

RANDOM FLIGHTS AND ANOMALOUS DIFFUSION: A NON-MARKOVIAN TAKE ON LORENTZ PROCESSES

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ABSTRACT. We study Lorentz processes in two different settings. Both cases are characterized by infinite expectation of the free-flight times, contrary to what happens in the classical Gallavotti-Spohn models. Under a suitable Boltzmann-Grad type scaling limit, they converge to non-Markovian random-flight processes with superdiffusive behavior. A further scaling limit yields another non Markovian process, i.e., a superdiffusion obtained by a suitable time-change of Brownian motion. Furthermore, we obtain the governing equations for our random flights and anomalous diffusion, which represent a non-local counterpart for the linear-Boltzmann and diffusion equations arising in the classical theory. It turns out that these equations have the form of fractional kinetic equations in both time and space. To prove these results, we develop a technique based on mixtures of Feller semigroups.

CONTENTS

1. Introduction	2
2. Preliminary notions	5
2.1. Isotropic transport processes	5
2.2. Para-Markov chains	8
2.3. The para-Markov counting process	11
3. Motion among random obstacles (Model 1)	12
3.1. A transport process with infinite mean flight times	13
3.2. Mittag-Leffler distribution of obstacles	17
3.3. The Boltzmann-Grad limit	20
3.4. Convergence to anomalous diffusion	28

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4. Averaging Feller Semigroups	31
5. Motion among random obstacles (Model 2)	36
5.1. Transport with infinite mean flight times and random speed	36
5.2. Boltzmann-Grad approximation and anomalous diffusion	39
Acknowledgements	42
Appendix A. Existence of Mittag-Leffler point process	42
Appendix B. Details on the proof of the Boltzmann-Grad limit	44
References	46

1. INTRODUCTION

A Lorentz process is a model for the motion of a particle among randomly located scatterers. It was firstly used to describe the transport of electrons through a conductor (see [33]). This process is difficult to handle, mostly due to the fact that the particle may collide with the same scatterer more than once, which produces a memory effect. So, one can imagine to approximate this process by means of more tractable processes.

The first to tackle this problem rigorously was Gallavotti [23]. In his model, of which we here consider the 3-dimensional version, the scatterers are hard spheres of radius R . Their centers are randomly distributed in space according to a Poisson point process with intensity ρ , and the particle interacts with the scatterers by means of elastic collisions. Each free flight time has a marginal exponential distribution, hence it is unlikely that long flights occur.

Intuitively, the memory effect due to recollisions tends to vanish if $R \rightarrow 0$. Indeed, Gallavotti studied this system in the Boltzmann-Grad limit: this is the limit in which $R \rightarrow 0$ and simultaneously $\rho \rightarrow \infty$, in such a way that the free flight time of the particle maintains a finite expectation. Under this limit, the Lorentz process converges to a Markovian random flight process, having independent, exponentially distributed free flight times. In this case, the position-velocity process is governed by the linear Boltzmann equation

$$\frac{\partial}{\partial t} f(x, v, t) = cv \cdot \nabla_x f(x, v, t) + \lambda \int_{S^2} (f(x, v', t) - f(x, v, t)) \mu(dv') \quad (1.1)$$

where μ denotes the uniform distribution of the unit sphere S^2 , c is the speed of the particle and λ is the mean number of direction changes per unit time. After the Boltzmann-Grad limit, a further scaling limit ($c \rightarrow \infty$, $\lambda \rightarrow \infty$ such that $c^2/\lambda \rightarrow 1$) leads to a diffusion process, which is governed by the heat equation

$$\frac{\partial}{\partial t} p(x, t) = \frac{1}{2} \Delta_x p(x, t). \quad (1.2)$$

Such a diffusion limit is based on central limit arguments, since the free flight times have finite mean and variance. This model was generalized by Spohn [59], where more general probability distributions for the scatterers are allowed, the scatterers are potentials with

finite range R , weak convergence in path space is proved. Other developments have been made, e.g., in [8; 9; 11; 17; 25]. A further progress is due to [36], where the Boltzmann-Grad and the diffusion limit are taken simultaneously.

However, even in these cases, convergence to a random flight under the Boltzmann-Grad limit requires that the expected value of the free flight times remains finite.

Cases in which the flight times have infinite expectation are excluded from these theories; more precisely, we here refer to flight times that are almost surely finite, i.e. they have a proper probability distribution, but their expected value is infinite. Yet, these cases are very interesting from a physical point of view, since, as clear in the literature, they are connected to anomalous diffusion, a motion such that the mean squared displacement does not grow asymptotically linearly in time. On this point, we stress that there is a wide literature concerning continuous time random walks and random flights where the heavy-tailed waiting times belong to the domain of attraction of a ν -stable law, $\nu \in (0, 1)$. See, e.g. [10; 41; 42; 43; 60] for continuous-time random walks and [52] for a recent random flight model. These processes and their anomalous diffusive limits fall in the theory of semi-Markov processes [35] and they are crucial models for anomalous diffusive phenomena related to fractional dynamics (see, e.g., [45] and references therein). Random flights with flight times having diverging moments have been extensively considered, in the physics literature: these processes are called Lévy walks (see, for example, [62] and references therein for a review) and they have plenty of applications for modeling anomalous phenomena (see, for example, [20; 21; 28; 34]).

One might ask, therefore, whether random walks/flights with infinite mean flight times and the related anomalous diffusions can be connected to mechanical models of Lorentz type.

In this paper we present two Lorentz process models. For both of them, the limit random flights and their related anomalous diffusion, together with their governing equations, are new. Such processes go beyond the semi-Markov setting and a useful tool in order to deal with them is the recent theory of the so-called para-Markov processes (see [19]).

In the first model, the Lorentz process is based on a new distribution of obstacles that we call the Mittag-Leffler point process. The reason of this name is that the free flight times have a so-called Mittag-Leffler distribution, a heavy tailed distribution belonging to the domain of attraction of a stable law, with interesting analytical properties. In this setting it is likely that long flights occur, contrary to what happens for the Poissonian case. We study this system under a limit of Boltzmann-Grad type. In this procedure, the free flight time of the particle has infinite mean both before and after the limit. The resulting random flight is non-Markovian and presents a long memory tail, in the sense that all the flight times are dependent; despite this, the process is easy to handle, since it is possible to explicitly write the joint distribution of its flight times, which happens to be of Schur constant type. After the Boltzmann-Grad limit, a further scaling limit gives an anomalous diffusion process, exhibiting self-similarity and also a super-diffusive behaviour, i.e. the particles spread faster than for

the Brownian motion. We shall call such a process the Mittag-Leffler anomalous-diffusion process.

In the second model, the distribution of obstacles is Poissonian (as in [23]), but the speed of the particle is heavy tailed distributed. A Boltzmann-Grad limit leads to another non Markovian random flight process with speed and flight times that depend on each other; however, a second scaling limit gives the same super-diffusion as before.

In addition to the kinetic limits, our paper also studies the governing equations of some of the processes involved. These are not first-order differential equations in time, like (1.1) and (1.2). Indeed, in Gallavotti-Spohn models, the limit processes are Markovian, and thus, according to Feller's theory, they are governed by the first order in time generator equations. In the present case, instead, the limit processes are non-Markovian. Despite this, we can derive their governing equations, that happen to be fractional kinetic equations, non-local in both time and space.

In order to rigorously get these equations, we face a problem which is in itself interesting from a probabilistic and analytical point of view and serves as a tool, in this context. Let $\{T_t\}_{t \geq 0}$ be a Feller semigroup on a Banach space \mathfrak{B} , with generator $(G, \text{Dom}(G))$. Moreover, consider the Bochner integral $Q_t = \int_0^\infty T_s l(ds)$, where $l(ds)$ denotes the so-called Lamperti distribution (see e.g. [32]). This is a distribution on $(0, \infty)$ with power law decay $t^{-\nu-1}$, $\nu \in (0, 1)$. The family $\{Q_t\}_{t \geq 0}$ defines a mixture of Feller semigroups. We prove that, for any $h \in \text{Dom}(G)$, the function $t \mapsto Q_t h$ solves the abstract Cauchy problem

$$\frac{\partial^\nu}{\partial t^\nu} g(t) = -(-G)^\nu g(t), \quad g(0) = h, \quad (1.3)$$

i.e. a fractional kinetic equation exhibiting fractional powers of operators. Such equation reduces to the well-known governing equation of Markov processes for $\nu = 1$. So, we shall see that the governing equations that emerge after the kinetic limits are of the type (1.3). They generalize equations (1.1) and (1.2) by introducing the ν -powers of operators.

Concerning the above models, a fundamental remark is needed: our assumption of heavy tailed, infinite mean flight times is not necessary to obtain non-Markovian random flights, anomalous diffusions and non-local governing equations. Indeed, by using different physical settings with respect to ours, other Lorentz processes converging to non-Markovian and anomalous diffusive limits, possibly governed by non-local equations, have already been presented in the literature. For example, there are recent papers concerning a bi-dimensional Lorentz gas under the action of a constant magnetic field (see e.g. [47; 48; 37]). In this case, the action of the external force prevents the position-velocity process from being Markovian. Hence a generalization of the Boltzmann equation, which is non-local in the time variable, appears (see e.g. formula (1.3) in [48]); also an anomalous diffusion equation arises, namely a non-local heat equation where the diffusion coefficient is replaced by an operator, the latter written in terms of the Green-Kubo formula (see formula (1.8) in [47]). Concerning the anomalous diffusive behaviour, another remarkable example regards the superdiffusion limit

arising in the context of the periodic Lorentz gas (see [40]) where the anomalous behavior is $t \log t$; for the periodic Lorentz gas, see also [13; 14; 24; 39].

2. PRELIMINARY NOTIONS

In the present paper we shall use results from very different theories, e.g., we shall combine tools from kinetic theory, the theory of non-Markov processes and non-local equations. Hence, for the reader convenience, we summarize the main facts in few pages.

2.1. Isotropic transport processes. In this paper we consider a class of transport processes, also called random flights, defined as follows. Let a particle start at a point $x \in \mathbb{R}^d$ and move along a unit vector v_0 with random speed \mathcal{C} , for a random waiting time J_1 . After the time interval J_1 , the particle undergoes scattering and continues its motion along a new unit vector v_1 , again with speed \mathcal{C} , for another random waiting time J_2 . This process repeats indefinitely. Here \mathcal{C} is assumed to be either a positive constant or a positive random variable. The directions $\{v_i, i \geq 0\}$ are assumed to be i.i.d. random vectors, with uniform law on the unit sphere

$$S^{d-1} = \{v \in \mathbb{R}^d : |v|_e^2 = 1\},$$

where $|\cdot|_e$ denotes the Euclidean norm. Moreover, we assume that the unit vectors $\{v_i, i \geq 0\}$ are independent of the waiting times $\{J_n, n \geq 1\}$ and of the speed \mathcal{C} . On the other hand, \mathcal{C} and $\{J_n, n \geq 1\}$ may be dependent. Let $\tau_n := \sum_{i=1}^n J_i$ be the instant of the n -th change of direction for $n \geq 1$, with the convention $\tau_0 := 0$, and let $\mathcal{N} = \{\mathcal{N}_t, t \geq 0\}$ be the process which counts the number of changes up to time t , i.e.

$$\mathcal{N}_t := \max\{n \geq 0 : \tau_n \leq t\}.$$

At time $t \geq 0$, the particle has unit velocity vector V_t and is located at the position X_t . To summarize the above description, we now give the rigorous definition of the joint process (X_t, V_t) .

Definition 2.1. *Let $(\Omega, \mathcal{A}, \mathbb{P}^{(x,v)})$, $v \in S^{d-1}$, $x \in \mathbb{R}^d$, be a family of probability spaces. Let $\{J_n\}_{n \geq 1}$, $\{v_i\}_{i \geq 0}$ and \mathcal{C} be random variables as defined above. Let \mathcal{N} be defined as above. The joint process $\{(X_t, V_t), t \geq 0\}$, such that*

$$V_t := v_{\mathcal{N}_t} \tag{2.1}$$

$$X_t := X_0 + \mathcal{C} \int_0^t V_s ds = X_0 + \mathcal{C} \sum_{i=1}^{\mathcal{N}_t} J_i v_{i-1} + \mathcal{C} (t - \tau_{\mathcal{N}_t}) v_{\mathcal{N}_t} \tag{2.2}$$

where $\mathbb{P}^{x,v}(X_0 = x, v_0 = v) = 1$, is said to be the isotropic transport process.

In practice, defining such a transport process requires specifying only the joint law of the vectors $(\mathcal{C}, J_1, J_2, \dots, J_n)$ for any n , from which the joint distribution for the flight times $\{J_n, n \geq 1\}$ and the distribution of the speed \mathcal{C} directly follow.

In Section 2.1.1 we recall some known facts on the Markovian case, i.e., the case where the J_n s are i.i.d. exponential r.v.'s and \mathcal{C} is a positive constant chosen independently on the J_n s. Then, in Sections 3.1 and 5.1, we shall define two non-Markovian models, where the waiting times are stochastically dependent and have infinite mean. Furthermore, their limit behaviour will be studied in Sections 3.4 and 5.2.

2.1.1. *The Markovian random flight and its diffusive limit.* Let the waiting times $\{J_n, n \geq 1\}$ be i.i.d. exponential random variables with mean λ^{-1} and let the speed \mathcal{C} be a positive constant, say $\mathcal{C} = c$ almost surely. Then the resulting transport process is the Markovian isotropic transport process, as studied, for example, in [30; 46; 49; 53; 61]; an analogous construction holds in non Euclidean spaces (see e.g. [50; 51]). In this setting, the counting process \mathcal{N} is a Poisson process with intensity λ and τ_n follows a Gamma distribution of parameters n, λ , also called Erlang distribution. Here the parameter λ can be interpreted as the mean number of direction changes per unit time.

In this case, we shall denote the random flight process of Definition 2.1 by

$$\left\{ \left(X_t^{(c,\lambda)}, V_t^{(c,\lambda)} \right), t \geq 0 \right\},$$

where the superscript (c, λ) indicates that the process depends on the two parameters c and λ . Here $\left\{ V_t^{(c,\lambda)}, t \geq 0 \right\}$ is a Markov process on S^{d-1} and $\left\{ \left(X_t^{(c,\lambda)}, V_t^{(c,\lambda)} \right), t \geq 0 \right\}$ is a Markov process on $\mathbb{R}^d \times S^{d-1}$.

Let us consider initial data $(x, v) \in \mathbb{R}^d \times S^{d-1}$. Moreover, consider the family of operators $\left\{ \mathcal{T}_t^{(c,\lambda)} \right\}_{t \geq 0}$ defined by

$$\mathcal{T}_t^{(c,\lambda)} h(x, v) := \mathbb{E}^{(x,v)} \left[h \left(X_t^{(c,\lambda)}, V_t^{(c,\lambda)} \right) \right] \quad h \in C_0 \left(\mathbb{R}^d \times S^{d-1} \right) \quad (2.3)$$

where $\mathbb{E}^{(x,v)}$ denotes the expectation under $\mathbb{P}^{x,v}$. Such a family defines a strongly continuous contraction semigroup on the space $C_0 \left(\mathbb{R}^d \times S^{d-1} \right)$ endowed with the sup-norm $\| \cdot \|$, i.e. a Feller semigroup (see [61, Lemma 2.1] or, for a very general approach, [51]).

Let us consider the semigroup $\left\{ e^{-\lambda t} P_t \right\}_{t \geq 0}$ where

$$P_t h(x, v) := h(x + vct, v) \quad h \in C_0 \left(\mathbb{R}^d \times S^{d-1} \right).$$

Then $\left\{ e^{-\lambda t} P_t \right\}_{t \geq 0}$ is strongly continuous and is generated by the operator $cv \cdot \nabla_x - \lambda \mathcal{I}$, where \mathcal{I} denotes the identity operator, on the domain

$$D = \left\{ h \in C_0 \left(\mathbb{R}^d \times S^{d-1} \right) : cv \cdot \nabla_x h \in C_0 \left(\mathbb{R}^d \times S^{d-1} \right) \right\}. \quad (2.4)$$

Furthermore, let us consider the scattering operator L defined by

$$Lh(x, v) := \int_{S^{d-1}} h(x, v') \mu(dv'), \quad h \in C_0 \left(\mathbb{R}^d \times S^{d-1} \right),$$

where μ denotes the uniform distribution on S^{d-1} . It is useful to see how [31, Theorem 1.9.2] applies here, giving the following result which is known in the literature.

Theorem 2.2. *The operator $\mathcal{T}_t^{(c,\lambda)}$ defined in Equation (2.3) has the following representation on D*

$$\mathcal{T}_t^{(c,\lambda)} = e^{-\lambda t} P_t + \sum_{n=1}^{\infty} \lambda^n e^{-\lambda t} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} P_{t-\tau_n} L P_{\tau_n-\tau_{n-1}} L \dots L P_{\tau_1} d\tau_1 \dots d\tau_n. \quad (2.5)$$

Moreover, for $h \in D$, one has that $t \mapsto \mathcal{T}_t^{(c,\lambda)} h$ is the unique (bounded) solution to the linear (abstract) Boltzmann equation on $C_0(\mathbb{R}^d \times S^{d-1})$

$$\frac{\partial}{\partial t} g(t) = c v \cdot \nabla_x g(t) + \lambda(L - \mathcal{I})g(t), \quad (2.6)$$

under the initial condition $g(0) = h$.

Proof. First, we observe that the operator L is bounded on $C_0(\mathbb{R}^d \times S^{d-1})$. Moreover, $c v \cdot \nabla_x - \lambda \mathcal{I}$ generates the semigroup $\{e^{-\lambda t} P_t\}_{t \geq 0}$, which is strongly continuous. Hence, we can apply [31, Theorem 1.9.2, (i)] to say that

$$G^{(c,\lambda)} := c v \cdot \nabla_x - \lambda \mathcal{I} + \lambda L, \quad (2.7)$$

with domain D , is the generator of a strongly continuous semigroup on $C_0(\mathbb{R}^d \times S^{d-1})$, which has the representation

$$\Phi_t = e^{-\lambda t} P_t + \sum_{n=1}^{\infty} \lambda^n e^{-\lambda t} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} P_{t-\tau_n} L P_{\tau_n-\tau_{n-1}} L \dots L P_{\tau_1} d\tau_1 \dots d\tau_n.$$

Moreover, [31, Theorem 1.9.2, (ii)] guarantees that $\Phi_t h$ is the unique solution of

$$\Phi_t h = e^{-\lambda t} P_t h + \int_0^t \lambda e^{-\lambda(t-s)} \Phi_s L P_{t-s} h ds. \quad (2.8)$$

We now observe that $\|\mathcal{T}_t^{(c,\lambda)} h\| \leq \|h\|$ and, by a standard conditioning argument, using the first renewal time, one has

$$\mathcal{T}_t^{(c,\lambda)} h(x, v) = e^{-\lambda t} h(x + cvt, v) + \int_0^t \int_{S^{d-1}} \lambda e^{-\lambda(t-s)} \mathcal{T}_s^{(c,\lambda)} h(x + cv(t-s), \omega) \mu(d\omega) ds,$$

i.e. $t \mapsto \mathcal{T}_t^{(c,\lambda)} h$ satisfies Equation (2.8). By the uniqueness of solution, we get the thesis. \square

We finally observe that the sum in Equation (2.5) converges in the sup-norm and can be explicitly written as

$$\mathcal{T}_t^{(c,\lambda)} h(x, v) = e^{-\lambda t} h(x + cvt, v) + \quad (2.9)$$

$$+ \sum_{n=1}^{\infty} \lambda^n e^{-\lambda t} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^{d-1})^n} h(x_t, \omega_n) d\tau_1 \dots d\tau_n \mu(d\omega_1) \dots \mu(d\omega_n)$$

where

$$x_t := x + c \sum_{j=1}^n (\tau_j - \tau_{j-1}) \omega_{j-1} + c(t - \tau_n) \omega_n, \quad \omega_0 := v,$$

and $v_t = \omega_n$ for $t \in [\tau_n, \tau_{n+1})$.

Another useful result we will use is the following explicit formula for the mean squared displacement of the Markovian random flight (details can be found in [2]). Denote by $\mathbb{P}^{x,\mu}$ the probability measure

$$\mathbb{P}^{x,\mu}(\cdot) := \int_{S^{d-1}} \mathbb{P}^{(x,v)}(\cdot) \mu(dv) \quad (2.10)$$

where μ is the uniform probability measure on the $d - 1$ -dimensional sphere and by $\mathbb{E}^{x,\mu}$ the corresponding expectation. Then

$$\mathbb{E}^{x,\mu} \left| X_t^{(c,\lambda)} - x \right|_e^2 = \frac{2c^2}{\lambda^2} (\lambda t - 1 + e^{-\lambda t}). \quad (2.11)$$

A remarkable fact is that the Markovian isotropic transport process is a finite speed and finite rate approximation of Brownian motion. Indeed, for Brownian motion both the speed of the particle and the rate of direction changes are infinite; but nevertheless, in many practical applications, Brownian motion is used to model random motions with large but finite speed and rate. This is allowed by the following result.

Proposition 2.3. *Let $B = \{B_t, t \geq 0\}$ be a d -dimensional Brownian motion, $d \geq 1$ under a probability measure \mathbb{P}^x , s.t. $\mathbb{P}^x(B_0 = x) = 1$. Then, under the scaling limit*

$$c \rightarrow \infty \quad \lambda \rightarrow \infty \quad \frac{c^2}{\lambda} = D > 0$$

we have that

$$\left\{ X_t^{(c,\lambda)} \right\} \xrightarrow{fdd} \{B_{Dt}\}.$$

The reader can consult [30; 51; 61] and references therein for details and an improved version of this statement showing weak convergence on the space of continuous functions with the uniform topology.

2.2. Para-Markov chains. In this section we recall the definition of a class of non-Markovian processes, called para-Markov [19], as these processes will be a useful tool in the following. These are right continuous processes on a countable state space whose waiting times between jumps have a suitable stochastic dependence, creating a long memory tail in the evolution.

They are proved to be equal in distribution to a continuous-time Markov chain whose time parameter is randomly scaled, hence the name para-Markov.

For a positive ν , we shall indicate with \mathcal{M}_ν the one-parameter Mittag-Leffler function, defined by

$$\mathcal{M}_\nu(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \nu k)} \quad x \in \mathbb{R}. \quad (2.12)$$

In the paper we shall use the notation

$$\mathcal{M}_\nu^{(k)}(-z^\nu) := \left(\frac{d}{dz}\right)^k \mathcal{M}_\nu(-z^\nu) \quad k \in \mathbb{N}. \quad (2.13)$$

Moreover, we recall that a non-negative random variable J is said to follow a Mittag-Leffler distribution with parameters $\nu \in (0, 1]$ and $\lambda \in (0, \infty)$ if it has cumulative distribution function

$$\mathbb{P}(J \leq t) = 1 - \mathcal{M}_\nu(-\lambda t^\nu), \quad t \geq 0.$$

For $\nu \in (0, 1)$, by using the asymptotic properties of the Mittag-Leffler function (see, e.g., [44]), one gets

$$\mathbb{P}(J > t) \sim \alpha t^{-\nu} \quad \text{as } t \rightarrow \infty, \quad (2.14)$$

being $\alpha > 0$ a constant. It follows that J has infinite expectation. Instead, for $\nu = 1$ we have $\mathcal{M}_1(x) = e^x$ and $J \sim \text{Exp}(\lambda)$.

Definition 2.4. Let $Y = \{Y_n, n \in \mathbb{N}\}$ be a discrete time Markov chain, on some probability space with a probability measure P , and a finite or countable state space \mathcal{S} . For $\nu \in (0, 1]$ and $\lambda : \mathcal{S} \rightarrow (0, \infty)$, let $\{J_n, n \geq 1\}$ be a sequence of non-negative random variables, such that, $\forall n \geq 1$,

$$P(J_1 > t_1, \dots, J_n > t_n | Y_0 = y_0, \dots, Y_{n-1} = y_{n-1}) = \mathcal{M}_\nu \left(- \left(\sum_{k=1}^n \lambda(y_{k-1}) t_k \right)^\nu \right) \quad (2.15)$$

where $t_k \geq 0$, $k \in \{1, \dots, n\}$. Let $T_n := \sum_{k=1}^n J_k$ and $T_0 := 0$. A continuous-time process $X^\nu = \{X_t^\nu, t \geq 0\}$ such that

$$X_t^\nu = Y_n \quad t \in [T_n, T_{n+1}), \quad n \in \mathbb{N}$$

is said to be a para-Markov chain.

From the above definition we have that each waiting time follows a Mittag-Leffler distribution:

$$P(J_k > t_k | Y_{k-1} = y_{k-1}) = \mathcal{M}_\nu(-(\lambda(y_{k-1}) t_k)^\nu).$$

Moreover, note that for $\nu = 1$ the joint survival function in Equation (2.15) is factorized as

$$P(J_1 > t_1, \dots, J_n > t_n | Y_0 = y_0, \dots, Y_{n-1} = y_{n-1}) = e^{-\sum_{k=1}^n \lambda(y_{k-1}) t_k}, \quad (2.16)$$

i.e. the waiting times are conditionally independent, exponential random variables and hence X^1 is a homogeneous, continuous-time Markov chain.

From [19], it is known that a para-Markov chain X^ν is equivalent, in terms of finite-dimensional distributions, to a Markov chain X^1 through a random time-change, as reported below. Specifically, this is achieved by substituting the time parameter t with $t\mathcal{L}$, where \mathcal{L} has a Lamperti distribution, which is defined as follows (see e.g. [32]).

Definition 2.5. *A non-negative random variable \mathcal{L} , on some probability space with probability P , follows a Lamperti distribution of parameter $\nu \in (0, 1]$ if its Laplace transform is given by*

$$E[e^{-\eta\mathcal{L}}] = \mathcal{M}_\nu(-\eta^\nu), \quad \eta \geq 0. \quad (2.17)$$

For $\nu = 1$ we have $E[e^{-\eta\mathcal{L}}] = e^{-\eta}$ which means $\mathcal{L} \stackrel{a.s.}{=} 1$, while for $\nu \in (0, 1)$ the variable \mathcal{L} is absolutely continuous, with density given by

$$\ell(dy) := \frac{\sin \pi\nu}{\pi} \frac{y^{\nu-1}}{y^{2\nu} + 2y^\nu \cos \pi\nu + 1} \mathbf{1}_{(0,+\infty)}(y) dy. \quad (2.18)$$

Consult [32, Equations (1.1) and (3.3)] with $\theta = 0$ and $\eta = z^{\frac{1}{\alpha}}$ for details. We also recall a property which will be useful later in the article:

$$\mathcal{L} \stackrel{d}{=} \frac{1}{\mathcal{L}}, \quad (2.19)$$

where $\stackrel{d}{=}$ denotes the equality in distribution.

Theorem 2.6. *Let us consider a para-Markov chain X^ν as in Definition 2.4. Let \mathcal{L} be a Lamperti random variable of parameter $\nu \in (0, 1]$. Then we have*

$$X_t^\nu \stackrel{fdd}{=} X_{\mathcal{L}t}^1 \quad \forall t \geq 0,$$

where $\stackrel{fdd}{=}$ denotes equality of finite dimensional distributions.

For a proof, see [19, Theorem 4].

A corner point is that the para-Markov chain X^ν is governed by an integro-differential equation. The operator acting on the time variable is the Caputo fractional derivative (see Equation (1.10) in [26] or [6] for a general introduction), defined by

$$\frac{\partial^\nu}{\partial t^\nu} f(t) := \begin{cases} \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t (f(s) - f(0))(t-s)^{-\nu} ds, & \nu \in (0, 1), \\ f'(t), & \nu = 1. \end{cases} \quad (2.20)$$

Let us consider a para-Markov chain X^ν and let us indicate with G the infinitesimal generator of the Markov chain X^1 . Let $P(t) = [p_{ij}(t)]$ be the transition matrix of X^ν . In [19] it is proven that in the case of finite states space \mathcal{S} , the transition matrix $P(t)$ is the solution of

$$\frac{\partial^\nu}{\partial t^\nu} P(t) = -(-G)^\nu P(t) \quad (2.21)$$

with initial condition $P(0) = I$, being I the identity matrix. We stress that for $\nu = 1$, Equation (2.21) formally reduces to the Kolmogorov backward equation which governs a Markov chain.

2.3. The para-Markov counting process. We here recall the definition of a counting process, which will be useful in the rest of the paper, as a special case of para-Markov chain with infinitely many states. For major details, see [19].

Let us consider a para-Markov chain $\mathcal{N}^\nu := \{\mathcal{N}_t^\nu, t \geq 0\}$ as in Definition 2.4. Let us assume a deterministic embedded chain

$$Y_n = n, \quad n \in \mathbb{N} \quad (2.22)$$

and $\lambda \in (0, \infty)$ constant. Then \mathcal{N}^ν is said to be the *Exchangeable fractional Poisson process* with parameters λ and ν .

In this case the sequence of waiting times $\{J_n^{(\nu)}, n \geq 1\}$ satisfies the following property: for each $n \geq 1$,

$$P\left(J_1^{(\nu)} > t_1, \dots, J_n^{(\nu)} > t_n\right) = \mathcal{M}_\nu\left(-\lambda^\nu \left(\sum_{k=1}^n t_k\right)^\nu\right) \quad \nu \in (0, 1], \quad \lambda \in (0, \infty). \quad (2.23)$$

From Equation (2.23) it follows that the distribution of \mathcal{N}^ν reads

$$P(\mathcal{N}_t^\nu = n) = \frac{(-t)^n}{n!} \mathcal{M}_\nu^{(n)}(-\lambda^\nu t^\nu) \quad n \in \mathbb{N}. \quad (2.24)$$

An important property of the sequence of waiting times described in Equation (2.23) is that it forms an infinite *Schur-constant* sequence. We recall that a sequence $\{Z_k\}_{k=1}^\infty$ of non-negative random variables is said to be an infinite *Schur-constant* sequence if, for any $n \geq 1$, the joint survival function satisfies

$$P(Z_1 > t_1, Z_2 > t_2, \dots, Z_n > t_n) = S(t_1 + t_2 + \dots + t_n),$$

for some function S that does not depend on n . This structure represents a particular model of *exchangeable* waiting times, as the function S depends on the t_k only through their sum. See [7; 15; 58] for details. This property justifies the name chosen for this counting process.

We note that for $\nu = 1$ we have that Equation (2.23) has the form

$$P[J_1 > t_1, \dots, J_n > t_n] = e^{-\lambda \sum_{k=1}^n t_k}, \quad (2.25)$$

namely the waiting times $\{J_n, n \geq 1\}$ are i.i.d. exponential and \mathcal{N}^1 reduces to a Poisson process of parameter λ .

Moreover, we emphasize that from Equations (2.17) and (2.23) one has

$$\left(J_1^{(\nu)}, \dots, J_n^{(\nu)}\right) \stackrel{d}{=} \frac{1}{\mathcal{L}}(J_1, \dots, J_n), \quad n \geq 1, \quad (2.26)$$

being $\{J_n, n \geq 1\}$ i.i.d. exponential of parameter λ . This also shows that $\{J_n^{(\nu)}, n \geq 1\}$ are exponential of random parameter $\lambda\mathcal{L}$.

We conclude this section by observing that λ can be interpreted as a measure of the intensity of the process, in the sense that it controls the probability to have at least one event in a small time interval Δt . Indeed, using the relation of Theorem 2.6, we have

$$\begin{aligned} P(\mathcal{N}_{t+\Delta t}^\nu - \mathcal{N}_t^\nu = 0) &= P(\mathcal{N}_{\mathcal{L}(t+\Delta t)}^1 - \mathcal{N}_{\mathcal{L}t}^1 = 0) \\ &= \int_0^\infty e^{-\lambda y \Delta t} P(\mathcal{L} \in dy) \\ &= \mathcal{M}_\nu(-\lambda^\nu (\Delta t)^\nu) \\ &= 1 - \frac{\lambda^\nu (\Delta t)^\nu}{\Gamma(1 + \nu)} + o((\Delta t)^\nu), \end{aligned}$$

which implies

$$P(\mathcal{N}_{t+\Delta t}^\nu - \mathcal{N}_t^\nu \geq 1) = \frac{\lambda^\nu (\Delta t)^\nu}{\Gamma(1 + \nu)} + o((\Delta t)^\nu). \quad (2.27)$$

We further observe that in the Poisson case λ also has the meaning of mean number of events per unit time. Indeed, by the Poisson distribution, we have $\mathbb{E}\mathcal{N}_t^1 = \lambda$ if $t = 1$. In contrast, the mean number of events per unit time in the exchangeable fractional Poisson process is infinite for $\nu \in (0, 1)$ for each λ .

3. MOTION AMONG RANDOM OBSTACLES (MODEL 1)

We recall that Lorentz processes are models for the motion of a particle among randomly distributed obstacles. In Gallavotti [23], the set of obstacle centres is a Poisson point process. Under the Boltzmann-Grad limit, this process converges to the Markovian random flight defined in Section 2.1.1. As seen in Proposition 2.3, a further scaling limit leads to Brownian motion.

In this section, we shall define a Lorentz process based on a more general distribution of obstacles. To this aim, we define the Mittag-Leffler point process, which includes the Poisson point process as a special case.

In this new setting, the obstacle system is more rarefied, making the model suitable for describing even more dilute gases. In particular, the free flight time of the particle has a distribution with power-law decay $Kt^{-\nu}$, $\nu \in (0, 1)$, $K > 0$, and thus has infinite expectation. This feature, in particular, has been previously ruled out by classical technical assumptions when studying Lorentz processes [59]. We shall prove that, under a Boltzmann-Grad type limit, the process converges to a random flight with infinite mean flight times. We will show that this random flight is superdiffusive with a mean square displacement spreading, for

$t \rightarrow +\infty$, as $\tilde{K}t^{2-\nu}$, $\tilde{K} > 0$ and furthermore it converges, under a second suitable limit, to an anomalous diffusive process with infinite mean square displacement.

For the sake of clarity, we believe it is appropriate to first describe the new random flight model in Section 3.1. Then, we shall define the Lorentz model in Section 3.2 and finally we shall state the limit theorems in Sections 3.3 and 3.4.

3.1. A transport process with infinite mean flight times. We consider an isotropic transport process with infinite mean and stochastically dependent flight times, which is defined as in Definition 2.1, with the following assumptions.

(A1). *Suppose that in Definition 2.1 the joint distribution of the flight times $\{J_n^{(\nu)}, n \geq 1\}$ is given, under all the probability measures $\mathbb{P}^{(x,v)}$, by Equation (2.23) for some $\nu \in (0, 1]$ and the (independent) speed is $\mathcal{C} = c$ almost surely, where c is a positive constant.*

In this case, we shall denote the random flight process of Definition 2.1 by

$$(X_{(1)}^\nu, V_{(1)}^\nu) = \{(X_{(1)}^\nu(t), V_{(1)}^\nu(t)), t \geq 0\}, \quad \nu \in (0, 1].$$

The assumption **(A1)** implies that the number of direction changes up to time t is \mathcal{N}_t^ν , i.e. the exchangeable fractional Poisson process defined in Section 2.3.

Let us consider initial data $(x, v) \in \mathbb{R}^d \times S^{d-1}$. We will use the family of operators $\{\mathcal{Z}_t\}_{t \geq 0}$ acting on suitable functions and defined by

$$\mathcal{Z}_t h(x, v) := \mathbb{E}^{(x,v)} [h(X_{(1)}^\nu(t), V_{(1)}^\nu(t))] \quad h \in C_0(\mathbb{R}^d \times S^{d-1}), \quad (3.1)$$

where $\mathbb{E}^{(x,v)}$ denotes the expectation under the measure $\mathbb{P}^{x,v}$.

We now obtain the counterpart of the evolution formula presented in Equation (2.5) (holding in the Markovian case) in the non-Markovian setting arising under **(A1)**.

Theorem 3.1. *Let $(X_t^{(c,\lambda)}, V_t^{(c,\lambda)})$ be the Markovian random flight defined in Section 2.1.1 and let \mathcal{L} follow a Lamperti distribution as in Definition 2.5. Then*

$$(X_{(1)}^\nu(t), V_{(1)}^\nu(t)) \stackrel{d}{=} (X_t^{(c,\lambda\mathcal{L})}, V_t^{(c,\lambda\mathcal{L})}). \quad (3.2)$$

Moreover, the following formula holds

$$\begin{aligned} \mathcal{Z}_t h &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) P_t h \\ &+ \sum_{n=1}^{\infty} (-\lambda)^\nu \mathcal{M}_\nu^{(n)}(-\lambda^\nu t^\nu) \int_{0 < \tau_1 < \tau_2 < \dots < t} P_{t-\tau_n} L P_{\tau_n-\tau_{n-1}} L \dots L P_{\tau_1} h d\tau_1 \dots d\tau_n \end{aligned}$$

where $\mathcal{M}_\nu^{(n)}$ is defined in Equation (2.13).

Proof. Under the Assumptions **(A1)**, the isotropic transport process in Definition 2.1 reads, for each $t \geq 0$,

$$\begin{pmatrix} V_{(1)}^\nu(t) \\ X_{(1)}^\nu(t) \end{pmatrix} = \begin{pmatrix} v_m \\ x + \sum_{k=1}^m cv_{k-1} J_k^{(\nu)} + cv_m \left(t - \sum_{k=1}^m J_k^{(\nu)} \right) \end{pmatrix} \quad \sum_{k=1}^m J_k^{(\nu)} \leq t < \sum_{k=1}^{m+1} J_k^{(\nu)}$$

where $m \geq 0$, which can compactly be re-written as

$$\begin{pmatrix} V_{(1)}^\nu(t) \\ X_{(1)}^\nu(t) \end{pmatrix} = \sum_{m=0}^{+\infty} \begin{pmatrix} v_m \\ x + \sum_{k=1}^m cv_{k-1} J_k^{(\nu)} + cv_m \left(t - \sum_{k=1}^m J_k^{(\nu)} \right) \end{pmatrix} \mathbb{1}_{\{\sum_{k=1}^m J_k^{(\nu)} \leq t < \sum_{k=1}^{m+1} J_k^{(\nu)}\}}.$$

From Equation (2.26) we have, for each $t \geq 0$,

$$\begin{pmatrix} V_{(1)}^\nu(t) \\ X_{(1)}^\nu(t) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} v_m \\ x + \sum_{k=1}^m cv_{k-1} \frac{J_k}{\mathcal{L}} + cv_m \left(t - \sum_{k=1}^m \frac{J_k}{\mathcal{L}} \right) \end{pmatrix} \quad \sum_{k=1}^m \frac{J_k}{\mathcal{L}} \leq t < \sum_{k=1}^{m+1} \frac{J_k}{\mathcal{L}}.$$

where $\{J_k/\mathcal{L}\}$ are exponential random variables of random parameter $\lambda\mathcal{L}$. Therefore the equality in distribution of Equation (3.2) follows.

Hence we can re-write Equation (3.1) as

$$\begin{aligned} \mathcal{Z}_t h(x, v) &= \int_0^\infty \mathbb{E}^{(x, v)} \left[h \left(X_t^{(c, \lambda)}, V_t^{(c, \lambda)} \right) \right] \ell(dl) \\ &= \int_0^\infty \mathcal{T}_t^{(c, \lambda)} h(x, v) \ell(dl) \end{aligned}$$

where $\{\mathcal{T}_t^{(c, \lambda)}, t \geq 0\}$ is the Markovian semigroup defined in (2.3) and $\ell(\cdot)$ is the Lamperti measure as in Equation (2.18). By using the representation of the Markovian operator in Equation (2.5) we obtain

$$\begin{aligned} \mathcal{Z}_t h &= \int_0^{+\infty} e^{-\lambda t} P_t h \ell(dl) \\ &+ \int_0^{+\infty} \left(\sum_{n=1}^{+\infty} e^{-\lambda t} (\lambda)^n \int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \dots LP_{\tau_1} h d\tau_1 \dots d\tau_n \right) \ell(dl). \end{aligned} \quad (3.3)$$

We now observe that we can interchange the summation and the integral in Equation (3.3), by using the dominated convergence theorem for Bochner integrals (see [3, Theorem 1.1.8]). Consider

$$f_k := \sum_{n=1}^k e^{-\lambda t} (\lambda)^n \int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \dots LP_{\tau_1} h d\tau_1 \dots d\tau_n.$$

We have to show that $\|f_k\|$ is dominated by an integrable function, which is independent of k . Indeed, as P_t and L are contraction operators

$$\begin{aligned} \|f_k\| &\leq \sum_{n=1}^{+\infty} \left\| e^{-\lambda t} (\lambda l)^n \int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \cdots L P_{\tau_1} h \, d\tau_1 \cdots d\tau_n \right\| \\ &\leq \sum_{n=1}^{+\infty} e^{-\lambda t} (\lambda l)^n \|h\| \int_{0 < \tau_1 < \dots < \tau_n < t} d\tau_1 \cdots d\tau_n \\ &= \sum_{n=1}^{+\infty} e^{-\lambda t} (\lambda l)^n \|h\| \frac{t^n}{n!} = \|h\| (1 - e^{-\lambda t}). \end{aligned}$$

Moreover,

$$\int_0^{+\infty} \|h\| (1 - e^{-\lambda t}) \ell(dl) = \|h\| (1 - \mathcal{M}_\nu(-(\lambda t)^\nu)).$$

Then [3, Theorem 1.1.8] applies here and from Equation (3.3) we get

$$\begin{aligned} \mathcal{Z}_t h &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) P_t h \\ &+ \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(e^{-\lambda t} (\lambda l)^n \int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \cdots L P_{\tau_1} h \, d\tau_1 \cdots d\tau_n \right) \ell(dl) \\ &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) P_t h \\ &+ \sum_{n=1}^{+\infty} \left(\int_0^{+\infty} e^{-\lambda t} (\lambda l)^n \ell(dl) \right) \left(\int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \cdots L P_{\tau_1} h \, d\tau_1 \cdots d\tau_n \right) \\ &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) P_t h \\ &+ \sum_{n=1}^{+\infty} \left((-\lambda)^n \int_0^{+\infty} \left(\frac{d}{d(\lambda t)} \right)^n e^{-\lambda t} \ell(dl) \right) \left(\int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \cdots L P_{\tau_1} h \, d\tau_1 \cdots d\tau_n \right) \end{aligned} \tag{3.4}$$

Hence, by the dominated convergence theorem we can exchange the derivative and the integral in the first term of the product to obtain

$$\begin{aligned} \mathcal{Z}_t h &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) P_t h \\ &+ \sum_{n=1}^{+\infty} (-\lambda)^n \mathcal{M}_\nu^{(n)}(-\lambda^\nu t^\nu) \int_{0 < \tau_1 < \dots < \tau_n < t} P_{t-\tau_n} L \cdots L P_{\tau_1} h \, d\tau_1 \cdots d\tau_n, \end{aligned}$$

where we used the notation in Equation (2.13). \square

In analogy with the Markovian case, it is possible to explicitly write the operator \mathcal{Z}_t as

$$\begin{aligned} \mathcal{Z}_t h(x, v) &= \mathcal{M}_\nu(-\lambda^\nu t^\nu) h(x + cvt, v) + \\ &+ \sum_{n=1}^{\infty} (-\lambda)^n \mathcal{M}_\nu^{(n)}(-\lambda^\nu t^\nu) \int_{0 < \tau_1 < \tau_2 < \dots < t} \int_{(S^{d-1})^n} h(x_t, \omega_n) d\tau_1 \dots d\tau_n \mu(d\omega_1) \dots \mu(d\omega_n) \end{aligned}$$

where

$$x_t := x + c \sum_{j=1}^n (\tau_j - \tau_{j-1}) \omega_{j-1} + c(t - \tau_n) \omega_n, \quad \omega_0 := v,$$

and $v_t = \omega_n$ for $t \in [\tau_n, \tau_{n+1})$.

We now aim to find the mean squared displacement of this non-Markovian random flight. We shall use the equality in distribution of Theorem 3.1, which makes the transport process analytically tractable.

Proposition 3.2. *Let $\left\{ \left(X_{(1)}^\nu(t), V_{(1)}^\nu(t) \right), t \geq 0 \right\}$ be the non-Markovian random flight as defined by Assumptions (A1). Denote $\mathbb{P}^{x,\mu} := \int_{S^{d-1}} \mathbb{P}^{x,v} \mu(dv)$ and by $\mathbb{E}^{x,\mu}$ the corresponding expectation. Then we have*

$$\mathbb{E}^{x,\mu} |X_{(1)}^\nu(t) - x|_e^2 = 2c^2 \int_0^t dw_1 \int_0^{w_1} dw_2 \mathcal{M}_\nu(-\lambda^\nu w_2^\nu), \quad (3.5)$$

and then for large t one has

$$\mathbb{E}^{x,\mu} |X_{(1)}^\nu(t) - x|_e^2 \sim \varepsilon t^{2-\nu}, \quad \nu \in (0, 1], \quad (3.6)$$

where ε is a positive constant.

Proof. Let us consider the mean squared displacement of the Markovian random flight in Equation (2.11). Then by Theorem 3.1 we get

$$\mathbb{E}^{x,\mu} |X_{(1)}^\nu(t) - x|_e^2 = 2c^2 \int_0^\infty \left(\frac{t}{\lambda l} - \frac{1}{\lambda^2 l^2} + \frac{e^{-\lambda t l}}{\lambda^2 l^2} \right) \ell(dl).$$

Considering the Laplace transform in the time variable ($t \mapsto s$), we obtain, for any $s > 0$,

$$2c^2 \int_0^\infty \left(\frac{1}{\lambda s^2} - \frac{1}{\lambda^2 l^2 s} + \frac{1}{\lambda^2 l^2 (s + \lambda l)} \right) \ell(dl) = 2c^2 \frac{1}{s^2} \int_0^\infty \frac{1}{s + \lambda l} \ell(dl).$$

By applying the Laplace transform to the second member of Equation (3.5) and using the definition of Lamperti distribution (2.17), one has

$$\begin{aligned} 2c^2 \frac{1}{s^2} \int_0^\infty e^{-sw_2} \mathcal{M}_\nu(-\lambda^\nu w_2^\nu) &= 2c^2 \frac{1}{s^2} \int_0^\infty e^{-sw_2} \int_0^\infty e^{-\lambda w_2 l} \ell(dl) \\ &= 2c^2 \frac{1}{s^2} \int_0^\infty \frac{1}{s + \lambda l} \ell(dl), \end{aligned}$$

and the first part of the thesis follows.

Now, denote $U(t) = \mathbb{E}^{x,\mu} \left| X_{(1)}^\nu(t) - x \right|_e^2$ and note that

$$\int_0^{+\infty} e^{-st} U'(t) dt = 2c^2 \frac{1}{s} \frac{s^{\nu-1}}{\lambda^\nu + s^\nu} \quad (3.7)$$

$$\sim K' s^{\nu-2} \quad \text{as } s \rightarrow 0 \quad (3.8)$$

for $K' > 0$, where we used the well-known Laplace transform of the Mittag-Leffler survival function

$$\int_0^\infty e^{-st} \mathcal{M}_\nu(-\lambda^\nu t^\nu) dt = \frac{s^{\nu-1}}{\lambda^\nu + s^\nu}. \quad (3.9)$$

It follows from the Karamata Tauberian Theorem [22, Theorem XIII.5.2] that

$$\mathbb{E}^{x,\mu} \left| X_{(1)}^\nu(t) - x \right|_e^2 \sim K t^{2-\nu} \quad (3.10)$$

for $K > 0$, as claimed. \square

From Equation (3.6) we conclude that the random flight is super diffusive for $\nu < 1$. Instead, for $\nu = 1$ we get the asymptotic behaviour of the Markovian random flight.

3.2. Mittag-Leffler distribution of obstacles. We preliminarily recall some basic notions on point processes. We follow the notation of [29]; another standard reference on point processes is [16].

Let $\mathcal{S} \in \mathcal{B}(\mathbb{R}^3)$ and denote $(\mathcal{S}, \mathcal{B}(\mathcal{S}), \mu)$ a measure space which is finite on Borel subsets with finite Lebesgue measure. We denote by Π a point process in \mathcal{S} , namely an integer-valued random measure. In particular, Π is a locally finite kernel from a probability space (Ω, \mathcal{A}, P) into $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$, i.e. a mapping $\Pi : (\Omega \times \mathcal{B}(\mathcal{S})) \mapsto \mathbb{N}$, such that $\Pi(\omega, B)$ is a locally finite integer-valued measure for fixed ω and an integer-valued random variable for fixed B .

From now on, let $\mu(B) = \rho|B|$, where $\rho \in (0, \infty)$ and $|B|$ denotes the Lebesgue measure of the Borel set B .

We recall that a point process is said to be a Poisson point process with parameter $\rho \in (0, \infty)$, if, for any collection of mutually disjoint, finite Borel sets $\{B_j, j \in \{1, \dots, n\}\}$, and any choice of non-negative integers $\{k_j, j \in \{1, \dots, n\}\}$, $n \geq 1$, we have

$$P \left(\bigcap_{j=1}^n \{\mathcal{N}(B_j) = k_j\} \right) = e^{-\sum_{j=1}^n \rho|B_j|} \prod_{j=1}^n \frac{(\rho|B_j|)^{k_j}}{k_j!} \quad (3.11)$$

where $\mathcal{N}(B)$ is the random variable $\omega \mapsto \Pi(\omega, B)$, i.e. the random number of points inside B . Equation (3.11) indicates that $\mathcal{N}(B_j)$, $j = 1, \dots, n$, are independent random variables, each having Poisson distribution with mean $\rho|B_j|$.

We are ready to introduce a new point process, whose distribution is described, in addition to ρ , by a fractional parameter $\nu \in (0, 1]$, such that the Poisson case is re-obtained when

$\nu = 1$. We use the notation $\mathcal{N}^\nu(B)$ to indicate the random number of points in $B \in \mathcal{B}(\mathcal{S})$. Moreover, for the Mittag-Leffler function $\mathcal{M}_\nu(\cdot)$, we will use the notation in Equations (2.12) and (2.13).

Definition 3.3. *A point process Π is said to be a Mittag-Leffler point process of parameters $\rho \in (0, \infty)$ and $\nu \in (0, 1]$ if, for any collection of mutually disjoint, finite Borel sets $\{B_j, j \in \{1, \dots, n\}\}$, and any choice of non-negative integers $\{k_j, j \in \{1, \dots, n\}\}$, $n \geq 1$, we have*

$$P\left(\bigcap_{j=1}^n \{\mathcal{N}^\nu(B_j) = k_j\}\right) = (-1)^k \mathcal{M}_\nu^{(k)}\left(-\left(\sum_{j=1}^n \rho |B_j|\right)^\nu\right) \prod_{j=1}^n \frac{(\rho |B_j|)^{k_j}}{k_j!} \quad (3.12)$$

where $k := \sum_{j=1}^n k_j$.

In Appendix A, the existence and uniqueness of the point process of Definition 3.3 are rigorously established. We further note that this point process has relationship with the one studied in [1].

At a first glance, from a purely formal perspective, the new point process is obtained by replacing the exponential function in Equation (3.11) with the Mittag-Leffler function in Equation (3.12). This implies the loss of independence among the number of points in disjoint sets.

Let us observe that for any finite Borel set $B \in \mathcal{B}(\mathcal{S})$, the random variable $\mathcal{N}^\nu(B)$ has distribution

$$P(\mathcal{N}^\nu(B) = k) = (-1)^k \mathcal{M}_\nu^{(k)}(-\rho^\nu |B|^\nu) \frac{(\rho |B|)^k}{k!} \quad k \in \mathbb{N}, \quad (3.13)$$

as a special case of Equation (3.12) for $n = 1$.

We further observe that, from a physical perspective, the Mittag-Leffler point process exhibits both similarities and differences compared to the Poisson case. Specifically, both are spatially homogeneous, meaning they are invariant under space translations. This follows from the fact that the distribution in Equation (3.12) only depends on the volumes of the B_j s.

On the other hand, the new point process is suitable to describe rarefied gases. To illustrate this point, we present the distribution of the random distance D_ν between a given point $y \in \mathbb{R}^3$ and its nearest point of Π . Such a distribution does not depend on y by space homogeneity. Note that y is not required to be a point of Π .

It will become clear that, for $x \rightarrow \infty$, the survival function $P(D_\nu > x)$ decays to zero as a power law, which is a substantially different behaviour from the Poisson case. Indeed, observe that $D_\nu > x$ if and only if there are no points within the ball of radius x centered at y , denoted as $B_x(y)$, for all $x \geq 0$. Hence, in the Poisson case, one has

$$P(D_1 > x) = P(\mathcal{N}(B_x(y)) = 0) = e^{-\rho \frac{4}{3}\pi x^3},$$

while, in the Mittag-Leffler case, one has

$$P(D_\nu > x) = P(\mathcal{N}^\nu(B_x(y)) = 0) = \mathcal{M}_\nu \left(-\rho^\nu \left(\frac{4}{3}\pi x^3 \right)^\nu \right), \quad \nu \in (0, 1). \quad (3.14)$$

Equation (3.14) gives that $P(D_\nu > x) \sim 1/x^{3\nu}$ as $x \rightarrow \infty$. This is consistent with the fact that the Mittag-Leffler model is better suited for situations where the gas exhibits a sparser structure, with large empty regions among particles. It means that the moving particles in the gas are more likely to experience longer free paths or travel greater distances without interacting. In contrast, the Poisson model describes a gas where interactions or collisions occur more frequently, meaning the empty regions are smaller.

Remark 3.4. *In view of the main theorem 3.8, where the case of large ρ will be considered, we observe that ρ controls the probability to have at least one point in a small region B . Indeed, we have*

$$\begin{aligned} P(\mathcal{N}^\nu(B) \geq 1) &= 1 - P(\mathcal{N}^\nu(B) = 0) \\ &= 1 - \mathcal{M}_\nu(-\rho^\nu |B|^\nu) \\ &= \frac{\rho^\nu}{\Gamma(1 + \nu)} |B|^\nu + o(|B|^\nu) \quad \nu \in (0, 1]. \end{aligned}$$

We further observe that in the Poisson case, ρ also has the meaning of expected number of points per unit volume; indeed, for the Poisson distribution, we have $\mathbb{E}\mathcal{N}(B) = \rho$ if $|B| = 1$. In contrast, the expected number of points per unit volume in the Mittag-Leffler case is infinite for each ρ when $\nu \in (0, 1)$.

Consider Π as in Definition 3.3. We now aim to determine the so-called Jánossy measure (see [27, Section 4.3]) associated to Π .

Lemma 3.5. *Let us consider $B \in \mathcal{B}(\mathbb{R}^3)$, $|B| < \infty$. Let B contain $\mathcal{N}^\nu(B)$ points of Π , which are located at the random positions $C_1, C_2, \dots, C_{\mathcal{N}^\nu(B)}$. Then*

$$P(C_1 \in dc_1, \dots, C_n \in dc_n, \mathcal{N}^\nu(B) = n) = \frac{1}{n!} \rho^n (-1)^n \mathcal{M}_\nu^{(n)}(-\rho^\nu |B|^\nu) dc_1 \cdots dc_n,$$

for each $n \geq 1$.

Proof. For each $n \geq 1$, consider a sequence $\{A_j, j \in \{1, \dots, n\}\}$ of mutually disjoint Borel sets s.t. $\cup_j A_j \subseteq B$. Then, using combinatorics arguments, we get

$$\begin{aligned} P(C_1 \in A_1, \dots, C_n \in A_n, \mathcal{N}^\nu(B) = n) &= \frac{1}{n!} P \left(\bigcap_{j=1}^n \{\mathcal{N}^\nu(A_j) = 1\}, \mathcal{N}^\nu(B \setminus \cup_{j=1}^n A_j) = 0 \right) \\ &= \frac{1}{n!} \rho |A_1| \cdots \rho |A_n| (-1)^n \mathcal{M}_\nu^{(n)}(-\rho^\nu |B|^\nu) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!} \rho^n (-1)^n \mathcal{M}_\nu^{(n)}(-\rho^\nu |B|^\nu) \int_{A_1} dc_1 \cdots \int_{A_n} dc_n \\
&= \frac{1}{n!} \int_{A_1} \cdots \int_{A_n} \rho^n (-1)^n \mathcal{M}_\nu^{(n)}(-\rho^\nu |B|^\nu) dc_1 \cdots dc_n,
\end{aligned}$$

where in the second equality we used Equation (3.12). Hence, the law of (C_1, \dots, C_n) is absolutely continuous and the thesis follows. \square

3.3. The Boltzmann-Grad limit. We now describe the new model of motion among random obstacles. The obstacles are assumed to be identical hard spheres of radius R , whose centres are randomly distributed according to the Mittag-Leffler point process, as in Definition 3.3. Some obstacles may overlap as it happens in the classical Gallavotti's model [23]. Outside the system of obstacles, the particle moves along a straight line with constant speed $c > 0$. When reaching the surface of an obstacle, an elastic collision occurs. This means that the particle is specularly reflected and the post-collisional speed is again equal to c . Each obstacle can be hit more than once, which leads to a long memory tail in the process.

We will analyze the system in a Boltzmann-Grad type limit, where the obstacle radius R approaches zero while the parameter ρ , representing the intensity of the process as described in Remark 3.4, diverges to infinity; these limits are taken in such a way that the product $R^2 \rho$ remains constant, that is

$$\rho \rightarrow \infty \quad R \rightarrow 0 \quad \rho c \pi R^2 \rightarrow \lambda \in (0, \infty). \quad (3.15)$$

Without loss of generality, we shall put $\rho = \lambda / c \pi R^2$ and consider the one variable limit as $R \rightarrow 0$.

The constraint in Equation (3.15) can be understood by studying the distribution of the free flight time. In Gallavotti's model, i.e. when $\nu = 1$, for $R \rightarrow 0$ the distribution of the free flight time converges to an exponential distribution with mean λ^{-1} . Hence, condition (3.15) means that the limit procedure is such that the free flight time of the particle maintains a strictly positive, finite mean.

In this regard, in Lemma 3.6 we shall see that this is not true if $\nu \in (0, 1)$, given that the free flight time has, in the present model, infinite mean both before and after the limit. Indeed, in this case the limit (3.15) simply ensures that the free flight time is almost surely positive and finite, meaning that it takes values in the open interval $(0, \infty)$ almost surely, and then it has a proper distribution.

So, suppose that a particle, which is initially located at $x \in \mathbb{R}^3$, is shot towards an arbitrary direction $v \in S^2$. Let T^R be the free flight time, i.e. the first hitting time with the system of obstacles.

Lemma 3.6. *The distribution of the free flight time T^R has a discrete component in 0, such that*

$$P(T^R = 0) = 1 - \mathcal{M}_\nu \left(-\rho^\nu \left(\frac{4}{3} \pi R^3 \right)^\nu \right)$$

and an absolutely continuous component on $(0, \infty)$, such that

$$P(T^R > t) = \mathcal{M}_\nu \left(-\rho^\nu \left(\pi R^2 ct + \frac{4}{3} \pi R^3 \right)^\nu \right) \quad t > 0.$$

Proof. Let $t > 0$. Then T^R is greater than t if and only if none of the obstacle centres lies in the set

$$\theta(x, v, t) := \{y \in \mathbb{R}^3 : |y - (x + cvs)|_e \leq R, \quad s \in [0, t]\} \quad (3.16)$$

i.e. a cylinder of height ct around the particle trajectory, capped by two hemispheres of radius R . So, for $t > 0$, using formula (3.13), we have

$$\begin{aligned} P(T^R > t) &= P(\mathcal{N}^\nu(\theta(x, v, t)) = 0) \\ &= \mathcal{M}_\nu \left(-\rho^\nu \left(\pi R^2 ct + \frac{4}{3} \pi R^3 \right)^\nu \right). \end{aligned}$$

The discrete mass at zero corresponds to the possibility that the starting point x is located inside some obstacles:

$$\begin{aligned} P(T^R = 0) &= P(\mathcal{N}^\nu(B_R(x)) \geq 1) \\ &= 1 - P(\mathcal{N}^\nu(B_R(x)) = 0) \\ &= 1 - \mathcal{M}_\nu \left(-\rho^\nu \left(\frac{4}{3} \pi R^3 \right)^\nu \right), \end{aligned}$$

where $B_R(x)$ is the ball of radius R centered at x . □

From Lemma 3.6, we observe that the distribution of T^R does not depend either on the initial position x or on the initial direction v , because the Mittag-Leffler random field is homogeneous and isotropic. Moreover, for large t we have $P(T^R > t) \sim Kt^{-\nu}$, $K > 0$, hence the free flight time T^R has infinite expectation.

Remark 3.7. *From Lemma 3.6, we observe that for $R \rightarrow 0$, the discrete component in 0 vanishes and the distribution of T^R converges to a Mittag-Leffler distribution, i.e.*

$$\lim_{R \rightarrow 0} P(T^R > t) = \mathcal{M}_\nu(-\lambda^\nu t^\nu) \quad t \geq 0. \quad (3.17)$$

As mentioned before, after the limit (3.15) the free flight time has a proper distribution with infinite expectation.

We are now ready to state the convergence theorem.

We denote by $\mathcal{X}_t^{(1)}$ the position of the particle at time t and by $\mathcal{V}_t^{(1)}$ the unit velocity vector at time t , i.e. the direction of the particle at time t . We call $(\mathcal{X}^{(1)}, \mathcal{V}^{(1)}) := \left\{ \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right), t \geq 0 \right\}$ the Lorentz process in $\mathbb{R}^3 \times S^2$, with initial data given by the random variables $(\mathcal{X}_0^{(1)}, \mathcal{V}_0^{(1)})$ in $\mathbb{R}^3 \times S^2$. Recall that the Lorentz process depends on the parameters of the Mittag-Leffler random field ρ and ν , and on the size of the obstacles R . However, to simplify the notation, we will not make this dependence explicit. On the other hand, the superscript indicates that it refers to the first model, while the second one will be discussed in the next section. Once again, for $\nu = 1$, the obstacles have a Poisson distribution, and such Lorentz process coincides with Gallavotti's model. In the following theorem we shall use the notation

$$P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv' \right) := P \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv' \mid \mathcal{X}_0 = x, \mathcal{V}_0 = v \right).$$

Theorem 3.8. *Consider the Lorentz process $(\mathcal{X}^{(1)}, \mathcal{V}^{(1)})$ defined above. Under the Boltzmann-Grad limit (3.15) the Lorentz process converges, in the sense of the single time distribution, to the random flight $(X_{(1)}^\nu, V_{(1)}^\nu)$ defined in Section 3.1, i.e.,*

$$P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv' \right) \rightarrow \mathbb{P}^{(x,v)} \left(X_{(1)}^\nu(t) \in dx', V_{(1)}^\nu(t) \in dv' \right)$$

weakly, for all $t > 0$ and for all $(x, v) \in \mathbb{R}^3 \times S^2$.

Proof. Let us write $E^{(x,v)}$ for the expectation under $P^{(x,v)}$. Then for suitable functions $h(x, v)$, we can write

$$E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) := \int_{S^2} \int_{B_{ct}(x)} h(x', v') P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv' \right),$$

where $B_{ct}(x)$ indicates the ball centered in x of radius ct . Indeed, due to finite speed of the particle, the distribution of $\mathcal{X}_t^{(1)}$ is supported on $B_{ct}(x)$.

To get the statement, using the Portmanteau lemma, we can prove that

$$\lim_{R \rightarrow 0} E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) = \mathbb{E}^{x,v} h \left(X_{(1)}^\nu(t), V_{(1)}^\nu(t) \right) \quad \forall h \in C_b(\mathbb{R}^3 \times S^2),$$

where $C_b(\mathbb{R}^3 \times S^2)$ is the set of continuous bounded functions from $(\mathbb{R}^3 \times S^2)$ to \mathbb{R} .

Among all possible paths, some allow each obstacle to be hit at most once, while others lead to recollisions. Therefore, we can partition the expectation as

$$E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) = E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbf{1}_{\mathcal{A}} + E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbf{1}_{\mathcal{A}^c}, \quad (3.18)$$

where \mathcal{A} is the event *absence of recollisions*.

We first compute the term $E^{(x,v)}\mathbb{1}_{\mathcal{A}}h\left(\mathcal{X}_t^{(1)},\mathcal{V}_t^{(1)}\right)$. To this aim, we use the decomposition

$$\mathbb{1}_{\mathcal{A}} = \mathbb{1}_{\mathcal{A}}\left(\mathbb{1}_{\{T^R=0\}} + \mathbb{1}_{\{T^R>t\}} + \mathbb{1}_{\{T^R\in(0,t)\}}\right),$$

where T^R is the first free flight time. Using the distribution of T^R in Lemma 3.6 we have

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{A}}\mathbb{1}_{\{T^R=0\}}h\left(\mathcal{X}_t^{(1)},\mathcal{V}_t^{(1)}\right)\right] = h(x,v)\left[1 - \mathcal{M}_\nu\left(-\rho^\nu\left(\frac{4}{3}\pi R^3\right)^\nu\right)\right]$$

and

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{A}}\mathbb{1}_{\{T^R>t\}}h\left(\mathcal{X}_t^{(1)},\mathcal{V}_t^{(1)}\right)\right] = h(x+cvt,v)\mathcal{M}_\nu\left(-\rho^\nu\left(\pi R^2ct + \frac{4}{3}\pi R^3\right)^\nu\right).$$

Finally, we consider the case of $T^R \in (0,t)$. In this time interval, the particle can hit the obstacles whose centres lie in the ball $B_{ct+R}(\mathcal{X}_0)$. Let M be the random number of obstacles inside $B_{ct+R}(\mathcal{X}_0)$ and let C_i , $i = 1, \dots, M$ be the random positions of their centres. In compact notation, $\underline{C} := (C_1, \dots, C_M)$. Lowercase letters shall indicate the realizations of such variables.

From now on, we shall write $\mathcal{X}_t^{(1)} = \mathcal{X}_t^{(1)}(\underline{C})$ and $\mathcal{V}_t^{(1)} = \mathcal{V}_t^{(1)}(\underline{C})$ to make the fact explicit that position and velocity can be written as a function of the location of the obstacles. Moreover, we shall indicate with \mathcal{H}_m the subset of $\left(B_{ct+R}(\mathcal{X}_0^{(1)})\right)^m$ such that, conditionally on (x,v) , the particle at least hits 1 obstacle up to time t and has no recollisions. By using Lemma 3.5 we can write

$$\begin{aligned} & E^{(x,v)}\left[\mathbb{1}_{\mathcal{A}}\mathbb{1}_{\{T^R\in(0,t)\}}h\left(\mathcal{X}_t^{(1)}(\underline{C}),\mathcal{V}_t^{(1)}(\underline{C})\right)\right] \\ &= \sum_{m=1}^{\infty} \int_{\mathcal{H}_m} h\left(\mathcal{X}_t^{(1)}(\underline{c}),\mathcal{V}_t^{(1)}(\underline{c})\right) P^{(x,v)}\left(C_1 \in dc_1, \dots, C_m \in dc_m, \mathcal{N}^\nu\left(B_{ct+R}(\mathcal{X}_0^{(1)})\right) = m\right) \\ &= \sum_{m=1}^{\infty} \int_{\mathcal{H}_m} h\left(\mathcal{X}_t^{(1)}(\underline{c}),\mathcal{V}_t^{(1)}(\underline{c})\right) \frac{\rho^{m(-1)^m}}{m!} \mathcal{M}_\nu^{(m)}\left(-\rho^\nu|B_{ct+R}(\mathcal{X}_0^{(1)})|^\nu\right) dc_1 dc_2 \dots dc_m. \end{aligned} \tag{3.19}$$

Let n be the number of hit obstacles out of m and let $\mathcal{F}^{(n)}$ be the subset of \mathcal{H}_m containing all the obstacle configurations such that the particle exactly hits n obstacles¹. Hence

$$\mathcal{H}_m = \bigcup_{n=1}^m \mathcal{F}^{(n)}$$

¹Notably, collisions must occur before time t , ensuring that the hit obstacles are not too far apart; otherwise, the particle would not have sufficient time to reach the first obstacle and then adjust its trajectory to hit the second.

and the above integral reads

$$\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \int_{\mathcal{H}_m} \delta_{\mathcal{F}^{(n)}}(\underline{c}) h\left(\mathcal{X}_t^{(1)}(\underline{c}), \mathcal{V}_t^{(1)}(\underline{c})\right) \frac{\rho^m (-1)^m}{m!} \mathcal{M}_\nu^{(m)}\left(-\rho^\nu \left|B_{ct+R}\left(\mathcal{X}_0^{(1)}\right)\right|^\nu\right) dc_1 dc_2 \dots dc_m,$$

where $\delta_{\mathcal{F}^{(n)}}(\underline{c}) = 1$ if $\underline{c} \in \mathcal{F}^{(n)}$ and 0 otherwise, and we exchanged the order of summations.

Now, for each n , there are $\binom{m}{n}$ possible ways to choose which obstacles are actually hit, i.e.

$$\mathcal{F}^{(n)} = \bigcup_{j=1}^{\binom{m}{n}} \mathcal{F}_j^{(n)}$$

where $\mathcal{F}_j^{(n)}$ is the j -th subset of $\mathcal{F}^{(n)}$ of all the possible configurations with n hit obstacles; hence $\delta_{\mathcal{F}_j^{(n)}} \delta_{\mathcal{F}^{(n)}} = \delta_{\mathcal{F}^{(n)}}$. Thus the above integral becomes

$$\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \sum_{j=1}^{\binom{m}{n}} \int_{\mathcal{H}_m} \delta_{\mathcal{F}_j^{(n)}}(\underline{c}) h\left(\mathcal{X}_t^{(1)}(\underline{c}), \mathcal{V}_t^{(1)}(\underline{c})\right) \frac{\rho^m (-1)^m}{m!} \mathcal{M}_\nu^{(m)}\left(-\rho^\nu \left|B_{ct+R}\left(\mathcal{X}_0^{(1)}\right)\right|^\nu\right) dc_1 dc_2 \dots dc_m.$$

Now, given that marginally each center is uniformly distributed on the sphere, all the possible $\binom{m}{n}$ choices give the same contribution. Hence, denoting by $\mathcal{F}_1^{(n)}$ the configuration such that the particle hits the first n obstacles of the sequence, we have

$$\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \binom{m}{n} \int_{\mathcal{H}_m} \delta_{\mathcal{F}_1^{(n)}}(\underline{c}) h\left(\mathcal{X}_t^{(1)}(\underline{c}), \mathcal{V}_t^{(1)}(\underline{c})\right) \frac{\rho^m (-1)^m}{m!} \mathcal{M}_\nu^{(m)}\left(-\rho^\nu \left|B_{ct+R}\left(\mathcal{X}_0^{(1)}\right)\right|^\nu\right) dc_1 dc_2 \dots dc_m.$$

Now, let $\underline{C} = (\tilde{C}, \hat{C})$, with $\tilde{C} = (C_1, \dots, C_n)$ and $\hat{C} = (C_{n+1}, \dots, C_m)$. We remark that actually in the above integral, the position $\mathcal{X}_t^{(1)}$ and the velocity $\mathcal{V}_t^{(1)}$ of the particle only depend on the hit obstacles \tilde{C} .

We indicate the tube-like flow induced by the hit obstacles \tilde{c}

$$\Theta(\tilde{c}) := \left\{ y \in B_{ct+R}\left(\mathcal{X}_0^{(1)}\right) \text{ s.t. } |\mathcal{X}_s^{(1)}(\tilde{c}) - y|_e \leq R, s \in [0, t] \right\}$$

while its complement in $B_{ct+R}\left(\mathcal{X}_0^{(1)}\right)$ is denoted by $\overline{\Theta(\tilde{c})}$. We use $\mathcal{K}^{(n)}$ for the set of feasible configurations of \tilde{c} and thus we can split the integral on $\mathcal{F}_1^{(n)}$ as follows

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \binom{m}{n} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n h\left(\mathcal{X}_t^{(1)}(\tilde{c}), \mathcal{V}_t^{(1)}(\tilde{c})\right) \\ & \int_{(\overline{\Theta(\tilde{c})})^{m-n}} dc_{n+1} \dots dc_m \frac{\rho^m (-1)^m}{m!} \mathcal{M}_\nu^{(m)}\left(-\rho^\nu \left|B_{ct+R}\left(\mathcal{X}_0^{(1)}\right)\right|^\nu\right). \end{aligned}$$

Using Equation (2.17) in the definition of the Lamperti random variable, we note that

$$\begin{aligned} \frac{\rho^m (-1)^m}{m!} \mathcal{M}_\nu^{(m)} \left(-\rho^\nu \left| B_{ct+R} \left(\mathcal{X}_0^{(1)} \right) \right|^\nu \right) &= \frac{\rho^m (-1)^m}{m!} \int_0^\infty (-1)^m l^m e^{-\rho |B_{ct+R}(\mathcal{X}_0^{(1)})|^l} P(\mathcal{L} \in dl) \\ &= \int_0^\infty \frac{(\rho l)^m}{m!} e^{-\rho |B_{ct+R}(\mathcal{X}_0^{(1)})|^l} P(\mathcal{L} \in dl). \end{aligned}$$

Taking into account that

$$\sum_{m=n}^{\infty} \frac{(\rho l)^{m-n}}{(m-n)!} \int_{(\Theta(\tilde{\mathcal{C}}))^{m-n}} dc_{n+1} \dots dc_m = e^{\rho l |\Theta(\tilde{\mathcal{C}})|},$$

and $|\Theta(\tilde{\mathcal{C}})| = \left| B_{ct+R} \left(\mathcal{X}_0^{(1)} \right) \right| - \left| \overline{\Theta(\tilde{\mathcal{C}})} \right|$, Equation (3.19) becomes

$$\begin{aligned} &E^{(x,v)} \left[\mathbf{1}_{\mathcal{A}} \mathbf{1}_{\{T^R \in (0,t)\}} h \left(\mathcal{X}_t^{(1)} \left(\tilde{\mathcal{C}} \right), \mathcal{V}_t^{(1)} \left(\tilde{\mathcal{C}} \right) \right) \right] \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \int_0^\infty P^{(x,v)}(\mathcal{L} \in dl) \frac{(\rho l)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) e^{-\rho l |B_{ct+R}(\mathcal{X}_0^{(1)})|} \\ &\quad \sum_{m=n}^{\infty} \frac{(\rho l)^{m-n}}{(m-n)!} \int_{(\Theta(\tilde{\mathcal{C}}))^{m-n}} dc_{n+1} \dots dc_m \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \int_0^\infty P^{(x,v)}(\mathcal{L} \in dl) \frac{(\rho l)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) e^{-\rho l |B_{ct+R}(\mathcal{X}_0^{(1)})|} e^{\rho l |\overline{\Theta(\tilde{\mathcal{C}})}|} \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \int_0^\infty P^{(x,v)}(\mathcal{L} \in dl) \frac{(\rho l)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) e^{-\rho l |\Theta(\tilde{\mathcal{C}})|}. \end{aligned}$$

Moreover, using Equation (2.17) again for the Laplace transform of the Lamperti distribution, the above equation can be written as

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \frac{(-\rho)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \int_0^\infty P(\mathcal{L} \in dl) (-l)^n e^{-\rho l |\Theta(\tilde{\mathcal{C}})|} \\ &= \sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \frac{(-\rho)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \left[\left(\frac{d}{dz} \right)^n \mathcal{M}_\nu(-z^\nu) \right] \Big|_{z=\rho |\Theta(\tilde{\mathcal{C}})|}. \end{aligned}$$

We now observe that the n obstacles can be hit in $n!$ different chronological orders. Let $\mathcal{K}_i^{(n)}$ denote the i -th such ordering, i.e.

$$\mathcal{K}^{(n)} = \bigcup_{i=1}^{n!} \mathcal{K}_i^{(n)}$$

and let $\mathcal{K}_1^{(n)}$ be the configuration in which the obstacle centered at c_j is the j -th one to be hit in chronological order. By using the notation for the derivative of the Mittag-Leffler function in Equation (2.13), Equation (3.19) can be written as

$$\begin{aligned}
& E^{(x,v)} \left[\mathbb{1}_{\mathcal{A}} \mathbb{1}_{\{TR \in (0,t)\}} h \left(\mathcal{X}_t^{(1)} \left(\tilde{\mathcal{C}} \right), \mathcal{V}_t^{(1)} \left(\tilde{\mathcal{C}} \right) \right) \right] \\
&= \sum_{n=1}^{\infty} \int_{\mathcal{K}^{(n)}} dc_1 \dots dc_n \frac{(-\rho)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\mathcal{C}})|^{\nu}) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n!} \int_{\mathcal{K}_i^{(n)}} dc_1 \dots dc_n \frac{(-\rho)^n}{n!} h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\mathcal{C}})|^{\nu}) \\
&= \sum_{n=1}^{\infty} \int_{\mathcal{K}_1^{(n)}} dc_1 \dots dc_n (-\rho)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\mathcal{C}})|^{\nu}). \tag{3.20}
\end{aligned}$$

We now perform a change of variables in the above integral, following [23]. Let τ_1, \dots, τ_n be the collision times, such that $0 < \tau_1 < \tau_2 < \dots < \tau_n < t$ and let v_1, v_2, \dots, v_n be the unit velocity vectors that emerge in the sequence of collisions.

Since no recollisions occur here, there is a one to one correspondence between $\tilde{\mathcal{C}}$ and $\tilde{\tau} := (\tau_1, \dots, \tau_n)$, $\tilde{\mathcal{V}} := (v_1, \dots, v_n)$. We thus perform a change of variables in the integral in Equation (3.20). The Jacobian of the transformation leading to the new variables has the form $\frac{c^n}{4^n} R^{2n}$. It means that Equation (3.20) reads

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int_{\mathcal{K}_1^{(n)}} dc_1 \dots dc_n (-\rho)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{C}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{C}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\mathcal{C}})|^{\nu}) \\
&= \sum_{n=1}^{\infty} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} (-\rho)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\tau}, \tilde{\mathcal{V}}), \mathcal{V}_t^{(1)}(\tilde{\tau}, \tilde{\mathcal{V}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\tau}, \tilde{\mathcal{V}})|^{\nu}) \\
&\quad \frac{c^n}{4^n} R^{2n} dv_1 \dots dv_n d\tau_1 \dots d\tau_n \\
&= \sum_{n=1}^{\infty} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} (-\rho)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\tau}, \tilde{\mathcal{V}}), \mathcal{V}_t^{(1)}(\tilde{\tau}, \tilde{\mathcal{V}}) \right) \mathcal{M}_{\nu}^{(n)}(-\rho^{\nu} |\Theta(\tilde{\tau}, \tilde{\mathcal{V}})|^{\nu}) \\
&\quad c^n \pi^n R^{2n} \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n
\end{aligned}$$

where in the last step we denoted by $\mu(dv_i) = dv_i/4\pi$ the uniform probability measure on the unit sphere.

Putting all together, the first summand of Equation (3.18) becomes

$$E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbb{1}_{\mathcal{A}} = h(x,v) \left[1 - \mathcal{M}_{\nu} \left(-\rho^{\nu} \left(\frac{4}{3} \pi R^3 \right)^{\nu} \right) \right] +$$

$$\begin{aligned}
& + h(x + cvt, v) \mathcal{M}_\nu \left(-\rho^\nu \left(\pi R^2 ct + \frac{4}{3} \pi R^3 \right)^\nu \right) + \\
& + \sum_{n=1}^{\infty} \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} (-\rho)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\tau}, \tilde{v}), \mathcal{V}_t^{(1)}(\tilde{\tau}, \tilde{v}) \right) \mathcal{M}_\nu^{(n)} \left(-\rho^\nu |\Theta(\tilde{\tau}, \tilde{v})|^\nu \right) \\
& \quad c^n \pi^n R^{2n} \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n. \tag{3.21}
\end{aligned}$$

Passing to the Boltzman-Grad limit (3.15) in Equation (3.21), we obtain that

$$\begin{aligned}
\lim_{R \rightarrow 0} E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbf{1}_{\mathcal{A}} & = h(x + cvt, v) \mathcal{M}_\nu(-\lambda^\nu t^\nu) + \sum_{n=1}^{\infty} (-1)^n \mathcal{M}_\nu^{(n)}(-\lambda^\nu t^\nu) \lambda^n \\
& \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} h(x_t, v_n) \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n, \tag{3.22}
\end{aligned}$$

where

$$x_t := x + c \sum_{j=1}^n (\tau_j - \tau_{j-1}) v_{j-1} + c(t - \tau_n) v_n, \quad v_0 := v.$$

and we have taken into account that the unit velocity vector at time t is v_n . For details on sufficient conditions to exchange the limit for $R \rightarrow 0$ and the sum over n in Equation (3.21) see Appendix B.

The right side of (3.22) coincides with the Duhamel expansion of $\mathbb{E}^{x,v} h \left(X_{(1)}^\nu(t), V_{(1)}^\nu(t) \right)$ of the random flight defined in Section 3.1, as shown in Theorem 3.1 in the special case $d = 3$.

To conclude this proof, it is sufficient to verify that the recollision term $E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbf{1}_{\mathcal{A}^c}$ in Equation (3.18) vanishes for $R \rightarrow 0$. Firstly, let us rewrite Equation (3.22) explicitly as

$$\begin{aligned}
& \lim_{R \rightarrow 0} \int_{S^2} \int_{B_{ct}(x)} h(x', v') P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv', \mathcal{A} \right) \\
& = \int_{S^2} \int_{B_{ct}(x)} h(x', v') P^{(x,v)} \left(X_{(1)}^\nu(t) \in dx', V_{(1)}^\nu(t) \in dv' \right). \tag{3.23}
\end{aligned}$$

The above considerations are true for each function $h \in C_b(\mathbb{R}^3 \times S^2)$, and in particular for a function h which is equal to 1 on the domain $B_{ct}(x) \times S^2$, in which case Equation (3.23) reads

$$\lim_{R \rightarrow 0} \int_{S^2} \int_{B_{ct}(x)} P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv', \mathcal{A} \right) = 1,$$

i.e.

$$\lim_{R \rightarrow 0} P^{(x,v)}(\mathcal{A}) = 1 \quad \text{and then} \quad \lim_{R \rightarrow 0} P^{(x,v)}(\mathcal{A}^c) = 0.$$

Finally, we observe that

$$\begin{aligned} & \lim_{R \rightarrow 0} \left| E^{(x,v)} h \left(\mathcal{X}_t^{(1)}, \mathcal{V}_t^{(1)} \right) \mathbf{1}_{\mathcal{A}^c} \right| \\ & \leq \sup_{B_{ct}(x) \times S^2} |h| \lim_{R \rightarrow 0} \int_{S^2} \int_{B_{ct}(x)} P^{(x,v)} \left(\mathcal{X}_t^{(1)} \in dx', \mathcal{V}_t^{(1)} \in dv', \mathcal{A}^c \right) \\ & = \sup_{B_{ct}(x) \times S^2} |h| \lim_{R \rightarrow 0} P^{(x,v)}(\mathcal{A}^c) = 0, \end{aligned}$$

and this concludes the proof. \square

Remark 3.9. *The above theorem shows that, under the Boltzmann-Grad limit (3.15) we obtain a resulting random flight which is easier to be handled with respect to the Lorentz process, because, as we have seen in the proof, the measure of paths having recollisions tends to zero. However, there is a crucial difference between the present model and Gallavotti's one. Specifically, in Gallavotti's model, the elimination of recollisions leads to Markovianity. Indeed, the resulting random flight is the Markovian transport process recalled in Section 2.1.1, which is in turn approximated by a diffusion process by a further scaling limit. Instead, when $\nu \in (0, 1)$, even if the probability of recollisions vanishes, the resulting random flight is not Markovian. This is due to the power-law behaviour induced by the Mittag-Leffler point process that generates infinite mean free flight times of the particles: these flight times remain with infinite expectation even in the limit. Moreover, we shall see in the next section that, under a proper scaling, it leads to an anomalous diffusive behaviour.*

3.4. Convergence to anomalous diffusion. We now aim to study the asymptotic behaviour of the random flight defined in Section 3.1. Specifically, we show that, under an appropriate limiting regime, the random flight weakly converges to an anomalous diffusive process. In the following, we define such process, by means of its finite dimensional distributions. We shall indicate with $\langle \cdot, \cdot \rangle$ the usual inner product.

Definition 3.10. *Consider a stochastic process $W = \{W_t, t \geq 0\}$ on \mathbb{R}^d , $d \geq 1$. We say that it is a Mittag-Leffler anomalous-diffusion process of parameter $\nu \in (0, 1]$ if, for any choice of times $0 \leq t_1 < \dots < t_n$, the vector $\Gamma = (W_{t_1}, \dots, W_{t_n})$ has the characteristic function*

$$\varphi_\nu^n(u) = \mathcal{M}_\nu \left(- \left(\frac{1}{2} \langle u, Qu \rangle \right)^\nu \right) \quad \forall u \in \mathbb{R}^{nd}, \quad n \geq 1, \quad (3.24)$$

where the matrix $Q \in \mathbb{R}^{nd \times nd}$ is a block diagonal matrix, and the j th block has the form

$$Q_j = [t_h \wedge t_k]_{h,k \in \{1, \dots, n\}} \quad j = 1, \dots, d. \quad (3.25)$$

We observe that for $\nu = 1$ the above process reduces to a standard Brownian motion on \mathbb{R}^d , with var-cov matrix Q . Indeed, $\mathcal{M}_1(x) = e^x$ and thus the characteristic function in Equation (3.24) becomes that of Brownian motion. See [57, page 23] for details.

From Equation (3.24), we see that $\{W_t, t \geq 0\}$ is self-similar with Hurst index equal to $1/2$. We indeed recall that a process $\{Y_t, t \geq 0\}$ is said to be self-similar with Hurst index $H > 0$ if, for any $a > 0$, we have that $\{Y_{at}, t \geq 0\}$ has the same finite dimensional distributions of $\{a^H Y_t, t \geq 0\}$. Furthermore, the process W turns out to be a particular case of the randomly scaled Gaussian processes studied in [18, Sect. 3], as we shall see later.

Putting $n = 1$ in Equation (3.24), we have that W_t has characteristic function

$$\varphi_\nu^1(u) = \mathcal{M}_\nu \left(-\frac{1}{2^\nu} |u|^{2\nu} t^\nu \right) \quad \forall u \in \mathbb{R}^d \quad (3.26)$$

from which it can be seen that each component of W_t has infinite variance. Hence this process can be characterized as a super-diffusion in physical terms.

Remark 3.11. We observe that W_t has a density $q(x, t)$, which is the fundamental solution to the fractional equation

$$\frac{\partial^\nu}{\partial t^\nu} q(x, t) = -\frac{1}{2^\nu} (-\Delta)^\nu q(x, t). \quad (3.27)$$

In Equation (3.27), the operator $\frac{\partial^\nu}{\partial t^\nu}$ denotes the Caputo derivative, while $-(-\Delta)^\nu$ denotes the fractional Laplacian. The fractional Laplacian has Fourier symbol $-|u|^{2\nu}$. The Caputo derivative of equation (3.26) is

$$\frac{\partial^\nu}{\partial t^\nu} \varphi_\nu^1(u) = -\frac{1}{2^\nu} |u|^{2\nu} \varphi_\nu^1(u).$$

Fourier inversion gives equation (3.27). We will make this rigorous in the next section (see Remark 4.2).

In the following proposition we shall see that the Mittag-Leffler anomalous-diffusion process is equal in distribution to a randomly scaled Brownian motion.

Proposition 3.12. Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion in \mathbb{R}^d , $d \geq 1$, with covariance matrix given by Equation (3.25). Then $\{W_t\} \stackrel{fdd}{=} \{B_{\mathcal{L}t}\}$, where \mathcal{L} is a Lamperti random variable independent of B .

Proof. Let us consider a sequence of times $0 \leq t_1 < \dots < t_n$, $n \geq 1$. Let \mathcal{L} follow a Lamperti distribution of parameter $\nu \in (0, 1]$. Consider the vector $\tilde{\Gamma} := (B_{\mathcal{L}t_1}, \dots, B_{\mathcal{L}t_n})$, on some probability space, with random var-cov matrix given by $\mathcal{L}Q$. Then, using \mathbb{E} for the expectation on this probability space, one can write

$$\begin{aligned} \mathbb{E} \left[e^{i\langle u, \tilde{\Gamma} \rangle} \right] &= \mathbb{E} \left[e^{-\frac{1}{2} \langle u, \mathcal{L}Q u \rangle} \right] \\ &= \mathbb{E} \left[e^{-\frac{1}{2} \langle u, Qu \rangle \mathcal{L}} \right] = \mathcal{M}_\nu \left(-\left(\frac{1}{2} \langle u, Qu \rangle \right)^\nu \right) \quad \forall u \in \mathbb{R}^{nd}, \end{aligned}$$

where the matrix $Q \in \mathbb{R}^{nd \times nd}$ is a block diagonal matrix, and the j th block has the form as in Equation (3.25). The thesis follows. \square

As a consequence, if Q is positive definite, then the process Γ as in Definition 3.10 has density given by

$$f_{\Gamma}(x) = \int_0^{\infty} \frac{1}{(2\pi t^{nd} \det Q)^{nd/2}} e^{-\frac{1}{2t} \langle x, Q^{-1}x \rangle} \ell(dl).$$

In the following theorem we consider analogous assumptions to those considered in Proposition 2.3, which deals with the Markovian case. Specifically, they involve letting the speed c and the parameter λ tend to infinity in such a way that the ratio c^2/λ remains finite. However, as noticed in Section 2.3, the interpretation of λ differs in the present framework: indeed, for $\nu \in (0, 1)$, the parameter λ characterizes the intensity of the counting process as described by Equation (2.27), whereas for $\nu = 1$, corresponding to the Markovian case, λ also represents the expected number of direction changes within a unit time interval.

Theorem 3.13. *Let W be a Mittag-Leffler anomalous-diffusion process as in Definition 3.10. Let B be the corresponding Brownian motion, in the sense of Proposition 3.12. Then, under the scaling limit*

$$c \rightarrow \infty \quad \lambda \rightarrow \infty \quad \frac{c^2}{\lambda} = 1 \quad (3.28)$$

we have that

$$\{X_{(1)}^{\nu}(t)\} \xrightarrow{fdd} \{W_t\}.$$

Proof. Without loss of generality, we shall put $c^2 = \lambda$ and consider the one variable limit as $c \rightarrow \infty$. The proof is based on the representation in Theorem 3.1. First, we observe that

$$\mathbb{P}^{(x,\nu)} \left(X_{(1)}^{\nu}(t_1) \in A_1, \dots, X_{(1)}^{\nu}(t_n) \in A_n \right) = \int_0^{\infty} \mathbb{P}^{(x,\nu)} \left(X_{t_1}^{(c,\lambda)} \in A_1, \dots, X_{t_n}^{(c,\lambda)} \in A_n \right) \ell(dl)$$

where A_i are continuity sets, $i = 1, \dots, n$, $n \geq 1$. We now apply the diffusive limit (3.28) to both members of the above equation. By the dominated convergence theorem,

$$\begin{aligned} & \lim_{c \rightarrow \infty} \mathbb{P}^{(x,\nu)} \left(X_{(1)}^{\nu}(t_1) \in A_1, \dots, X_{(1)}^{\nu}(t_n) \in A_n \right) \\ &= \lim_{c \rightarrow \infty} \int_0^{\infty} \mathbb{P}^{(x,\nu)} \left(X_{t_1}^{(c,\lambda)} \in A_1, \dots, X_{t_n}^{(c,\lambda)} \in A_n \right) \ell(dl) \\ &= \int_0^{\infty} \left[\lim_{c \rightarrow \infty} \mathbb{P}^{(x,\nu)} \left(X_{t_1}^{(c,\lambda)} \in A_1, \dots, X_{t_n}^{(c,\lambda)} \in A_n \right) \right] \ell(dl) \\ &= \int_0^{\infty} \mathbb{P}^x \left(B_{\frac{t_1}{c}} \in A_1, \dots, B_{\frac{t_n}{c}} \in A_n \right) \ell(dl) \\ &= \mathbb{P}^x \left(B_{\frac{t_1}{c}} \in A_1, \dots, B_{\frac{t_n}{c}} \in A_n \right) \end{aligned}$$

where we used the Proposition 2.3 with $D = 1$. Using the property of the Lamperti distribution in Equation (2.19) we have

$$\mathbb{P}^x \left(B_{\frac{t_1}{\mathcal{L}}} \in A_1, \dots, B_{\frac{t_n}{\mathcal{L}}} \in A_n \right) = \mathbb{P}^x (B_{t_1 \mathcal{L}} \in A_1, \dots, B_{t_n \mathcal{L}} \in A_n).$$

Finally, by Proposition 3.12 we get

$$\lim_{c \rightarrow \infty} \mathbb{P}^{(x,v)} (X_{(1)}^\nu(t_1) \in A_1, \dots, X_{(1)}^\nu(t_n) \in A_n) = \mathbb{P}^x (W_{t_1} \in A_1, \dots, W_{t_n} \in A_n).$$

□

4. AVERAGING FELLER SEMIGROUPS

Both the anomalous diffusion defined in Section 3.4 and the para-Markov chains recalled in Section 2 are governed by equations of the form $\partial_t^\nu q = -(-G)^\nu q$. One can conjecture that this theory can be extended to more general cases. In particular, we might think that, whenever a non-Markovian process $X = \{X_t, t \geq 0\}$ is equal in distribution to $\{M_{\mathcal{L}t}, t \geq 0\}$, where $M = \{M_t, t \geq 0\}$ is a Markovian process and \mathcal{L} is a Lamperti random variable, then its governing equation is of the type $\partial_t^\nu q = -(-G)^\nu q$, where G is the infinitesimal generator of M . In this section we make this idea rigorous and we present the connection between random scaling of Markov processes and non-local operators. This connection, which is interesting in itself, will be used later in the paper.

Let us consider a Polish space (E, \mathcal{E}) endowed by its Borel σ -algebra. Let $(\Omega, \mathcal{F}, \mathbb{P}^x)$, $x \in E$, be a family of probability spaces equipped with a filtration $\{\mathcal{F}_t, t \geq 0\}$. Let $M = \{M_t, t \geq 0\}$ be an adapted Feller process, which takes value on (E, \mathcal{E}) , associated with the Feller semigroup of operators

$$\mathcal{T}_t h(x) = \mathbb{E}^x h(M_t)$$

on the Banach space $(C_0(E), \|\cdot\|)$, endowed with the sup-norm. Feller semigroups are strongly continuous, i.e. $\|\mathcal{T}_t h - h\| \rightarrow 0$ as $t \rightarrow 0$ (for more on Feller semigroups see [12]). It is known that the mapping $t \mapsto \mathcal{T}_t h$ is the unique solution to the abstract Cauchy problem

$$\frac{\partial}{\partial t} g(t) = Gg(t) \quad g(0) = h \quad h \in \text{Dom}(G)$$

where G is the infinitesimal generator of the semigroup.

We here consider the operators $\{\mathcal{Q}_t\}_{t \geq 0}$ defined through the Bochner integrals

$$\mathcal{Q}_t h = \int_0^\infty \mathcal{T}_{ty} h \ell(y) dy \tag{4.1}$$

where $\ell(\cdot)$ denotes a density of a Lamperti distribution as in Equation (2.18).

We now show that the mapping $t \mapsto \mathcal{Q}_t h$ is the unique solution of the following (non-local) abstract Cauchy problem

$$\frac{\partial^\nu}{\partial t^\nu} g(t) = -(-G)^\nu g(t) \quad g(0) = h \in \text{Dom}(G),$$

where the operators are defined as follows. The fractional derivative on the left-hand side is

$$\frac{\partial^\nu}{\partial t^\nu} g(t) := \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t (g(s) - g(0))(t-s)^{-\nu} ds \quad (4.2)$$

for any g such that the integral makes sense as a Bochner integral on $C_0(E)$ and the differentiation is possible in the strong sense. The operator defined in Equation (4.2) is the abstract analogue of the Caputo fractional derivative in Equation (2.20). In order to define the operator $-(-G)^\nu$ we use Bochner subordination technique, as follows. We define the family of operators $\{\mathcal{U}_t\}_{t \geq 0}$ such that

$$\mathcal{U}_t h := \int_0^{+\infty} \mathcal{T}_s h \mu_t(s) ds \quad (4.3)$$

where $\mu_t(\cdot)$ is the density of a stable subordinator $\{H(t), t \geq 0\}$ of stability index $\nu \in (0, 1)$, i.e., a strictly increasing Lévy process with Laplace transform

$$e^{-t\lambda^\nu} = \int_0^{+\infty} e^{-\lambda s} \mu_t(s) ds. \quad (4.4)$$

It is known that the family $\{\mathcal{U}_t\}_{t \geq 0}$ forms a strongly continuous semigroup on $C_0(E)$ [56, Proposition 13.1] and that it is generated by the operator $G^{(\nu)}$ such that [56, Theorem 13.6]

$$G^{(\nu)} h := \int_0^{+\infty} (\mathcal{T}_s h - h) \frac{\nu s^{-\nu-1}}{\Gamma(1-\nu)} ds, \quad h \in \text{Dom}(G), \quad (4.5)$$

a Bochner integral on $C_0(E)$. It is further true that (see [54])

$$G^{(\nu)}|_{\text{Dom}(G)} = -(-G)^\nu. \quad (4.6)$$

In the following, we shall denote by $\{L(t), t \geq 0\}$ the inverse process of H , i.e.

$$L(t) := \inf \{s \geq 0 : H(s) > t\} \quad t \geq 0. \quad (4.7)$$

The random variable $L(t)$ is known to be absolutely continuous for each $t > 0$ and belongs to the class of inverse subordinators, crucial processes in the context of anomalous diffusion related to fractional kinetic (see more on inverse subordinators in [4]).

Theorem 4.1. *Let $\{\mathcal{T}_t\}_{t \geq 0}$ be a Feller semigroup as above, generated by $(G, \text{Dom}(G))$. Consider the operators $\{\mathcal{Q}_t\}_{t \geq 0}$ defined in Equation (4.1) and the abstract Cauchy problem*

$$\frac{\partial^\nu}{\partial t^\nu} g(t) = -(-G)^\nu g(t) \quad g(0) = h, \quad (4.8)$$

where the operators appearing in Equation (4.8) are defined as in Equations (4.2) and (4.5). It is true that:

(1) $\{\mathcal{Q}_t\}_{t \geq 0}$ is a family of uniformly bounded linear operators on $C_0(E)$ such that

$$\mathcal{Q}_t : C_0(E) \mapsto C_0(E) \quad \mathcal{Q}_t : \text{Dom}(G) \mapsto \text{Dom}(G)$$

for any $t \geq 0$;

(2) For any $h \in C_0(E)$ the mapping $t \mapsto \mathcal{Q}_t h$ is strongly continuous, i.e., $\|\mathcal{Q}_t h - \mathcal{Q}_s h\| \rightarrow 0$, as $t \rightarrow s$ for any $s, t \geq 0$;

(3) For any $h \in \text{Dom}(G)$ the mapping $t \mapsto \mathcal{Q}_t h$ solves Equation (4.8).

(4) If $q(t)$ is a bounded (strongly) continuous solution on $C_0(E)$ to Equation (4.8) such that $t \mapsto -(-G)^\nu q(t)$ is bounded and (strongly) continuous, then $q(t) = \mathcal{Q}_t h$ for any $t \geq 0$.

Proof. For the proof the following representation for \mathcal{Q}_t :

$$\mathcal{Q}_t h = \int_0^{+\infty} \mathcal{U}_s h \lambda_t(s) ds \quad (4.9)$$

is used, where $\lambda_t(\cdot)$ is the density of the inverse stable subordinator defined in Equation (4.7) and \mathcal{U}_s is defined in Equation (4.3). In order to derive representation (4.9) we use the following arguments. Consider $H_i := \{H_i(t), t \geq 0\}$, $i = 1, 2$, two i.i.d. stable subordinators, having Laplace transform as in Equation (4.4) and let us use the notation $\{L_2(t), t \geq 0\}$ for the inverse process of H_2 . Note that

$$\mathbb{E} e^{-\eta H_1(L_2(t))} = \mathbb{E} e^{-\eta^\nu L_2(t)} = \mathcal{M}_\nu(-t^\nu \eta^\nu) \quad \eta \geq 0,$$

where we used the Laplace transform in Equation (4.4) and the fact that (see Remark 3.1 in [41])

$$\int_0^{+\infty} e^{-\gamma s} \lambda_t(ds) = \mathcal{M}_\nu(-\gamma t^\nu), \quad \gamma \geq 0. \quad (4.10)$$

We further observe that the Definition 2.5 of Lamperti distribution gives

$$\mathbb{E} e^{-\eta t \mathcal{L}} = \mathcal{M}_\nu(-t^\nu \eta^\nu), \quad \eta \geq 0,$$

so that

$$t \mathcal{L} \stackrel{d}{=} H_1(L_2(t)). \quad (4.11)$$

We now use the notation $g_t(\cdot)$ for the density of $H_1(L_2(t))$, i.e.

$$g_t(z) := \int_0^{+\infty} \mu_s(z) \lambda_t(s) ds \quad z > 0. \quad (4.12)$$

It follows that

$$\begin{aligned}
\mathcal{Q}_t h &= \int_0^\infty \mathcal{T}_{ty} h \ell(y) dy \\
&= \int_0^\infty \mathcal{T}_z h g_t(z) dz \\
&= \int_0^\infty \mathcal{U}_s h \lambda_t(s) ds
\end{aligned} \tag{4.13}$$

where in the last step we used Equation (4.12), the definition of \mathcal{U}_t in Equation (4.3) and the Fubini theorem for Bochner integrals (see [3, Theorem 1.1.9]).

We are now ready to prove the Theorem.

- (1) The linearity simply follows from the definition (4.1). Similarly, the fact that $\mathcal{Q}_t h \in C_0(E)$ for each $h \in C_0(E)$ and $t \geq 0$ comes from the definition of Bochner integral provided that the integral exists. This can be seen by using [3, Theorem 1.1.4] that applies here since $\|\mathcal{T}_{ty} h\| \leq \|h\|$. The same argument leads to contractivity, indeed,

$$\begin{aligned}
\|\mathcal{Q}_t h\| &= \left\| \int_0^{+\infty} \mathcal{T}_{ty} h \ell(dy) \right\| \\
&\leq \int_0^{+\infty} \|\mathcal{T}_{ty} h\| \ell(dy) \\
&\leq \|h\|,
\end{aligned} \tag{4.14}$$

which implies that $\{\mathcal{Q}_t\}_{t \geq 0}$ is uniformly bounded. So, the operators $\{\mathcal{Q}_t\}$ are also well-defined and contractive.

We have that $\{\mathcal{T}_t\}_{t \geq 0}$ is a Feller semigroup, therefore we can apply [12, Lemma 4.5] to conclude that the subordinate semigroup $\{\mathcal{U}_t\}_{t \geq 0}$ is also a Feller semigroup. It follows that \mathcal{U}_t maps $C_0(E)$ and $\text{Dom}(G)$ into themselves and it is strongly continuous.

We now prove that $\mathcal{Q}_t h \in \text{Dom}(G)$ if $h \in \text{Dom}(G)$. We shall use that $\text{Dom}(G) \subseteq \text{Dom}(G^{(\nu)})$ and that $G^{(\nu)} \mathcal{U}_y h = \mathcal{U}_y G^{(\nu)} h$ for $h \in \text{Dom}(G)$. To get this result, it is sufficient to check the assumptions of [3, Proposition 1.1.7]. Specifically, one has that $\mathcal{Q}_t h = \int_0^\infty \mathcal{U}_s h \lambda_t(s) ds$ is well defined, i.e.

$$s \mapsto \mathcal{U}_s h \lambda_t(s)$$

is Bochner integrable for any $t > 0$. Moreover, $\mathcal{U}_s h \lambda_t(s)$ falls in the domain of $G^{(\nu)}$ because $G^{(\nu)}$ is the generator of \mathcal{U}_s . Furthermore, we know that $G^{(\nu)} \mathcal{U}_s h \lambda_t(s) = \mathcal{U}_s (G^{(\nu)} h) \lambda_t(s)$ and $\mathcal{U}_s (G^{(\nu)} h) \lambda_t(s)$ is Bochner integrable so that

$$s \mapsto G^{(\nu)} \mathcal{U}_s h \lambda_t(s)$$

is Bochner integrable as well, and the first item of the thesis follows.

- (2) Now we prove strong continuity. Again, the subordinate semigroup $\{\mathcal{U}_t\}_{t \geq 0}$ is a Feller semigroup and it is strongly continuous. Furthermore, since $L_2(t)$ is the inverse of a strictly increasing subordinator, then it is path-continuous and then $\{\mathcal{U}_{L_2(t)}\}_{t \geq 0}$ is strongly continuous.

By the representation (4.9) we have that

$$\mathcal{Q}_t h = \mathbb{E} \mathcal{U}_{L_2(t)} h,$$

which gives

$$\|\mathcal{Q}_{t_2} h - \mathcal{Q}_{t_1} h\| \leq \mathbb{E} \|\mathcal{U}_{L_2(t_2)} h - \mathcal{U}_{L_2(t_1)} h\| \quad t_1, t_2 \geq 0.$$

Consider the limit $t_1 \rightarrow t_2$. Using $\|\mathcal{U}_{L_2(t)} h\| \leq \|h\|$ to apply the dominated convergence for Bochner integrals ([3, Theorem 1.1.8]), we get strong continuity.

- (3) As a consequence of [5, Theorem 3.1] a Feller semigroup $\{\mathcal{U}_t\}_{t \geq 0}$ generated by $G^{(\nu)}|_{\text{Dom}(G)} = -(-G)^\nu$, integrated against the distribution of an inverse stable subordinator defines a mapping that solves Equation (4.8). Consistently, from Items (1) and (2) above, the functions

$$t \mapsto \frac{\partial^\nu}{\partial t^\nu} \mathcal{Q}_t h \quad \text{and} \quad t \mapsto -(-G)^\nu \mathcal{Q}_t h$$

are well defined in the strong sense.

- (4) First, we prove that $t \mapsto G^{(\nu)} \mathcal{Q}_t h$ is bounded and (strongly) continuous for each $h \in \text{Dom}(G)$. Indeed,

$$G^{(\nu)} \mathcal{Q}_t h = \int_0^{+\infty} \mathcal{U}_s G^{(\nu)} h \lambda_t(s) ds = \mathcal{Q}_t G^{(\nu)} h \quad (4.15)$$

by the same arguments used in the proof of Item (1) and the strong continuity follows from the strong continuity and boundedness of $t \mapsto \mathcal{Q}_t G^{(\nu)} h$, since $G^{(\nu)} h \in C_0(E)$.

Uniqueness follows from a Laplace transform argument. Note that $v(t) = q(t) - \mathcal{Q}_t h$ solves the problem (4.8) with $v(0) = 0$. We can take the Laplace transform, $t \mapsto s$, of Equation (4.8), which is well defined by the boundedness of both sides.

Denoting by \tilde{v} the Laplace transform of v , one gets

$$s^\alpha \tilde{v}(s) = G^{(\nu)} \tilde{v}(s) \iff (s^\alpha - G^{(\nu)}) \tilde{v}(s) = 0. \quad (4.16)$$

Since the resolvent $(s^\alpha - G^{(\nu)})^{-1}$ exists and is bounded, we have $\tilde{v}(s) = 0$ for any $s \geq 0$ being $\tilde{v}(s)$ continuous.

□

In the next section we shall apply the theory of averaging Feller semigroups to transport processes. Before doing that, in the following remark we clarify the statement of Remark 3.11.

Remark 4.2. From Proposition 3.12 and the previous Theorem, the process W_t is such that the function $q(x, t) = \mathbb{E}^x u(W_t)$ solves a fractional kinetic equation of the form of Equation (4.8). In this case, the corresponding Feller process is a Brownian motion whose generator is such that $G|_{C_0^2(\mathbb{R}^d)} = \Delta$, i.e., the Laplace operator. Therefore (4.8) reduces to a fractional diffusion equation in both time and space. It follows that

$$q(x, t) = \int_{\mathbb{R}^d} u(y) p_t(y - x) dy, \quad (4.17)$$

where $p_t(y - x)$ is a density for the random variable W_t , satisfies Equation (4.8) with $G^\nu|_{C_0^2(\mathbb{R}^d)} = -(-\Delta)^\nu$. Furthermore, $p_t(y - x) dy \rightarrow \delta_x(dy)$ weakly as $t \rightarrow 0$ and thus we can conclude that $p_t(y - x)$ satisfies the equation in the sense of fundamental solutions.

5. MOTION AMONG RANDOM OBSTACLES (MODEL 2)

5.1. Transport with infinite mean flight times and random speed. We here consider another non-Markovian, isotropic transport process, which differs from Model 1 because the speed of the particle is not fixed anymore. Specifically, we here assume that the speed is a random variable which is stochastically dependent on the random flight times. Again, the random number of direction changes up to time t is \mathcal{N}_t^ν , i.e. the exchangeable fractional Poisson process defined in Section 2.3. In more detail, the model is defined by the following assumptions.

(A2). Suppose that in Definition 2.1 the vectors $(\mathcal{C}, J_1^{(\nu)}, \dots, J_n^{(\nu)})$, $n \geq 1$, under all the probability measures $\mathbb{P}^{(x, \nu)}$, have density given by

$$\mathbb{P}^{(x, \nu)} \left[\mathcal{C} \in d\tau, J_1^{(\nu)} \in dx_1, \dots, J_n^{(\nu)} \in dx_n \right] = \left(\lambda \frac{\tau}{c} \right)^n e^{-\lambda \frac{\tau}{c} \sum_{i=1}^n x_i} \ell \left(\frac{\tau}{c} \right) \frac{1}{c} d\tau dx_1 \dots dx_n \quad (5.1)$$

where $\ell(\cdot)$ denotes the Lamperti distribution as in Equation (2.18), which is equivalent to say that

$$\left(\mathcal{C}, J_1^{(\nu)}, \dots, J_n^{(\nu)} \right) \stackrel{d}{=} \left(c\mathcal{L}, \frac{1}{\mathcal{L}} J_1, \dots, \frac{1}{\mathcal{L}} J_n \right) \quad (5.2)$$

where $\{J_n, n \geq 1\}$ are i.i.d. exponential variables of mean $1/\lambda$.

This assumption has the following implications:

- (1) The joint distribution of the flight times $\{J_n^{(\nu)}, n \geq 1\}$ is given by Equation (2.23) for some $\nu \in (0, 1]$.
- (2) The speed is $\mathcal{C} = c\mathcal{L}$ where c is a positive constant and \mathcal{L} follows a Lamperti distribution of parameter ν , as in Definition 2.5.
- (3) The speed increases with \mathcal{L} but at the same time each flight time decreases, such that the free path remains unchanged.

We shall denote the random flight process of Assumptions **(A2)** by

$$(X_{(2)}^\nu(t), V_{(2)}^\nu(t)), \quad \nu \in (0, 1].$$

In this setting, the flight times $\{J_n^{(\nu)}, n \geq 1\}$ are dependent and they all have the same Mittag-Leffler distribution, so that $\mathbb{E}J_n^{(\nu)} = \infty$ for each $n \geq 1$. Note that for $\nu = 1$ one re-obtains the isotropic Markovian random flight of Section 2.1.1.

The process is non-Markovian but, as shown in the following theorem, it is equal in distribution to a mixture of Markovian random flights of the type defined in Section 2.1.1. In the same theorem we also determine the governing equation of $(X_{(2)}^\nu(t), V_{(2)}^\nu(t))$ which involves fractional power of the operators appearing in the linear Boltzmann equation (2.6).

We will use the family of operators $\{\tilde{\mathcal{Q}}_t\}_{t \geq 0}$ defined by

$$\tilde{\mathcal{Q}}_t h(x, v) := \mathbb{E}^{(x, v)} [h(X_{(2)}^\nu(t), V_{(2)}^\nu(t))] \quad h \in C_0(\mathbb{R}^d \times S^{d-1}). \quad (5.3)$$

Specifically, we will consider the Boltzmann semigroup $\{\mathcal{T}_t^{(c, \lambda)}\}_{t \geq 0}$ as defined in Equation (2.3) with representation as in Theorem 2.2. Then, by applying the Phillips formula (4.5) to it, we are able to properly define the fractional power of the Boltzmann generator (2.7) as

$$-\left(-cv \cdot \nabla - \lambda(L - \mathcal{I})\right)^\nu h := \int_0^\infty (\mathcal{T}_s^{(c, \lambda)} h - h) \frac{\nu s^{-\nu-1}}{\Gamma(1-\nu)} ds, \quad h \in D, \quad (5.4)$$

where D is the domain of the Boltzmann generator as in Equation (2.4).

Theorem 5.1. *Let $(X_t^{(c, \lambda)}, V_t^{(c, \lambda)})$ be the Markovian random flight defined in Section 2.1.1 and let \mathcal{L} be a Lamperti random variable. Then*

$$(X_{(2)}^\nu(t), V_{(2)}^\nu(t)) \stackrel{d}{=} (X_{\mathcal{L}t}^{(c, \lambda)}, V_{\mathcal{L}t}^{(c, \lambda)}). \quad (5.5)$$

Moreover, for $h \in D$, the function $t \mapsto \tilde{\mathcal{Q}}_t h$ defined in Equation (5.3) is the unique solution of the (abstract) fractional Boltzmann equation

$$\frac{\partial^\nu}{\partial t^\nu} g(t) = -\left(-cv \cdot \nabla - \lambda(L - \mathcal{I})\right)^\nu g(t)$$

subject to $g(0) = h$, in the sense of Theorem 4.1, Item (4).

Proof. Under the Assumptions **(A2)**, the isotropic transport process in Definition 2.1 reads

$$\begin{pmatrix} V_{(2)}^\nu(t) \\ X_{(2)}^\nu(t) \end{pmatrix} = \begin{pmatrix} v_m \\ x + \sum_{k=1}^m \mathcal{C}v_{k-1} J_k^{(\nu)} + \mathcal{C}v_m \left(t - \sum_{k=1}^m J_k^{(\nu)}\right) \end{pmatrix} \quad \sum_{k=1}^m J_k^{(\nu)} \leq t < \sum_{k=1}^{m+1} J_k^{(\nu)}$$

for $m \geq 0$, which can compactly be re-written as

$$\begin{pmatrix} V_{(2)}^\nu(t) \\ X_{(2)}^\nu(t) \end{pmatrix} = \sum_{m=0}^{+\infty} \begin{pmatrix} v_m \\ x + \sum_{k=1}^m \mathcal{C}v_{k-1}J_k^{(\nu)} + \mathcal{C}v_m \left(t - \sum_{k=1}^m J_k^{(\nu)} \right) \end{pmatrix} \mathbb{1}_{\{\sum_{k=1}^m J_k^{(\nu)} \leq t < \sum_{k=1}^{m+1} J_k^{(\nu)}\}}.$$

From Equation (5.2) we have, for each $t \geq 0$,

$$\begin{pmatrix} V_{(2)}^\nu(t) \\ X_{(2)}^\nu(t) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} v_m \\ x + \sum_{k=1}^m \mathcal{C}v_{k-1}J_k + \mathcal{C}v_m (\mathcal{L}t - \sum_{k=1}^m J_k) \end{pmatrix} \quad \sum_{k=1}^m J_k \leq \mathcal{L}t < \sum_{k=1}^{m+1} J_k$$

and the equality in distribution of Equation (5.5) follows.

Hence we can re-write (5.3) as

$$\begin{aligned} \tilde{\mathcal{Q}}_t h(x, v) &= \int_0^\infty \mathbb{E}^{(x, v)} \left[h \left(X_{lt}^{(c, \lambda)}, V_{lt}^{(c, \lambda)} \right) \right] \ell(dl) \\ &= \int_0^\infty \mathcal{T}_{lt}^{(c, \lambda)} h(x, v) \ell(dl) \end{aligned}$$

where $\mathcal{T}_t^{(c, \lambda)}$ is the Markovian semigroup defined in Equation (2.3). We recall that the family $\left\{ \mathcal{T}_t^{(c, \lambda)} \right\}_{t \geq 0}$ is a Feller semigroup, whose generator (2.7) has domain D , hence $\tilde{\mathcal{Q}}_t$ is an averaged Feller semigroup of the form of Equation (4.1). We fall under the assumptions of Theorem 4.1 from which the result immediately follows. \square

Remark 5.2. *By using analogous arguments as in the above proof, recalling Equation (5.2) and observing that each $\frac{1}{\mathcal{L}}J_i$ follows an exponential distribution of random mean $1/(\lambda\mathcal{L})$, we also get*

$$\left(X_{(2)}^\nu(t), V_{(2)}^\nu(t) \right) \stackrel{d}{=} \left(X_t^{(c\mathcal{L}, \lambda\mathcal{L})}, V_t^{(c\mathcal{L}, \lambda\mathcal{L})} \right).$$

The dependence of Equation (5.1) guarantees that each flight time is inversely proportional to the speed; therefore, the length of each free path remains constant under the scaling with the Lamperti variable.

Remark 5.3. *Under Assumptions (A1), the random variable $X_{(1)}^\nu(t)$ is supported on a ball centered at the starting position x of radius ct , and thus the mean square displacement, computed in Proposition 3.2, is finite. Instead, in the case of Assumptions (A2) the speed is random with a distribution supported on $(0 + \infty)$, and thus the radius of the ball can assume any positive value. It follows that the random position $X_{(2)}^\nu(t)$ is not supported anymore on a set with finite measure. Furthermore, from the mean squared displacement of the Markovian random flight in Equation (2.11) and using the representation of the process in Theorem 5.1, it is not hard to see that the random flight is superdiffusive with infinite mean square displacement.*

5.1.1. *Other types of fractional Boltzmann equations.* In some recent papers, other fractional Boltzmann equations have been studied. In [52], the authors considered a semi-Markov transport process. Here, the flight times are i.i.d. Mittag-Leffler random variables, i.e. the counting process is a renewal process, also known as fractional Poisson process, see [38]. Such a transport process is governed by the fractional Boltzmann equation

$$\left(\frac{\partial}{\partial t} - cv \cdot \nabla_x\right)^\nu g(x, v, t) - \frac{t^{-\nu}}{\Gamma(1-\nu)} h(x + cvt, v) = \lambda(L - \mathcal{I})g(x, v, t)$$

under the condition $g(x, v, 0) = h(x, v)$. The authors also proved that a suitable scaling limit leads to an anomalous diffusive process with continuous trajectories and finite velocity, say $\{\widetilde{W}_t, t \geq 0\}$, having a super-diffusive behavior, with the mean-squared displacement growing as $\mathbb{E} \left| \widetilde{W}_t \right|_e^2 \sim Ct^2, C > 0$.

Another fractional Boltzmann equation with the form

$$\frac{\partial^\nu}{\partial t^\nu} g(x, v, t) = cv \cdot \nabla_x g(x, v, t) + \lambda(L - \mathcal{I})g(x, v, t) \quad (5.6)$$

has been considered in [2]. In this case, the transport process is constructed as the Markovian random flight, time-changed with the inverse of a ν -stable subordinator.

5.2. Boltzmann-Grad approximation and anomalous diffusion. We here show that the same anomalous diffusion as in Section 3.4, i.e., the Mittag-Leffler anomalous-diffusion process, can approximate a different Lorentz model. We indeed consider a Lorentz process with a Poissonian distribution of obstacle centres, defined as in Gallavotti [23], except for the fact that the speed of the particle is here assumed to be random. Under a Boltzmann-Grad procedure, this process is proved to converge to the non-Markovian random flight defined as in Model 2. A further scaling limit leads to the Mittag-Leffler anomalous-diffusion process.

The speed of such Lorentz process is assumed to follow a Lamperti distribution as in Definition 2.5, which is characterized by a power law decaying density. From a physical point of view this means that the gas particles are not in thermal equilibrium, in which case the speed has an exponentially decaying distribution. Indeed, it is known that a perfect gas in thermal equilibrium has a Maxwell-Boltzmann distribution for the particle velocities.

To fix the notation, consider a Poisson point process with intensity $\rho \in (0, \infty)$ for the obstacle centres. All the obstacles are spheres of radius R . We shall indicate with $(\mathcal{X}_t^c, \mathcal{V}_t^c)$ the Lorentz process as in Gallavotti [23]. The particle speed is given by the constant $c > 0$, as indicated by the superscript. It is known that under the Boltzmann Grad limit

$$\rho \rightarrow \infty \quad R \rightarrow 0 \quad \rho c \pi R^2 \rightarrow \lambda \in (0, \infty), \quad (5.7)$$

we have that

$$(\mathcal{X}_t^c, \mathcal{V}_t^c) \xrightarrow{d} (X_t^{(c,\lambda)}, V_t^{(c,\lambda)}) \quad (5.8)$$

where $(X^{(c,\lambda)}, V^{(c,\lambda)})$ is the Markovian random flight recalled in Section 2.1.1.

Now, consider another Lorentz process defined analogously as by Gallavotti [23], where the particle speed is given by $\mathcal{C} = c\mathcal{L}$, being \mathcal{L} a Lamperti random variable. It can be proved that the free-flight time distribution has infinite mean analogously to what happens in Lemma 3.6. For this new process, let us denote the position of the particle at time t by $\mathcal{X}_t^{(2)}$ and the unit velocity vector at time t , i.e. the direction of the particle at time t , by $\mathcal{V}_t^{(2)}$. We call $(\mathcal{X}^{(2)}, \mathcal{V}^{(2)}) := \left\{ \left(\mathcal{X}_t^{(2)}, \mathcal{V}_t^{(2)} \right), t \geq 0 \right\}$ this Lorentz process in $\mathbb{R}^3 \times S^2$, with initial data $(\mathcal{X}_0^{(2)}, \mathcal{V}_0^{(2)})$. The superscript indicates that this is the second Lorentz model considered in this paper. As in Section 3.3 we denote $P^{(x,v)}(\cdot) := P(\cdot | \mathcal{X}_0^{(2)} = x, \mathcal{V}_0^{(2)} = v)$. Under any $P^{(x,v)}$, we can write

$$\left(\mathcal{X}_t^{(2)}, \mathcal{V}_t^{(2)} \right) \stackrel{d}{=} \left(\mathcal{X}_t^{c\mathcal{L}}, \mathcal{V}_t^{c\mathcal{L}} \right), \quad \forall t \geq 0. \quad (5.9)$$

This Lorentz process also depends on the parameters ρ and ν , and on the size of the obstacles R . Note that here the dependence on ν is only due to the random speed of the particle, being the random field Poissonian. As in the previous section, to simplify the notation, we will not make this dependence explicit. Once again, for $\nu = 1$, the speed is constant since $\mathcal{L} = 1$ almost surely, and such Lorentz process coincides with that of Gallavotti's model.

Theorem 5.4. *Consider the Lorentz process $(\mathcal{X}^{(2)}, \mathcal{V}^{(2)})$ defined above. Under the Boltzmann-Grad limit 5.7 we have*

$$P^{(x,v)} \left(\mathcal{X}_t^{(2)} \in dx', \mathcal{V}_t^{(2)} \in dv' \right) \rightarrow \mathbb{P}^{(x,v)} \left(X_{(2)}^\nu(t) \in dx', V_{(2)}^\nu(t) \in dv' \right)$$

weakly, for all $t > 0$ and for all $(x, v) \in \mathbb{R}^3 \times S^2$, where the limit process is the random flight defined in Section 5.1.

Proof. Let us write

$$E^{(x,v)} h \left(\mathcal{X}_t^{(2)}, \mathcal{V}_t^{(2)} \right) = \int_{S^2} \int_{B_{ct}(x)} h(x', v') P^{(x,v)} \left(\mathcal{X}_t^{(2)} \in dx', \mathcal{V}_t^{(2)} \in dv' \right).$$

where $B_{ct}(x)$ indicates the ball centered in x of radius ct . To get the statement, we shall prove that

$$\lim_{R \rightarrow 0} E^{(x,v)} h \left(\mathcal{X}_t^{(2)}, \mathcal{V}_t^{(2)} \right) = E^{(x,v)} h \left(X_t^{(c\mathcal{L}, \lambda\mathcal{L})}, V_t^{(c\mathcal{L}, \lambda\mathcal{L})} \right) \quad \forall h \in C_b(\mathbb{R}^3 \times S^2), \quad (5.10)$$

where $(X^{(c,\lambda)}, V^{(c,\lambda)})$ is the Markovian random flight recalled in Section 2.1.1 and the thesis will follow from the representation of the second random flight as in Remark 5.2.

Specifically, we have from Equation (5.9)

$$\lim_{R \rightarrow 0} E^{(x,v)} h \left(\mathcal{X}_t^{(2)}, \mathcal{V}_t^{(2)} \right) = \lim_{R \rightarrow 0} E^{(x,v)} h \left(\mathcal{X}_t^{c\mathcal{L}}, \mathcal{V}_t^{c\mathcal{L}} \right)$$

$$= \lim_{R \rightarrow 0} \int_0^\infty \mathbb{E}^{(x,v)} h(\mathcal{X}_t^{lc}, \mathcal{V}_t^{lc}) P(\mathcal{L} \in dl).$$

Now we observe that h is bounded and so is the expected value. Using the dominated convergence theorem in the previous Equation then yields

$$\int_0^\infty \left(\lim_{R \rightarrow 0} \mathbb{E}^{(x,v)} h(\mathcal{X}_t^{cl}, \mathcal{V}_t^{cl}) \right) P(\mathcal{L} \in dl) = \int_0^\infty \mathbb{E}^{(x,v)} h(X_t^{(cl,\lambda)}, V_t^{(cl,\lambda)}) P(\mathcal{L} \in dl)$$

where we used Equations (5.7) and (5.8), i.e. the same convergence theorem of Gallavotti's model, where the speed c is replaced by cl and hence λ is replaced by λl . The equality (5.10) immediately follows. \square

In the following theorem we prove that under an appropriate scaling limit the random flight of model 2 converges to the same anomalous diffusion as in Section 3.4.

Theorem 5.5. *Let W be a Mittag-Leffler anomalous-diffusion process as in Definition 3.10. Let B be the corresponding Brownian motion, in the sense of Proposition 3.12. Then, under the scaling limit*

$$c \rightarrow \infty \quad \lambda \rightarrow \infty \quad \frac{c^2}{\lambda} = 1 \quad (5.11)$$

we have that

$$\{X_{(2)}^\nu(t)\} \xrightarrow{fdd} \{W_t\}.$$

Proof. Without loss of generality, we shall put $c^2 = \lambda$ and consider the one-variable limit as $c \rightarrow \infty$. The proof is based on the representation in Theorem 5.1. Firstly, we observe that

$$\mathbb{P}^{(x,v)}(X_{(2)}^\nu(t_1) \in A_1, \dots, X_{(2)}^\nu(t_n) \in A_n) = \int_0^\infty \mathbb{P}^{(x,v)}(X_{lt_1}^{(c,\lambda)} \in A_1, \dots, X_{lt_n}^{(c,\lambda)} \in A_n) \ell(dl)$$

where A_i are continuity sets, $i = 1, \dots, n$, $n \geq 1$. We now apply the diffusive limit (5.11) to both members of the above equation. By the dominated convergence theorem,

$$\begin{aligned} & \lim_{c \rightarrow \infty} \mathbb{P}^{(x,v)}(X_{(2)}^\nu(t_1) \in A_1, \dots, X_{(2)}^\nu(t_n) \in A_n) \\ &= \lim_{c \rightarrow \infty} \int_0^\infty \mathbb{P}^{(x,v)}(X_{lt_1}^{(c,\lambda)} \in A_1, \dots, X_{lt_n}^{(c,\lambda)} \in A_n) \ell(dl) \\ &= \int_0^\infty \left[\lim_{c \rightarrow \infty} \mathbb{P}^{(x,v)}(X_{lt_1}^{(c,\lambda)} \in A_1, \dots, X_{lt_n}^{(c,\lambda)} \in A_n) \right] \ell(dl) \\ &= \mathbb{P}^x(B_{t_1\mathcal{L}} \in A_1, \dots, B_{t_n\mathcal{L}} \in A_n) \end{aligned}$$

where we used Proposition 2.3 with $D = 1$. Finally, by Proposition 3.12 we get

$$\lim_{c \rightarrow \infty} \mathbb{P}^{(x,v)}(X_{(2)}^\nu(t_1) \in A_1, \dots, X_{(2)}^\nu(t_n) \in A_n) = \mathbb{P}^x(W_{t_1} \in A_1, \dots, W_{t_n} \in A_n).$$

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APPENDIX A. EXISTENCE OF MITTAG-LEFFLER POINT PROCESS

As a first step, we demonstrate the existence of Π on finite Borel subsets of \mathbb{R}^3 .

Lemma A.1. *Let S be a Borel set of \mathbb{R}^3 with finite Lebesgue measure $|\cdot|: |S| < +\infty$. Then define*

$$\Pi_S := \sum_{k=1}^{\mathcal{N}^\nu} \delta_{C_k} \quad (\text{A.1})$$

where $\{C_k\}_{k \in \{1, \dots, \mathcal{N}^\nu\}}$ is a sequence of i.i.d. r.v.'s uniform on S and independent of \mathcal{N}^ν which, in its turn, is a r.v. with a fractional Poisson distribution

$$P(\mathcal{N}^\nu = n) = \frac{1}{n!} (\rho|S|)^n (-1)^n \mathcal{M}_\nu^{(n)}(-\rho^\nu|S|^\nu), \quad (\text{A.2})$$

where $\rho \in \mathbb{R}^+$. Then Π_S is a measure as in Definition 3.3 with $\mu(\cdot) = |\cdot \cap S|/|S|$.

Proof. We show that Π_S defined as above is a random measure that satisfies the Definition 3.3 for $\mu = |\cdot \cap S|/|S|$. Hence, denoting by Π'_S a random measure as in Definition 3.3, we use [29, Corollary 10.1] and thus we show that

$$Ee^{-\Pi_S f} = Ee^{-\Pi'_S f} \quad (\text{A.3})$$

for any f positive, continuous with compact support. We have that

$$\begin{aligned} Ee^{-\Pi_S f} &= Ee^{-\sum_{k=1}^{\mathcal{N}^\nu} f(C_k)} = E \left(Ee^{-f(C_1)} \right)^{\mathcal{N}^\nu} \\ &= \sum_m \left(\frac{1}{|S|} \int_S e^{-f(s)} ds \right)^m P(\mathcal{N}^\nu = m) \end{aligned}$$

$$\begin{aligned}
&= \sum_m \left(\frac{1}{|S|} \int_S e^{-f(s)} ds \right)^m \frac{1}{m!} (|S|\rho)^m (-1)^m \mathcal{M}_\nu^{(m)}(-(|S|\rho)^\nu) \\
&= \int_0^{+\infty} \sum_m \left(\frac{1}{|S|} \int_S e^{-f(s)} ds \right)^m \frac{l^m (\rho|S|)^m}{m!} e^{-l\rho|S|} \mathcal{L}(dl) \\
&= \int_0^{+\infty} e^{-l|S|\rho \frac{1}{|S|} \int_S (1-e^{-f(s)}) ds} \mathcal{L}(dl) \\
&= \mathcal{M}_\nu \left(-\rho^\nu \left(\int_S (1-e^{-f(s)}) ds \right)^\nu \right), \tag{A.4}
\end{aligned}$$

where l is the realisation of a Lamperti random variable \mathcal{L} of parameter ν . Now, take a simple function $f = \sum_j a_j \mathbb{1}_{A_j}$. Then we have that

$$\begin{aligned}
Ee^{-\Pi'_S f} &= Ee^{-\sum_j a_j \Pi'_S \mathbb{1}_{A_j}} = E \prod_j e^{-a_j \Pi'_S \mathbb{1}_{A_j}} = \\
&\sum_{k_1 \dots k_j} \prod_j \frac{1}{k_1! \dots k_j!} e^{-a_j k_j} \rho^{k_j} (|A_j \cap S|)^{k_j} (-1)^{\sum_j k_j} \mathcal{M}_\nu^{(\sum_j k_j)} \left(-\rho^\nu \left(\sum_j (|A_j \cap S|)^{k_j} \right)^\nu \right) \\
&= \sum_{k_1 \dots k_j} \prod_j \frac{1}{k_1! \dots k_j!} e^{-a_j k_j} \rho^{k_j} (|A_j \cap S|)^{k_j} \int_0^{+\infty} e^{-\rho \sum_j (|A_j \cap S|)^{k_j} s} s^{\sum_j k_j} (-1)^{\sum_j k_j} \mathcal{L}(ds) \\
&= \int_0^{+\infty} e^{-s\rho \sum_j |A_j \cap S| (1-e^{-a_j})} \mathcal{L}(ds) = \mathcal{M}_\nu \left(-\rho^\nu \left(\sum_j |A_j \cap S| (1-e^{-a_j}) \right)^\nu \right) \\
&= \mathcal{M}_\nu \left(-\rho^\nu \left(\int_S (1-e^{-f(s)}) ds \right)^\nu \right). \tag{A.5}
\end{aligned}$$

Pick now f such that $f_k \uparrow f$ for f_k simple. Then, by monotone convergence we get that $\Pi'_S f_k \uparrow \Pi'_S f$ as well as

$$\int_S (1-e^{-f_k(s)}) ds \uparrow \int_S (1-e^{-f(s)}) ds. \tag{A.6}$$

Hence, by dominated convergence we obtain $Ee^{-\Pi'_S f} = Ee^{-\Pi_S f}$. \square

It turns out that Π is a Cox process in the sense of [29, Page 180]. Take a random measure on $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$, say Ξ . A point process Π on \mathbb{R}^3 is said to be a Cox process directed by Ξ if it is conditionally Poisson with the intensity Ξ , i.e., $\mathbb{E}(\Pi \mid \Xi) = \Xi$ a.s..

Lemma A.2. *Define the measure on $\mathcal{B}(\mathbb{R}^3)$ as $\Xi(\cdot) := \mathcal{L}\rho|\cdot|$ where \mathcal{L} is a Lamperti random variable and $\rho \in \mathbb{R}^+$. A random measure on $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3), |\cdot|)$ as in Definition 3.3 on \mathbb{R}^3 is a Cox point process directed by Ξ . The restriction of Π on a Borel set B with $|B| < +\infty$ is*

a random measure of the same type on $S = B$ and thus it has the representation given in Lemma A.1.

Proof. Denote by \mathcal{P}^ξ a Poisson random measure with the intensity $\xi(\cdot)$, for a measure $\xi(\cdot)$ on $\mathcal{B}(\mathbb{R}^3)$. Use [29, Corollary 10.1] to see that Π has the same distribution as \mathcal{P}^Ξ where Ξ is independent of \mathcal{P}^ξ . Indeed note that, for any positive f continuous with compact support, we have that

$$Ee^{-\Pi f} = Ee^{-\mathcal{P}^\xi f} = \int_0^{+\infty} Ee^{-\mathcal{P}^s f} \ell(ds). \quad (\text{A.7})$$

Observe that \mathcal{P}^s is, for any $s > 0$ fixed, a Poisson random measure with intensity $\rho|\cdot|$. Then

$$Ee^{-\Pi f} = \int_0^\infty e^{-\rho \int_{\mathbb{R}^3} (1-e^{-f(s)}) ds} \ell(dl) = \mathcal{M}_\nu \left(-\rho^\nu \int_{\mathbb{R}^3} (1-e^{-f(s)}) ds \right). \quad (\text{A.8})$$

For Π' to be a measure as in Definition 3.3 we can compute the Laplace functional for a simple function f as in (A.5) and then the Cox property follows from monotone and dominated convergence. It follows that

$$\mathbb{E}[\Pi | \Xi] = \mathbb{E}[\mathcal{P}^\Xi | \Xi] = \Xi \quad (\text{A.9})$$

where we used the independence of Ξ and \mathcal{P}^ξ together with the freezing Lemma (see, e.g., [55, Lemma A.3]).

Consider now the restriction $\Pi|_B$, we have that, for any f positive, continuous and compactly supported that

$$Ee^{-\Pi|_B f} = \mathcal{M}_\nu \left(-\rho^\nu \left(\int_B (1-e^{-f(s)}) ds \right)^\nu \right) \quad (\text{A.10})$$

proceeding as in (A.5) and using monotone and dominated convergence. \square

APPENDIX B. DETAILS ON THE PROOF OF THE BOLTZMANN-GRAD LIMIT

Following Gallavotti's intuition, the tube-like flow is the union of non-disjoint cylinders, and its volume is given by

$$|\Theta(\tilde{\mathcal{I}}, \tilde{\mathcal{V}})| = \pi R^2 ct + o(R^2). \quad (\text{B.1})$$

We note that by Equation (3.15) $\rho^n c^n \pi^n R^{2n} = \lambda^n$. Now, let us call $\Theta := \Theta(\tilde{\mathcal{I}}, \tilde{\mathcal{V}})$ and

$$a_n(R) := \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} (-\lambda)^n h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{I}}, \tilde{\mathcal{V}}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{I}}, \tilde{\mathcal{V}}) \right) \mathcal{M}_\nu^{(n)}(-\rho^\nu |\Theta|^\nu) \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n.$$

We need it to be absolutely dominated by b_n for each $n \geq 1$, the last one being independent of R and summable.

Firstly, we observe that

$$\left| h \left(\mathcal{X}_t^{(1)}(\tilde{\mathcal{I}}, \tilde{\nu}), \mathcal{V}_t^{(1)}(\tilde{\mathcal{I}}, \tilde{\nu}) \right) \right| < \zeta_1 \quad \text{for some } \zeta_1 \in (0, \infty).$$

Moreover, using the notation for the derivative of the Mittag-Leffler function in Equation (2.13) we get

$$\begin{aligned} & \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} \left| \mathcal{M}_\nu^{(n)}(-\rho^\nu |\Theta|^\nu) \right| \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n \\ &= \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} \left| \left(\frac{d}{d(\rho|\Theta|)} \right)^n \int_0^\infty e^{-\rho|\Theta|l} P(\mathcal{L} \in dl) \right| \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n. \end{aligned}$$

We now observe that the correction term in the approximation of the volume of the tube-like flow (B.1) is negative. Moreover, for R sufficiently small, we can write

$$\lambda t + \rho o(R^2) \geq \frac{\lambda t}{2} \implies e^{-(\lambda t + \rho o(R^2))} \leq e^{-\frac{\lambda t}{2}}.$$

Then, by using dominated convergence, the above equation reads

$$\begin{aligned} & \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} \int_0^\infty l^n e^{-l(\lambda t + \rho o(R^2))} P(\mathcal{L} \in dl) \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n \\ & \leq \int_{0 < \tau_1 < \tau_2 < \dots < \tau_n < t} \int_{(S^2)^n} \int_0^\infty l^n e^{-\frac{\lambda t}{2} l} P(\mathcal{L} \in dl) \mu(dv_1) \dots \mu(dv_n) d\tau_1 \dots d\tau_n \\ & = \frac{t^n}{n!} \int_0^\infty l^n e^{-\frac{\lambda t}{2} l} P(\mathcal{L} \in dl), \end{aligned}$$

where the inequality holds for sufficiently small R . Finally,

$$|a_n(R)| \leq \zeta_1 \frac{1}{2^n} \int_0^\infty \frac{(\lambda l t)^n}{n!} e^{-\lambda l t} P(\mathcal{L} \in dl) =: b_n$$

and

$$\begin{aligned} \sum_{n \geq 1} b_n &= \sum_{n \geq 1} \zeta_1 \frac{1}{2^n} \int_0^\infty \frac{(\lambda l t)^n}{n!} e^{-\lambda l t} P(\mathcal{L} \in dl) \\ &\leq \zeta_1 \int_0^\infty e^{-\lambda l t} (e^{\lambda l t} - 1) P(\mathcal{L} \in dl) \\ &= \zeta_1 \left(1 - \int_0^\infty e^{\lambda l t} P(\mathcal{L} \in dl) \right) < \infty. \end{aligned}$$

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