Rearrangements and polar factorisation of countably degenerate functions

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Abstract

This paper proves some extensions of Brenier's [**1**] theorem that an integrable vector-valued function u, satisfying a non-degeneracy condition, admits a unique polar factorisation $u = u^{\#} \circ s$. Here $u^{\#}$ is the *monotone rearrangement* of u, equal to the gradient of a convex function almost everywhere on a bounded connected open set Y with smooth boundary, and s is a measure-preserving mapping.

We show that two weaker alternative hypotheses are sufficient for the existence of the factorisation; that $u^{\#}$ be *almost injective* (in which case s is unique), or that u be *countably degenerate* (which allows u to have level sets of positive measure). We allow Y to be any set of finite positive Lebesgue measure.

Our construction of the measure preserving map s is especially simple.

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1 Introduction

The notions of increasing and decreasing rearrangement of a real function of a real variable are classical, see for example Hardy, Littlewood and Pólya [**5**]. Some novel ideas were introduced into rearrangement theory by Ryff in the 1960s. In particular he showed [**9**] that any real integrable function on a bounded interval could be written as the composition of its increasing rearrangement with a measure-preserving map; this expression has become known as the *polar factorisation*. In recent years Brenier [**1**] and McCann [**7**] have taken up the vector-valued case; the *monotone rearrangement*, which is the gradient of a convex function, plays the part of the increasing rearrangement.

Brenier [**1**] proved the existence and uniqueness of the monotone rearrangement, defined on a bounded connected open set with smooth boundary, of an integrable vector-valued function. For such domains he also proved the existence and uniqueness of the polar factorisation for a special class of functions, which he called *non-degenerate*, possessing the property that the inverse image of every set of zero measure has zero measure. McCann's [**7**] main result ensures the existence and uniqueness of the monotone rearrangement of an integrable vector-valued function on a general set of finite positive Lebesgue measure, but he did not consider the polar factorisation. Our purpose here is to extend this work on vector-valued functions by establishing the polar factorisation for a larger class of functions than Brenier considered, and on general sets of finite positive Lebesgue measure.

In our situation, the monotone rearrangement $u^{\#}$ of u on a set Y always exists as a consequence of McCann's work. We obtain the measure-preserving mapping by a simple and direct method involving the inverse function of $u^{\#}$. This construction is most natural when $u^{\#}$ is injective off a set of measure zero; we then say $u^{\#}$ is *almost injective*. We show that if u is non-degenerate then $u^{\#}$ is almost injective, thus our results are indeed an extension of Brenier's. We are also able to extend the existence of the polar factorisation to a still wider class of functions, that are rendered non-degenerate by the removal of countably many level sets of positive measure. Before stating the results, we require some definitions and notation.

Definition Let (X, μ) and (Y, ν) be finite positive measure spaces with $\mu(X) = \nu(Y)$. Two vector-valued functions $f \in L^1(X, \mu, \mathbb{R}^n)$ and $g \in L^1(Y, \nu, \mathbb{R}^n)$ are *rearrangements* if

$$\mu\left(f^{-1}(S)\right) = \nu\left(g^{-1}(S)\right) \text{ for every } S \in \mathfrak{B}(\mathbb{R}^n),$$

where $\mathfrak{B}(\mathbb{R}^n)$ denotes the Borel field of $\mathbb{R}^n$. (Brenier [**1**] uses a different definition which is shown to be equivalent to the one above in Douglas [**2**].)

Definition A *measure-preserving mapping* from a finite positive measure space (U, μ) to a positive measure space (V, ν) with $\mu(U) = \nu(V)$ is a mapping $s : U \to V$ such that for each ν-measurable set $A \subset V$, $\mu(s^{-1}(A)) = \nu(A)$.

Throughout this paper we will denote n-dimensional Lebesgue measure by λ_n, and the extended real numbers, that is the set $\mathbb{R} \cup \{-\infty, \infty\}$, by $\overline{\mathbb{R}}$.

Definition A finite measure space (U, μ) is a *measure interval* if it is isomorphic to $[0, \mu(U)]$ with Lebesgue measure. (We defer the definition of isomorphism of measure spaces to the next section.)

Definition Let $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure interval. Let $Y \subset \mathbb{R}^n$ be such that $\lambda_n(Y) = \mu(X)$. The *monotone rearrangement of* u *on* Y is the unique function $u^{\#} : Y \to \mathbb{R}^n$ that is a rearrangement of u, and satisfies $u^{\#} = \nabla\psi$ almost everywhere in Y for some proper lower semicontinuous convex function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$.

In the case when Y is a bounded, connected, open subset of $\mathbb{R}^n$, with smooth boundary, and we make the stronger hypothesis that $u \in L^p(X, \mu, \mathbb{R}^n)$, where $1 < p \leq \infty$, Brenier [**1**, Theorem 1.1] yields that ψ belongs to $W^{1,p}(Y, \lambda_n, \mathbb{R}^n)$.

Definition Let $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure interval. Let $Y \subset \mathbb{R}^n$ be such that $\lambda_n(Y) = \mu(X)$, and let $u^{\#}$ denote the monotone rearrangement of u on Y. We say u has a *polar factorisation through* Y if there exists a measure-preserving mapping $s : X \to Y$ such that $u = u^{\#} \circ s$. While $u^{\#}$ must be unique, s need not be unique.

Existence (and uniqueness) of a polar factorisation has been proved under certain restrictions.

Definition Let u be a vector-valued integrable function defined on a measure interval. We say u is *non-degenerate* if $\mu(u^{-1}(E)) = 0$ for each set $E \subset \mathbb{R}^n$ of zero Lebesgue measure, otherwise u is *degenerate*. It is easily seen that all rearrangements of a non-degenerate function are non-degenerate.

For Y a bounded connected open subset of $\mathbb{R}^n$, with smooth boundary, and $\lambda_n(Y) = \mu(X)$, Brenier [**1**, Theorem 1.2] showed that any non–degenerate function u has a unique polar factorisation through Y. Gangbo [**3**] subsequently gave an elementary proof of this result.

Definition A mapping $\sigma : U \to V$, where (U, μ) is a finite positive measure space, is *almost injective* if there exists a set $U_0 \subset U$ such that σ restricted to U_0 is injective, and $\mu(U \setminus U_0) = 0$.

We prove the following results.

Theorem 1 *Suppose $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure interval. Let $Y \subset \mathbb{R}^n$ satisfy $\lambda_n(Y) = \mu(X)$, and suppose that the monotone rearrangement of u on Y, denoted $u^\#$, is almost injective. Then u has a unique polar factorisation $u = u^\# \circ s$, where $s : X \to Y$ is a measure-preserving mapping.*

Remark: We will show in Lemma 5 that if u is non-degenerate then $u^\#$ is almost injective, for any admissible Y; thus Theorem 1 applies to non-degenerate functions.

Theorem 2 *Suppose $u \in L^1(X, \mu, \mathbb{R}^n)$ where (X, μ) is a measure interval. Let $Y \subset \mathbb{R}^n$ satisfy $\lambda_n(Y) = \mu(X)$, and let $u^\#$ be the monotone rearrangement of u on Y. Suppose further that there exists a countable set B with*
(i) $\lambda_n((u^\#)^{-1}(b)) > 0$ for each $b \in B$, and
(ii) $u^\#$ restricted to $(u^\#)^{-1}(\mathbb{R}^n \setminus B)$ is almost injective.
Then u has a polar factorisation $u = u^\# \circ s$, where $s : X \to Y$ is a measure-preserving mapping.

Definition Let u be an integrable vector-valued function defined on a measure interval. We say u is *countably degenerate* if there exists a countable set $B \subset \mathbb{R}^n$ (called the *degenerate set* of u) such that
(i) $\mu(u^{-1}(b)) > 0$ for each $b \in B$, and
(ii) u restricted to $u^{-1}(\mathbb{R}^n \setminus B)$ is a non-degenerate function.

It is immediate that a rearrangement of a countably degenerate function is countably degenerate with the same degenerate set. Unlike a non-degenerate function, a countably degenerate function may have (countably many) level sets of positive measure.

Corollary 3 *Let $u \in L^1(X, \mu, \mathbb{R}^n)$ be countably degenerate, where (X, μ) is a measure interval. Let $Y \subset \mathbb{R}^n$ satisfy $\lambda_n(Y) = \mu(X)$, and let $u^\#$ denote the monotone rearrangement of u on Y. Then u admits a polar factorisation $u = u^\# \circ s$ for some measure preserving mapping $s : X \to Y$, and if in addition u is non–degenerate, then the polar factorisation is unique.*

We now proceed to the proofs of these results and some of their ramifications. In Section 3 we discuss existence and uniqueness of the polar factorisation in the context of some examples.

2 Proofs and Further Results

Definition A mapping $s : U \to V$, where (U, μ) and (V, ν) are finite measure spaces, is a *measure-preserving transformation* if
(i) $s : U \setminus L \to V \setminus M$ is a bijection, where L and M are some sets of zero (respectively μ and ν) measure, and
(ii) s and s^{-1} are measure preserving mappings.

Measure-preserving mappings are surjective (up to sets of measure zero), but need not be injective.

Definition Two finite measure spaces (U, μ) and (V, ν) are *isomorphic* if there exists a measure-preserving transformation $T : U \to V$.

We justify our earlier assertion that the monotone rearrangement exists and is unique. Let u be an integrable vector-valued function defined on a measure interval (X, μ), and let $Y \subset \mathbb{R}^n$ satisfy $\lambda_n(Y) = \mu(X)$. Define η, ν by

$$\eta(S) = \lambda_n(Y \cap S),\ \nu(S) = \mu(u^{-1}(S))$$

for each $S \in \mathfrak{B}(\mathbb{R}^n)$. Now $\eta/\lambda_n(Y)$ and $\nu/\lambda_n(Y)$ are probability measures on $\mathbb{R}^n$, with η vanishing on (Borel) subsets of $\mathbb{R}^n$ having Hausdorff dimension $n-1$, therefore the main theorem of McCann [**7**] yields the existence of a convex function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ whose derivative $\nabla\psi$ at its points of differentiability is such that

$$\eta\left((\nabla\psi)^{-1}(S)\right) = \nu(S) \tag{1}$$

for each $S \in \mathfrak{B}(\mathbb{R}^n)$. Then $\nabla\psi : Y \to \mathbb{R}^n$ is uniquely determined λ_n-almost everywhere in Y; moreover ψ can be chosen to be proper and lower semicontinuous. Rewriting (1) we obtain

$$\lambda_n\left((\nabla\psi)^{-1}(S)\right) = \mu\left(u^{-1}(S)\right)$$

for each $S \in \mathfrak{B}(\mathbb{R}^n)$, which is exactly the statement that $\nabla\psi$ and u are rearrangements. The monotone rearrangement is unique in the sense that if $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex function, and $\nabla\phi$ (as a function defined on Y) is a rearrangement of u, then $\nabla\phi(x) = \nabla\psi(x)$ for almost every $x \in Y$.

Our first lemma is a small modification of the standard result that any complete separable metric space with a finite positive continuous Borel measure is a measure interval.

Lemma 4 *Let X be a set of finite positive Lebesgue measure a in $\mathbb{R}^n$. Then there is a bijection of X onto $[0,a]$ that is an isomorphism (endowing $[0,a]$ with λ_1).*

Proof. We can find a $\mathcal{G}_\delta$-set W with $X \subset W \subset \mathbb{R}^n$ such that $W \setminus X$ has zero measure. Then W is homeomorphic to a complete separable metric space (see for example Kechris [**6**, Theorem 3.11]) and therefore there is a bijection of W onto $[0,a]$ that is an isomorphism relative to the respective measures λ_n and λ_1 (see for example Kechris [**6**, Theorem 17.41]). To complete the proof, it will be enough to show that if $M \subset [0,a]$ has zero measure, then there exists a bijection from $[0,a]\setminus M$ to $[0,a]$ which is an isomorphism. We choose a set $C \subset [0,a]\setminus M$, of zero measure and cardinal $\mathfrak{c}$; this is possible since $[0,a]\setminus M$ contains a compact set of positive measure which is therefore isomorphic to an interval, which in turn contains a Cantor set of zero measure and cardinal $\mathfrak{c}$. Now choose τ to map $M \cup C$ bijectively onto C and let τ leave all points of $[0,a]\setminus(M \cup C)$ fixed, to obtain an isomorphism of $[0,a]$ onto $[0,a]\setminus M$. □

Proof of Theorem 1

We begin by disposing of a measurability question. After discarding a set of zero measure, we can assume $u^\# = \nabla\psi$ throughout Y, where $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is some proper lower semicontinuous convex function, that $u^\#$ is injective on Y, and that Y is σ-compact. Since $\nabla\psi$ is continuous relative to its domain of existence (see for example Rockafellar [**8**, Theorem 25.5]), it follows that $u^\#$ is continuous relative to Y. Hence $u^\#$ maps each compact set $K \subset Y$ homeomorphically onto $u^\#(K)$, and since Y is σ-compact it follows that $u^\#$ maps Borel subsets of Y to Borel subsets of $\mathbb{R}^n$. Moreover the range $\mathcal{R}$ of $u^\#$ is σ-compact.

It follows, since u is a rearrangement of $u^\#$, that by deleting a set of zero measure from X we can assume the range of u lies in $\mathcal{R}$. Let $v : \mathcal{R} \to Y$ be the inverse function of $u^\# : Y \to \mathcal{R}$, and define $s = v \circ u : X \to Y$. Evidently $u^\# \circ s = u$ throughout X; we now check that s is measure-preserving.

It will suffice to consider a Borel set $S \subset Y$, and prove that $\mu(s^{-1}(S)) = \lambda_n(S)$. Let $T = u^\#(S)$. Then T is Borel, $(u^\#)^{-1}(T) = S$, and

$$s^{-1}(S) = u^{-1}(v^{-1}(S)) = u^{-1}(u^\#(S)) = u^{-1}(T).$$

Now $(u^{\#})^{-1}(T)$ and $u^{-1}(T)$ have equal measure, so $s^{-1}(S)$ and S have equal measure, which shows that s is a measure-preserving map.

It remains to prove the uniqueness of s. If a measure-preserving map $t : X \to Y$ differs from s on a set of positive measure, then since $u^{\#}$ is injective, $u^{\#} \circ t$ differs from $u^{\#} \circ s$ on a set of positive measure, so $u^{\#} \circ t$ differs from u on a set of positive measure. This shows that the measure-preserving map in the polar factorisation of u on Y is unique. □

Proof of Theorem 2

Enumerate $B = \{b_i\}_{i \in I}$, where I is a countable index set. Define $X_0 = u^{-1}(\mathbb{R}^n \setminus B)$, $Y_0 = (u^{\#})^{-1}(\mathbb{R}^n \setminus B)$, and for each $i \in I$, $X_i = u^{-1}(b_i)$, $Y_i = (u^{\#})^{-1}(b_i)$. For $i \in I$, X_i and Y_i are measurable sets with $\mu(X_i) = \lambda_n(Y_i) > 0$. Lemma 4 ensures that (X_i, μ) and (Y_i, λ_n) are measure intervals, therefore for each $i \in I$ there exists a measure-preserving transformation $s_i : X_i \to Y_i$. It is immediate that $u = u^{\#} \circ s_i$ almost everywhere in X_i.

Suppose $\mu(X_0) > 0$. (If not, defining $s(x) = s_i(x)$ when $x \in X_i$ yields the result.) Then $u^{\#}$ restricted to Y_0 is the monotone rearrangement of u restricted to X_0, and furthermore $u^{\#}$ is almost injective on Y_0. Theorem 1 gives the existence of a (unique) measure-preserving mapping $s_0 : X_0 \to Y_0$ such that $u(x) = u^{\#} \circ s_0(x)$ for almost every $x \in X_0$. Define $s : X \to Y$ by $s(x) = s_i(x)$ when $x \in X_i$ for $i \in I \cup \{0\}$. Countable additivity of the measure yields that s is a measure-preserving mapping, and $u = u^{\#} \circ s$ follows from above. □

If $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ then $\psi^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ denotes the (Legendre-Fenchel) *conjugate convex function* of ψ, defined by

$$\psi^*(x) = \sup\{x \cdot y - \psi(y) | y \in \mathbb{R}^n\}.$$

A $\overline{\mathbb{R}}$-valued function is called *proper* if it is not identically ∞, and nowhere has the value $-\infty$.

Lemma 5 *Let $Y \subset \mathbb{R}^n$ be a set of finite positive Lebesgue measure, let $u : Y \to \mathbb{R}^n$ be integrable, and suppose there is a proper lower semicontinuous convex function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $u = \nabla\psi$ almost everywhere on Y. Suppose further that u is non-degenerate. Then u is almost injective on Y.*

Proof. We can suppose, after discarding a set of zero measure, that ψ is differentiable with gradient u throughout Y. Define

$$K = \{k \in \mathbb{R}^n | \psi^*(k) < \infty \text{ and } \psi^* \text{ is non-differentiable at } k\},$$

so K has zero measure by Rademacher's Theorem on the differentiability of convex functions (see for example Rockafellar [**8**, Theorem 25.5]). Suppose $x, z \in Y$, $z \neq x$, are such that $u(x) = u(z) = k$ say. Then $k \in \partial\psi(x) \cap \partial\psi(z)$. Since ψ is a proper lower semicontinuous convex function we have $\psi^{**} = \psi$ and therefore $x, z \in \partial\psi^*(k)$ (see for example Rockafellar [**8**, Theorems 12.2 and 23.5]). Hence $k \in K$. It follows that the restriction of u to $Y \setminus u^{-1}(K)$ is injective; moreover $u^{-1}(K)$ has zero measure since u is non-degenerate. □

Proof of Corollary 3

The existence of a polar factorisation for a countably degenerate function follows from Theorem 2 and Lemma 5, and uniqueness for a non–degenerate function from Theorem 1 and Lemma 5. □

We show in Proposition 7 following, that if u is non–degenerate, then any two monotone rearrangements of u are related by a unique measure-preserving transformation. We need the following preliminary result:

Lemma 6 *Let (X, μ) and (Y, ν) be measure intervals of equal measure, and suppose $\tau : X \to Y$ is a measure preserving map. Suppose further that τ is almost injective on X. Then τ is a measure-preserving transformation.*

Proof. We lose nothing by assuming that X and Y are real intervals, that $\mu = \nu = \lambda_1$, and that τ is injective on the whole of X. It will suffice to show that τ^{-1} is measure-preserving. Let $S \subset X$ be measurable; then by Egoroff's Theorem we can write $H_1 \subset S \subset H_2$ where H_1 and $X \setminus H_2$ are countable unions of compact sets relative to each of which τ is continuous, and $\lambda_1(H_1) = \lambda_1(S) = \lambda_1(H_2)$. Then $\tau(H_1)$ and $\tau(H_2)$ are measurable, being respectively $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ relative to Y. Moreover $\lambda_1(\tau(H_i)) = \lambda_1(\tau^{-1}(\tau(H_i))) = \lambda_1(H_i)$ for $i = 1, 2$. Hence $\tau(S)$ is measurable with $\lambda_1(\tau(S)) = \lambda_1(S)$. □

Proposition 7 *Let X and Y be sets of equal finite positive Lebesgue measure in $\mathbb{R}^n$, let $u \in L^1(X, \lambda_n, \mathbb{R}^n)$ and $v \in L^1(Y, \lambda_n, \mathbb{R}^n)$ be rearrangements of each other, and suppose that u, v are equal almost everywhere in X, Y to the gradients of proper lower semicontinuous convex functions $\phi, \psi : \mathbb{R}^n \to \overline{\mathbb{R}}$. Suppose further that u and v are almost injective. Then there is a unique measure-preserving transformation $\tau : X \to Y$ such that $u = v \circ \tau$ almost everywhere in X.*

Proof. We can suppose, after removing sets of zero measure, that u, v are injective on the whole of X, Y respectively. Since v is the monotone rearrangement of u on Y and is injective, we can by Theorem 1 choose a unique measure-preserving mapping $\tau : X \to Y$ such that $u = v \circ \tau$ almost everywhere in X. Discarding a further set of zero measure from X we can suppose that $u = v \circ \tau$ throughout X. Suppose $x, y \in X$ with $\tau(x) = \tau(y)$. Then $u(x) = v(\tau(x)) = v(\tau(y)) = u(y)$, hence $x = y$ by injectivity of u. Thus τ is injective, and is therefore a measure-preserving transformation by Lemma 6. □

Let (X, μ) be a measure interval, and let $\Omega \subset \mathbb{R}^n$ be an open connected bounded set with smooth boundary, satisfying $\lambda_n(\Omega) = \mu(X)$. Brenier [**1**, Theorem 1.2 (b)] shows that for a non-degenerate function u defined on X, the unique maximiser of $\int_X u \cdot s$ over s in the set of measure-preserving mappings from X to Ω is the s satisfying $u = u^{\#} \circ s$. We prove an extension of this result; for any integrable u, any measure-preserving mapping which arises from a polar factorisation of u through Ω is a maximiser of $\int_X u \cdot s$. We have not been able to prove that such measure-preserving mappings are the only maximisers.

We recall (see Halmos [**4**, Theorem C, page 163]) that for a measure-preserving mapping s from (U, μ) to (V, ν), for every ν-integrable (scalar) function f (defined on V), $f \circ s$ is μ-integrable and

$$\int_U f \circ s \, d\mu = \int_V f \, d\nu.$$

Proposition 8 *Let (X, μ) be a measure interval, and u be an integrable function defined on X. Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, with smooth boundary, satisfying $\lambda_n(\Omega) = \mu(X)$. Let $u^{\#}$ denote the monotone rearrangement of u on Ω. Then measure-preserving mappings $s : X \to \Omega$ which satisfy $u = u^{\#} \circ s$ maximise $\int_X u(x) \cdot s(x) d\mu(x)$ over the set of measure-preserving mappings from X to Ω.*

Proof. Let $s : X \to \Omega$ be a measure-preserving mapping such that $u = u^{\#} \circ s$. Lemma 4 ensures (X, μ) and (Ω, λ_n) are isomorphic, therefore we can find a measure-preserving transformation

$\tau : \Omega \to X$. Now

$$\begin{aligned} \int_X u(x) \cdot s(x) d\mu(x) &= \int_\Omega u^\#(x) \cdot x d\lambda_n(x) \\ &\geq \int_\Omega (u \circ \tau)(x) \cdot \sigma(x) d\lambda_n(x) \qquad (2) \\ &= \int_X u(x) \cdot \sigma \circ \tau^{-1}(x) d\mu(x) \end{aligned}$$

for every measure-preserving mapping $\sigma : \Omega \to \Omega$, where inequality (2) follows by Douglas [**2**, Lemma 4]. Furthermore for every measure-preserving mapping $\eta : X \to \Omega$, $\eta \circ \tau : \Omega \to \Omega$ is a measure-preserving mapping. Combining this with the above inequality we obtain,

$$\int_X u(x) \cdot s(x) d\mu(x) \geq \int_X u(x) \cdot \eta(x) d\mu(x)$$

for every measure-preserving mapping $\eta : X \to \Omega$. □

Remark For a measure-preserving mapping $\sigma : X \to \Omega$, the monotone rearrangement $\sigma^\#$ is the identity function, and a corollary to the proof of the above proposition is that for u as above,

$$\int_\Omega u^\#(x) \cdot \sigma^\#(x) d\lambda_n(x) \geq \int_X u(x).\sigma(x) d\mu(x).$$

This inequality fails for general $\sigma \in L^\infty(X, \mu, \mathbb{R}^n)$ if $n \geq 2$ (see Brenier [**1**]).

3 Special Cases and Examples

The one-dimensional case.

The situation in dimension one is well-understood. Consider $f \in L^1(I)$ where $I = [0,1]$. Ryff [**9**] showed that f has a polar factorisation $f = f^\# \circ \sigma$ almost everywhere, where $f^\# \in L^1(I)$ is the increasing rearrangement of f, and $\sigma : I \to I$ is a measure-preserving mapping.

Suppose f has at least one level set of positive measure, and is therefore degenerate. If $\tau : I \to I$ is chosen to act as a non-trivial measure-preserving transformation on one interval of constancy of $f^\#$ and to fix all other points of I, then $\tau \circ \sigma$ is measure-preserving and $f = f^\# \circ \tau \circ \sigma$. Thus the polar factorisation is not generally unique in the countably degenerate case.

Suppose on the other hand that f has no level sets of positive measure, which ensures $f^\#$ is increasing but has no intervals of constancy, and therefore $f^\#$ is injective. The uniqueness of the polar decomposition now follows easily, as in the proof of Theorem 1.

It should be noted that this uniqueness of the polar factorisation holds even for degenerate functions that have no level sets of positive measure. An example of such a function can be constructed as follows. Let $C \subset I$ denote the Cantor ternary set and let C' denote C with the left-hand end-points of its contiguous intervals removed. There is a well-known construction of a continuous increasing map g of I onto I, that is constant on the intervals contiguous to C, and maps C' bijectively onto I. Let $f : I \to C'$ be the inverse function of g. Then f is strictly increasing (and therefore has no level sets of positive measure) but degenerate, because C' is a set of zero measure whose inverse image I has positive measure.

A two-dimensional example.

In higher dimensions, for a countably degenerate function having a level set of positive measure, the polar factorisation is not unique, as can be seen by the same argument given above in the one-dimensional case. However, when, in higher dimensions, a function lacks level sets of positive measure, the polar factorisation, if it exists, may not be unique, as may be seen from the following example. Let $u : D \to \mathbb{R}^2$ be defined on the unit disc $D \subset \mathbb{R}^2$ by

$$u(x) = \begin{cases} |x|_e^{-1} x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $|\cdot|_e$ denotes the Euclidean norm on $\mathbb{R}^2$. Then u is equal throughout $D \setminus \{0\}$ to the gradient of the convex function defined by $\psi(x) = |x|_e$. The range of u comprises the unit circle together with 0, and the inverse image under u of any point consists of 0 or a line-segment. It follows that u is degenerate but not countably degenerate. Since ψ is convex, a polar factorisation exists trivially, but the measure-preserving map is not unique, as may be seen by considering any nontrivial area-preserving map of the disc that leaves each radius invariant.

Conjecture.

We conjecture that if (X, μ) is a measure interval, $u : X \to \mathbb{R}^n$ an integrable function, and $Y \subset \mathbb{R}^n$ a measurable set having $\lambda_n(Y) = \mu(X)$ then the polar factorisation of u through Y exists, and it is unique only if the monotone rearrangement of u on Y is almost injective.

An alternative view of non-degeneracy.

Consider $u \in L^1(X, \mu, \mathbb{R}^n)$. Then u gives rise to a Borel measure ν on $\mathbb{R}^n$, by

$$\nu(S) = \mu(u^{-1}(S)) \ \ \forall S \in \mathfrak{B}(\mathbb{R}^n),$$

called the *push-forward of* μ *by* u by McCann [**7**]. Non-degeneracy of u is equivalent to absolute continuity of ν with respect to Lebesgue measure on $\mathfrak{B}(\mathbb{R}^n)$. More generally, ν has a Lebesgue decomposition $\nu = \nu_{ac} + \nu_s$ where ν_{ac} is absolutely continuous and ν_s is singular with respect to Lebesgue measure; ν_{ac} and ν_s are concentrated on disjoint Borel sets. The function u is countably degenerate if and only if ν_s is concentrated on a countable set (the degenerate set of u), and in this case ν_s is purely atomic.

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