# Stationary vortices in three-dimensional quasigeostrophic shear flow

**G.R. BURTON**[1] **and J. NYCANDER**[2]

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1. School of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, U.K.
2. Defence Research Establishment, Inst. 64, 172 90 Stockholm, Sweden

**Abstract**

An existence theorem for localised stationary vortex solutions in an external shear flow is proved for three-dimensional quasigeostrophic flow in an unbounded domain. The external flow is a linear shear flow whose strength varies linearly with height. The flow conserves an infinite family of Casimir integrals. Flows that have the same value of all Casimir integrals are called *isovortical flows*, and the potential vorticity- (PV-) fields of isovortical flows are *stratified rearrangements* of one another. The theorem guarantees the existence of a maximum energy flow in any family of isovortical flows that statisfies the following conditions: the PV-anomaly must have compact support, it must have the same sign everywhere, and this sign must be the same as the sign of the external shear over the vertical interval to which the support of the PV-anomaly is confined. This flow represents a stationary and localised vortex, and the maximum-energy property implies that it is formally stable.

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# 1 Introduction

Coherent vortices are common in most large-scale geophysical flows, particularly in regions of strong shear. In such regions, the vorticity anomaly of the vortices almost invariably has the same sign as the shear of the background flow ("cooperative shear"). Many examples of this are given by the long-lived vortices found in the zonal flow on the giant planets.

It has also been demonstrated in many laboratory experiments and numerical simulations that such vortices can be generated by shear flow instabilities, and that they have a long lifetime (sometimes infinite), maintaining themselves by merger with smaller vortices of the same sign. Vortices in "adverse shear" (i.e. with opposite signs of the background shear and the vorticity anomaly), on the other hand, are rarely seen in real flows or numerical simulations. Yet there exist theoretical solutions describing stationary and linearly stable vortices in adverse shear (Moore & Saffman, 1971). In these explicit solutions, however, the background shear is much smaller than the vorticity anomaly.

One explanation of the difference between cooperative and adverse shear is provided by the existence theorem of Nycander (1995). This theorem states that in every family of "isovortical flows" (to be defined below) that consists of a background linear shear flow and a compact region of additional vorticity with the same sign as the background shear, there exists a maximum energy flow, representing a localized and stationary vortex. The vorticity decreases monotonically outward from the vortex center (assuming that the shear and the vorticity anomaly are positive). The fact that such a vortex is a maximum energy state guarantees that it is stable both linearly and non-linearly (albeit in an informal sense).

Nothing could be proved about the existence of a stationary vortex in adverse shear, but it is clear from the proof that if such a solution exists, it corresponds to a saddle point of the energy. It can therefore be expected to be unstable, at least nonlinearly.

Another explanation is that a vortex in cooperative shear is a "maximum entropy state", according to the statistical-mechanical theory of Miller (1990) and Robert & Sommeria (1991). However, the underlying mathematical structure explaining this is again the fact that it is also a maximum energy state.

These theories apply to ideal two-dimensional flow governed by the Euler equation, which is a highly simplified model of geophysical flows. In the present paper we extend the existence theorem of Nycander (1995) to three-dimensional quasigeostrophic flow, which is a more realistic model. In this model the stream-function for the horizontal velocity field is obtained from the potential vorticity (PV) field by inversion of a three-dimensional elliptic operator. The PV is a Lagrangian invariant (i.e. it is conserved along fluid trajectories), which implies the conservation of an infinite family of Casimir integrals (whose integrands are functions of $z$ and the PV). Flows that have the same value of all Casimirs are called *isovortical flows*. We also call the PV-fields of such isovortical flows *stratified rearrangements* of one another. A stratified rearrangement may be generated by a horizontal incompressible deformation of the PV-field that preserves the area inside any contour line of PV at any fixed height level.

We assume the background flow to be a linear shear flow at every fixed level, and the shear to vary linearly with height. We then superimpose on this flow a compact region of additional PV ("PV-anomaly"), with the same sign as the background shear. We will prove that in the set of stratified rearrangements of such a given flow, there exists a maximum energy flow. This energy maximiser is a localized stationary vortex. As in the case of two-dimensional flows, the fact that this flow maximises the energy also implies that it is stable.

Usually, the three-dimensional quasigeostrophic equation is studied in a domain which is bounded vertically. However, we have not been able to prove the existence theorem for this case, and instead assume that there are no boundaries. Effectively, this means that we study vortices that are small compared to the height of the atmosphere or the ocean. The difficulty with the bounded case appears to be technical, and we believe that the corresponding theorem is valid for that case as well.

The article is organized as follows. In section 2 the basic equations and invariants are given, and a simple heuristic argument for the existence theorem is presented. In section 3 we present the notation and the central theorem to be proved (Theorem 1), and also give an outline of the proof. Section 4 contains some basic theory and inequalities concerning rearrangements, and some theory of convex sets. In section 5 we prove some inequalities that are needed later to prove that the energy maximiser has finite extent. Section 6 contains the proof of Theorem 1. In section 7 some standard results on spaces of rearrangements are extended to the stratified case. In section 8, finally, we discuss possible generalisations of the theory and its relation to recent numerical simulations of three-dimensional quasigeostrophic turbulence.

## 2 Basic equations and heuristic argument

Three-dimensional quasigeostrophic flow is described by the equation

$$\frac{d}{dt}\left[\Delta_\perp p + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial p}{\partial z}\right)\right] = 0, \tag{1}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla,$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$, p is the pressure, $\mathbf{v}_g = \rho^{-1}\hat{\mathbf{z}} \times \nabla p$ is the geostrophic velocity, and the quantity in square brackets is the potential vorticity (PV), which is a Lagrangian invariant of the flow. We will neglect the latitudinal dependence of the Coriolis parameter $f$. For simplicity, we will also assume the buoyancy frequency $N$ to be constant, which does not principally alter the character of the problem. With these assumptions, equation (1) can be written

$$\frac{\partial}{\partial t}\Delta\Psi + J(\Delta\Psi, \Psi) = 0, \tag{2}$$

where $\Delta$ is the three-dimensional Laplacian, the Jacobian is defined by $J(f,g) = \partial_x f \partial_y g - \partial_y f \partial_x g$, $\Psi$ is the stream-function, the flow being given by $\mathbf{v} = \nabla\Psi \times \hat{\mathbf{z}}$, and $-\Delta\Psi$ is the PV. The dimensionless variables have been chosen so that the ratio between the vertical and horizontal length scales is $f/N$.

We now assume that the background flow is given by $\mathbf{V} = -2y(c_0 + c_1 z)\hat{\mathbf{x}}$, corresponding to the stream-function $-(c_0+c_1 z)y^2$ and the PV $2(c_0+c_1 z)$. Here $c_0$ and $c_1$ are arbitrary constants. This represents a linear shear flow whose strength varies linearly with height. Decomposing the total stream-function as $\Psi = -(c_0 + c_1 z)y^2 + \psi$, equation (2) can be written

$$\frac{\partial}{\partial t}\Delta\psi + J(\Delta\psi, -(c_0 + c_1 z)y^2 + \psi) = 0, \tag{3}$$

which is the equation we will study in what follows. The PV-anomaly $q = -\Delta\psi$ is assumed to have compact support, and the domain of the flow is infinite in all directions.

Equation (3) conserves the infinite family of Casimir integrals,

$$C_F = \int_{\mathbb{R}^3} F(z,q)d\mathbf{r},$$

where $F$ is an arbitrary function of both arguments. The Casimirs are also conserved by any horizontal incompressible deformation of the PV-field $q(\mathbf{r})$, giving rise to a “stratified rearrangement”. Equation (3) further conserves the total energy,

$$E(q) = W(q) - J(q), \tag{4}$$

where

$$W(q) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{q(\mathbf{r})q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}d\mathbf{r}',$$

$$J(q) = \int_{\mathbb{R}^3}(c_0 + c_1 z)y^2 q(\mathbf{r})d\mathbf{r}.$$

We call $W$ the *perturbation energy*, since it is quadratic in the vortex PV-anomaly $q$. Note that $W$ is not conserved by the flow.

Stationary solutions of equation (3) are given by $J(\Delta\psi, \Psi) = 0$, which expresses a functional dependence between $\Delta\psi$ and $\Psi$. They can also be obtained formally from the following variational property. A general isovortical first order perturbation of a given PV-field $q$ (i.e. one satisfying $\delta C_F = 0$ for any $F$) is given by $\delta q = J(\xi, q)$, where $\xi(\mathbf{r})$ is arbitrary. The variation of the energy caused by such a perturbation is $\delta E = -\int \Psi J(\xi, q)d\mathbf{r}$. Hence, if $\delta E = 0$ for any $\xi$, then $J(\Delta\psi, \Psi) = 0$. In particular, a flow that maximises the energy in the set of all stratified rearrangements of some given PV-field $q$ must be stationary. The purpose of the present work is to prove that such an energy maximiser exists, and to give an exact derivation of the steady-state equation. For the proof to be valid it is necessary that $q$ has the same sign everywhere, and that it is the same as the sign of the external vorticity $2(c_0 + c_1 z)$ at all height levels where $q \neq 0$.

We first give a simple intuitive argument. If we change the sign of the expression (4), it has exactly the same form as the potential energy due to the force of gravity of some mass distribution with the density $q$. The first term $W$ then represents the interaction energy between the mass elements, and the second term $J$ the contribution from an external gravitational field. Arbitrary stratified rearrangements are obtained by displacing the mass elements horizontally, assuming that the matter in incompressible. No vertical displacement is allowed.

If $c_0 = c_1 = 0$ (i.e. in the absence of external flow) the minimum potential energy is obviously attained by putting the densest matter at the centre at each height level $z = const$. The corresponding flow is an axisymmetric vortex $q(r, z)$, with $q$ being a monotonic decreasing function of $r = (x^2 + y^2)^{1/2}$, and $q \geq 0$ everywhere (or monotonic increasing and $q \leq 0$ everywhere). The functional dependence on $z$ is determined by the given vertical distribution, and in principle arbitrary. Such a vortex is trivially stationary, and the present consideration demonstrates that it is also a maximum energy flow. This helps explain the tendency toward horizontal axisymmetrisation and vertical alignment of the vortices that has been seen in recent numerical simulations of three-dimensional quasigeostrophic turbulence (McWilliams 1989, Viera 1995, Sutyrin et al. 1996).

For nonzero external flow, the term $J(q)$ in equation (4) means that the matter is placed in a one-dimensional external potential well, with the minimum at $y = 0$ if $c_0 + c_1 z$ is positive. One intuitively expects that a state of minimum potential energy then still exists, with the densest matter near $y = 0$. This would correspond to a vortex with monotonic radial profile of potential vorticity, in this case flattened in the $y$-direction, i.e. elongated in the direction of the external flow. Below, we will present a rigorous proof for this conjecture.

# 3 Statement of results

## 3.1 Notation and Terminology

Throughout, *measure* will refer to Lebesgue measure on $\mathbb{R}^N$, and will be called *area* in dimension 2, or *volume* in dimension 3. If $S \subset \mathbb{R}^N$ is measurable then $|S|$ will denote the measure of $S$.

When $f$ and $g$ are real integrable functions defined on a bounded measurable set $\Omega \subset \mathbb{R}^N$, we say $f$ is a *rearrangement* of $g$ if

$$|\{\mathbf{r} \in \Omega | f(\mathbf{r}) \geq s\}| = |\{\mathbf{r} \in \Omega | g(\mathbf{r}) \geq s\}| \ \ \forall s \in \mathbb{R}.$$

A definition of rearrangements on unbounded domains makes most sense for one-signed functions. We say $f : \mathbb{R}^N \to \mathbb{R}$ is *admissible* if $f$ is measurable, non-negative almost everywhere, and satisfies $|\{\mathbf{r} \in \mathbb{R}^N | f(\mathbf{r}) > s\}| < \infty$ for some $s > 0$. Two admissible functions $f$ and $g$ defined on $\mathbb{R}^N$ will be called *rearrangements* of each other if

$$|\{\mathbf{r} \in \mathbb{R}^N | f(\mathbf{r}) \geq s\}| = |\{\mathbf{r} \in \mathbb{R}^N | g(\mathbf{r}) \geq s\}| \ \ \forall s > 0.$$

When $f$ is square-integrable on bounded measurable $\Omega \subset \mathbb{R}^N$, the set of all rearrangements of $f$ on $\Omega$ is denoted $\mathcal{R}_\Omega(f)$, and the closed convex hull in $L^2(\Omega)$ of $\mathcal{R}_\Omega(f)$ is denoted $\mathcal{C}_\Omega(f)$ (see §4.6 for the definition). We will omit the subscript $\Omega$ when there is no ambiguity.

Consider a bounded measurable $\Omega \subset \mathbb{R}^3$ and $q_0 \in L^2(\Omega)$. Now $q_0(\cdot, z)$ is square-integrable on $\Omega_z := \{(x, y) \in \mathbb{R}^2 | (x, y, z) \in \Omega\}$ for almost every real $z$. Hence we can define

$$\mathfrak{R}_\Omega(q_0) = \{q \in L^2(\Omega) | q(\cdot, z) \in \mathcal{R}_{\Omega_z}(q_0(\cdot, z)) \text{ for a.e. real } z\}$$

$$\mathfrak{C}_\Omega(q_0) = \{q \in L^2(\Omega) | q(\cdot, z) \in \mathcal{C}_{\Omega_z}(q_0(\cdot, z)), \text{ for a.e. real } z\}$$

and we refer to elements of $\mathfrak{R}_\Omega(q_0)$ as *stratified rearrangements* of $q_0$. The subscript $\Omega$ will again be omitted when appropriate.

To extend the definition to functions on the unbounded domain $\mathbb{R}^3$, for non-negative functions $q$, $q_0 \in L^2(\mathbb{R}^3)$ having compact support, we say $q$ is a *stratified rearrangement* of $q_0$ if $q(\cdot, z)$ is a rearrangement of $q_0(\cdot, z)$ for almost every real $z$.

We write points in $\mathbb{R}^3$ as $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', z')$ and so on, abbreviating the volume element to $d\mathbf{r} = dxdydz$ where convenient. We fix positive constants $c_0$ and $c_1$. For non-negative $q \in L^2(\mathbb{R}^3)$ having compact support, we define

$$Kq(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} |\mathbf{r} - \mathbf{r}'|^{-1} q(\mathbf{r}') d\mathbf{r}' = \psi(\mathbf{r}) \ \ \forall \mathbf{r} \in \mathbb{R}^3,$$

$$W(q) = \frac{1}{2} \int_{\mathbb{R}^3} q(\mathbf{r}) Kq(\mathbf{r}) d\mathbf{r} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2,$$

where the second form follows from the Divergence Theorem, since $\psi(\mathbf{r}) = O(|\mathbf{r}|^{-1})$ and $\nabla\psi(\mathbf{r}) = O(|\mathbf{r}|^{-2})$ as $|\mathbf{r}| \to \infty$.

The energy $E = W - J$ is defined in equation (4).

**Theorem 1** *Let $0 < z_0 < z_1$ and let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support lying in $z_0 < z < z_1$. Let $c_0$ and $c_1$ be positive numbers. Then there exists a maximiser $\bar{q}$ for $E$ relative to the stratified rearrangements of $q_0$, and $\psi := K\bar{q}$ satisfies*

$$-\Delta\psi(x, y, z) = \varphi(\psi(x, y, z) - (c_0 + c_1 z)y^2, z) \ \ a.e. \ in \ \mathbb{R}^3$$

*for some function $\varphi : \mathbb{R}^2 \to [0, \infty)$ such that $\varphi(\cdot, z)$ is increasing for almost every real $z$.*

**Remark** Moreover $\bar{q}$ can be assumed doubly Steiner-symmetric; for the definition see §4.3.

### 3.2 Outline of proof of Theorem 1

A complete proof of Theorem 1 will be given in §6, but since a number of preliminaries are required, we digress at this stage to explain the strategy, which is modelled on the plan sketched by Benjamin (1976) in his theory of steady vortex-rings.

The first step is to prove the existence of a maximiser for $E$ relative to the stratified rearrangements of $q_0$ defined on a bounded box $\Omega$. Here the arguments of Benjamin prove difficult to realise in detail, and we follow instead the approach of Burton (1987a), Theorem 7. A weak compactness argument is employed, but since the set $\mathfrak{R}(q_0)$ is not weakly compact in general, we extend the class of admissible functions for our maximization. We work in the set $\mathfrak{C}(q_0)$, which is closed, bounded and convex in $L^2(\Omega)$ and therefore weakly compact, in the sense that any sequence in $\mathfrak{C}(q_0)$ has a subsequence converging weakly in $L^2(\Omega)$ to an element of $\mathfrak{C}(q_0)$. This weak compactness, together with the weak continuity of the energy $E$, easily leads to the existence of an energy maximiser $\bar{q}$ in the class $\mathfrak{C}(q_0)$. To complete the first step, we have to show that $\bar{q}$ in fact belongs to $\mathfrak{R}(q_0)$. To this end, the necessary condition at the maximiser $\bar{q}$ is studied, and is found to require that $\bar{q}$ be the unique maximiser of a certain linear functional (defined in terms of $\bar{q}$) relative to $\mathfrak{C}(q_0)$. We then show that the maximum of this linear functional relative to $\mathfrak{C}(q_0)$ is realised by some element of $\mathfrak{R}(q_0)$. The uniqueness then shows that $\bar{q} \in \mathfrak{R}(q_0)$.

The mathematics of this first step is more involved than in the corresponding proof for two-dimensional flow by Nycander (1995). In that case the fact that the maximiser must be symmetric decreasing in $x$ and $y$ could be used to prove that a maximising sequence of rearrangements is totally bounded, and that the sequence is therefore strongly convergent. In the present three-dimensional case, however, the rearrangements in a maximising sequence may oscillate rapidly in $z$ (this is possible even if they are symmetric decreasing in $x$ and $y$), and the sequence is therefore not totally bounded a priori. The weak compactness argument is therefore necessary.

The second step is to show that increasing the size of the confining box $\Omega$ indefinitely does not affect the maximiser, i.e. that the support of the maximiser does not touch the boundary of the confining box if the latter is large enough.

Since the contribution $K\bar{q}$ to the stream-function from the vortex vanishes at infinity, the streamline $\Psi = 0$ for fixed $z$ comes arbitrarily close to the $y$-axis for $|x| \to \infty$. This streamline is therefore a separatrix. Inside of it the streamlines are closed, and outside they are open. (This is an important difference between the present case and the two-dimensional problem treated by Nycander (1995). In that case the corresponding contribution to the stream-function diverges logarithmically at infinity. There is therefore no separatrix, and all streamlines are closed. The same is true for 3D quasigeostrophic flow in a domain which is bounded vertically.)

¿From the far-field behaviour of $\Psi$ is is possible to show that the area inside the separatrix at any fixed $z$ is unbounded (i.e. that it can be made arbitrarily large by increasing the size of the box, cf. Lemma 4). To estimate the far-field behaviour we first show that the maximiser must have positive energy, cf. Lemma 1, and that as a consequence of this the volume of its support must be finite in some finite box, cf. Lemma 3. Together with the necessary condition for a maximum, which says that $\bar{q}$ is an increasing function of the stream-function

$\Psi := K\bar{q} - (c_0 + c_1 z)y^2$ for (almost every) fixed $z$, the unbounded area inside the separatrix implies that the support of the maximiser lies entirely in the interior of the box, if it is large enough. Hence, if we choose $\Omega$ large enough for fixed $q_0$, the maximiser $\bar{q}$ is also a maximiser for all larger domains $\Omega$, thus completing the proof.

Noteworthy features of the method are that no smoothness of $q_0$ is assumed (hence vortex patches can be treated), and that the variations performed in deriving the steady-state equation are exact rather than first-order approximations.

# 4 Rearrangements, inequalities, and convexity

Here we summarise some of the theory of rearrangements that we will need to prove Theorem 1, without giving proofs. Some properties of spaces of stratified rearrangements are deferred until §7, since these are not standard and proofs must be given. Some theory of convex sets and weak convergence is also presented.

## 4.1 General properties

If $f$ is integrable on a bounded measurable $\Omega \subset \mathbb{R}^N$, and $g$ is a rearrangement of $f$ on $\Omega$, then $g$ is integrable on $\Omega$ and

$$\int_\Omega f = \int_\Omega g.$$

If $f \in L^2(\Omega)$ and $g \in \mathcal{R}(f)$ then $g^2$ is a rearrangement of $f^2$ and therefore $\|g\|_2 = \|f\|_2$. The convexity of $\|\cdot\|_2$ now ensures $\|g\|_2 \leq \|f\|_2$ for all $g \in \mathcal{C}(f)$.

Consequently, if $q_0 \in L^2(\Omega)$ for bounded measurable $\Omega \subset \mathbb{R}^3$ then $\|q\|_2 = \|q_0\|_2$ for all $q \in \mathfrak{R}(q_0)$, and $\|q\|_2 \leq \|q_0\|_2$ for all $q \in \mathfrak{C}(q_0)$.

## 4.2 Increasing rearrangements

Any real integrable function $f$ defined on a bounded measurable set $\Omega \subset \mathbb{R}^N$ has an *increasing rearrangement* $f^*$ defined on the interval $(0, m)$ where $m$ is the measure of $\Omega$, which is an increasing function satisfying

$$|\{\xi \in (0,m) | f^*(\xi) \geq s\}| = |\{\mathbf{r} \in \Omega | f(\mathbf{r}) \geq s\}| \;\; \forall s > 0.$$

Then $f^*$ is uniquely defined except for the values at its discontinuities.

If $f,\ g \in L^2(\Omega)$, then the inequality

$$\int_\Omega fg \leq \int_0^m f^* g^* \tag{5}$$

is classical; for a proof see for example Theorem 1 of Burton (1987a). ¿From it may be deduced the inequality

$$\int_\Theta f \geq \int_0^\theta f^* \quad \text{for } \Theta \subset \Omega \text{ measurable, } \theta = |\Theta| \tag{6}$$

by setting $g(\mathbf{t}) = -1$ if $\mathbf{t} \in \Theta$, and $g(\mathbf{t}) = 0$ if $\mathbf{t} \in \Omega \setminus \Theta$.

Ryff (1965), Lemma 2, showed that any integrable function on an interval can be expressed as the composition of its increasing rearrangement with a measure-preserving transformation; see our Lemma 5 for further explanation.

### 4.3 Steiner-symmetrisation

Any integrable function $f$ defined on a symmetric interval $(-s, s) \subset \mathbb{R}$ has a *symmetric decreasing rearrangement* $f^\triangle$; that is a rearrangement as an even function on on $(-s, s)$, decreasing on $(0, s)$. The inequality analogous to (5) holds for symmetric decreasing rearrangements, that is,

$$\int_{-s}^{s} fg \leq \int_{-s}^{s} f^\triangle g^\triangle \quad \forall f, g \in L^2(-s, s). \tag{7}$$

If now $S := (-s, s) \times (-s, s) \times (-s, s)$ denotes a cube in $\mathbb{R}^3$ and $f \in L^1(S)$, then $f(\cdot, y, z)$ is an integrable function on $(-s, s)$ for almost every $(y, z)$ in the square $Q := (-s, s) \times (-s, s)$; the *Steiner-symmetrisation* $f^s$ *of* $f$ *in the* $x$*-direction* is defined to be such that $f^s(\cdot, y, z)$ is the symmetric decreasing rearrangement of $f(\cdot, y, z)$ for almost every $(y, z) \in Q$. ¿From (7) we deduce

$$\int_S fg \leq \int_S f^s g^s \quad \forall f, g \in L^2(S). \tag{8}$$

Steiner-symmetrisation in the $y$-direction is similarly defined (we will not need it in the $z$-direction). A function that is invariant under Steiner-symmetrisation in both the $x$- and $y$-directions will be called *doubly Steiner-symmetric.* The two operations of Steiner-symmetrisation in the $x$- and $y$-directions do not commute. If however a function $f$ is subjected to Steiner-symmetrisation in both the $x$- and $y$-directions (in either order), the resulting rearrangement of $f$ is doubly Steiner-symmetric.

### 4.4 Riesz's inequality

The notion of Steiner-symmetrisation extends to certain non-negative functions on the whole of $\mathbb{R}^3$. Any function $f$ that is admissible (in the sense of §3.1) admits Steiner-symmetrisations; if $f^s$ denotes its Steiner-symmetrisation in the $x$-direction, then for almost every $(y,z) \in \mathbb{R}^2$ the function $f^s(\cdot,y,z)$ is the symmetric decreasing rearrangement of $f(\cdot,y,z)$. If $f$, $g$ and $h$ are admissible functions then a variant of *Riesz's inequality* asserts that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\mathbf{r})g(\mathbf{r}-\mathbf{r}')h(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^s(\mathbf{r})g^s(\mathbf{r}-\mathbf{r}')h^s(\mathbf{r}')d\mathbf{r}d\mathbf{r}', \tag{9}$$

where either side may be infinite. For a proof of a very general version of (9), see Brascamp, Lieb & Luttinger (1974), Lemma 3.2. Clearly the above remarks apply also to Steiner-symmetrisation in the $y$-direction.

### 4.5 Consequences for energy functionals

Suppose $q \in L^2(\mathbb{R}^3)$ is non-negative and has compact support. It follows from Riesz's inequality (9) that Steiner-symmetrisation in either the $x$- or $y$-direction does not reduce $W(q)$.

Steiner-symmetrisation in the $x$-direction leaves $J(q)$ unchanged, whereas inequality (8) ensures that Steiner-symmetrisation in the $y$-direction does not increase $J(q)$.

Consequently $E(q)$ is not reduced by Steiner-symmetrisation in either the $x$- or $y$-directions. Thus $q$ has a doubly Steiner-symmetric rearrangement $\bar{q}$ satisfying $E(\bar{q}) \geq E(q)$.

### 4.6 Convex sets

As observed in §3.2, the necessity for studying the convex sets $\mathcal{C}(q_0)$ and $\mathfrak{C}(q_0)$ arises from the sets $\mathcal{R}(q_0)$ and $\mathfrak{R}(q_0)$ not being weakly closed in general. We review here some of the essentials of convex analysis, in the context of the Hilbert space $L^2(\Omega)$, where $\Omega$ is a measurable subset of $\mathbb{R}^N$. This material can be found, in a more general setting, in Yosida (1980), especially the discussion of reflexivity on p.91, Theorem $3'$ on p.109, and Theorem 1 on p.126.

A set $S \subset L^2(\Omega)$ is called *convex* if $S$ contains the straight line-segment joining each pair of its points. For any set $S \subset L^2(\Omega)$, the *convex hull* of $S$ consists of all the *convex combinations* of points of $S$, that is, the (finite) linear combinations whose coefficients are non-negative and sum to 1. The convex hull of $S$ is itself a convex set, and is the smallest (in the sense of set inclusion) convex set containing $S$. The *closed convex hull* of $S$ consists of all limits of (strongly) convergent sequences in the convex hull. Again, the closed convex hull of $S$ is a convex set. If

$M > 0$ and $\|x\|_2 \le M$ for all $x \in S$, then the same is true for all $x$ lying in the convex hull of $S$, and for all $x$ lying in the closed convex hull of $S$. Thus the closed convex hull of a bounded set is also bounded.

Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ in $L^2(\Omega)$ *converges weakly* to $f \in L^2(\Omega)$ if

$$\int_\Omega f_n g \to \int_\Omega fg \text{ as } n \to \infty, \ \ \forall g \in L^2(\Omega).$$

The following one-dimensional example is illuminating: define $f_n(\xi) = \sin n\xi$ for $\xi \in (0, 2\pi)$. The $f_n$ are all rearrangements of each other, and $f_n \to 0$ weakly as $n \to \infty$. In this case the weak limit is not a rearrangement. This construction therefore shows that the set of rearrangements of $f_1$ is not weakly closed in $L^2(0, 2\pi)$.

However, it is a well-known consequence of the Hahn-Banach Theorem that a closed convex set in $L^2(\Omega)$ is weakly closed and therefore contains the weak limits of all its weakly convergent sequences. Thus a simple way to extend a set in $L^2(\Omega)$ to a weakly closed set is to take its closed convex hull.

It is a consequence of the Banach-Alaoglu Theorem that every bounded sequence in $L^2(\Omega)$ has a subsequence converging weakly to some point of $L^2(\Omega)$. It follows that if $C \subset L^2(\Omega)$ is closed, convex and bounded, then every sequence in $C$ has a subsequence converging to an element of $C$. The application of this observation to a set $\mathfrak{C}(q_0)$ (introduced in §3.1) plays a crucial part in the proof of Theorem 1.

## 5 Preliminary estimates

We now perform some calculations of the energy and stream-function due to a stratified rearrangement of $q_0$ that will be used in the proof of Theorem 1.

**Lemma 1** *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Then some stratified rearrangement $q$ of $q_0$ with compact support satisfies $E(q) > 0$.*

*Proof.* Consider the rearrangement $q$ of $q_0$ defined by $q(x, y, z) = q_0(\alpha x, \alpha^{-1} y, z)$ where $0 < \alpha \le 1$. We make a linear change of variable to obtain

$$\begin{aligned} W(q) &= \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q_0(\alpha x, \alpha^{-1} y, z) q_0(\alpha x', \alpha^{-1} y', z') d\mathbf{r} d\mathbf{r}'}{((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}} \\ &= \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q_0(x, y, z) q_0(x', y', z') d\mathbf{r} d\mathbf{r}'}{(\alpha^{-2}(x - x')^2 + \alpha^2 (y - y')^2 + (z - z')^2)^{1/2}} \\ &= \frac{\alpha}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{q_0(x, y, z) q_0(x', y', z') d\mathbf{r} d\mathbf{r}'}{((x - x')^2 + \alpha^4 (y - y')^2 + \alpha^2 (z - z')^2)^{1/2}} \ \ge \ \alpha W(q_0), \end{aligned}$$

and

$$
\begin{aligned}
J(q) &= \int_{\mathbb{R}^3} (c_0 + c_1 z) y^2 q_0(\alpha x, \alpha^{-1} y, z) d\mathbf{r} \\
&= \int_{\mathbb{R}^3} (c_0 + c_1 z)(\alpha y)^2 q_0(x, y, z) d\mathbf{r} \;=\; \alpha^2 J(q_0),
\end{aligned}
$$

whence $E(q) \geq \alpha W(q_0) - \alpha^2 J(q_0) > 0$ for sufficiently small $\alpha$. □

**Remark** The next lemma is adapted from Burton (1987b) Lemma 4, and its proof makes use of the observation that if $f$ is a non-negative decreasing function on $(0, \infty)$, then for $0 < \alpha < x$ we have

$$
\int_{x-\alpha}^{x} f \leq \frac{\alpha}{x} \int_0^x f, \tag{10}
$$

which is easily proved by a linear change of variables.

**Lemma 2** *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Then there is a positive constant $C$ (depending on $q_0$ only) such that*

$$
Kq(x, y, z) \leq C(x^2 + y^2)^{-1/6} \text{ whenever } x^2 + y^2 \geq 2,
$$

*for every doubly Steiner-symmetric stratified rearrangement $q$ of $q_0$.*

*Proof.* Let $\rho$ be the radius of the ball having the same volume as the set $\{\mathbf{r}' \in \mathbb{R}^3 | q_0(\mathbf{r}') > 0\}$. Let $\mathbf{r} = (x, y, z)$ and suppose $x^2 + y^2 = 2a^2$ where $a > 1$. Then $|x| \geq a$ or $|y| \geq a$; without loss of generality we assume $x \geq a$. Let $0 < b < a$. Fix a doubly Steiner-symmetric stratified rearrangement $q$ of $q_0$. Then

$$
\begin{aligned}
Kq(\mathbf{r}) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(\mathbf{r}') d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|} \\
&= \frac{1}{4\pi} \left( \int_{|x'-x|>b} + \int_{|x'-x|<b} \right) \frac{q(\mathbf{r}') d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|} \\
&\leq \frac{1}{4\pi b} \int_{\mathbb{R}^3} q + \frac{1}{4\pi} \left( \int_{q(\mathbf{r}')>0} \frac{d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|^2} \right)^{1/2} \left( \int_{|x'-x|<b} q^2(\mathbf{r}') d\mathbf{r}' \right)^{1/2} \\
&\leq \frac{1}{4\pi b} \int_{\mathbb{R}^3} q + \frac{1}{4\pi} \left( \int_{|\mathbf{r}'-\mathbf{r}|<\rho} \frac{d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|^2} \right)^{1/2} \left( \frac{b}{x} \int_{\mathbb{R}^3} q^2(\mathbf{r}') d\mathbf{r}' \right)^{1/2} \\
&\leq \frac{1}{4\pi b} \|q\|_1 + \frac{(4\pi\rho)^{1/2}}{4\pi} \left( \frac{b}{a} \right)^{1/2} \|q\|_2 \,,
\end{aligned}
$$

where the Steiner-symmetry in $x$ has been in conjunction with (10) used to derive the penultimate line. We now choose $b = a^{1/3}$ to obtain

$$
Kq(\mathbf{r}) \leq C 2^{-1/6} a^{-1/3} = C(x^2 + y^2)^{-1/6},
$$

for some positive constant $C$ depending only on $q_0$. $\square$

**Lemma 3** *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and have compact support. Let $a$ and $\gamma$ be positive numbers. Then there is a positive number $\beta$ such that for every stratified rearrangement $q$ of $q_0$ satisfying $E(q) \geq \gamma$, there is a cube $A$ of side $a$ for which*

$$|\{\mathbf{r}' \in A | q(\mathbf{r}') > 0\}| \geq \beta.$$

*Proof.* Consider a positive number $\beta$, and suppose there exists a stratified rearrangement $q$ of $q_0$ such that $E(q) \geq \gamma$, but

$$|\{\mathbf{r}' \in A | q(\mathbf{r}') > 0\}| < \beta$$

for every cube $A$ of side $a$. We show that for a sufficiently small choice of $\beta$ this leads to a contradiction. Let $\rho$ denote the radius of the ball of volume $\beta$.

Fix $\mathbf{r} \in \mathbb{R}^3$ and let $X$ denote a cube with centre $\mathbf{r}$ and side $na$, where $n$ is a positive integer to be chosen later. Then $X$ can be covered by cubes $A(1), \ldots, A(n^3)$ of side $a$. Hence

$$\begin{aligned}
Kq(\mathbf{r}) &= \frac{1}{4\pi}\left(\sum_{i=1}^{n^3}\int_{A(i)} + \int_{\mathbb{R}^3\setminus X}\right)\frac{q(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|} \\
&\leq \frac{1}{4\pi}\sum_{i=1}^{n^3}\left(\int_{A(i)} q^2(\mathbf{r}')d\mathbf{r}'\right)^{1/2}\left(\int_{\mathbf{r}'\in A(i),\, q(\mathbf{r}')>0}\frac{d\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|^2}\right)^{1/2} + \frac{2}{4\pi na}\int_{\mathbb{R}^3\setminus X} q(\mathbf{r}')d\mathbf{r}' \\
&\leq \frac{n^3}{4\pi}\left(\int_{\mathbb{R}^3} q^2(\mathbf{r}')d\mathbf{r}'\right)^{1/2}\left(\int_{|\mathbf{r}'-\mathbf{r}|<\rho}\frac{d\mathbf{r}'}{|\mathbf{r}'-\mathbf{r}|^2}\right)^{1/2} + \frac{2}{4\pi na}\int_{\mathbb{R}^3} q(\mathbf{r}')d\mathbf{r}' \\
&= n^3\rho^{1/2}(4\pi)^{-1/2}\|q\|_2 + 2(4\pi na)^{-1}\|q\|_1.
\end{aligned}$$

Consequently

$$W(q) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q(\mathbf{r})Kq(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \leq 2^{-2}n^3\pi^{-1/2}\rho^{1/2}\|q\|_2\|q\|_1 + (4\pi na)^{-1}\|q\|_1^2.$$

We now choose $n$ large enough to ensure $(4\pi na)^{-1}\|q_0\|_1^2 < \gamma/2$ and then choose $\beta$ (and therefore $\rho$) small enough to ensure $2^{-2}n^3\pi^{-1/2}\rho^{1/2}\|q_0\|_2\|q_0\|_1 < \gamma/2$, choices that depend on $a$, $\gamma$ and $q_0$ but not on the particular rearrangement $q$. We find that $E(q) \leq W(q) < \gamma$, and this contradiction shows that $\beta$ has the desired properties. $\square$

**Lemma 4** *Let $q_0 \in L^2(\mathbb{R}^3)$ be non-negative and vanish outside a cube of side $\alpha$ and centre $\mathbf{o}$. Let $a$, $\beta$ be positive numbers, $a < \alpha$. Then there is a positive number $\delta$ such that, if $q$ is any* *doubly Steiner-symmetric stratified rearrangement of* $q_0$ *satisfying*

$$|\{\mathbf{r}' \in A | q(\mathbf{r}') > 0\}| \geq \beta \tag{11}$$

*for some cube* $A$ *of side* $a$*, then*

$$|\{(x,y) \in \mathbb{R}^2 | Kq(x,y,z) - (c_0 + c_1 z)y^2 > \delta\}| > \alpha^2 \quad \forall z \in [-\alpha, \alpha].$$

*Proof.* Consider a cube $A$ of side $a$ and a doubly Steiner-symmetric stratified rearrangement $q$ of $q_0$ satisfying (11). Since $q_0(x,y,z)$ vanishes when $|z| > \alpha$, there is no loss of generality in assuming $A$ lies in the region defined by $-\alpha < z < \alpha$. Moreover, the symmetry of $q$ ensures that symmetrising $A$ in the $x$- and $y$-directions does not reduce the volume in (11); we may therefore assume $A$ is centred on the $z$-axis.

Suppose $\mathbf{r} = (x,y,z)$ with $|z| < \alpha$. Then, using (6),

$$\begin{aligned} Kq(\mathbf{r}) &\geq \frac{1}{4\pi} \int_A \frac{q(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|} \\ &\geq \frac{1}{4\pi}((|x| + a)^2 + (|y| + a)^2 + 4\alpha^2)^{-1/2} \int_A q(\mathbf{r}')d\mathbf{r}' \\ &\geq \frac{1}{8\pi}(x^2 + y^2)^{-1/2} \int_0^\beta q_0^* \;=:\; \kappa(x^2 + y^2)^{-1/2} \end{aligned}$$

say, provided that $(x^2+y^2)^{1/2} \geq \xi := (2a^2+4\alpha^2)^{1/2}$, where $^*$ denotes increasing rearrangement onto the real interval $(0, v)$ with $v = |\{\mathbf{r}' \in \mathbb{R}^3 | q_0(\mathbf{r}') > 0\}|$. Therefore

$$Kq(\mathbf{r}) - (c_0 + c_1\alpha)y^2 \geq \kappa(x^2 + y^2)^{-1/2} - (c_0 + c_1\alpha)y^2$$

whenever $(x^2 + y^2)^{-1/2} > \xi$. Now the planar region defined by the inequality

$$\kappa(x^2 + y^2)^{-1/2} - (c_0 + c_1\alpha)y^2 > 0$$

has infinite area, because it contains the region defined by the inequalities

$$0 < y < x, \quad y < \kappa^{1/2} 2^{-1/2} (c_0 + c_1\alpha)^{-1/2} x^{-1/2}$$

which has infinite area. We can therefore choose $\delta > 0$ such that the region defined by

$$\kappa(x^2 + y^2)^{-1/2} - (c_0 + c_1\alpha)y^2 > \delta, \quad x^2 + y^2 > \xi^2$$

has area at least $\alpha^2$. Then

$$|\{(x,y) | Kq(x,y,z) - (c_0 + c_1 z)y^2 > \delta\}| > \alpha^2$$

for $|z| < \alpha$, where $\delta > 0$ depends on $q_0$ but not on $q$. □

## 6 Proof of Theorem 1

Consider a rectangular domain $\Omega = Q \times I$ where $Q$ is a square centred at the origin in the $xy$-plane and $I = [z_0, z_1]$. Choose $\alpha \geq 2z_1$ so that if $Q$ has side at least $\alpha$ then $\Omega$ contains the support of $q_0$, and define

$$e = \sup\{E(q) | q \in \mathfrak{C}_\Omega(q_0)\}.$$

Let $\{q_n\}_{n=1}^\infty$ be a maximizing sequence, that is, a sequence in $\mathfrak{C}(q_0)$ for which $E(q_n) \to e$. Now $\mathfrak{C}(q_0)$ is a closed bounded convex set in the Hilbert space $L^2(\Omega)$, hence $\mathfrak{C}(q_0)$ is weakly compact, so $\{q_n\}_{n=1}^\infty$ has a subsequence $\{q_{n_j}\}_{j=1}^\infty$ that converges weakly in $L^2(\Omega)$ to some limit $\bar{q} \in \mathfrak{C}(q_0)$. The compactness of $K$ as a linear operator on $L^2(\Omega)$ (which follows from the square-integrability of $|\mathbf{r} - \mathbf{r}'|^{-1}$ over $\Omega \times \Omega$) ensures that $Kq_{n_j} \to K\bar{q}$ strongly in $L^2(\Omega)$, hence $E(q_{n_j}) \to E(\bar{q})$ as $j \to \infty$, and therefore $E(\bar{q}) = e$. This proves the existence of a maximiser $\bar{q}$ of $E$ relative to the extended class of functions $\mathfrak{C}(q_0)$, confined to the rectangle $\Omega$.

To derive the first-variation condition satisfied by $\bar{q}$, we use the strict convexity of $E$. Consider any $q \in \mathfrak{C}(q_0)$, $q \neq \bar{q}$. Then we have

$$\begin{aligned} E(\bar{q}) \geq E(q) &= E(\bar{q}) + \int_\Omega (q(\mathbf{r}) - \bar{q}(\mathbf{r}))(K\bar{q}(\mathbf{r}) - (c_0 + c_1 z)y^2)d\mathbf{r} + W(q - \bar{q}) \\ &> E(\bar{q}) + \int_\Omega (q - \bar{q})\Psi, \end{aligned}$$

where $\Psi(\mathbf{r}) = K\bar{q}(\mathbf{r}) - (c_0 + c_1 z)y^2$, hence

$$\int_\Omega q\Psi < \int_\Omega \bar{q}\Psi.$$

This shows that $\bar{q}$ is the unique maximiser relative to $\mathfrak{C}(q_0)$ of the bounded linear functional

$$q \mapsto \int_\Omega q\Psi.$$

Since Lemma 6 assures us that the supremum of any bounded linear functional relative to $\mathfrak{C}(q_0)$ is attained by at least one element of $\mathfrak{R}(q_0)$ we can deduce that $\bar{q} \in \mathfrak{R}(q_0)$. Lemma 7 provides a function $\varphi : \mathbb{R}^2 \to [0, \infty)$ such that $\bar{q}(x, y, z) = \varphi(\Psi(x, y, z), z)$ almost everywhere in $\mathbb{R}^3$, and $\varphi(\cdot, z)$ is increasing for almost every $z$. Thus, our maximiser relative to the extended set of functions $\mathfrak{C}(q_0)$ turns out to be a stratified rearrangement, and is an increasing function of $\Psi$ for almost every fixed $z$.

The above argument was conducted on a bounded domain $\Omega = Q \times I$, and in principle $\bar{q}$ could depend on the choice of $Q$. We now proceed to show that if $Q$ is chosen large enough, it ceases to have any influence whatever on the problem. This is achieved using the estimates developed in §5, which are $Q$-independent. We begin by recalling our observation in §4.5 that Steiner-symmetrisation of $q$ in either the $x$- or $y$-directions does not reduce $E(q)$. We therefore assume that $\bar{q}$ is doubly Steiner-symmetric.

By Lemma 1 we can choose $l \geq \alpha \geq 2z_1$ and $\gamma > 0$ such that if $Q$ has side at least $l$ then $e \geq \gamma$. Next an application of Lemmas 3 and 4 shows that $\delta > 0$ may be chosen, independent of $Q$ (having side at least $l$), such that $\Psi(\cdot, z) > \delta$ occurs on a set of area at least $\alpha^2$, for almost every $z \in I$. Since, for almost every $z$, the set where $\bar{q}(\cdot, z) > 0$ has area at most $\alpha^2$, and $\bar{q}(\cdot, z)$ is an increasing function of $\Psi(\cdot, z)$, it follows that $\bar{q}$ is positive only when $\Psi(\cdot) > \delta$, except on a set of measure zero.

Finally, the estimate of Lemma 2 shows that if $z \in I$ and $\Psi(x, y, z) > \delta$ then $x^2 + y^2 \leq \max\{2, (C/\delta)^6\}$, where $C$ is independent of $Q$. Let $Q_0$ denote the square whose side is $\max\{2^{1/2}, l, (C/\delta)^3\}$, and let $\Omega_0 = Q_0 \times I$. If $Q$ is larger than $Q_0$ then $\bar{q}$ vanishes outside $\Omega_0$; hence if $\bar{q}$ denotes the maximiser for $\Omega_0$, then $\bar{q}$ maximises $E$ over all stratified rearrangements of $q_0$. If the corresponding $\varphi$ is extended so that $\varphi(u, z) = 0$ for $u \leq \delta$, then each $\varphi(\cdot, z)$ is increasing, and $\bar{q} = \varphi(\Psi, z)$ almost everywhere. □

## 7 The space of stratified rearrangements

Here we extend some standard results on spaces of rearrangements to the stratified case. The issue that arises is whether the operations we perform at each $z$-level, fit together in a measurable way. Lemmas 6 and 7 are the stratified counterparts of Theorems 4 and 5 of Burton (1987a); we have taken the opportunity to simplify the proofs.

We begin with a result that was proved by Ryff (1965), Lemma 2, for functions on an interval. We omit the proof, since Ryff's argument carries over to our case with only a slight modification, concerning level sets having positive area. We indicate the necessary modificaation by giving a formula in the statement of Lemma 5.

If $\Theta \subset \mathbb{R}^N$ is a bounded measurable set and $\theta = |\Theta|$, a map $\sigma : \Theta \to (0, \theta)$ is called *measure-preserving* if $|\{\mathbf{t} \in \Theta | \sigma(\mathbf{t}) \leq \xi\}| = \xi$ for every $0 < \xi < \theta$. If $\sigma$ is a measure-preserving map then $|\sigma^{-1}(B)| = |B|$ for every measurable set $B \subset (0, \theta)$. A measure-preserving map need not be invertible.

**Lemma 5** *Let $U \subset \mathbb{R}^2$ be a bounded measurable set, with $|U| = m$ say, and let $f$ be a real* *integrable function on $U$. For $(x,y) \in U$ define*

$$\sigma(x,y) = |\{(x',y') \in U | f(x',y') < f(x,y)\}| + |\{(x',y') \in U | f(x',y') = f(x,y) \text{ and } x' < x\}|$$

*Then $\sigma : U \to [0,m]$ is a measure-preserving map and $f = f^* \circ \sigma$ almost everywhere in $U$.*

**Lemma 6** *Let $\Omega = Q \times I$ be a rectangular domain where $Q = (-\alpha,\alpha) \times (-\alpha,\alpha) \subset \mathbb{R}^2$ and $I = (z_0, z_1) \subset \mathbb{R}$. Let $q_0 \in L^2(\Omega)$ and $\psi \in L^2(\Omega)$, and let $q_0^*(\cdot,z)$ and $\psi^*(\cdot,z)$ be the increasing rearrangements of $q_0(\cdot,z)$ and $\psi(\cdot,z)$ respectively on $[0,\alpha^2]$, which exist for almost every $z \in I$. Then there is a measurable function $\sigma : \Omega \to [0,\alpha^2]$ such that for almost every $z \in I$, the map $\sigma(\cdot,z) : Q \to [0,\alpha^2]$ is measure-preserving, and $\psi(\cdot,z) = \psi^*(\sigma(\cdot,z),z)$ almost everywhere in $Q$.*

*Further $\tilde{q}(x,y,z) := q_0^*(\sigma(x,y,z),z)$ for $(x,y,z) \in \Omega$ defines $\tilde{q} \in \mathfrak{R}(q_0)$ that realises the supremum of $\int_\Omega q\psi$ relative to $\mathfrak{C}(q_0)$.*

*Proof.* For almost every $z \in I$, we have $q_0(\cdot,z),\ \psi(\cdot,z) \in L^2(Q)$, and for any rearrangement $\chi$ of $q_0(\cdot,z)$ we have

$$\int_Q \chi(x,y)\psi(x,y,z)dxdy \leq \int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)dt; \tag{12}$$

note that $q_0^*, \psi^* \in L^2((0,\alpha^2) \times I)$. The left-hand side of (12) defines a bounded linear functional of $\chi$; the inequality (12) therefore holds when $\chi$ belongs to the closed convex hull of the rearrangements of $q_0(\cdot,z)$. Now taking $q \in \mathfrak{C}(q_0)$ we can set $\chi = q(\cdot,z)$ in (12) and integrate with respect to $z$ to obtain

$$\int_\Omega q\psi \leq \int_{z_0}^{z_1} \int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)dtdz \quad \forall q \in \mathfrak{C}(q_0). \tag{13}$$

We now construct $q \in \mathfrak{R}(q_0)$ that realises equality in (13). Define

$$\sigma(x,y,z) = |\{(x',y') | \psi(x',y',z) < \psi(x,y,z)\}| + |\{(x',y') | x' < x \text{ and } \psi(x',y',z) = \psi(x,y,z)\}|.$$

Then $\sigma : \Omega \to [0,\alpha^2]$ is a measurable function. Moreover Lemma 5 assures us that for almost every fixed $z$, the map $\sigma(\cdot,z)$ is measure-preserving and satisfies $\psi(\cdot,z) = \psi^*(\sigma(\cdot,z),z)$. If we choose $\tilde{q}(\cdot,z) = q_0^*(\sigma(\cdot,z),z)$ then $\tilde{q} \in \mathfrak{R}(q_0)$, and for almost every $z$,

$$\int_Q \tilde{q}(x,y,z)\psi(x,y,z)dxdy = \int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)dt.$$

Now integrating with respect to $z$ yields equality in (13) as desired. □

**Lemma 7** *Let $\Omega = Q \times I$ be a rectangular domain where $Q = (-\alpha, \alpha) \times (-\alpha, \alpha) \subset \mathbb{R}^2$ and $I = (z_0, z_1) \subset \mathbb{R}$. Let $q_0 \in L^2(\Omega)$ and $\psi \in L^2(\Omega)$. Suppose $\int_\Omega q\psi$ attains its maximum relative to $\mathfrak{R}(q_0)$ at a unique element $\bar{q}$. Then there is a real function $\varphi$ defined on $\mathbb{R} \times I$ such that $\bar{q}(x,y,z) = \varphi(\psi(x,y,z),z)$ for almost every $(x,y,z) \in \Omega$, and such that $\varphi(\cdot,z)$ is increasing for almost every $z \in I$.*

*Proof.* Let $\psi^*$, $q_0$ and $\sigma$ be as in Lemma 6. Then $\psi(x,y,z) = \psi^*(\sigma(x,y,z),z)$ and, by uniqueness and Lemma 6, $\bar{q}(x,y,z) = q_0^*(\sigma(x,y,z),z)$, for almost every $(x,y,z) \in \Omega$.

Now for almost every $z \in I$, the functions $q_0^*(\cdot,z)$ and $\psi^*(\cdot,z)$ are increasing on $[0,\alpha^2]$. In order to show that $q_0^*(\cdot,z)$ is almost everywhere an increasing function of $\psi^*(\cdot,z)$, it will be enough to show that on any open interval where $\psi^*(\cdot,z)$ is constant, $q_0^*(\cdot,z)$ is constant also, for almost every $z \in I$.

Consider rational numbers $r < s$ and let $Z(r,s)$ denote the set of $z \in I$ such that $\psi^*(\cdot,z)$ is constant on the open interval $(r,s)$ but $q_0^*(\cdot,z)$ is non-constant on $(r,s)$. Then $Z(r,s)$ is measurable; we show $Z(r,s)$ has measure zero. Consider the possibility that $Z(r,s)$ has positive measure. Define

$$\hat{q}(t,z) = \begin{cases} q_0^*(r+s-t,z) & \text{if } t \in (r,s) \text{ and } z \in Z(r,s), \\ q_0^*(r,s) & \text{if } t \notin (r,s) \text{ or } z \notin Z(r,s). \end{cases}$$

Then, for almost every $z \in I$, $\hat{q}(\cdot,z)$ is a rearrangement of $q_0^*(\cdot,z)$. Hence

$$q_1(x,y,z) = q_0^*(\sigma(x,y,z,)z) \ \forall (x,y,z) \in \Omega$$

defines $q_1 \in \mathfrak{R}(q_0)$, and moreover the constancy of $\psi(\cdot,z)$ on $(r,s)$ for $z \in Z(r,s)$ ensures that

$$\int_\Omega q_1\psi = \int_{z_0}^{z_1}\int_0^{\alpha^2} \hat{q}(t,z)\psi(t,z)dtdz = \int_{z_0}^{z_1}\int_0^{\alpha^2} q_0^*(t,z)\psi^*(t,z)dtdz = \int_\Omega \bar{q}\psi.$$

But $q_0^*(\cdot,z)$ is increasing and nonconstant on $(r,s)$ for all $z \in Z(r,s)$ hence $\hat{q}$ differs from $q_0^*$ on a set of positive measure, hence $q_1$ differs from $\bar{q}$ on a set of positive measure. This contradicts the uniqueness of the maximiser $\bar{q}$. Hence $Z(r,s)$ has zero measure as desired.

Now let

$$Z = \bigcup_{r,s\in\mathbb{Q},\ r<s} Z(r,s)$$

which has zero measure, being a countable union of sets of zero measure (here $\mathbb{Q}$ denotes the set of all rational numbers). Consider $z \in I \setminus Z$. If $\psi^*(\cdot,z)$ is constant on an interval $(p,q)$, then $q_0^*(\cdot, z)$ is constant on $(r, s)$ for all rationals $r$ and $s$ with $p < r < s < q$, hence $q_0^*(\cdot, z)$ is constant on $(p, q)$. Therefore $q_0^*(\cdot, z) = \varphi(\psi^*(\cdot, z), z)$ almost everywhere on $[0, \alpha^2]$ for some increasing function $\varphi(\cdot, z)$; then by composing with $\sigma$ we obtain $\bar{q}(\cdot, z) = \varphi(\psi(\cdot, z), z)$ almost everywhere on $Q$.

Thus $\bar{q}(x, y, z) = \varphi(\psi(x, y, z), z)$ for almost all $(x, y, z) \in \Omega$ as desired. □

## 8 Discussion

In Theorem 1 we have proved the existence of a stationary vortex solution of equation (3) in the set of stratified rearrangements of any given PV-anomaly field $q_0$ (i.e. in any family of isovortical flows) that satisfies the following conditions: $q_0$ must have compact support, it must have the same sign everywhere, and this sign must be the same as the sign of the background shear $2(c_0 + c_1 z)$ over the interval in $z$ to which the support of $q_0$ is confined. If $q_0 \geq 0$ the PV-field of the maximiser is symmetric decreasing in $x$ and $y$ for every fixed $z$ (symmetric increasing if $q_0 \leq 0$).

The fact that a flow maximises the energy implies that it is linearly stable (Nycander, 1995). It should also mean that the flow is nonlinearly stable in a practical sense, as argued by Benjamin (1976). This is analogous to Lyapunov stability for a system with a finite number of degrees of freedom. However, we cannot formalize this to a statement of stability in some norm.

In one case the shape of the stationary vortex can be found analytically. If the PV-anomaly is constant inside an ellipsoidal surface, and vanishes outside this surface, and if the stream-function of the background flow is a quadratic function, then the discontinuity surface will always remain ellipsoidal, and the general time-dependent solution can be found (Meacham et al 1994). Steady solutions of this kind can be found both in adverse shear and cooperative shear, and the present result implies that those in cooperative shear are stable.

One possible generalisation of Theorem 1 is to add a term $-(d_0 + d_1 z)x^2$ to the stream-function of the background flow, which is then a general strain flow. In accordance with the heuristic argument of section 2, we expect an energy maximiser to exist if this stream-function is sign-definite, i.e. if the origin is an elliptic stagnation point of the background flow. If the origin is a hyperbolic stagnation point, on the other hand, the external potential of the heuristic argument has no minimum, and it is clear that no maximiser exists.

Another generalisation is to add a constant vertical shear to the background flow of equation

(3). This case can be recovered from equation (3) by a shift of the coordinate system in the $y$-direction. In effect, this means that the energy maximiser is a vortex located at the $y$-value where the vertical shear vanishes.

In our model we assumed that there are no vertical boundaries. Often, however, equation (2) is solved with the boundary conditions $\partial\Psi/\partial z = 0$ at $z = 0$ and $z = H$. We believe that the corresponding existence theorem is true for this bounded case as well, but we have not been able to prove this. The problem is that the energy functional is not convex in the bounded case. Thus, the theorem proved here is relevant only for vortices that are small (both vertically and horizontally) compared to the total height of the atmosphere.

An important difference between the two cases is that a stationary vortex has a separatrix in the unbounded case considered here, but not in the bounded case. The reason is that the Green's function diverges logarithmically in the in the bounded case, while it behaves as $1/|\mathbf{r}|$ in the case considered here.

As shown in Lemma 1, the energy maximiser must have positive energy. This makes it possible to estimate the amplitude necessary for a stationary vortex to have approximately spherical shape, as opposed to a strongly elongated shape. If we assume, for simplicity, that $q = q_0 = const.$ inside and $q = 0$ outside the sphere $|\mathbf{r}| < a$, and that the background flow is independent of $z$ (i.e. that $c_1 = 0$), it is straightforward to calculate that the energy is $E = (2/15)\pi q_0 a^5(2q_0 - c_0)$. The first term represents the perturbation energy $W$ and the second term the external contribution $J$ in equation (4). Hence, if $q_0 < c_0/2$ the stationary vortex must be significantly elongated, and is therefore probably less robust than if $q_0 > c_0/2$. (The perturbation energy $W$ can be thought of as a "binding energy" of the vortex.) This crude estimate is perhaps supported by the observation in the turbulence simulations by McWilliams (1989) that coherent vortices emerge in regions where the vorticity is larger than the local strain rate.

Numerical simulations of three-dimensional quasigeostrophic flow have revealed a tendency for vortices to align vertically (if they have the same sign) and to axisymmetrise horizontally (McWilliams 1989, Viera 1995, Sutyrin et al 1996). Both these processes can be interpreted as a tendency to approach the maximum energy state, which is a vertically aligned axisymmetric vortex (in the absence of background flow).

It is typical for many nonlinear infinite-dimensional systems that conditional extreme points of conserved quantities act as attractors in this way. In dissipative systems this is often interpreted as a "selective decay" of the invariants. In the ideal model used here, the conservation of PV and energy of course prevents an unsteady flow from evolving into a maximum energy state. However, the excitation of small scales (i.e. filamentation of the PV-field) can effectively act like dissipation, and in a course-grained sense move the flow to a different isovortical family where it is close to a maximum energy state. This is the basic idea behind the statistical mechanical theory for ideal two-dimensional flow of Miller (1990) and Robert & Sommeria (1991). It seems likely that this theory can be generalised to the model studied in the present article.

The vertical alignment and horizontal axisymmetrisation are irreversible, nonlinear processes. However, it has also been observed in simulations that columns of uniform PV can perform a reversible and almost periodic motion (Viera 1995, Dritschel & Ambaum 1996, Sutyrin et al 1996). This can be interpreted as an essentially linear wave on the axisymmetric stationary state. The dispersion relation for these waves is $\omega = mQ(1/2 - I_m(ka)K_m(ka))$, where $Q$ is the PV and $a$ the radius of the column, $I_m$ and $K_m$ modified Bessel functions, and $m$ and $k$ the azimuthal and vertical wavenumbers, respectively. The nonlinear, irreversible behaviour sets in only if the wave amplitude (i.e. the deviation from the axisymmetric state) is large enough, as studied in detail by Sutyrin et al (1996). We caution, however, that a column of uniform PV can probably tolerate oscillations of larger amplitude before the nonlinear behaviour sets in than smoother vortices. If for example, the PV is a strictly decreasing (or strictly increasing) function of $r$, no normal modes exist, as can be shown similarly as in Appendix B of Åkerstedt et al (1996). This means that any infinitesimal perturbation will be sheared away, and that the vortex approaches axisymmetry as $t \to \infty$.

In numerical simulations of three-dimensional quasigeostrophic turbulence that use the boundary conditions $\partial\Psi/\partial z = 0$ at $z = 0$ and $z = H$, a very clear preference is seen for coherent vortices to form at the top or the bottom of the domain (McWilliams 1989, Dritschel & Ambaum 1996). This can be understood in terms of the maximum energy argument employed in the present work. Poisson's equation can in this case be solved by introducing mirror vortices outside the boundaries of the domain, with the same sign as the real vortices. If a vortex touches the boundary it also touches a mirror vortex, in effect forming a "virtual vortex" twice the size of the real vortex. The energy is therefore much larger than if the real vortex were situated in the middle of the domain. This makes vortices at the boundaries more robust.

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