New mathematical tools for the Lagrangian description of atmospheric dynamics

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Abstract

Applications of the mathematical theory of rearrangements to atmosphere and ocean dynamics are discussed. Results in the existing literature, mainly for two-dimensional incompressible flow, are reviewed. We prove that a state of geostrophic and hydrostatic balance minimises the energy subject to conservation of absolute momentum and potential temperature, and show that a steady state of a semigeostrophic flow can be characterised as a stationary point of the energy subject to rearrangements of the potential vorticity. By restricting the class of rearrangements to those consistent with the evolution equations, it is suggested that the enstrophy cascade may be prevented in semigeostrophic flow.

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1 Introduction

The description of weather systems in the atmosphere has always drawn a great deal on a Lagrangian description of the flow. Obvious examples are the description of synoptic developments in terms of air masses, the parcel theory of convection, and the description of the dynamics of precipitation systems in terms of conveyor belts, (e.g. Browning and Roberts(1994)). The importance of Lagrangian concepts in dynamical understanding has been highlighted in recent years by the extensive use of potential vorticity diagnostics, (Hoskins et al. (1985)). The power of Lagrangian methods in numerical simulation has been demonstrated by contour dynamics algorithms (Dritschel (1988ab)). Unfortunately, however, much theory still uses a Eulerian description of the flow because more mathematical tools are available in that case. For instance, the analysis of numerical algorithms for advection is very incomplete because, in a Eulerian sense, the problem is nonlinear unless the advecting velocity is uniform. In general, little can be said about the solution of nonlinear problems, apart from verifying global conservation properties, so that the analysis stops with the linear case. However, the process being described is quite simple, the advected quantity is transported along trajectories with no change in value. This is exploited by Lagrangian numerical methods, and should be susceptible to analysis.

A concept that arises naturally when considering the Lagrangian description of incompressible fluid motion is that of a rearrangement. Let σ represent a quantity, associated with a fluid, which is conserved following the motion of a fluid particle, such as potential vorticity in adiabatic flow. Assume we are working in a bounded region. As time advances, the fluid particles will, in general, exchange their spatial positions, but each particle retains its original value of σ. Then we can say that, for any given time t, σ considered as a function $\sigma(\mathbf{x}, t)$ is a *rearrangement* of the initial function $\sigma(\mathbf{x}, 0)$, where $\mathbf{x}$ is the typical Eulerian position vector. Alternatively we say that σ is *rearrangement preserved* by the flow. In this paper, we define this concept precisely and then show how some tools available from rearrangement theory can be used in analysis of atmospheric (or oceanic) flows. Further mathematical details and proofs are given by Douglas (1996).

Three practical results are illustrated. The first is the characterisation of minimum energy states subject to conservation of properties following particles. It is shown that it is necessary to consider whether the extreme state can be reached by mixing the fluid, in which case the use of conservation following particles may not be appropriate. The second is the behaviour of the time evolution of a flow subject to (potential) vorticity conservation. Again it is crucial whether the long time evolution approximates mixing, equivalent to the enstrophy cascade of two-dimensional turbulence, or whether this is prohibited. The third is the study of the stability of steady states, following the work of Nycander (1995). If we work in a class of perturbations which are rearrangements of the potential vorticity, we can show that steady states can be characterised as stationary points of the energy. This class of perturbations is a subclass of all possible perturbations and is chosen to be consistent with the time dependent dynamics, which conserve potential vorticity. The result is not true if all perturbations are allowed.

2 Equivalent definitions of rearrangement of vector valued functions

2.1 Introduction

In this section we establish four equivalent definitions of rearrangement for vector valued functions, and give a characterisation of the set of rearrangements of a prescribed vector valued function. We define rearrangement for vector valued functions on finite measure spaces (U, μ) which are isomorphic to $(0, \mu(U))$ endowed with Lebesgue measure λ. By isomorphic, we mean there exists a measure preserving transformation $T : U \to (0, \mu(U))$. We recall the definition of measure preserving transformation in the next section. The restriction to finite measure spaces (U, μ) isomorphic to $(0, \mu(U))$ with Lebesgue measure is not severe: Royden [**12**] yields that any separable complete metric space U, equipped with a Borel measure μ such that $\mu(U) < \infty$ and $\mu(\{x\}) = 0$ for each $x \in U$, is isomorphic to $((0, \mu(U)), \lambda)$.
Definition Let (U, μ) be a measure space which is isomorphic to $((0, \mu(U)), \lambda)$. Let $f, g \in L^p(U, \mu, \mathbf{R}^d)$, for $1 \le p < \infty$. Then f is a *rearrangement* of g if

$$\mu\left(f^{-1}(B)\right) = \mu\left(g^{-1}(B)\right)$$

for every Borel subset B of $\mathbf{R}^d$.
We prove the following theorem.

Theorem 1 *Let (U, μ) be as above. Let $f, g \in L^p(U, \mu, \mathbf{R}^d)$, for $1 \le p < \infty$. Then the following are equivalent.*
(i) f is a rearrangement of g.
(ii) For each $F \in C(\mathbf{R}^d)$ such that $|F(\xi)| \le K(1 + |\xi|_2^p)$ (where $|.|_2$ denotes Euclidean distance on $\mathbf{R}^d$, and K is a constant), the following equation is satisfied:

$$\int_U F(f(x))d\mu(x) = \int_U F(g(x))d\mu(x).$$

(iii) $\mu(f^{-1}(C)) = \mu(g^{-1}(C))$ for each set $C \in \{\prod_{i=1}^d [\alpha_i, \infty) : \alpha_i \in \mathbf{R}$ for each $i = 1, ..., d\} \bigcup \{\emptyset, \mathbf{R}^d\}$.
(iv) For each $\sigma \in \mathbf{R}^d$, $\alpha > 0$,

$$\int_U (|g - \sigma|_\infty - \alpha)_+ d\mu = \int_U (|f - \sigma|_\infty - \alpha)_+ d\mu$$

where $|.|_\infty$ denotes the infinity norm on $\mathbf{R}^d$, and the + subscript denotes the positive part of the function.

Brenier [**2**] used property (ii) to define rearrangement of vector valued functions, whilst Cullen, Norbury and Purser used property (iii). This theorem shows that their definitions are equivalent. Property (iv) is a vector valued extension of the characterisation of the set of rearrangements of a given real valued function by Eydeland, Spruck and Turkington [**6**]: for non–negative $f_0 \in L^p(U, \mu)$,

$$R(f_0) = \{w \text{ measurable}, w \ge 0 : \int_U (w - \alpha)_+ = \int_U (f_0 - \alpha)_+, \forall \alpha > 0\}.$$

It follows from (iv) that for $f_0 \in L^p(U, \mu, \mathbf{R}^d)$, where $1 \le p < \infty$ and (U, μ) is as in Theorem 1

$$\begin{aligned} R(f_0) \quad = \quad \{ \; w \; \mu\text{–measurable} \; : \int_U (|w - \sigma|_\infty - \alpha)_+ \, d\mu \\ = \int_U (|f_0 - \sigma|_\infty - \alpha)_+ \, d\mu, \forall \sigma \in \mathbf{R}^d, \forall \alpha > 0\}. \end{aligned}$$

It may be shown that $R(f_0)$ is closed, and using the characterisation above, that for $w \in R(f_0)$, $||w||_p = ||f_0||_p$, where

$$||w||_p = \left\{ \int_U |w|_\infty^p d\mu \right\}^{\frac{1}{p}}.$$

We omit the proofs, which are elementary.

2.2 Measure preserving mappings and transformations

We recall the concept of a measure preserving mapping.

Definition A *measure preserving mapping* from a finite measure space (U, μ) to a measure space (V, ν) with $\mu(U) = \nu(V)$ is a mapping $s : U \to V$ such that for each ν–measurable set $A \subset V$, $\mu(s^{-1}(A)) = \nu(A)$.

Halmos [**7**, Theorem 2, page 163] yields that this is equivalent to requiring that for every ν–integrable function f, $f \circ s$ is μ–integrable and

$$\int_U f \circ s d\mu = \int_V f d\nu.$$

Measure preserving mappings are surjective (up to sets of measure zero), but not necessarily injective. If a measure preserving mapping s is injective, and s maps μ–measurable sets to ν–measurable sets, then s^{-1} exists and is a measure preserving mapping. Such an s is called a *measure preserving transformation.*

2.3 Analytic set theory

We proceed with the proof of Theorem 1 in stages. We require a result from the theory of analytic sets. As a preliminary, we establish some notation. Let H be a family of subsets of a given set X. Define

$$\begin{aligned} H_{\sigma\delta} &= \{ \text{ countable disjoint unions of elements of } H\} \\ H_C &= \{ \text{ complements (relative to } X\text{) of elements of } H\} \end{aligned}$$

$B_{cd}(H)$ will denote the smallest family H^*, with $H \subset H^*$, such that $H^*_C = H^*_{\sigma\delta} = H^*$.

Kechris [**9**, page 65, Theorem 10.1 (iii)] yields the following result.

Theorem *Let H be a family of subsets of X such that (i) $X \in H$ (ii) $H_1 \bigcap H_2 \in H$ whenever $H_1, H_2 \in H$. Then $B_{cd}(H)$ is a σ–algebra.*

Lemma 1 *Let f, g be as in Theorem 1. Define*

$$\begin{aligned} \mathcal{M} &= \{A \subset \mathbf{R}^d : \mu(f^{-1}(A)) = \mu(g^{-1}(A))\} \\ H &= \{\prod_{i=1}^d [a_i, b_i] : a_i, b_i \in \mathbf{R}, a_i \le b_i, \text{ for } i = 1, ..., d\} \bigcup \{\emptyset, \mathbf{R}^d\} \end{aligned}$$

Suppose $H \subset \mathcal{M}$. Then $\mathcal{M}$ contains the Borel sets of $\mathbf{R}^d$.

Proof H is closed under finite intersection, therefore the above theorem yields that $B_{cd}(H)$ is a σ–algebra. H generates the Borel sets, therefore it follows that the Borel sets are contained in $B_{cd}(H)$. $\mathcal{M}$ is closed under countable disjoint union and complementation (relative to $\mathbf{R}^d$). Given that $H \subset \mathcal{M}$ we have $B_{cd}(H) \subset B_{cd}(\mathcal{M}) = \mathcal{M}$, so $\mathcal{M}$ contains the Borel sets. This completes the proof.

2.4 Proof of Theorem 1

We begin by showing that (i) implies (ii). Let $F \in C(\mathbf{R}^d)$ satisfy $|F(\xi)| \leq K\{1 + |\xi|_2^p\}$ for each $\xi \in \mathbf{R}^d$, where K is some constant. We assume that F is non–negative. (If not we work with the positive and negative parts of F.) F is measurable, therefore the fundamental approximation lemma yields the existence of a sequence of simple functions (φ_n) such that
(i) $0 \leq \varphi_n(\xi) \leq \varphi_{n+1}(\xi)$ for each $\xi \in \mathbf{R}^d$.
(ii) $\varphi_n(\xi) \to F(\xi)$ for each $\xi \in \mathbf{R}^d$.

We demonstrate that

$$\int_U \varphi_n(f(x))d\mu(x) = \int_U \varphi_n(g(x))d\mu(x). \tag{1}$$

A simple function is a finite linear combination of indicator (characteristic) functions of measurable sets, therefore it is sufficient to show

$$\int_U 1_A(f(x))d\mu(x) = \mu(f^{-1}(A)) = \mu(g^{-1}(A)) = \int_U 1_A(g(x))d\mu(x), \tag{2}$$

for each Lebesgue measurable set $A \subset \mathbf{R}^d$, where 1_A denotes the indicator function of A. Noting that a Lebesgue set is the disjoint union of a Borel set and a Lebesgue negligible set, we need only show (2) for Borel sets. This is immediate from (i). Thus we have verified (1).

We have that $\varphi_n \circ f(x) \to F \circ f(x)$ for each $x \in U$, and that $|\varphi_n \circ f(x)| \leq K\{1+|f(x)|_2^p\}$ for each $x \in U$ and $n \in \mathbf{N}$, and analogous statements hold if we replace f with g. Applying the Dominated Convergence theorem we obtain

$$\begin{aligned}
\int_U F(f(x))d\mu(x) &= \lim_{n\to\infty} \int_U \varphi_n(f(x))d\mu(x) \\
&= \lim_{n\to\infty} \int_U \varphi_n(g(x))d\mu(x) \\
&= \int_U F(g(x))d\mu(x).
\end{aligned}$$

This verifies (ii).

We show that (ii) implies (i). Let families of sets H and $\mathcal{M}$ be as in Lemma 1. Let $H_1 \in H$. There exists a sequence $(\varphi_n) \subset C(\mathbf{R}^d)$ such that $|\varphi_n(y)| \leq 1+|y|_2^p$ for each $y \in \mathbf{R}^d$ and $n \in \mathbf{N}$, with $\varphi_n(y) \to 1_{H_1}(y)$ for each $y \in \mathbf{R}^d$. It follows that $\varphi_n \circ f(x) \to 1_{H_1} \circ f(x)$ for each $x \in U$ and $|\varphi_n \circ f(x)| \leq 1 + |f(x)|_2^p$ for each $x \in U$ and $n \in \mathbf{N}$, with analogous statements holding if we replace f by g. Noting that (ii) holds, we apply the Dominated Convergence theorem to obtain

$$\begin{aligned}
\mu(f^{-1}(H_1)) &= \int_U 1_{H_1} \circ f(x)d\mu(x) \\
&= \lim_{n\to\infty} \int_U \varphi_n \circ f(x)d\mu(x) \\
&= \lim_{n\to\infty} \int_U \varphi_n \circ g(x)d\mu(x) \\
&= \int_U 1_{H_1} \circ g(x)d\mu(x) = \mu(g^{-1}(H_1)).
\end{aligned}$$

Thus $H_1 \in \mathcal{M}$. It follows that $H \subset \mathcal{M}$. Lemma 1 yields that $\mathcal{M}$ contains the Borel sets of $\mathbf{R}^d$, therefore f and g are rearrangements.

All elements of the family $\{\prod_{i=1}^d[\alpha_i,\infty) : \alpha_i \in \mathbf{R} \text{ for each } i = 1,...,d\}$ are Borel sets of $\mathbf{R}^d$, therefore (i) implies (iii). To see the converse, we show that $H \subset \mathcal{M}$, given that (iii) holds. We proceed by induction. Let $\mathcal{P}(k)$ be the proposition that all sets of the form $\prod_{i=1}^k[a_i,b_i] \times \prod_{i=k+1}^d[a_i,\infty) \in \mathcal{M}$, where $a_i, b_i \in \mathbf{R}$. We demonstrate $\mathcal{P}(1)$. Now

$$[a_1,b_1] \times \prod_{i=2}^d[a_i,\infty) = \prod_{i=1}^d[a_i,\infty)\backslash\left(\bigcup_{n=1}^{\infty}\left([b_1+1/n,\infty)\times\prod_{i=2}^d[a_i,\infty)\right)\right),$$

and noting that $\mathcal{M}$ is closed under countable increasing union, and differences of two ordered elements (with respect to the partial order $\subset$), we obtain that $[a_1,b_1] \times \prod_{i=2}^d[a_i,\infty) \in \mathcal{M}$. This shows $\mathcal{P}(1)$. We demonstrate that $\mathcal{P}(k+1)$ is true given that $\mathcal{P}(k)$ holds. We have that

$$\begin{aligned}\prod_{i=1}^{k+1}[a_i,b_i] \times \prod_{i=k+2}^{d}[a_i,\infty) &= \prod_{i=1}^{k}[a_i,b_i] \times \prod_{i=k+1}^{d}[a_i,\infty)\\ &\backslash \left(\bigcup_{n=1}^{\infty}\left(\prod_{i=1}^{k}[a_i,b_i]\times[b_{k+1}+1/n,\infty)\times\prod_{i=k+2}^{d}[a_i,\infty)\right)\right).\end{aligned}$$

We are given that $\mathcal{P}(k)$ holds, therefore $\prod_{i=1}^k[a_i,b_i]\times\prod_{i=k+1}^d[a_i,\infty) \in \mathcal{M}$, and $\prod_{i=1}^k[a_i,b_i]\times[b_{k+1}+1/n,\infty)\times\prod_{i=k+2}^d[a_i,\infty) \in \mathcal{M}$ for each $n \in \mathbf{N}$. Noting that $\mathcal{M}$ is closed under countable increasing union and differences of ordered elements, we obtain that $\prod_{i=1}^{k+1}[a_i,b_i] \times \prod_{i=k+2}^d[a_i,\infty) \in \mathcal{M}$. This verifies $\mathcal{P}(k+1)$. By induction $\mathcal{P}(d)$ holds, that is all sets of the form $\prod_{i=1}^d[a_i,b_i] \in \mathcal{M}$ for $a_i, b_i \in \mathbf{R}$, $i = 1,...,d$. It is immediate that $\emptyset, \mathbf{R}^d \in \mathcal{M}$, therefore $H \subset \mathcal{M}$. Lemma 1 yields that $\mathcal{M}$ contains the Borel sets of $\mathbf{R}^d$. This shows (i).

Let (iv) hold. The characterisation of the set of rearrangements of a scalar valued function by Eydeland, Spruck and Turkington [**6**] yields that $|g-\sigma|_\infty \in R(|f-\sigma|_\infty)$ in the scalar valued sense for each $\sigma \in \mathbf{R}^d$. Therefore we have

$$\mu\{x : |g(x)-\sigma|_\infty \geq \alpha\} = \mu\{x : |f(x)-\sigma|_\infty \geq \alpha\}$$

for each positive $\alpha \in \mathbf{R}$ or equivalently,

$$\mu\{x : |g(x)-\sigma|_\infty < \alpha\} = \mu\{x : |f(x)-\sigma|_\infty < \alpha\}.$$

Therefore we have $\mu(g^{-1}(C_\alpha(\sigma))) = \mu(f^{-1}(C_\alpha(\sigma)))$, where $C_\alpha(\sigma)$ denotes the open cube of side 2α about $\sigma \in \mathbf{R}^d$. Let K denote the set of all d–dimensional open cubes. We have shown that $K \subset \mathcal{M}$. We now demonstrate that this implies that all open subsets of $\mathbf{R}^d$ belong to $\mathcal{M}$. Recall that $\mathcal{M}$ is closed under countable decreasing intersections, increasing countable unions, and differences of ordered elements of $\mathcal{M}$. For $j = 0,...,d$ every j–dimensional closed cube is a countable decreasing intersection of j–dimensional open cubes. Further, for $j = 1,..,d$ every j–dimensional open cube with one $(j-1)$–dimensional open face attached is an increasing countable union of j– dimensional closed cubes. Now, for $j = 1,...,d$, every $(j-1)$–dimensional open cube is the difference of a set of the type described in the preceding sentence, and a j dimensional open cube contained in it. It follows by induction that open and closed cubes of dimensions $0,...,d$ belong to $\mathcal{M}$. Every open subset of $\mathbf{R}^d$ is a countable disjoint union of open cubes of dimensions $0,...,d$, therefore such sets belong to $\mathcal{M}$. The methods of Lemma 1, (noting that the intersection of two open sets is open,) yield that $\mathcal{M}$ contains the Borel sets. Thus (iv) implies (i). The converse follows because (i) implies that $\mu(g^{-1}(C_\alpha(\sigma))) = \mu(f^{-1}(C_\alpha(\sigma)))$ for each positive $\alpha \in \mathbf{R}$, $\sigma \in \mathbf{R}^d$. This completes the proof.

3 Energy minimising solutions of atmospheric and oceanic flow

3.1 Introduction

This section studies a variational problem over the set of rearrangements of a prescribed vector valued function, which arises from an energy minimising principle. We study the semigeostrophic equations, (recalled in the next section,) a standard model for slowly varying flows constrained by rotation and stratification, using the methods of Cullen, Norbury and Purser [**5**]. At any given time, $\mathbf{X}$, which describes the state of the fluid, is known on particles. The *Cullen–Norbury–Purser* principle states that for a solution, the particles are arranged to minimise geostrophic energy. This yields a variational problem, minimise energy over the set of rearrangements of a prescribed fluid configuration. We verify the conjecture of Cullen, Norbury and Purser [**5**, Section 5] that the energy minimum is uniquely attained, and that the minimiser is equal to the gradient of a convex function. We prove the following theorem.

Theorem 2 *Let Ω be a bounded connected closed subset of $\mathbf{R}^3$, with smooth boundary. Define, for $\mathbf{X} = (X, Y, Z) \in L^p(\Omega, \mu, \mathbf{R}^3)$, where $2 \le p \le \infty$ and μ denotes 3–dimensional Lebesgue measure,*

$$E(\mathbf{X}) = \frac{1}{2}\int_{\Omega} X^2 + x^2 + Y^2 + y^2 d\mu(\mathbf{x}) - \int_{\Omega} \mathbf{X}.\mathbf{x} d\mu(\mathbf{x})$$

where $\mathbf{x} = (x, y, z) \in \Omega$. Suppose $\mathbf{X_0} \in L^p(\Omega, \mu, \mathbf{R}^3)$, for p as above. Then there exists $\mathbf{X_0}^* \in R(\mathbf{X_0})$ such that
(i) $E(\mathbf{X_0}^*) < E(\mathbf{X})$ for each $\mathbf{X} \in R(\mathbf{X_0}) \backslash \{\mathbf{X_0}^*\}$.
(ii) $\mathbf{X_0}^* = \nabla \Psi$ for some convex function $\Psi \in W^{1,p}(\Omega)$.
(iii) $\mathbf{X_0}^*$ is a cyclically monotone function.

The functional E represents the Geostrophic energy of the fluid. We define E and $\mathbf{X}$ in the next section. The unique energy minimiser is the monotone rearrangement of the prescribed function: this concept was introduced by Brenier [**2**], and is recalled in section 3.3. The proof uses an approximation argument, with the strict inequality following by the uniqueness of the monotone rearrangement.

3.2 The semigeostrophic equations, and the Cullen–Norbury–Purser principle

We state the three dimensional Boussinesq equations of semigeostrophic theory on an f plane. These are a standard model for slowly varying flows constrained by rotation and stratification, and are used to study front formation in meteorology. We state the equations in the form used by Hoskins [**8**].

$$\frac{Du_g}{Dt} - f v_{ag} = 0, \frac{Dv_g}{Dt} + f u_{ag} = 0, \tag{3}$$

$$\frac{D\theta}{Dt} = 0, \tag{4}$$

$$\nabla.\mathbf{u} = 0,$$

$$\nabla \phi = \left(f v_g, -f u_g, \frac{g\theta}{\theta_0} \right) \tag{5}$$

where

$$\begin{aligned} \mathbf{u} &\equiv (u,v,w) \equiv \mathbf{u_g} + \mathbf{u_{ag}}, \\ \mathbf{u_g} &\equiv (u_g, v_g, 0), \\ \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla \end{aligned}$$

f is the Coriolis parameter, assumed constant, g denotes the acceleration due to gravity, θ_0 is a reference value of the potential temperature θ, and ϕ is a pressure variable. Subscripts g and ag denote geostrophic and ageostrophic velocity (or wind) components respectively, where the geostrophic velocity is defined to be the horizontal component of velocity in balance with the pressure gradient. This definition is included in equation (5), as is the statement of hydrostatic balance. We solve the equations (for the velocity $\mathbf{u}$) in a closed bounded connected set $\Omega \subset \mathbf{R}^3$, with normal velocity $\mathbf{u}.\mathbf{n}$ given on $\partial\Omega$. For $\mathbf{x} = (x,y,z) \in \Omega$, by making the the substitution

$$\mathbf{X} \equiv (X,Y,Z) \equiv (x + v_g/f, y - u_g/f, (g/f^2\theta_0)\theta),$$

it is shown in Purser and Cullen [**11**] that we may replace (3) and (4) by

$$\frac{D\mathbf{X}}{Dt} = \mathbf{u_g}.$$

We think of $\mathbf{X}$ as a function of the physical space co–ordinates $\mathbf{x}$. Rewriting in terms of $\mathbf{X}$ and $\mathbf{x}$, we have

$$\frac{DX}{Dt} = f(y - Y) \tag{6}$$

$$\frac{DY}{Dt} = f(X - x) \tag{7}$$

$$\frac{DZ}{Dt} = 0. \tag{8}$$

The geostrophic energy E is defined as

$$\begin{aligned} E &= \int_\Omega \frac{1}{2}u_g^2 + \frac{1}{2}v_g^2 - \frac{g\theta z}{\theta_0} d\mu(\mathbf{x}) \\ &= f^2\frac{1}{2}\int_\Omega X^2 + x^2 + Y^2 + y^2 d\mu(\mathbf{x}) - f^2\int_\Omega \mathbf{x}.\mathbf{X} d\mu(\mathbf{x}) \end{aligned}$$

Henceforth we ignore the constant f^2. At any time t, $\mathbf{X}$ is found on particles by predicting (X,Y,Z) on particles using the equations (6), (7) and (8). The *Cullen–Norbury–Purser* principle states that for a solution, the particles are arranged to minimise geostrophic energy. Suppose one possible state of the fluid is described by values $\mathbf{X_0} = (X_0, Y_0, Z_0)$ which are known on particles. The Cullen–Norbury–Purser principle yields the energy minimisation problem

$$\inf_{\mathbf{X} \in R(\mathbf{X_0})} E(\mathbf{X}),$$

where the energy minimiser (if it exists and is unique) gives the actual state of the fluid. In this way, solutions can be viewed as a sequence of minimum energy states.

We make some (physically reasonable) assumptions to enable us to use vector valued rearrangement theory. Let Ω be a closed, bounded, connected subset of $\mathbf{R}^3$, with smooth boundary. Suppose the possible fluid configuration $\mathbf{X_0} \in L^p(\Omega, \mu, \mathbf{R}^3)$, for $2 \leq p < \infty$, where μ denotes 3–dimensional Lebesgue measure. (Choosing $p \geq 2$ ensures finite geostrophic energy.)

3.3 Monotone rearrangement of vector valued functions

We recall the concept of the monotone rearrangement of a vector valued function: essentially, this is the vector valued analogue of the increasing rearrangement of a real valued function. Let Ω and μ be as in the last paragraph of the previous section. The following theorem is due to Brenier [**2**, section 1.2, theorem 1.1].

Theorem 1.1 *For each $u \in L^p(\Omega, \mu, \mathbf{R}^3)$, where $1 \leq p < \infty$, there is a unique $u^* \in R(u)$ such that*

$$u^* \in \{\nabla\Psi : \Psi \in W^{1,p}(\Omega, \mu), \Psi \text{ convex}\},$$

and the mapping $u \to u^*$ is continuous.

When Ω is not convex, Ψ is understood to be the restriction to Ω of a convex function defined on $\mathbf{R}^3$. We call u^* the *monotone rearrangement of* u. The name comes from the fact that u^* is a cyclically monotone function. We note that McCann [**10**] has generalised the first part of this result (concerning the existence of an essentially unique rearrangement equal to the gradient of a convex function) to more general measures than Lebesgue measure.

Definition A function $u \in L^p(\Omega, \mu, \mathbf{R}^3)$ is *non–degenerate* if $\mu(u^{-1}(E)) = 0$ for each set $E \subset \mathbf{R}^3$ with Lebesgue measure zero. We say that a function which fails to be non–degenerate is *degenerate.*

Brenier established further properties of the monotone rearrangement of a non–degenerate function in the following theorem [**2**, section 1.2, theorem 1.2]

Theorem 1.2 *For each non–degenerate $u \in L^p(\Omega, \mu, \mathbf{R}^3)$ there exists a unique pair (u^*, s), where u^* is the monotone rearrangement of u, and s is a measure preserving mapping from (Ω, μ) to (Ω, μ), such that*

(i) $u = u^* \circ s$.

(ii) *s is the unique measure preserving mapping that maximises $\int_\Omega u(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})$.* Note that Theorem 1.2 is not true if u is degenerate: the measure preserving mapping is not unique, nor do we have uniqueness in property (ii). The author is not aware of any corresponding result for degenerate functions.

3.4 Existence and uniqueness of energy minimiser

Recall that we are studying the energy minimisation problem

$$\inf_{\mathbf{X} \in R(\mathbf{X_0})} \int_\Omega x^2 + X^2 + y^2 + Y^2 d\mu(\mathbf{x}) - \int_\Omega \mathbf{x}.\mathbf{X} d\mu(\mathbf{x}),$$

where $\mathbf{X_0} \in L^p(\Omega, \mu, \mathbf{R}^3)$ for $2 \leq p < \infty$, and $\mathbf{X} = (X, Y, Z)$. We show that the first integral is conserved under rearrangements.

Lemma 2 *Let $\mathbf{X_0}$ be as in Theorem 2. Let $\mathbf{X_1} \in R(\mathbf{X_0})$. Then*

$$\int_\Omega x^2 + X_1^2 + y^2 + Y_1^2 d\mu(\mathbf{x}) = \int_\Omega x^2 + X_0^2 + y^2 + Y_0^2 d\mu(\mathbf{x})$$

where $\mathbf{X_0} = (X_0, Y_0, Z_0)$ and $\mathbf{X_1} = (X_1, Y_1, Z_1)$.

Proof $\mathbf{X_1} \in R(\mathbf{X_0})$ implies that $X_1 \in R(X_0)$. It follows that

$$\int_\Omega X_1^2 d\mu(\mathbf{x}) = \int_\Omega X_0^2 d\mu(\mathbf{x}).$$

A similar result holds for Y_0 and Y_1. The result follows.

To show that there is a unique energy minimiser, it remains to show that

$$\sup_{\mathbf{X}\in R(\mathbf{X_0})} \int_\Omega \mathbf{x}.\mathbf{X} d\mu(\mathbf{x})$$

is uniquely attained. If $\mathbf{X_0}$ is non–degenerate, the result follows easily using Theorem 1.2. Our method of proof is to approximate degenerate functions with a sequence of non–degenerate functions. This shows that the monotone rearrangement is an energy minimiser. We demonstrate that an energy minimiser is the gradient of a convex function: the monotone rearrangement is the unique such amongst the set of rearrangements, therefore the result follows.

Lemma 3 *Let $\mathbf{X} \in L^p(\Omega, \mu, \mathbf{R}^3)$ (where Ω, μ and p are as in section 3.2). Then there exists a sequence of non–degenerate functions $(\mathbf{X_n})$ such that $\mathbf{X_n} \to \mathbf{X}$ in $L^p(\Omega, \mu, \mathbf{R}^3)$.*

Proof For each $n \in \mathbf{N}$, choose a simple function φ_n such that $||\mathbf{X} - \varphi_n||_p \leq 1/n$. Now for each $n \in \mathbf{N}$, define $\mathbf{X_n}$ by $\mathbf{X_n}(\mathbf{x}) = \varphi_n(\mathbf{x}) + (1/n)\mathbf{x}$ for $\mathbf{x} \in \Omega$. It is immediate that $\mathbf{X_n} \to \mathbf{X}$ in $L^p(\Omega, \mu, \mathbf{R}^3)$. It remains to show that $\mathbf{X_n}$ is non–degenerate for each $n \in \mathbf{N}$. Fix $n \in \mathbf{N}$. φ_n is a simple function, therefore it takes finitely many values which we enumerate $\{\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_m}\}$. Define $A_i = {\varphi_n}^{-1}(\mathbf{b_i})$ for each $i = 1, ..., m$. Write ${\mathbf{X_n}}^i$ for $\mathbf{X_n}|_{A_i}$. For a given i, ${\mathbf{X_n}}^i = \mathbf{b_i} + (1/n)\mathbf{x}$. Let E be a Lebesgue negligible subset of $\mathbf{R}^3$. Then

$$\begin{aligned} \mu\left(({\mathbf{X_n}}^i)^{-1}(E)\right) &= \mu\left(A_i \bigcap (nE - n\mathbf{b_i})\right) \\ &\leq \mu(nE - n\mathbf{b_i}) \\ &= \mu(nE) = 0. \end{aligned} \qquad (9)$$

By way of explanation, we have used translation invariance of Lebesgue measure to obtain the first equality in (9), and properties of Lebesgue measure to obtain the second. This demonstrates that ${\mathbf{X_n}}^i$ is non–degenerate (as an element in $L^p(A_i, \mu, \mathbf{R}^3)$), for each $i = 1, ..., m$.

Let E be a Lebesgue negligible subset of $\mathbf{R}^3$. Then

$$\begin{aligned} \mu\left({\mathbf{X_n}}^{-1}(E)\right) &= \mu\left(\bigcup_{i=1}^m ({\mathbf{X_n}}^i)^{-1}(E)\right) \\ &= \sum_{i=1}^m \mu\left(({\mathbf{X_n}}^i)^{-1}(E)\right) = 0. \end{aligned} \qquad (10)$$

To obtain (10) we have used the countable additivity of μ, and the fact that ${\mathbf{X_n}}^i$ is non–degenerate for each $i = 1, ..., m$. This shows that $\mathbf{X_n}$ is non–degenerate, and completes the proof.

Lemma 4 *Let $\mathbf{X_0}$ be as in Theorem 2. Then*

$$\int_\Omega {\mathbf{X_0}}^*(\mathbf{x}).\mathbf{x} d\mu(\mathbf{x}) \geq \int_\Omega \mathbf{X}(\mathbf{x}).s(\mathbf{x}) d\mu(\mathbf{x})$$

for each $\mathbf{X} \in R(\mathbf{X_0})$ and each $s : \Omega \to \Omega$ a measure preserving mapping.

Proof Let $\mathbf{X} \in R(\mathbf{X_0})$ and let $s : \Omega \to \Omega$ be a measure preserving mapping. From the previous lemma we may choose a sequence $(\mathbf{X_n})$ of non–degenerate functions such that $\mathbf{X_n} \to \mathbf{X}$ in $L^p(\Omega, \mu, \mathbf{R}^3)$. For each $n \in \mathbf{N}$, Theorem 1.2 (i) yields the existence of a unique measure preserving mapping $s_n : \Omega \to \Omega$ such that $\mathbf{X_n} = \mathbf{X_n}^* \circ s_n$. Applying Theorem 1.1 we have $\mathbf{X_n}^* \to \mathbf{X}^* = \mathbf{X_0}^*$. Now

$$\begin{aligned}
\int_\Omega \mathbf{X_0}^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) &= \lim_{n\to\infty} \int_\Omega \mathbf{X_n}^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) \\
&= \lim_{n\to\infty} \int_\Omega \mathbf{X_n}^* \circ s_n(\mathbf{x}).s_n(\mathbf{x})d\mu(\mathbf{x}) \qquad (11) \\
&= \lim_{n\to\infty} \int_\Omega \mathbf{X_n}(\mathbf{x}).s_n(\mathbf{x})d\mu(\mathbf{x}) \\
&\geq \lim_{n\to\infty} \int_\Omega \mathbf{X_n}(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x}) \qquad (12) \\
&= \int_\Omega \mathbf{X}(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})
\end{aligned}$$

as required. By way of explanation, (11) holds because s_n is a measure preserving map, and (12) follows because Theorem 1.2(ii) yields that

$$\int_\Omega \mathbf{X_n}(\mathbf{x}).s_n(\mathbf{x})d\mu(\mathbf{x}) \geq \int_\Omega \mathbf{X_n}(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})$$

for each measure preserving mapping $s : \Omega \to \Omega$, and for each $n \in \mathbf{N}$. This completes the proof.

Lemma 5 *Let* $\mathbf{X_0}$ *be as in Theorem 2. Then*

$$\int_\Omega \mathbf{X_0}^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) > \int_\Omega \mathbf{X}(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x})$$

for each $\mathbf{X} \in R(\mathbf{X_0})\backslash\{\mathbf{X_0}^*\}$.

Proof Applying the previous lemma for the identity mapping, we have

$$\int_\Omega \mathbf{X_0}^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) \geq \int_\Omega \mathbf{X}(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x})$$

for each $\mathbf{X} \in R(\mathbf{X_0})\backslash\{\mathbf{X_0}^*\}$. It remains to show strict inequality. Suppose there exists $\mathbf{X_1} \in R(\mathbf{X_0})$ such that $\int_\Omega \mathbf{X_1}.\mathbf{x}d\mu = \int_\Omega \mathbf{X_0}^*.\mathbf{x}d\mu$. Applying the previous lemma to $\mathbf{X_1} \in R(\mathbf{X_0})$ we obtain

$$\begin{aligned}
\int_\Omega \mathbf{X_1}(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) &= \int_\Omega \mathbf{X_0}^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) \\
&\geq \int_\Omega \mathbf{X_1}(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})
\end{aligned}$$

for each measure preserving mapping $s : \Omega \to \Omega$. Brenier [**2**, Proposition 2.1] yields that $\mathbf{X_1} \in \{\nabla\Psi : \Psi \in W^{1,2}(\Omega), \Psi \text{ convex}\}$. However Theorem 1.1 states that $\mathbf{X_0}^*$ is the unique member of $R(\mathbf{X_0})$ belonging to $\{\nabla\Psi : \Psi \in W^{1,2}(\Omega), \Psi \text{ convex}\}$, therefore $\mathbf{X_1} = \mathbf{X_0}^*$. This completes the proof.

Proof of Theorem 2

Follows from Lemmas 2 and 5.

4 Discussion

We have discussed a number of applications of Lagrangian mathematics, particularly the use of rearrangements arising from an area or volume preserving map. We have shown that in various balanced models described in terms of potential vorticity and an invertibility principle that useful nonlinear and large amplitude information can be obtained about the flow evolution. This includes an important identification of flows where the potential enstrophy cascade can be prevented, and variational arguments to characterise steady states. We have shown that the most powerful results occur in cases where we can use a rearrangement inequality to extremise a function over all possible rearrangements. Examples quoted used monotone or symmetrising rearrangements. It is important to identify more such situations and exploit them.

Other possible applications are to numerical methods and data assimilation. It is clearly desirable to satisfy the rearrangement property in a numerical solution of the evolution equation for potential vorticity, and to preserve it as well as possible in formulations where potential vorticity is not used as a model variable. This should form part of the assessment of competing numerical methods. If the non-chaotic dynamics associated with advection by a convex streamfunction is important, then a property equivalent to this has to be preserved in the numerical approximation, so that the enstrophy cascade is absent there also and no enstrophy sink has to be provided by adding artificial viscosity. In data assimilation, it may be appealing to try and work with displacements or rearrangements of fields in the analysis procedure, rather than with Eulerian perturbations. This could include minimisation over rearrangements, possibly restricted to those consistent with convexity conditions, as part of a general variational framework.

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5 Figure Captions

Figure 1. Graphs of $f(x) = x, x \in [0,1]$ and

$$g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{13}$$

illustrating the reason why f is a rearrangement of g.

Figure 2. Graphs of the inverse images of f and g, with $f = f(x)$, $g = g(x)$ defined as in Fig.1. The sets $f^{-1}[1/2,1]$ and $g^{-1}[1/2,1]$ are illustrated.

Figure 3. Graphs of two different rearrangements of $f(x) = x, x \in [0,1]$. The shaded areas are the same.

Figure 4. Values of the vector valued functions $f = f(x,y)$ and $g = g(x,y)$ defined on $[0,1] \times [0,1]$ by (**??**) and (**??**) respectively.

Figure 5. The rearrangements $f_3 = f_3(x)$ and $f_8 = f_8(x)$ defined by equation (**??**) of the function $f_0 = f_0(x)$ defined by equation (**??**).

Figure 6. The effect on the surface $P = P(x)$ of local concentrations of its curvature.