaction compared to a vacuum value. The result is that the nonlinear field with screened interactions is asymptotically localized *regardless* of the β value, provided the lattice is infinite and the medium is in some sense random.

More explicitly, we start with DANSE (1), where we interpret the β value as the intrinsic value of the nonlinear field. We refer to this value as the bare value. Next

we add small dielectric coupling to the environment and wonder about the strength of nonlinear interaction under this coupling. We show that in the presence of randomness the nonlinearity becomes dispersive, that is the actual β value starts to depend on position coordinate n . It is the dispersive character of dielectric response from the surrounding random medium that localizes the nonlinear wave field in 1D independently of the β value.

The theory approach designed above is rooted in (and incidentally inspired by [26]) quantum electrodynamics, where one distinguishes between the bare (predicted by fundamental theory) electric charge, and the actually measured charge [27, 28]. The latter is somewhat smaller than the former as a result of screening by short-living electron-positron pairs surrounding the electric charge in the vacuum. Our analysis suggests the situation behind DANSE (1) is just similar.

In fact, assume for concreteness that the wave function ψ_n refers to an electromagnetic wave. It is understood that the electric field of the wave perturbs the surrounding dielectric medium causing its polarization (we work everywhere with frequencies below the ionization frequency). The action of the polarization field is that it partially screens the bare electric field of the wave, with that consequence that the actual strength of nonlinear interaction is decreased by a factor of

$$\Delta\beta/\beta \simeq -P_\omega/D_\omega, \quad (3)$$

where β is the bare value, and $\Delta\beta$ is a variation caused by polarization. In the above, P_ω is the amplitude of the polarization field, and D_ω is the electric displacement, both frequency dependent in general. In a basic theory of dielectrics one writes the polarization and displacement fields respectively as $P_\omega = \varepsilon_0\chi E_\omega$ and $D_\omega \equiv \varepsilon_0 E_\omega + P_\omega$, where ε_0 is the permittivity of free space, and χ is the dielectric susceptibility. Upon substituted in Eq. (3) this yields $\Delta\beta/\beta \simeq -\chi/(1+\chi)$, from which the effective control parameter is $\beta_{\text{eff}} = \beta + \Delta\beta \simeq \beta/(1+\chi)$ [29].

It is understood that χ is, in general, a complex quantity, i.e., $\chi = \chi_1 + i\chi_2$, whose components can depend on the frequency of the applied field. These frequency-dependent components are related to each other via the Kramers-Kronig (Hilbert) transform, i.e., $\chi_1(\omega) = (1/\pi)\text{P} \int \chi_2(\omega')d\omega'/(\omega' - \omega)$. In this context, one writes the effective control parameter as $\beta_{\text{eff}} = \beta/[1 + \chi_1(\omega)]$, where $\chi_1(\omega) \equiv \text{Re}\chi(\omega)$ is the real part of $\chi(\omega)$. The imaginary part $\chi_2(\omega) \equiv \text{Im}\chi(\omega)$ is expressed as $\chi_2(\omega) = \sigma_{\text{a.c.}}(\omega)/\varepsilon_0\omega$, where $\sigma_{\text{a.c.}}(\omega)$ is the real part of the complex frequency-dependent a.c. (alternating-current) conductivity of the medium. We should stress that the frequency dependence of $\chi(\omega)$ and $\sigma_{\text{a.c.}}(\omega)$ is particularly relevant for random systems where spatial inhomogeneity dictates a distribution of time delays to the field P_ω . Putting all the various pieces together, we have

$$\text{Re}\chi(\omega) = \frac{1}{\pi\varepsilon_0}\text{P} \int \frac{\sigma_{\text{a.c.}}(\omega')}{\omega'} \frac{d\omega'}{\omega' - \omega}. \quad (4)$$

At this point, one needs a model of a.c. conduction for the surrounding medium. It is understood that the charge carriers—whose displacement is held responsible for the frequency-dependent polarization response P_ω —are subject to the same random potential in Eq. (2) that scatters the wave function ψ_n . At low temperatures, the actual charge conduction mechanisms fall into two main categories [30, 31], that is transport of a carrier by quantum-mechanical tunneling through [32], or classical hopping over [33], the potential barrier separating two localized states. These both lead to linear frequency dependence of a.c. conductivity versus frequency—the former ensures that a.c. conductivity stays finite as temperature vanishes, the latter accounts for some increase of a.c. conductivity as temperature rises [34]. We ought to note that temperature and frequency dependence of a.c. conductivity in disordered semiconductors has been the subject of very intense theoretical and experimental treatments (e.g., Refs. [30, 31, 35–37]). What is important for our purposes here is that the linear scaling $\sigma_{\text{a.c.}}(\omega) \propto \omega$ applies universally for $T \rightarrow 0$ [37–40], where T is temperature, and that it is confirmed experimentally that in the absence of measurable d.c. conductivity the a.c. counterpart behaves linearly with ω [31, 37].

Summarizing the above reasoning, we write the a.c. conductivity as $\sigma_{\text{a.c.}}(\omega) = A\omega$, where A is a coefficient, which can depend on temperature T , but not on the frequency ω . In material science [31, 36], it is customary to express this coefficient as $A = \varepsilon_0\varepsilon_r \tan\delta$, where ε_r is relative permeability, and $\tan\delta$ is dielectric loss of the material. With these definitions, Eq. (4) reduces to $\text{Re}\chi(\omega) = \mu\text{P} \int d\omega'/(\omega' - \omega)$, where we have introduced $\mu \equiv A/\pi\varepsilon_0 = (\varepsilon_r/\pi) \tan\delta$.

In the absence of meaningful d.c. (direct-current) conductivity, the integration in Eq. (4) can be performed from $\sim \omega$ (which is assumed to be comparable to, yet somewhat greater than the dielectric loss peak frequency, ω_m) and up to the actual frequency spread of the wave packet, which we shall denote by $\Delta\omega$. Assuming a broad wave packet, with $\Delta\omega \gg \omega \gtrsim \omega_m$, from Eq. (4) one gets, with logarithmic accuracy, $\text{Re}\chi(\omega) \simeq \mu \log|\Delta\omega/\omega|$. Translating frequencies into wave numbers using $\omega = v/n$, where v is the speed of the wave, one also gets $\text{Re}\chi(n) \simeq \mu \log|\Delta n/n|$. If, however, the spread $\Delta\omega$ is large compared to ω , then the ratio $|\Delta n|/n$ is also large, i.e., $|\Delta n| \gg n$, permitting one to write the real part of χ as $\text{Re}\chi(\Delta n) \simeq \mu \log|\Delta n|$, without keeping an eye on n . To this end, the effective β value becomes $\beta_{\text{eff}} = \beta/[1 + \text{Re}\chi(\Delta n)] \simeq \beta/(1 + \mu \log|\Delta n|)$. Writing, under the logarithm sign, the spread Δn as $|n - n'|$, where $|n - n'| \gg 1$, one gets $\text{Re}\chi(n, n') \simeq \mu \log|n - n'|$, leading to

$$\beta_{\text{eff}} = \beta_{\text{eff}}(n, n') \simeq \beta/(1 + \mu \log|n - n'|). \quad (5)$$

One sees, from Eq. (5), that the effective β value is dispersive (depends on position coordinate n'). To this end,

the DANSE model in Eq. (1)—which assigns one single value of β to all sites—becomes insufficient. That is one needs to employ a more general dynamical equation that would directly take into account the dispersive character of interactions taking place. Here, we work with the nonlinear Schrödinger model with dispersive nonlinearity [21, 41]

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \sum_{n'} \beta_{n,n'} \frac{|\psi_{n'}|^2}{|n-n'|} \psi_n, \quad (6)$$

where $\beta_{n,n'} \equiv \beta(|n-n'|) > 0$ is the position-dependent nonlinearity parameter, the nonlocal kernel $\sim 1/|n-n'|$ represents the Poisson potential (here associated with electrostatic interaction), and \hat{H}_L is the Anderson Hamiltonian of the type given by Eq. (2). Remark that Eq. (6) differs from the familiar Schrödinger-Newton equation (SNE) [42–46] in that the SNE invokes neither randomness nor dispersive interactions. Identifying, in dynamical Eq. (6), the position-dependent $\beta_{n,n'}$ with the effective value in Eq. (5), one obtains

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \sum_{n'} \frac{\beta}{1 + \mu \log |n-n'|} \frac{|\psi_{n'}|^2}{|n-n'|} \psi_n. \quad (7)$$

If $\mu \ll 1$, i.e., the medium is a poor conductor, then one can substitute the logarithmic form $1 + \mu \log |n-n'|$ with the algebraic power $|n-n'|^\mu$, i.e., $1 + \mu \log |n-n'| \approx |n-n'|^\mu$, leading to

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta \sum_{n'} \frac{|\psi_{n'}|^2}{|n-n'|^{1+\mu}} \psi_n, \quad (8)$$

which is equivalent to some modification of the Poisson potential in the medium. For a broad wave packet, characterized by $\Delta\omega \gg \omega$, the model in Eq. (8) simplifies to a superquadratic modification of DANSE (1), i.e.,

$$i\hbar \frac{\partial \psi_n}{\partial t} = \hat{H}_L \psi_n + \beta |\psi_n|^{2s} \psi_n, \quad (9)$$

where $s = 1 + \mu$. In fact, if the nonlinear field is spread over a large number of states $\Delta n \gg 1$, and if the variation is so smooth that $|\psi_{n+1} - \psi_{n-1}|/2|\psi_n| \ll 1$, then one can replace the actual separation $|n-n'|$ in Eq. (8) with the characteristic spread Δn , take $1/|\Delta n|^{1+\mu}$ out of the summation sign, and sum over n' using the conservation of the total probability $\sum_{n'} |\psi_{n'}|^2 = 1$. On the other hand, if $\Delta n \gg 1$, then $\sum_{n'} |\psi_{n'}|^2 = 1$ dictates $|\psi_n|^2 \sim 1/\Delta n$, from which $1/|\Delta n|^{1+\mu} \sim |\psi_n|^{2(1+\mu)}$. Eliminating Δn from the nonlinear term with the aid of $\beta/|\Delta n|^{1+\mu} \sim \beta |\psi_n|^{2(1+\mu)}$, one is led to Eq. (9), where $s = 1 + \mu > 1$.

Equation (9) predicts that the nonlinear field can spread along the lattice up to a certain finite distance only, beyond which it is Anderson localized similar to the linear field. This is because the nonlinear term in Eq. (9) acts as a stochastic pump [8, 47] whose strength decays

while spreading [17, 48]. In fact, if the nonlinear field is spread over Δn states, then the nonlinear frequency shift is $\Delta\omega_{\text{NL}} \simeq \beta |\psi_n|^{2s} \simeq \beta/|\Delta n|^s$, while the distance between the eigenstates approaches zero as $\delta\omega \sim 1/|\Delta n|$. Hence it follows that the Chirikov overlap parameter [49, 50] decays with Δn as $K \equiv \Delta\omega_{\text{NL}}/\delta\omega \simeq \beta/|\Delta n|^\mu$. If $K \gg 1$, then the nonlinear field in Eq. (9) behaves as a chaotic system, meaning the integrals of motion are destroyed, and the nonlinear field can expand in the n direction via a stochastic process. On the contrary, if $K \ll 1$, then the behavior is regular, and the field is localized. Assume the nonlinear field is initially so narrow that $K \gg 1$. Then it starts to spread along the lattice driven by the nonlinear term. On the other hand, because K decreases with Δn increasing, this spreading process comes to a halt as soon as $K \sim 1$. Further spreading is forbidden by the transition to regularity and the K values becoming smaller than 1. One sees that the nonlinear field automatically (without tuning of parameters) evolves into a marginally regular state at the border of chaos and saturates exactly at that border. By the time regularity is reached the field has spread over $|\Delta n|_{\text{max}} \sim \exp[(1/\mu) \ln \beta]$ states. In a 1D model, this is equivalent to a nonlinear localization scale

$$\Lambda_{\text{loc}} \simeq \exp[(\pi/\varepsilon_r \tan \delta) \ln \beta], \quad (10)$$

where $\tan \delta$ is dielectric loss tangent. If losses are absent, then $\tan \delta$ vanishes, leading to $\Lambda_{\text{loc}} \rightarrow +\infty$. That means that the nonlinear field without dielectric coupling to environment can spread to arbitrarily long spatial scales, provided $K \gg 1$. This recovers the type of dynamics considered previously [8–10, 14, 16] in the framework of DANSE (1). Remark that the dependence of Λ_{loc} on $\mu \rightarrow +0$ is nonperturbative, as it should [16, 17].

Let us now explore a possibility that the “medium” to which the wave field is dielectrically coupled is the nonlinear field itself. The problem—which makes sense for broad enough wave packets—leads to a phenomenon that one could characterize as self-screening. The self-screening occurs because the nonlinear term $\sim \beta |\psi_n|^2$ causes the components of the wave field to overlap. This destabilizes the wave function at some locations, around which the wave field turns into a chaotic state. If such states occur sufficiently close to each other, then the nonlinear field can propagate between the different chaotic states by either tunneling through, or hopping over, the regular states in-between. The process is similar to the hopping mechanism considered by Pollak [32] and Elliott [30, 31], with chaotic states thought of as the analog localized states. This gives rise to a non-zero a.c. conductivity throughout the body of the wave field ψ_n and, as a result, to some form of self-screening and self-localization.

In fact, in the basis of linearly localized modes, the evolution of the components a_k of the wave field is caused by their nonlinear coupling, that is $i\dot{a}_k \sim \beta a_{m_1} a_{m_2}^* a_{m_3}$, where star denotes complex conjugate, the upper dot

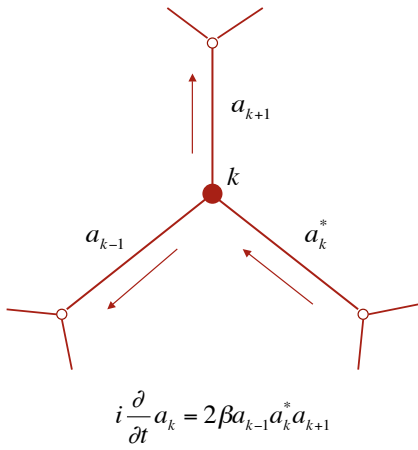


FIG. 1: Mapping dynamical equations (11) on a Cayley graph.

means time differentiation, $a_{-m} = a_m^*$ by way of temporal translation symmetry, and we have set $\hbar = 1$ for simplicity. If all couplings are next-neighbor-like (a situation of prime interest here as such couplings have the largest cross-section), then the evolution equation for a_k reduces to (for symmetry reasons)

$$i\dot{a}_k \sim 2\beta a_{k-1} a_k^* a_{k+1}, \quad (11)$$

which is obtained from the above by letting $m_1 = k \mp 1$, $m_2 = k$, and $m_3 = k \pm 1$. If $k \rightarrow \pm\infty$, then the set of dynamical equations (11) defines an infinite chain process with the topology of a Cayley tree. This is proven by constructing a one-to-one correspondence between (11) and a Cayley graph as follows. Each cyclic term $i\dot{a}_k$ with the wave number k and the eigenfrequency $\omega_k = \omega(|k|)$ is represented as a node in the graph space labeled by the coordinate k . The coupling links between the different nodes are represented as bonds. From the structure of the nonlinear term in Eq. (11) one sees that there will be three and only three bonds at each node—one ingoing bond representing the wave process with amplitude a_k^* , and two outgoing bonds representing the processes with amplitudes a_{k-1} and a_{k+1} , respectively (see Fig. 1).

It is assumed, following Refs. [16, 21], that each nonlinear oscillator with the equation of motion (11) can reside in either chaotic (with probability p) or regular (with probability $1-p$) state. The probability for an oscillator to be in chaotic state is given by the Boltzmann factor $p = \exp(-\delta\omega/\Delta\omega_{\text{NL}})$, where $\delta\omega \sim 1/\Delta n$ is the distance between Anderson eigenstates in the frequency domain, and $\Delta\omega_{\text{NL}} = \beta|\psi_n|^2$ is the nonlinear frequency shift. Using here that $\sum_{n'} |\psi_{n'}|^2 = 1$ implies $|\psi_n|^2 \sim 1/\Delta n$ for $\Delta n \gg 1$, one gets $p = \exp(-1/\beta)$ independently of Δn , that is p is preserved through dynamics.

If p is not-too-close to zero yet smaller than the percolation threshold probability $p_c = 1/2$ [52], then the

chaotic states are clustered and do *not* constitute a percolation path throughout the field (and hence do not contribute to the d.c. conduction). If, however, hops can occur between the adjacent chaotic states only, and if this process occurs near its percolation threshold, then the dependence of a.c. conductivity over frequency is given by a power law $\sigma_{\text{a.c.}}(\omega) \propto \omega^\eta$, where $\eta = (\theta + d - d_f)/(2 + \theta)$ [39, 53], d_f is the Hausdorff dimension of chaotic clusters, and θ is the connectivity index [54]. Using known values $d = 6$, $d_f = 4$, and $\theta = 4$ for mean-field percolation on Cayley trees [55, 56], one gets $\eta = 1$ exactly, hence $\sigma_{\text{a.c.}}(\omega) \propto \omega$. This linear scaling extends the classic result of Elliott [30, 31] to DANSE (1).

Once the linear scaling $\sigma_{\text{a.c.}}(\omega) \propto \omega$ is established, one repeats the renormalization procedure described above starting from the Kramers-Kronig transform (4), to obtain, in place of DANSE (1), the nonlinear Schrödinger equation with superquadratic nonlinearity, Eq. (9), where the exponent of superquadratic power is written as $2s = 2(1 + \mu_0)$, $\mu_0 = A_0/\pi\varepsilon_0 \ll 1$, and A_0 is the coefficient of a.c. conduction pertaining to the hopping motion. This type of nonlinearity steers automatically the nonlinear field to the margins of regularity via the dependence of the Chirikov overlap parameter $K \equiv \Delta\omega_{\text{NL}}/\delta\omega \simeq \beta/|\Delta n|^{\mu_0}$ on the number of states. The end result is that the nonlinear field is asymptotically localized, with the nonlinear localization length

$$\Lambda_{\text{loc}} \sim \exp[(1/\mu_0) \ln \beta]. \quad (12)$$

We define the range of validity of this result from the p values lying beneath the percolation point—provided the chaotic states can form connected clusters—to the onset of percolation at $p_c = 1/2$. Remark that the percolation point $p = p_c$ is included: the reason is that the d.c. conductivity is still zero for $p = p_c$ exactly [55, 56]. That is the nonlinear field of the type given by the DANSE (1) is Anderson localized both at ($p = p_c$) and below ($p < p_c$) the percolation threshold. This is different from a conclusion drawn previously [16] that the nonlinear field in Eq. (1) is delocalized as soon as $p = p_c$. Here, we argue it is asymptotically localized through self-screening and the linear dependence $\sigma_{\text{a.c.}}(\omega) \propto \omega$ in proximity to percolation. With use of $p = \exp(-1/\beta)$ one can translate $p_c = 1/2$ into a critical β value yet permitting asymptotic localization: $\beta_c = 1/\ln 2$. For the β values greater than this, asymptotic localization is only possible through coupling to external medium. Within the localization length, one obtains the dispersion of field spreading along the lattice as the mean-squared displacement of the random walker on the incipient percolation cluster, leading to $\langle(\Delta n)^2(t)\rangle \sim t^{2/(2+\theta)}$ [53, 56]. Using $\theta = 4$ for percolation on trees, one obtains $\langle(\Delta n)^2(t)\rangle \sim t^{1/3}$. This scaling law has been observed numerically in Refs. [9, 10, 14] and associated with weak chaos.

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