THE q-SCHUR² ALGEBRA

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Abstract. We study a class of endomomorphism algebras of certain q-permutation modules over the Hecke algebra of type B, whose summands involve both parabolic and quasi-parabolic subgroups, and prove that these algebras are integrally free and quasi-hereditary, and are stable under base change. Some consequences for decomposition numbers are discussed.

The notion of a q-Schur algebra was introduced by Dipper and James [DJ2], who used these algebras to parametrize the irreducible representations of the finite general linear groups in non-describing characteristics. With hindsight, these algebras had already appeared earlier in an entirely different quantum group context [Ji] inspired by physics. In [Ji] Jimbo considered the endomorphism algebras of tensor spaces as Hecke algebra modules. In his context, a q-Schur algebra can be viewed as a quotient of the quantized enveloping algebra associated to $\mathfrak{gl}_n$. In [PW], these algebras were shown to be quasi-hereditary. The quasi-heredity property is an embodiment in classical algebra of the geometric derived category stratification exhibited by perverse sheaves [PS]. It means more applications can be deduced from a ring-theoretic point of view; see e. g., [DPS3], and the possibility is raised of even deeper results in the future, as suggested by [CPS2].

Certainly, these algebras play a central role in the representation theories of the finite and quantum general linear groups. Naturally, one asks: Are there such algebras for types other than A? Our paper [DS] showed that there were similar quasi-hereditary quotients of quantized enveloping algebras for all types of root systems. However, no connection with Hecke algebras and finite groups of Lie type was found there.

This paper aims at the same question and constructs possible algebras directly from Hecke algebras (hence from finite groups of Lie type). We restrict attention to the type B case. Imitating the definition of a q-Schur algebra, we introduce

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the notion of a q-Schur^2 algebra. These algebras are the endomorphism algebras of certain modules over the Hecke algebra of type B — called "tensor" spaces — whose summands involve not only parabolic subgroups but also quasi-parabolic subgroups of the Weyl group of type B. A main result of our paper shows that q-Schur^2 algebras are quasi-hereditary. We speculate that similar constructions exist for other classical types (only type D remains, actually, since type C is equivalent to type B in our context), while a weaker variation, at least, applies in general. See our papers [DPS1] and [DPS2] with Brian Parshall.

We outline the contents of the paper. Section 1 collects some facts about Young tableaux and bitableaux. The notion of a semi-standard bitableau is new. In section 2, we introduce the notion of a quasi-parabolic subgroup of the Weyl group of type B in the restricted sense. We describe the distinguished coset and double coset representatives for these subgroups. The notion of a q-Schur^2 algebra is given in section 3. In section 4, we first characterize q-permutation modules associated to quasi-parabolic subgroups in terms of certain eigenspaces in §4.1. Then we prove the freeness of the q-Schur^2 algebras and their base change property. In section 5, we generalize the classical Young rule to the type B case, that is, we prove that the number of semi-standard bitableaux is equal to the multiplicity of a Specht module in a permutation module. Integral Specht filtrations for permutation modules are discussed after introducing "Murphy" type bases for these modules. Finally, in section 6, we prove the quasi-heredity of a q-Schur^2 algebra. We also discuss some consequences for decomposition numbers for related algebras of known interest, and make further remarks. Over a field our theory applies in all characteristics, without exception.

The major part of the work, that is, the freeness and quasi-heredity of the integral q-Schur^2 algebras, was completed in late 1995. Parts of the ideas and results have been communicated since then, both privately and publicly, and were announced briefly by the second author at the AMS (summer institute) conference in Seattle, July, 1996. All results here were presented by the first author at a seminar of University of Chicago in November, 1996. We thank R. Dipper for letting us know in late September about a base change property he and James had obtained in their work on an endomorphism algebra of type B. We obtained our result (4.1.2), which is crucial in proving the base change property, after knowing their result existed. After an earlier version of this manuscript was completed, we received a preprint by R. Dipper, G. James and A. Mathas entitled "The (Q,q)-Schur algebra", evidently influenced by our work (for example, through [D2]). It turns out that their algebra is Morita equivalent to the q-Schur^2 algebra.

1. Young tableaux

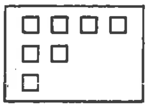
In this section we collect some definitions and results on partitions and Young tableaux.

1.1 Standard and semi-standard tableaux. A *composition* α of a nonnegative[1] integer r, denoted $\alpha \models r$, is a finite sequence $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers with sum $|\alpha| = \sum_i \alpha_i = r$. A composition α is *tight* if all zero parts appear

[1] It is useful to allow $r = 0$ here and in the definitions below, to avoid special cases.

at the right hand side of the sequence, and α is called a *partition*, denoted $\alpha \vdash r$, if the sequence is non-decreasing. We denote by $\Lambda(n,r)$ the set of all compositions of r with n parts (counting zeros). Elements of $\Lambda(n,r)$ may be identified with elements of $\Lambda(m,r)$, if $m \geq n$, by adding zeros on the right, though often we will want to keep track of n.

Let $\trianglelefteq$ be the dominance order on compositions. Thus, $\alpha \trianglelefteq \beta$ if and only if $\sum_{i=1}^{j} \alpha_i \leq \sum_{i=1}^{j} \beta_i$ for all j. For a composition α of r, we may identify α with its corresponding *diagram* which consists of boxes arranged in a manner as illustrated by the example $\alpha = (421)$, for which we have

$\alpha =$ 

Let α' be the *dual* partition of $\alpha \models r$: thus $\alpha'_i = \#\{\alpha_j \geq i\}$. The double dual is denoted α''; if α is a partition, then $\alpha = \alpha''$.

An *α-tableau* $\mathbf{t}$ is obtained by replacing boxes by positive integers. We will call the α-tableau *regular* if its entries are the numbers $1, 2, \cdots, r$ with no repeats. The symmetric group $\mathfrak{S}_r$ acts on the set of regular α-tableaux by permuting the entries. A regular tableau $\mathbf{t}$ is called *row-standard* if each row of $\mathbf{t}$ is an increasing sequence, and *standard* if both row and columns of $\mathbf{t}$ are increasing. Let $\mathbf{t}^\alpha$ be the α-tableau in which the numbers $1, 2, \cdots, r$ appear in order along successive rows. Let $\mathfrak{S}_\alpha$ be the row stabilizer of $\mathbf{t}^\alpha$ and put $\bar{\mathcal{D}}_\alpha = \{w \in \mathfrak{S}_r \mid \mathbf{t}^\alpha w \text{ is row-standard}\}$. Thus, we have a bijection $\delta = \delta_\alpha$ from the set of all row-standard α-tableaux to $\bar{\mathcal{D}}_\alpha$, satisfying $\mathbf{s} = \mathbf{t}^\alpha \delta(\mathbf{s})$ for any $\mathbf{s}$. We remark that $\bar{\mathcal{D}}_\alpha$ is the distinguished cross section of minimal length for $\mathfrak{S}_\alpha \backslash \mathfrak{S}_r$, in the sense of Coxeter groups. (As a permutation, $\delta(\mathbf{s})$ has the smallest number of order inversions among elements in its right coset.)

For a partition β of r, let $\mathbf{t}_\beta$ be the standard β-tableau in which the numbers $1, 2, \cdots, r$ appear in the same order down successive *columns*. Let w_β be the element in $\mathfrak{S}_r$ defined by $\mathbf{t}^\beta w_\beta = \mathbf{t}_\beta$. Thus, by [DJ1; (1.5)], the set of all standard β-tableaux consists of all $\mathbf{t}^\beta d$ where $dx = w_\beta$ for some x with $\ell(d) + \ell(x) = \ell(w_\beta)$.

A β-tableau of type α is a β-tableau such that, for each i, the number of entries i is equal to α_i. A β-tableau $\mathbf{t}$ of type α is called *semi-standard* if its entries are nondecreasing along each row and increasing along each column. We denote by $\mathfrak{T}(\beta,\alpha)$ the set of all β-tableaux of type α and by $\mathfrak{T}^{ss}(\beta,\alpha)$ the set of all semi-standard β-tableaux of type α. Clearly, $\mathfrak{T}^{ss}(\beta,\omega)$ is the set of all standard β-tableaux, where $\omega = (1^r)$.

We now define a map $\delta(*,*) : W \times \mathfrak{T}(\beta,\alpha) \to \bar{\mathcal{D}}_\alpha$ as follows: If $w \in W$ and $\mathfrak{s} \in \mathfrak{T}(\beta,\alpha)$, we define $\delta(w,\mathfrak{s}) \in \bar{\mathcal{D}}_\alpha$ by letting $\mathbf{t}^\alpha \delta(w,\mathfrak{s})$ be the row-standard α-tableau for which i belongs to row a if the place occupied by i in $\mathbf{t}^\beta w$ is occupied by a in $\mathfrak{s}$.

(1.1.1) The map $\delta(1,*)$ gives a bijection between $\mathfrak{T}(\beta,\alpha)$ and $\bar{\mathcal{D}}_\alpha$, and $\mathfrak{s}$ has non-decreasing rows if and only if $\delta(1,\mathfrak{s}) \in \bar{\mathcal{D}}_{\alpha\beta} = \bar{\mathcal{D}}_\alpha \cap \bar{\mathcal{D}}_\beta^{-1}$ ([DJ1; (1.7)]).

As a Coxeter group, $\mathfrak{S}_r$ is generated by basic transpositions $(1,2), \cdots, (r-1,r)$ and every $\mathfrak{S}_\alpha$ is generated by the subset consisting of those $(i,i+1)$ which stabilize the rows of $\mathbf{t}^\lambda$. If α is tight, it may be recovered from, and identified with, this subset. For $d \in \bar{\mathcal{D}}_{\alpha\beta}$, let $\alpha d \cap \beta$ and $\alpha \cap d\beta$ denote the subsets defined by the Young subgroups

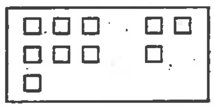
$$\mathfrak{S}_{\alpha d\cap\beta} = d^{-1}\mathfrak{S}_\alpha d \cap \mathfrak{S}_\beta \text{ and } \mathfrak{S}_{\alpha\cap d\beta} = \mathfrak{S}_\alpha \cap d\mathfrak{S}_\beta d^{-1}, \tag{1.1.2}$$

respectively. We also identify $\alpha d \cap \beta$ and $\alpha \cap d\beta$ with the corresponding tight compositions.

1.2 Bitableaux. A *multi-composition* of r is a finite sequence $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ of compositions $\lambda^{(i)} \models r_i$ such that the total sum $|\lambda| = \sum_i r_i = r$. When $m = 2$, λ is called a *bicomposition.* A bicomposition λ is *tight* if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are tight, and λ is called a *bipartition* if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bicomposition. Thus, $\lambda^{(i)} = (\lambda_1^{(i)}, \cdots, \lambda_{m_i}^{(i)})$ $(i = 1, 2)$ is a finite sequence of non-negative integers with sum $|\lambda^{(1)}| + |\lambda^{(2)}| = r$, where $|\lambda^{(i)}| = \sum_j \lambda_j^{(i)}$. Let Π_r be the set of all bicompositions of r and Π_r^+ the set of all bipartitions of r. Later in §3, we shall consider for a positive integer n the set $\Pi(n, r)$ of bicompositions λ such that each $\lambda^{(i)}$ has n parts (some possibly zero), i.e., $m_1 = m_2 = n$. Clearly, when $n \geq r$, Π_r^+ can be viewed as a subset of $\Pi(n, r)$ naturally. Note that $\Pi(n, r)$ identifies the orbits of the set $I^2(n, r) = \{(i_1, \cdots, i_r) \mid -n \leq i_j \leq n, i_j \neq 0, \forall j\}$ on which the symmetric group $\mathfrak{S}_r$ acts by "place permutations".

For a bicomposition λ, we sometimes identify λ with its corresponding *diagram* which consists of boxes arranged in a manner as illustrated by the example $\lambda = (331, 21)$, for which we have

$\lambda =$ 

(1.2.1) Definition and Notation. Let $\lambda \in \Pi_r$.

(a) The *dual* bipartition λ' of λ is defined as $\lambda' = (\lambda^{(2)\prime}, \lambda^{(1)\prime})$ where $\lambda^{(i)\prime}$ denote the dual partition of $\lambda^{(i)}$. The double dual is denoted λ''; if the composition λ is a bipartition, then $\lambda = \lambda''$. Let $\trianglelefteq$ denote the dominance order on bicompositions, that is, $\lambda \trianglelefteq \mu$ iff $\sum_{i=1}^j \lambda_i^{(1)} \leq \sum_{i=1}^j \mu_i^{(1)}$ for all j and $|\lambda^{(1)}| + \sum_{i=1}^{j'} \lambda_i^{(2)} \leq |\mu^{(1)}| + \sum_{i=1}^{j'} \mu_i^{(2)}$ for all j'. Then we clearly have that, for bipartitions λ, μ, $\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$.

(b) For compositions α and β, let $\alpha \vee \beta$ be the composition obtained by concatenating α and β, i. e., $\alpha \vee \beta = (\alpha_1, \alpha_2 \cdots, \beta_1, \beta_2, \cdots)$. Let $\bar{\lambda} = \lambda^{(1)} \vee \lambda^{(2)}$. Then $\bar{\lambda}$ is a composition of r. Sometimes, we also identify $\bar{\lambda}$ with the bicomposition $(-, \bar{\lambda})$ in Π_r. (We could make a similar identification for any composition.) Thus, for $\lambda, \mu \in \Pi(n, r)$, we have $\lambda \trianglelefteq \mu$ if and only if $\bar{\lambda} \trianglelefteq \bar{\mu}$.

(c) Let $\underline{\lambda} = (\underbrace{1 \cdots 1}_{|\lambda^{(1)}|}, \underbrace{1 \cdots 1}_{|\lambda^{(2)}|})$ and $\hat{\lambda}$ the bicomposition $(|\lambda^{(1)}|, \lambda^{(2)})$.

As in §1.1, replacing boxes by positive integers, we obtain a λ-*bitableau.* A *regular* λ-bitableau $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ is obtained by replacing each box by one of the numbers $1, 2, \cdots, r$, allowing no repeats. We call λ the *shape* of $\mathbf{t}$. A regular bitableau $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ is called *row-standard* if each row of each $\mathbf{t}_i$ is an increasing sequence, and *standard* if both rows and columns of the $\mathbf{t}_i$ are increasing. We define $\mathbf{t}^\lambda$ to be the standard λ-bitableau in which the numbers $1, 2, \cdots, r$ appear in the same order down successive rows in the first diagram of λ and then in the second diagram, and $\mathbf{t}_\lambda$ the standard λ-bitableau in which the numbers $1, 2, \cdots, r$ appear in the same order down successive columns in the second diagram of λ and then in the first diagram. For example, if $\lambda = (331, 21)$, then

$$\mathbf{t}^\lambda = \boxed{\begin{array}{lll} 1\,2\,3 & & 8\;9 \\ 4\,5\,6 & & 10 \\ 7 & & \end{array}} \quad \text{and} \quad \mathbf{t}_\lambda = \boxed{\begin{array}{lll} 4\,7\,9 & & 1\;3 \\ 5\,8\,10 & & 2 \\ 6 & & \end{array}}$$

Let $\mathfrak{S}_r$ act on the set of regular bitableaux by permuting the entries and define w_λ to be the element of $\mathfrak{S}_r$ satisfying $\mathbf{t}^\lambda w_\lambda = \mathbf{t}_\lambda$.

For a λ-bitableau $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$, we define $\bar{\mathbf{t}} = \begin{array}{c}\mathbf{t}_1 \\ \mathbf{t}_2\end{array}$. This is a tableau of shape $\bar{\lambda}$. Note that $\bar{\mathbf{t}}^\lambda = \mathbf{t}^{\bar{\lambda}}$. We define $\mathbf{t}' = (\mathbf{t}_2', \mathbf{t}_1')$, where $\mathbf{t}_i'$ denotes the transpose of $\mathbf{t}_i$. So we have $(\mathbf{t}_\lambda)' = \mathbf{t}^{\lambda'}$ and $(\mathbf{t}^\lambda)' = \mathbf{t}_{\lambda'}$.

We now define semi-standard bitableaux. Let μ be a bipartition and λ a bicomposition. A *μ-bitableau of type* λ (indeed, of type $\bar{\lambda}$) is a μ-bitableau with possibly repeated positive integer entries such that, for each i, the number of entries i is equal to $\bar{\lambda}_i$ where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots)$. (Recall $\bar{\lambda}$ is *defined* as the composition $\lambda^{(1)} \vee \lambda^{(2)}$.)

(1.2.2) Definition. A μ-bitableau $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ is called *semi-standard* if

(ss1) $\mathbf{t}$ is a μ-bitableau of type λ, for some composition λ

(ss2) both $\mathbf{t}_1, \mathbf{t}_2$ are semi-standard (i.e., have nondecreasing rows and increasing columns), and

(ss3) $\mathbf{t}_1$ contains an α-tableau of type $\lambda^{(1)}$ as a subtableau for some partition α. This subtableau must appear at the top-left corner in $\mathbf{t}_1$.

For example, if $\mu = (321, 21)$ and $\lambda = (211, 32)$, the following are semi-standard μ-bitableaux of type λ:

$$\boxed{\begin{array}{lll} 1\,1\,5 & & 4\;4 \\ 2\,4 & & 5 \\ 3 & & \end{array}} \qquad \boxed{\begin{array}{lll} 1\,1\,3 & & 4\;5 \\ 2\,4 & & 5 \\ 4 & & \end{array}}$$

The first one has a semi-standard (211)-tableau of type (211) as a subtableau, and the second has a semi-standard (31)-tableau of type (211) as a subtableau.

Let $\mathfrak{T}(\mu, \lambda)$ be the set of all μ-bitableaux of type λ and $\mathfrak{T}^{ss}(\mu, \lambda)$ the set of all semi-standard μ-bitableaux of type λ.

For a bipartition $\mu \in \Pi_r^+$, a bicomposition $\lambda \in \Pi_r$ and each semi-standard bitableau $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu, \lambda)$, the induced tableau $\bar{\mathfrak{s}}$ is a $\bar{\mu}$-tableau of type $\bar{\lambda}$ with nondecreasing rows. So, it defines a distinguished double coset representative $\delta(1, \bar{\mathfrak{s}}) \in \mathcal{D}_{\bar{\lambda}\bar{\mu}}$ (cf. (1.1.1)). We write $\delta(\mathfrak{s}) = \delta(1, \bar{\mathfrak{s}})$. For example, if $\mu = (321, 21)$ and $\lambda = (211, 32)$, we take

$$\mathfrak{s} = \boxed{\begin{array}{lll} 1\,1\,5 & & 4\;4 \\ 2\,4 & & 5 \\ 3 & & \end{array}}$$

Then

$$\mathbf{t}^{\mu} = \boxed{\begin{array}{ll} 1\,2\,3 & 7\,8 \\ 4\,5 & 9 \\ 6 & \end{array}} \quad \text{and} \quad \mathbf{t}^{\lambda}\delta(\mathfrak{s}) = \boxed{\begin{array}{ll} 1\,2 & 5\,7\,8 \\ 4 & 3\,9 \\ 6 & \end{array}}$$

Note that $\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}$ can be obtained by replacing all the numbers i in $\mathfrak{s}$ by the sequence obtained by reading the i-th row in $\mathbf{t}^{\bar{\lambda}}$. The replacements in $\mathfrak{s}$ are made from left to right, down successive rows. (For a proof, observe that replacing $\mathbf{t}^{\lambda}$ by $\mathbf{t}^{\lambda}\delta(\mathfrak{s})$ in this procedure gives $\mathbf{t}^{\mu}$.) Thus, in the previous example, we have

$$\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1} = \boxed{\begin{array}{ll} 1\,2\,8 & 6\,7 \\ 3\,5 & 9 \\ 4 & \end{array}}$$

Let μ be a bipartition of r, and let $\mathbf{T}^{s}(\mu)$ be the set of all standard μ-bitableaux. We define $\mathfrak{f} : \mathbf{T}^{s}(\mu) \to \mathfrak{T}(\mu, \lambda)$ by sending $\mathbf{t}$ to $\mathfrak{f}(\mathbf{t})$ which is obtained by replacing each number in $\mathbf{t}$ by its row index in $\mathbf{t}^{\bar{\lambda}}$. Thus, $\mathfrak{f}(\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}) = \mathfrak{s}$ from the above. We put

$$\mathbf{T}_{\mathfrak{s}} = \mathbf{T}_{\mathfrak{s}}(\mu, \lambda) = \mathfrak{f}^{-1}(\mathfrak{s})$$

for any $\mathfrak{s} \in \mathfrak{T}(\mu, \lambda)$. Note that, if $\mathfrak{f}(\mathbf{t})$ is a semi-standard μ-tableau of type λ, then there are no two elements in the same row of $\mathbf{t}^{\bar{\lambda}}$ and in the same column of $\mathbf{t}$. The following result will be useful in §5.

(1.2.3) Lemma. *Keep the notation introduced above and let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{T}^{ss}(\mu, \lambda)$.*

(a) *We have $\{1, \cdots, |\lambda^{(1)}|\}\delta(\mathfrak{s}) \subseteq \{1, \cdots, |\mu^{(1)}|\}$.*

(b) *Let $\alpha = \bar{\lambda} \cap \delta(\mathfrak{s})\bar{\mu}$. Then $\mathfrak{S}_{\bar{\lambda}} \cap \mathcal{D}_{\alpha} = \{x \in \mathfrak{S}_{\bar{\lambda}} \mid \mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}x$ is standard$\}$.*

(c) *$\mathbf{T}_{\mathfrak{s}} \cap \mathbf{T}_{\mathfrak{s}'} = \emptyset$ if $\mathfrak{s} \neq \mathfrak{s}'$, and $\mathbf{T}_{\mathfrak{s}} = \{\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}x \mid x \in \mathfrak{S}_{\bar{\lambda}} \cap \mathcal{D}_{\alpha}\}$.*

Proof. (a) is obvious from the definition of semi-standard μ-bitableaux of type λ. To prove (b), we first note that $\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}$ is standard, as $\mathfrak{s}$ has strictly increasing columns. So, for $x \in \mathfrak{S}_{\bar{\lambda}}$, if $\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}x$ is standard, then $\delta(\mathfrak{s})^{-1}x \in \bar{\mathcal{D}}_{\bar{\mu}}$, forcing $x \in \bar{\mathcal{D}}_{\alpha}$ since $\delta(\mathfrak{s}) \in \bar{\mathcal{D}}_{\bar{\lambda}\bar{\mu}}$. Conversely, for $x \in \mathcal{D}_{\alpha} \cap \mathfrak{S}_{\bar{\lambda}}$, write $x = w_1 \cdots w_k$ with $w_i \in S$ and $\ell(x) = k$, and put $\mathbf{t}_m = \mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}w_1 \cdots w_m$. We apply induction on k. (Note that $w_1 \cdots w_{k-1} \in \bar{\mathcal{D}}_{\alpha} \cap \mathfrak{S}_{\bar{\lambda}}$, so we may apply induction to this element.) The result is clear if $k = 0$. Suppose $w_k = (j, j+1)$. Then j and $j+1$ are not in the same row of $\bar{\mathbf{t}}_{k-1}$, as $\mathbf{t}_k = \mathbf{t}_{k-1}(j, j+1)$ is row-standard. So j and $j+1$ belong to distinct rows of $\bar{\mathbf{t}}_{k-1}$. Since $\mathfrak{f}(\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}) = \mathfrak{s}$ and $w_1 \cdots w_{k-1} \in \mathfrak{S}_{\bar{\lambda}}$, we have $\mathfrak{s} = \mathfrak{f}(\mathbf{t}_{k-1})$. So, $\mathfrak{f}(\mathbf{t}_{k-1})$ is semistandard. Now, the fact that j and $j+1$ are in the same row of $\mathbf{t}^{\bar{\lambda}}$ forces that j and $j+1$ are not in the same column of $\mathbf{t}_{k-1}$. Therefore, $\mathbf{t}_k$ is standard by induction, proving (b). Finally, the inclusion "$\supseteq$" in (c) is obvious, since $\mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1} \in \mathbf{T}_{\mathfrak{s}}$, and $x \in \mathfrak{S}_{\bar{\lambda}}$. Thus, $\mathbf{T}_{\mathfrak{s}}$ is a subset of $\{\mathbf{t} \in \mathbf{T}^{s}(\mu) \mid \mathbf{t} = \mathbf{t}^{\mu}\delta(\mathfrak{s})^{-1}x$ for some $x \in W_{\bar{\lambda}}\}$ which is contained in the set at the right hand side. $\square$

Suppose $\lambda^{(2)} \in \Lambda(n, b)$. If d is a nonnegative integer, and $b \geq d$, put $\Pi(\lambda^{(2)}) = \Pi(\lambda^{(2)}, d) = \{(\alpha, \beta) \mid \alpha \in \Lambda(n, b-d), \beta \in \Lambda(n, d), \alpha + \beta = \lambda^{(2)}\}$; otherwise, put $\Pi(\lambda^{(2)}) = \emptyset$. The following lemma needed in §5 gives the relation between semistandard tableaux and bitableaux.

(1.2.4) Lemma. *Let $\mu \in \Pi_r^+$ and $\lambda \in \Pi(n,r)$. Put $b = |\lambda^{(2)}|$ and $d = |\mu^{(2)}|$. If $b \geq d$, let $\rho = (b-d, d) \models b$. Then we have*

(a) $\#\bar{\mathcal{D}}_{\lambda^{(2)}\rho} = \#\Pi(\lambda^{(2)})$ if $b \geq d$, and in general

(b) $\#\mathfrak{T}^{ss}(\mu,\lambda) = \sum_{(\alpha,\beta)\in\Pi(\lambda^{(2)})} \#\mathfrak{T}^{ss}(\mu^{(1)}, \lambda^{(1)} \vee \alpha)\#\mathfrak{T}^{ss}(\mu^{(2)}, \beta)$.

Proof. (a) follows from [JK; (1.3.10)], and (b) is obvious since

$$\mathfrak{T}^{ss}(\mu,\lambda) = \cup_{(\alpha,\beta)\in\Pi(\lambda^{(2)})}\mathfrak{T}^{ss}(\mu^{(1)}, \lambda^{(1)} \vee \alpha) \times \mathfrak{T}^{ss}(\mu^{(2)}, \mathbf{0} \vee \beta)$$

where $\mathbf{0} = (0, \cdots, 0)$ with the number of zeros equal to the number of parts of $\lambda^{(1)}$. □

2. The Weyl group W of type B

2.1 The function n_0. Let $W = W_r = W(B_r)$ be the Weyl group of type B_r. Then W has the following equivalent characterizations.

(W1) W is the group with generators $s_0, s_1, \cdots, s_{r-1}$ and relations $s_k^2 = 1$ and $(s_is_j)^{m_{ij}} = 1$, where $0 \leq k \leq r-1$, $m_{01} = 4$, $m_{ij} = 2$ if $|i-j| \geq 2$ and $m_{ij} = 3$ if $j = i+1, 0 < i < r-1$. As usual, we denote the set of these generators by S.

(W2) W is isomorphic to the wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_r$. So $\mathfrak{S}_r$ is a subgroup of W, which will be denoted by $\bar{W} = \bar{W}_r$ in the sequel.

(W3) W is the reflection group consisting of orthogonal transformations on $\mathbb{R}^r$ defined by all permutations and sign changes of an orthonormal basis. So we may identify W with the group of permutations

$$\begin{pmatrix} r & r-1 & \cdots & 1 & -1 & \cdots & -(r-1) & -r \\ i_1 & i_2 & \cdots & i_r & -i_r & \cdots & -i_2 & -i_1 \end{pmatrix}.$$

We shall write $w = (i_1, \cdots, i_r, -i_r, \cdots, -i_1)$ for simplicity.

If we put $t_1 = s_0$ and $t_i = s_{i-1}t_{i-1}s_{i-1}$, $2 \leq i \leq r$, then $t_i^2 = 1$ and $t_it_j = t_jt_i$ and the subgroup $C \cong C_2^r$ generated by t_i is a normal subgroup of W isomorphic to the subgroup $(\mathbb{Z}/2\mathbb{Z})^r$ in (W2). Clearly, $W/C \cong \bar{W}$. We shall call a subgroup of C generated by a subset of $T = \{t_1, \cdots, t_r\}$ a *parabolic subgroup* of C. This definition agrees with the definition of parabolic subgroups of W which are generated by a subset of $S = \{s_0, s_1, \cdots, s_{r-1}\}$ if we view C as a Coxeter group. Note that the mapping sending t_1 to the permutation $(r, r-1, \cdots, 2, -1, 1, -2, \cdots, -(r-1), -r)$ and s_i to $(r, \cdots, i, i+1, \cdots, \cdots, -i-1, -i, \cdots, -r)$ gives an isomorphism between the groups described in (W1) and (W3). For $1 \leq i \leq r$ we shall view $W_i = \langle s_0, s_1, \cdots, s_{i-1}\rangle$, $\bar{W}_i = \langle s_1, \cdots, s_{i-1}\rangle$ and $C_{[1,i]} = \langle t_1, \cdots, t_i\rangle$ as subgroups of W, $\bar{W}$ and C, respectively, in a natural way.

As a Coxeter group, we have the length function ℓ and Bruhat-Chevalley order $\leq$ on W. For $w \in W$ the expression $w = w_1 \cdots w_m$ with $w_i \in S$ is called *reduced* if $m = \ell(w)$. Note that each t_i, as given recursively above, is reduced, as follows using the exchange condition. Also, note that the longest element w_0 of W is $t_1 \cdots t_r$. Multiplied by any s_i, it becomes shorter. Therefore, we have for any subset T' of T, $\ell(\prod_{x\in T'} x) = \sum_{x\in T'} \ell(x)$. (One way to prove all the assertions of this paragraph is to notice $t_1 \cdots t_r$ is central and not equal to 1, so must be w_0. The later has length $r^2 = 1 + 3 + \cdots + (2r-1)$ by an easy root system argument.)

We *define a function* $n_0 : W \to \mathbb{N}$ such that $n_0(w)$ is the number of times s_0 occurs in some (any — see below) reduced expression of w.

(2.1.1) Lemma. *(a) The number $n_0(w)$ $(w \in W)$ is the same for any reduced expression of w, and the image of the function n_0 is the subset $\{0, 1, \cdots, r\}$.*

(b) If $\ell(yw) = \ell(y) + \ell(w)$ then $n_0(yw) = n_0(y) + n_0(w)$.

(c) $n_0(xwy) = n_0(w)$ for all $x, y \in \bar{W}$

(d) If $w = (i_1, i_2, \cdots, i_r, -i_r, \cdots, -i_2, -i_1)$ then $n_0(w)$ is the number of negative numbers in the sequence $i_1, i_2, \cdots, i_r$.

Proof. (a) follows from the relations in (W1) and the fact that any two reduced expressions can be transformed to each other by a sequence of relations (with two s_0's appearing on each side or none). (b) follows from (a). To see (c), it suffices to show $n_0(sw) = n_0(w)$ for any $s \in S$ and $s \neq s_0$. It is obvious from (b) if $\ell(sw) = \ell(w) + 1$. Suppose now $\ell(sw) = \ell(w) - 1$ and $n_0(sw) < n_0(w)$. Since $w = s(sw)$ and $\ell(w) = \ell(sw) + 1$, we have from (b) $n_0(w) = n_0(s) + n_0(sw) = n_0(sw)$, a contradiction. We now prove (d). Let m be the number of negative numbers in the sequence $i_1, i_2, \cdots, i_r$. Then there exists an element $y \in \bar{W}$ such that yw sends i to $-i$ for $1 \leq i \leq m$ and fixes the others. That is, we have $yw = t_1 \cdots t_m$. Now the assertion follows from (c). □

2.2 Quasi-parabolic subgroups and distinguished coset representatives.

Associated to each bicomposition $\lambda \in \Pi_r$, let $\bar{W}_{\bar{\lambda}}$ be the Young subgroup of $\bar{W}$, i.e., $\bar{W}_{\bar{\lambda}}$ is the row stabilizer of $\mathbf{t}^\lambda$ in $\bar{W}$. We define $W_\lambda = C_\lambda \bar{W}_{\bar{\lambda}}$ where $C_\lambda = C_{[1,a]}$ is the subgroup generated by $\{t_1, \cdots, t_a\}$ with $a = |\lambda^{(1)}|$. We sometimes use $\bar{W}_\lambda$ or $W_{\bar{\lambda}}$ to denote the "top" part $\bar{W}_{\bar{\lambda}}$ of W_λ. Thus, using the notation in (1.2.1b,c), we have $\bar{W}_\lambda = W_{\bar{\lambda}} = \bar{W}_{\bar{\lambda}}$, $C_\lambda = W_{\underline{\lambda}}$ and $W_\lambda = W_{\underline{\lambda}} W_{\bar{\lambda}}$ with top part $W_{\bar{\lambda}}$ and "bottom" part $W_{\underline{\lambda}}$. Clearly we have the "sandwich": $W_{\tilde{\lambda}} \subseteq W_\lambda \subseteq W_{\hat{\lambda}}$, where $W_{\tilde{\lambda}}$ (resp. $W_{\hat{\lambda}}$) is the largest (resp. smallest) parabolic subgroup of W contained (resp. containing) W_λ. If $W_\lambda \neq W_{\hat{\lambda}}$ (see (1.2.1c)), then W_λ is not a parabolic subgroup of W. However, W_λ is a Weyl subgroup of type $B_{\lambda^{(1)}_1} \times B_{\lambda^{(1)}_2} \times \cdots \times A_{\lambda^{(2)}_1} \times \cdots$. We call W_λ a *quasi-parabolic subgroup* of W.

(2.2.1) Remark. It would also be possible to consider a more general notion of a quasi-parabolic subgroup in which the factors B_m did not all come at the beginning. This would be a perfectly reasonable approach, even having the advantage that many intersections of conjugates one works with would be again of this type. The disadvantage would be in a more complicated notion of distinguished coset representative, a notion which is quite simple for our restricted quasi-parabolic subgroups above.

Recall that, in the classical parabolic subgroup case, a *distinguished* representative of a coset or double coset is an element of minimal length. We begin in (2.2.2) with left cosets, but later apply the corollary (2.2.3) to right cosets in (2.2.4) and (2.2.5).

(2.2.2) Lemma. *For $w \in C$, write $w = d'w'$ with $w' \in \bar{W}$ and d' distinguished* *(as a left coset representative for the parabolic subgroup $\bar{W}$). Then $\ell(d') \geq \ell(w')$ with equality iff $w = 1$.*

Proof. We first note that if $j < i$ then $s_j t_{i+1} = t_{i+1} s_j$ (obvious, in terms of permutations). So if $w = t_{i_1+1} \cdots t_{i_m+1}$ with $i_1 < \cdots < i_m$ then $w = d'w'$ where $d' = s_{i_1} \cdots s_1 s_0 \cdots s_{i_m} \cdots s_1 s_0$ and $w' = s_1 \cdots s_{i_1} \cdots s_1 \cdots s_{i_m}$. Clearly, $\ell(d's_i) = \ell(d') + 1$ for all i with $1 \leq i \leq r-1$. This can be seen by induction on m and the fact that $xs > x, sy > y \Rightarrow xsy > xy$ ([Sh; Thm1]). This proves that d' is distinguished and $\ell(d') = m + i_1 + \cdots + i_m > i_1 + \cdots + i_m = \ell(w')$ if $d \neq 1$. □

(2.2.3) Corollary. *For $w \in C$ and $y \in \bar{W}$, we have $\ell(wy) \geq \ell(y)$ with equality iff $w = 1$.*

Proof. Write $w = d'w'(\neq 1)$ as in (2.2.2). Then $\ell(wy) = \ell(d') + \ell(w'y) \geq \ell(d') + |\ell(y) - \ell(w')| > \ell(w') + |\ell(y) - \ell(w')| \geq \ell(y)$. □

For $\lambda \in \Pi_r$ with $|\lambda^{(1)}| = a$, let $\bar{\mathcal{D}}_{\lambda^{(1)}}$ denote the distinguished cross section of the *right* $\mathfrak{S}_{\lambda^{(1)}}$-cosets in $\mathfrak{S}_a = \bar{W}_a$ and $\mathcal{D}_{\hat{\lambda}}$ the distinguished cross section of the right cosets of the parabolic subgroup $W_{\hat{\lambda}}$ in W.

(2.2.4) Corollary. *If $\lambda = (\lambda^{(1)}, -)$, then every coset $W_\lambda d$ has a unique element of minimal length and $\bar{\mathcal{D}}_{\lambda^{(1)}}$ is the set of minimal length representatives for the cosets $W_\lambda \backslash W$.*

Proof. Clearly, every right coset of W_λ has a unique representative in $\bar{\mathcal{D}}_{\lambda^{(1)}}$. For $d \in \bar{\mathcal{D}}_{\lambda^{(1)}}$ and $1 \neq x = wy \in W_\lambda$ with $w \in C$ and $y \in W_{\hat{\lambda}}$, we have that, if $w \neq 1$, then $\ell(xd) = \ell(wyd) > \ell(yd) \geq \ell(d)$ by the corollary above. Also, if $y \neq 1$, then $\ell(xd) = \ell(wyd) \geq \ell(yd) > \ell(d)$ as $\ell(yd) = \ell(y) + \ell(d)$. So $\ell(xd) > \ell(d)$ whenever $x \neq 1$, and d is the shortest element in the coset $W_\lambda d$. The uniqueness is obvious. □

(2.2.5) Theorem. *Keep the notation introduced above. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bicomposition of r. Then every coset $W_\lambda d$ has a unique element of minimal length and $\mathcal{D}_\lambda \stackrel{\text{def}}{=} \bar{\mathcal{D}}_{\lambda^{(1)}} \mathcal{D}_{\hat{\lambda}}$ is the set of minimal length representatives for the cosets $W_\lambda \backslash W$.*

Proof. This reduces to the previous case. For any $d = xy \in \mathcal{D}_\lambda$ with $x \in \bar{\mathcal{D}}_{\lambda^{(1)}}$ and $y \in \mathcal{D}_{\hat{\lambda}}$, we have $\ell(d) = \ell(x) + \ell(y)$ as $x \in W_{\hat{\lambda}}$. Now if $w \in W_\lambda$ and $w \neq 1$ then $wx \in W_{\hat{\lambda}}$. So $\ell(wd) = \ell(wx) + \ell(y) > \ell(x) + \ell(y) = \ell(xy) = \ell(d)$, by the corollary above. Therefore, d is the shortest element in $W_\lambda d$. The fact that $W_\lambda d$ contains a unique element in $\mathcal{D}_\lambda$ can be proved similarly. □

For parabolic subgroups W_J and W_K, it is well known that $\mathcal{D}_{JK} = \mathcal{D}_J \cap \mathcal{D}_K^{-1}$ is the set of distinguished representatives of double cosets $W_J \backslash W / W_K$. So, one might expect that the same is valid for quasi-parabolic subgroups W_λ and $\mathcal{D}_\lambda$. The following theorem confirms this.

(2.2.6) Theorem. *For bicompositions λ and μ of r, let $\mathcal{D}_{\lambda\mu} \stackrel{\text{def}}{=} \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$. Then*
(a) each double coset $W_\lambda w W_\mu$ contains a unique element of $\mathcal{D}_{\lambda\mu}$;

(b) *if* $d \in \mathcal{D}_{\lambda\mu}$ *then* d *is the unique element of minimal length in its coset* $W_\lambda d W_\mu$.

Proof. Let w be an element of minimal length in its double coset $W_\lambda w W_\mu$. Then w has minimal length in $W_\lambda w$ and in wW_μ. Thus $w \in \mathcal{D}_{\lambda\mu}$. So part (a) will imply part (b).

Each double coset certainly contains an element of minimal length, hence an element of $\mathcal{D}_{\lambda\mu}$. We must show the element is unique. So let $d_1, d_2 \in \mathcal{D}_{\lambda\mu}$ have the property that $D = W_\lambda d_1 W_\mu = W_\lambda d_2 W_\mu$. Then $W_\lambda d_i W_\mu$ is a union of the subsets $W_{\bar\lambda} d_i W_{\bar\mu}$, $C_\lambda d_i W_{\bar\mu}$, $W_{\bar\lambda} d_i C_\mu$ and $C_\lambda d_i C_\mu$ where $C_\nu = C \cap W_\nu$. Clearly, the mapping n_0 on $W_{\bar\lambda} d_i W_{\bar\mu}$ is constant, say m. We claim that

$$D_m := \{w \in W_\lambda d_i W_\mu \mid n_0(w) = m\} = W_{\bar\lambda} d_i W_{\bar\mu}. \tag{2.2.6.1}$$

Thus, we have $W_{\bar\lambda} d_1 W_{\bar\mu} = W_{\bar\lambda} d_2 W_{\bar\mu}$, and therefore, $d_1 = d_2$ by [C; (2.7.3)].

We now prove our claim. Let $d = d_i$ and write $d = u\hat{d}v$, where $\hat{d} \in \mathcal{D}_{\hat\lambda\hat\mu}$ and $u \in \bar{\mathcal{D}}_{\lambda(i)}$ as in (2.2.5) and some $v \in \bar{W}_{\hat\mu}$. (By Howlett's result [C; (2.7.5)] discussed above (2.3.1) below, we can write $d = u\hat{d}v$ for some $u \in W_{\bar\lambda}, v \in W_{\hat\mu}$ with additivity of lengths. Applying (2.2.3) forces $u, v \in \bar{W}$. Additivity now also shows $u \in \bar{\mathcal{D}}_\lambda \cap W_{\hat\lambda} = \bar{\mathcal{D}}_{\lambda(1)}$.) If $x = cdw \in C_\lambda d_i W_{\bar\mu}$ with $c \in C_\lambda$, $w \in W_{\bar\mu}$ and $c \neq 1$, then $n_0(cu\hat{d}) = n_0(cu) + n_0(\hat{d})$ by (2.1.1b). Since $vw \in \mathfrak{S}_r$ it follows from (2.1.1c) that $n_0(x) = n_0(cu\hat{d}vw) = n_0(cu\hat{d}) = n_0(cu) + n_0(\hat{d}) = n_0(c) + n_0(d) > n_0(d)$, as $n_0(c) > 1$. This proves that $D_m \cap C_\lambda d_i W_{\bar\mu} = d_i W_{\bar\mu} \subseteq W_{\bar\lambda} d_i W_{\bar\mu}$. Similarly, one has $D_m \cap W_{\bar\lambda} d_i C_\mu \subseteq W_{\bar\lambda} d_i W_{\bar\mu}$.

It remains to show that $D_m \cap C_\lambda d_i C_\mu \subseteq W_{\bar\lambda} d_i W_{\bar\mu}$. Take $x = c_1 d c_2 \in D_m \cap C_\lambda d_i C_\mu$. As C is normal in W we have $x = c_1 u\hat{d}v c_2 = uc_1'\hat{d}c_2'v$ for some $c_1' \in C_\lambda$ and $c_2' \in C_\mu$. Since $n_0(x) = n_0(\hat{d})$, we find, using (2.2.3) and (2.1.1b), that $c_2' \in W_{\hat\nu} = W_{\hat\lambda}^{\hat{d}} \cap W_{\hat\mu}$. So $c_2' = \hat{d}^{-1}c_2''\hat{d}$ for some $c_2'' \in W_{\hat\lambda} \cap C$ and $x = uc_1'c_2''\hat{d}v$. Now $uc_1'c_2'' \in W_{\hat\lambda}$ and we must have $c_1'c_2'' = 1$. (Otherwise, $n_0(x) = n_0(uc_1'c_2'') + n_0(\hat{d}) > m$.) Consequently, we have $x = d \in W_{\bar\lambda} d W_{\bar\mu}$. Our claim is proved. □

We shall call the elements in $\mathcal{D}_\lambda$ and $\mathcal{D}_{\lambda\mu}$ *distinguished* coset and double coset representatives for quasi-parabolic subgroups. If, for $\lambda \in \Pi_r$, we view $\bar\lambda$ as a bicomposition (see (1.2.1)), then $W_{\bar\lambda}$ is a parabolic subgroup of W. Thus, we have a reversed "sandwich": $\mathcal{D}_{\hat\lambda\hat\mu} \subseteq \mathcal{D}_{\lambda\mu} \subseteq \mathcal{D}_{\bar\lambda\bar\mu}$ for all $\lambda, \mu \in \Pi_r$. Also, $\mathcal{D}_{\bar\lambda} \cap \bar{W} = \mathcal{D}_\lambda \cap \bar{W}$ and $\mathcal{D}_{\bar\lambda\bar\mu} \cap \bar{W} = \mathcal{D}_{\lambda\mu} \cap \bar{W}$. We denote these sets by $\bar{\mathcal{D}}_\lambda$ and $\bar{\mathcal{D}}_{\lambda\mu}$, respectively. Thus, with the notation introduced in §1.1 for symmetric groups, we have $\bar{\mathcal{D}}_\lambda = \bar{\mathcal{D}}_{\bar\lambda}$ and $\bar{\mathcal{D}}_{\lambda\mu} = \bar{\mathcal{D}}_{\bar\lambda\bar\mu}$. The following result will be useful in §4.2.

(2.2.7) Corollary. *Let* $x \in \bar{W}_{\hat\lambda}$ *and* $y \in \bar{W}_{\hat\mu}$. *Then* $d \in \mathcal{D}_{\underline{\lambda}\underline{\mu}} \Leftrightarrow xdy \in \mathcal{D}_{\underline{\lambda}\underline{\mu}}$.

Proof. Since $W_\lambda = C_\lambda$ is normal in $W_{\hat\lambda}$, we have $W_{\underline{\lambda}} xdW_{\underline{\mu}} = xW_{\underline{\lambda}} dW_{\underline{\mu}}$. So $n_0(xd) = n_0(d)$ is minimal in this double coset. Apply (2.2.6.1) to $\underline{\lambda}$ and $\underline{\mu}$, we see that xd is the shortest element in $W_{\underline{\lambda}} xdW_{\underline{\mu}}$. Therefore, $xd \in \mathcal{D}_{\underline{\lambda}\underline{\mu}}$ by (2.2.6). Now our assertion follows easily. □

(2.2.8) Remark. We remark that, for $d \in \mathcal{D}_{\lambda\mu}$, $W_\lambda^d \cap W_\mu$ is not in general a quasi-parabolic subgroup in our restricted sense (2.2.1). This subgroup is, however, generated by a set of reflections contained in $\{s_1, \cdots, s_{r-1}\} \cup \{t_1, \cdots, t_r\}$. We will call this set $\lambda d \cap \mu$ and define $\lambda \cap d\mu$ similarly as in (1.1.2). In case $W_\lambda^d \cap W_\mu$ is quasi-parabolic, $\lambda d \cap \mu$ may be identified with the corresponding tight bicomposition. Note that this case is equivalent to that where $W_\lambda \cap W_\mu^{d^{-1}}$ is quasi-parabolic. If V is a subgroup of W (e.g., $V = C$) and J is a set of reflections, it is useful to write V_J for the subgroup of V generated by the elements of J contained in V.

2.3 Distinguished decomposition. For any $d \in \mathcal{D}_{\lambda\mu}$, there is a unique element $\hat{d} \in \mathcal{D}_{\hat{\lambda}\hat{\mu}}$ such that $d \in W_{\hat{\lambda}} \hat{d} W_{\hat{\mu}}$. Thus, by a result of Howlett [C; (2.7.5)], there exist (unique) $u \in W_{\hat{\lambda}}$ and $v \in \mathcal{D}_{\hat{\lambda}\hat{d}\cap\hat{\mu}} \cap W_{\hat{\mu}}$ such that $d = u\hat{d}v$, and then $\hat{d}v \in \mathcal{D}_{\hat{\lambda}}$ and $\ell(d) = \ell(u) + \ell(\hat{d}) + \ell(v)$. By the definition in (2.2.5), we have $u \in \bar{\mathcal{D}}_{\lambda^{(1)}}$. We shall call such a decomposition the *right distinguished decomposition* of d. Similarly, we have also a *left* distinguished decomposition $d = u_0\hat{d}v_0$ with $v_0 \in W_{\hat{\mu}}$ and $u_0 \in \mathcal{D}^{-1}_{\hat{\lambda}\cap\hat{d}\hat{\mu}} \cap W_{\hat{\lambda}}$. Thus, $u_0\hat{d} \in \mathcal{D}^{-1}_{\hat{\mu}}$.

(2.3.1) Lemma. *For $d \in \mathcal{D}_{\lambda\mu}$, let $d = u\hat{d}v$ be a right distinguished decomposition of d. Then $v^{-1} \in \bar{\mathcal{D}}_{\mu^{(1)}}$. A similar result holds for left distinguished decompositions.*

Proof. Let $d = u\hat{d}v$ as above, and let $d = u_0\hat{d}v_0$ be a left distinguished decomposition of d. Then $(v_0)^{-1} \in \bar{\mathcal{D}}_{\mu^{(1)}}$ and $(u_0\hat{d})^{-1} \in \mathcal{D}_{\hat{\mu}}$. So $x = v_0v^{-1} = (u_0\hat{d})^{-1}u\hat{d} \in W_{\hat{\lambda}\hat{d}\cap\hat{\mu}} = W_{\hat{\lambda}}^{\hat{d}} \cap W_{\hat{\mu}}$. Since $v \in \mathcal{D}_{\hat{\lambda}\hat{d}\cap\hat{\mu}} \cap W_{\hat{\mu}}$, it follows that $v_0 = xv$ and $\ell(v_0) = \ell(x) + \ell(v)$. Therefore, $v \in \bar{W}$, and $v^{-1} \in \bar{\mathcal{D}}_{\mu^{(1)}}$. □

Let $d = u\hat{d}v$ be as above. As $\hat{d}$ is a distinguished representative of a double coset of parabolic subgroups, there exists a (tight) bicomposition $\hat{\nu}$ such that $W_{\hat{\lambda}}^{\hat{d}} \cap W_{\hat{\mu}} = W_{\hat{\nu}}$. Clearly, we have $C_{\hat{\lambda}}^{\hat{d}} \cap C_{\hat{\mu}} = C_{\hat{\nu}}$, and every element in $C_{\hat{\nu}}$ is fixed under conjugation by $\hat{d}$.

Certainly, for any $d \in W$, we have $d^{-1}t_id = t_j$ for some j. The subgroup $C_{\lambda d\cap\mu} = d^{-1}C_\lambda d \cap C_\mu$ is a parabolic subgoup of C. Since $v^{-1}C_\mu v = C_\mu$, we have $C_{\lambda d\cap\mu} = v^{-1}C_{\hat{\nu}}v$. Thus, we have a decomposition

$$C_\mu = C_{\lambda d\cap\mu} \times C_{\mu\backslash\lambda d\cap\mu} \tag{2.3.2}$$

where $C_{\mu\backslash\lambda d\cap\mu}$ is the "parabolic" complement of $C_{\lambda d\cap\mu}$ in C_μ generated by t_i's. Similarly, we have $C_{\lambda\cap d\mu} = C_\lambda \cap dC_\mu d^{-1} = uC_{\hat{\nu}}u^{-1}$ and a decomposition as above for C_λ.

Note also that, as $W_{\bar{\lambda}}$ is parabolic and $d \in \mathcal{D}_{\bar{\lambda}\bar{\mu}}$, $W_{\bar{\lambda}}^d \cap W_{\bar{\mu}}$ is parabolic by a well-known result of Kilmoyer [C; (2.7.4)], call it $W_{\bar{\nu}}$. Clearly, $W_\lambda^d \cap W_\mu = C_{\lambda d\cap\mu}W_{\bar{\nu}}$. The following result is an easy group-theoretic consequence.

(2.3.3) Proposition. *Maintain the notation introduced above, and let $d \in \mathcal{D}_{\lambda\mu}$. Then every element $w \in W_\lambda dW_\mu$ is uniquely expressible in the form $w = xdcy$ where $x \in W_\lambda$, $c \in C_{\mu\backslash\lambda d\cap\mu}$ and $y \in \mathcal{D}_{\bar{\nu}} \cap W_{\bar{\mu}}$.*

We shall call the decomposition in (2.3.3) the *generalized distinguished decomposition* of w with respect to quasi-parabolic subgroups in the restricted sense (2.2.1).

3. q-SCHUR^2 ALGEBRAS

3.1 The Hecke algebra $\mathcal{H} = \mathcal{H}(W)$. Let $\mathcal{Z} = \mathbb{Z}[q, q^{-1}, q_0, q_0^{-1}]$ be the ring of Laurent polynomials in two variables, and K the quotient field of $\mathcal{Z}$. Let $\mathcal{H} = \mathcal{H}(W)$ be the Hecke algebra over $\mathcal{Z}$ associated with W. Thus, $\mathcal{H}$ is a free $\mathcal{Z}$-module with basis $\{T_w \mid w \in W\}$ and multiplication defined by all the rules:

$$T_wT_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) = \ell(w) + 1 \\ (q_s - 1)T_w + q_sT_{ws}, & \text{if } \ell(ws) = \ell(w) - 1 \end{cases}$$

for all $w \in W$ and $s \in S$, where $q_{s_i} = \begin{cases} q_0 \text{ if } i = 0 \\ q \text{ if } i \neq 0. \end{cases}$ For any $\mathcal{Z}$-module M and any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, we write $M_{\mathcal{Z}'} = M \otimes_{\mathcal{Z}} \mathcal{Z}'$. In particular, we write $\mathcal{H}'$ for $\mathcal{H}_{\mathcal{Z}'}$ and T_w for $T_w \otimes 1$ by abuse of notation. Note that the images of q_0 and q are invertible in $\mathcal{Z}'$. The defining basis of $\mathcal{H}'$ has the following useful property.

(3.1.1) Lemma. *(a) For $x, y \in W$, put $T_xT_y = \sum_{z \in W} f_{x,y,z}T_z$. Then $xy \leq z$ whenever $f_{x,y,z} \neq 0$. Also, $f_{x,y,xy} = q_0^a q^b$ for some nonnegative integers a and b.*

(b) For $x, z \in \bar{W}$, $y \in W$, put $T_xT_yT_z = \sum_{w \in W} f_{x,y,z,w}T_w$. Then $n_0(w) = n_0(y)$ whenever $f_{x,y,z,w} \neq 0$.

Proof. The proof of (a) for $\mathcal{Z}$ is similar to the proof of the case $q = q_0$ given by Shi in [Sh; Thm8]. For general $\mathcal{Z}'$, it follows by base change. The second statement follows from (a) and (2.1.1) □

(3.1.2) Corollary. *If $W = G_1G_2$ with $W = |G_1||G_2|$, then the set*

$$\{T_{x_1}T_{x_2} \mid x_j \in G_j\}$$

forms a basis for $\mathcal{H}'$. In particular, the set $\{T_xT_y \mid x \in W_\lambda, y \in \mathcal{D}_\lambda\}$ forms a basis for $\mathcal{H}'$.

Proof. Choose an order $w_1, w_2, \cdots$ on W satisfying $w_i \leq w_j$ implies $i < j$. Then we see, by the previous lemma, the transition matrix from the set $\{T_{x_1}T_{x_2} \mid x_j \in G_j\}$ to the basis $\{T_w \mid w \in W\}$ is upper triangular with invertible diagonal product. Hence the set forms a basis. □

3.2 q-Permutation modules. For a subset X of W, we define $T_X = \sum_{w \in X} T_w$. Clearly, if $X = W_\lambda$ is a parabolic subgroup of W, the submodule $\mathcal{H}'_\lambda = \mathcal{H}'(W_\lambda)$ generated by all $T_w, w \in W_\lambda$, is a subalgebra and $\mathcal{Z}'T_{W_\lambda}$ is a free $\mathcal{H}'_\lambda$-module of rank 1. This is the q-analogue of the trivial representation of W_λ. For the non-parabolic subgroup C of W, we introduce the element $\pi_r = \prod_{i=1}^r (q^{i-1} + T_{t_i})$, following [DJ4; (3.2)]. This element is central in $\mathcal{H}'$ and $\pi_rT_{t_{j+1}} = q_0\pi_rT_{s_j \cdots s_1}T_{s_1 \cdots s_j}$. We also note that $(\pi_r)^2 = z_r\pi_r$ where z_r is central in $\mathcal{H}(\bar{W})$ and invertible in $\mathcal{H}_K(\bar{W})$ (see [DJ4; (4.5)]). In general, for a bicomposition $\lambda \in \Pi_r$ with $a = |\lambda^{(1)}|$, we define

$$\pi_\lambda = \pi_a = \prod_{i=1}^{a} (q^{i-1} + T_{t_i}). \tag{3.2.1}$$

Clearly, π_λ is in the center of $\mathcal{H}'(W_a)$. Following [DJM; §4], we define the element $x_\lambda = \pi_\lambda x_{\bar\lambda}$ where $x_{\bar\lambda} = T_{W_{\bar\lambda}}$. x_λ serves as (as a generator for) the "trivial representation" for W_λ and put $\mathcal{T}_\lambda = x_\lambda\mathcal{H}$, $\mathcal{T}'_\lambda = x_\lambda\mathcal{H}'$. Recall that the q-permutation module $\mathcal{T}_\lambda$ for a parabolic subgroup W_λ is free of rank $\#\mathcal{D}_\lambda$. The following result generalizes this to quasi-parabolic subgroups.

(3.2.2) Proposition. *(a) The $\mathcal{H}'$-module $\mathcal{T}'_\lambda$ is free with basis $\{x_\lambda T_w \mid w \in \mathcal{D}_\lambda\}$.*
(b) If W_λ is parabolic, then $\mathcal{T}'_\lambda = T_{W_\lambda}\mathcal{H}'$.
(c) $\mathcal{T}'_\lambda \cong \mathcal{T}'_{\lambda''}$. (Recall λ' is the dual bipartition of λ, and λ'' is the double dual.)
(d) Let K be the quotient field of $\mathcal{Z}$. Then $\mathcal{T}_\lambda = \mathcal{T}_{\lambda K} \cap \mathcal{H}$.

Proof. To see (a), we first note that the linear independence of the set follows from Corollary 3.1.2. To prove the set spans it suffices to prove the corresponding statement for the case $W = W_{\hat\lambda}$, in view of the factorization $\mathcal{D}_\lambda = \bar{\mathcal{D}}_{\lambda^{(1)}}\mathcal{D}_{\hat\lambda}$. (Note that $W_{\hat\lambda} \cap \mathcal{D}_\lambda = \bar{\mathcal{D}}_{\lambda^{(1)}}$.) Since $x_\lambda T_s = qx_\lambda$ and $x_\lambda T_{t_{i+1}} = q_0 x_\lambda T_{s_i\cdots s_1}T_{s_1\cdots s_i}$ for all $s \in W_{\bar\lambda} \cap S$ and $t_{i+1} \in C_\lambda$, this follows easily (rewrite the product of T's as T_uT_d with $u \in \bar W_\lambda$ and $d \in \bar{\mathcal{D}}_{\lambda^{(1)}}$), proving (a).

The statement (b) follows from the claim that $\pi_r T_{\bar W} = q^{\binom{r}{2}}T_W$. Indeed, for a subsequence $\underline{i} = \{i_1, \cdots, i_m\}$ of $\{0, 1, \cdots, r-1\}$, let $t_{\underline{i}} = t_{i_1+1}\cdots t_{i_m+1}$ and $\underline{i}^c = \{0, 1, \cdots, r-1\}\backslash\{i_1, \cdots, i_m\}$, the complementary subsequence of $\underline{i}$. Since

$$t_{\underline{i}} = t_{i_1+1}\cdots t_{i_m+1} = (s_{i_1}\cdots s_1t_1\cdots s_{i_m}\cdots s_1t_1)(s_1\cdots s_{i_m}\cdots s_1\cdots s_{i_1}) = d_{\underline{i}}w$$

where the first product $d_{\underline{i}}$ is in $\mathcal{D}^{-1}_{(-,r)}$ (as in the proof of (2.2.2)), $\ell(w)$ is the composition sum-of-parts $|\underline{i}|$, and $\pi_r = \sum_{\underline{i}} q^{|\underline{i}^c|}T_{t_{\underline{i}}}$, it follows that

$$\pi_r T_{\bar W} = \sum_{\underline{i}} q^{|\underline{i}^c|}T_{t_{\underline{i}}}T_{\bar W} = \sum_{\underline{i}} q^{|\underline{i}^c|+|\underline{i}|}T_{d_{\underline{i}}}T_{\bar W} = q^{\binom{r}{2}}\sum_{\underline{i}} T_{d_{\underline{i}}}T_{\bar W} = q^{\binom{r}{2}}T_W,$$

proving (b).

Since $|\lambda^{(1)}| = |\lambda^{(1)''}|$ and $|\lambda^{(2)}| = |\lambda^{(2)''}|$, the partitions obtained by reordering $\lambda^{(1)}$ and $\lambda^{(2)}$ are just $\lambda^{(1)''}$ and $\lambda^{(2)''}$. So there exists $d \in \mathcal{D}_{\bar\lambda\bar\lambda''} \cap \bar W$ such that $T_d^{-1}x_{\bar\lambda}T_d = x_{\bar\lambda''}$ and $T_d^{-1}\pi_\lambda T_d = \pi_{\lambda''} = \pi_\lambda$. Therefore, $T_d^{-1}x_\lambda T_d = x_{\lambda''}$, and the map sending $x_\lambda h$ to $T_d^{-1}x_\lambda h$ gives the required isomorphism for (c).

We leave (d) as an exercise, (using (3.1.2)). □

3.3 The q-Schur^2 algebras $\mathcal{S}^2_q(n,r)$. Recall from §1.2 that $\Pi(n,r)$ is the set of bicompositions of r in which each single composition has n parts. Then $\Pi(n,r)$ is a poset with the dominance order $\trianglelefteq$, and we have $\lambda \trianglelefteq \mu$ if and only if $\bar\lambda \trianglelefteq \bar\mu$ (cf. (1.2.1b)). For our later use, we list some interesting subsets of $\Pi(n,r)$. Let

$$(3.3.1)\qquad \begin{cases} \hat\Pi(n,r) = \{\lambda \in \Pi(n,r) \mid W_\lambda = W_{\hat\lambda}\} \\ \bar\Pi(n,r) = \{\lambda \in \Pi(n,r) \mid W_\lambda = W_{\bar\lambda}\} \\ \Omega_m = \{(\lambda^{(1)},\lambda^{(2)}) \in \Pi(n,r) \mid \#\text{ of parts of }\lambda^{(2)} \le m\} \end{cases}$$

Clearly, we have $\bar\Pi(n,r) \subset \hat\Pi(n,r) \subset \Pi(n,r)$, and all Ω_m are (order) coideals of $\Pi(n,r)$. Note that $\bar\Pi(n,r)$ is an ideal. Let $\Pi^+(n,r)$ be the subset of all bipartitions in $\Pi(n,r)$ and define $\hat\Pi^+(n,r)$ and $\bar\Pi^+(n,r)$ similarly. Note that $\Pi^+_r = \Pi^+(r,r)$.

Recall from §3.2 the q-permutation modules $\mathcal{T}'_\lambda = x_\lambda\mathcal{H}'$ for any $\lambda \in \Pi(n,r)$. We now consider their direct sums — the "*tensor spaces*" — and associated endomorphism algebras

$$(3.3.2)\qquad \begin{cases} \mathcal{T}' = \mathcal{T}(n,r)' = \oplus_{\lambda\in\Pi(n,r)}\mathcal{T}'_\lambda, & \mathcal{S}^2_q(n,r,\mathcal{Z}') = \mathrm{End}_{\mathcal{H}'}(\mathcal{T}'); \\ \hat{\mathcal{T}}' = \hat{\mathcal{T}}(n,r)' = \oplus_{\lambda\in\hat{\Pi}(n,r)}\mathcal{T}'_\lambda, & \hat{\mathcal{S}}^2_q(n,r,\mathcal{Z}') = \mathrm{End}_{\mathcal{H}'}(\hat{\mathcal{T}}'); \\ \bar{\mathcal{T}}' = \bar{\mathcal{T}}(n,r)' = \oplus_{\lambda\in\bar{\Pi}(n,r)}\mathcal{T}'_\lambda, & \bar{\mathcal{S}}^2_q(n,r,\mathcal{Z}') = \mathrm{End}_{\mathcal{H}'}(\bar{\mathcal{T}}'). \end{cases}$$

By (3.2.2a), we see that $\hat{\mathcal{S}}^2_q(n,r,\mathcal{Z}')$ is the endomorphism algebra of the tensor space involving only parabolic subgroups. We will call $\mathcal{S}^2_q(n,r,\mathcal{Z}')$ *the q-Schur² algebra* (pronounced as q-schur-two-algebra) of degree (n,r). It is also convenient to name $\hat{\mathcal{S}}^2_q(n,r,\mathcal{Z}')$ and $\bar{\mathcal{S}}^2_q(n,r,\mathcal{Z}')$ the *Hecke endomorphism algebras* of type B as in [DPS1] and *q-Schur′ algebras* (compare [GH]). For simplicity, we write $\mathcal{S}^2_q(n,r)$ for $\mathcal{S}^2_q(n,r,\mathcal{Z})$.

Comparing with q-Schur algebras, it is natural to ask the following questions: (1) Is a q-Schur² algebra $\mathcal{Z}'$-free? (2) Is a q-Schur² algebra quasi-hereditary? (3) Does base change induce an isomorphism $\mathcal{S}^2_q(n,r)_{\mathcal{Z}'} \cong \mathcal{S}^2_q(n,r,\mathcal{Z}')$? In next three sections, we shall give affirmative answers to all three questions.

3.4 Bistandard bases and twisted Specht modules. We recall some recent results obtained in [DJM] in this subsection. Recall, for a bipartition μ of r, the set $\mathbf{T}^s(\mu)$ of all standard μ-bitableaux. For $\mathbf{t} \in \mathbf{T}^s(\mu)$, let $\delta(\mathbf{t}) \in \mathcal{D}_{\bar{\mu}} \cap \bar{W}$ be given by $\mathbf{t}^\mu\delta(\mathbf{t}) = \mathbf{t}$. The element $\delta(\mathbf{t})$ should not be confused with the coset representative $\delta(\mathfrak{s})$ associated to a semi-standard bitableau defined in §1.2.

Define, for any $\mathbf{s},\mathbf{t} \in \mathbf{T}^s(\mu)$, the elements $x_{\mathbf{st}} = x^\mu_{\mathbf{st}} = T^\iota_{\delta(\mathbf{s})}x_\mu T_{\delta(\mathbf{t})}$ where $(-)^\iota$ is the anti-involution on $\mathcal{H}'$ satisfying $T^\iota_w = T_{w^{-1}}$. By [DJM; (4.14)], the set $\{x_{\mathbf{st}} \mid \mathbf{s},\mathbf{t} \in \mathbf{T}^s(\mu), \mu \in \Pi^+_r\}$ forms a basis for $\mathcal{H}'$. We shall call the basis $\{x_{\mathbf{st}}\}$ the *bistandard basis* or *Murphy basis* (or *Green-Murphy basis*) of $\mathcal{H}$. For $\lambda \in \Pi^+_r$, let $\mathcal{H}'^{\geq\lambda}$ (resp. $\mathcal{H}'^{>\lambda}$) be spanned by all $x^\mu_{\mathbf{st}}$ with $\mu \trianglerighteq \lambda$ (resp. $\mu \vartriangleright \lambda$). Then both $\mathcal{H}'^{\geq\lambda}$ and $\mathcal{H}'^{>\lambda}$ are ideals of $\mathcal{H}'$ ([DJM; (4.18)]). We also record the following useful fact which is implicit in [DJM; (4.11)].

(3.4.1) Proposition. *For any $h, h' \in \mathcal{H}'$ and $\mathbf{s},\mathbf{t} \in \mathbf{T}^s(\lambda)$, write for some $\xi_{\mathbf{s}'}, \zeta_{\mathbf{t}'} \in \mathcal{Z}'$*

$$hx^\lambda_{\mathbf{st}} \equiv \sum_{\mathbf{s}'\in\mathbf{T}^s(\lambda)} \xi_{\mathbf{s}'}x^\lambda_{\mathbf{s}'\mathbf{t}} \bmod(\mathcal{H}'^{>\lambda}) \quad \textit{and} \quad x^\lambda_{\mathbf{st}}h' \equiv \sum_{\mathbf{t}'\in\mathbf{T}^s(\lambda)} \zeta_{\mathbf{t}'}x^\lambda_{\mathbf{st}'} \bmod(\mathcal{H}'^{>\lambda}).$$

Then $\xi_{\mathbf{s}'}$ and $\zeta_{\mathbf{t}'}$ are independent of $\mathbf{t}$ *and* $\mathbf{s}$ *respectively.*

Further, for a bipartition λ of r, let $\mathcal{T}'^+_\lambda = \mathcal{T}'_\lambda \cap \mathcal{H}'^{>\lambda}$ and $S^{\natural\prime}_\lambda = \mathcal{T}'_\lambda/\mathcal{T}'^+_\lambda$. Then $S^{\natural\prime}_\lambda$ is $\mathcal{Z}'$-free with basis

$$(3.4.2)\qquad \{x_{\lambda\mathbf{t}} + \mathcal{T}^+_\lambda \mid \mathbf{t} \in \mathbf{T}^s(\lambda)\},$$

where $x_{\lambda\mathbf{t}} = x_{\mathbf{t}^\lambda\mathbf{t}} = x_\lambda T_{\delta(\mathbf{t})}$. Moreover, if F is a field which is also a $\mathcal{Z}$-algebra such that $\mathcal{H}_F$ is semi-simple. Then $\{S^\natural_{\lambda F} \mid \lambda \in \Pi^+_r\}$ is a complete set of simple

$\mathcal{H}_F$-modules and $\mathcal{H}_F^{\trianglerighteq\lambda}/\mathcal{H}_F^{\triangleright\lambda} \cong S_{\lambda F}^{\natural\oplus d_\lambda}$ with $d_\lambda = \#\mathbf{T}^s(\lambda)$. In particular, we have a decomposition

$$\mathcal{T}_{\lambda F} = S_{\lambda F}^{\natural} \bigoplus (\oplus_{\lambda \lhd \mu} S_{\mu F}^{\natural\oplus m_{\lambda\mu}}) \tag{3.4.3}$$

for some integers $m_{\lambda\mu} \in \mathbb{Z}$.

We remark that the modules $S_\lambda^\natural$ are actually the so-called "twisted" Specht modules. They appear at the top of permutation modules. In a later paper, we will realize them as *submodules* of the twisted permutation modules. Thus, we are able to define the notion of Specht modules and their bistandard bases (compare [DJ1]).

4. Freeness of the intertwining modules $\mathrm{Hom}_{\mathcal{H}'}(\mathcal{T}'_\mu, \mathcal{T}'_\lambda)$

In the rest of the paper, we aim at answering the questions raised at the end of §3.3, especially the quasi-heredity of q-Schur^2 algebras. The approach we shall adopt is the direct constructions of two bases — the natural "T-type" basis and the "bistandard" basis — for a q-Schur^2 algebra. The latter may be regarded as a kind of generalization of the method of Green [G]. A second approach, when q_0 is a fractional power of q as in [DPS2], will be sketched later. We follow the notation introduced in last section. Thus, $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra, $\mathcal{H}' = \mathcal{H} \otimes_{\mathcal{Z}} \mathcal{Z}'$ and $\mathcal{T}' = \mathcal{T}_{\mathcal{Z}'}$, etc.

4.1 Characterization of q-permutation modules. Recall from (3.2.2) that, for any $0 \le a \le r$, the module $\pi_a\mathcal{H}'$ is free with basis $\{\pi_a T_w \mid w \in \mathcal{D}_{\langle t_1,\cdots,t_a\rangle}\}$. The following lemma gives another basis for $\pi_a\mathcal{H}'$ like that given in (3.1.2) for $\mathcal{H}'$.

(4.1.1) Lemma. *For any non-negative integers a, i with $a+i \le r$, the set*

$$\{\pi_a T_{t_{a+1}}^{\varepsilon_1} \cdots T_{t_{a+i}}^{\varepsilon_i} T_w \mid w \in \mathcal{D}_{\langle t_1,\cdots,t_{a+i}\rangle}, \varepsilon_j \in \{0,1\}\}$$

is a basis for $\pi_a\mathcal{H}'$.

Proof. Clearly, by (3.1.2), the set is linearly independent. (Note: $T_{t_{a+1}}^{\varepsilon_1} \cdots T_{t_{a+i}}^{\varepsilon_i} = T_{t_{a+1}^{\varepsilon_1} \cdots t_{a+i}^{\varepsilon_i}}$.) So it generates a free submodule M of $\pi_a\mathcal{H}'$. Now, for any field k which is also a $\mathcal{Z}'$-algebra, $M_k = \pi_a\mathcal{H}'_k$ by comparison of dimensions. Therefore, $M = \pi_a\mathcal{H}'$ by [CPS1; (3.3.1)]. □

The following result is the key to characterizing the q-permutation modules $\mathcal{T}'_\lambda$.

(4.1.2) Theorem. *For $0 \le a < r$ we have*

$$\pi_{a+1}\mathcal{H}' = (1+T_{t_1})\mathcal{H}' \cap T_{s_1}(1+T_{t_1})\mathcal{H}' \cap T_{s_2 s_1}(1+T_{t_1})\mathcal{H}' \cap \cdots \cap T_{s_a s_{a-1}\cdots s_1}(1+T_{t_1})\mathcal{H}'.$$

Proof. We apply induction on a. Clearly, the result is true for $a = 0$. Assume now $a > 0$ and the result is true for all numbers $\le a-1$. Thus, we need to prove that

$$\pi_a\mathcal{H}' \cap T_{s_a s_{a-1}\cdots s_1}(1+T_{t_1})\mathcal{H}' = \pi_{a+1}\mathcal{H}'.$$

Let $h = T_{s_a s_{a-1}\cdots s_1}(1+T_{t_1})h_1 \in \pi_a\mathcal{H}'$. Then $T_{s_{a-1}\cdots s_1}(1+T_{t_1})h_1 = T_{s_a}^{-1}h \in \pi_{a-1}\mathcal{H}'$; that is, $T_{s_a}^{-1}h \in \pi_{a-1}\mathcal{H}' \cap T_{s_{a-1}\cdots s_1}(1+T_{t_1})\mathcal{H}'$. Thus, $T_{s_a}^{-1}h \in \pi_a\mathcal{H}'$, or $h \in T_{s_a}(\pi_a\mathcal{H}')$ by induction. Therefore, it suffices to prove $\pi_a\mathcal{H}' \cap T_{s_a}(\pi_a\mathcal{H}') = \pi_{a+1}\mathcal{H}'$.

By Lemma 4.1.1, $\pi_a\mathcal{H}'$ has a basis $\{\pi_a T_w, \pi_a T_{t_{a+1}}T_w \mid w \in \mathcal{D}_{\langle t_1,\cdots,t_{a+1}\rangle}\}$ while $\pi_{a+1}\mathcal{H}'$ has a basis $\{\pi_a(q^a + T_{t_{a+1}})T_w \mid w \in \mathcal{D}_{\langle t_1,\cdots,t_{a+1}\rangle}\}$. It follows that

$$\pi_a\mathcal{H}' = \pi_{a+1}\mathcal{H}' \oplus \sum_{w\in\mathcal{D}} \mathcal{Z}'\pi_a T_{t_{a+1}}T_w,$$

where $\mathcal{D} = \mathcal{D}_{\langle t_1,\cdots,t_{a+1}\rangle}$, is a direct sum of free submodules. Thus, we obtain

$$T_{s_a}(\pi_a\mathcal{H}') = T_{s_a}(\pi_{a+1}\mathcal{H}') \oplus M = \pi_{a+1}(T_{s_a}\mathcal{H}') \oplus M = \pi_{a+1}\mathcal{H}' \oplus M$$

where

$$\begin{aligned} M &= T_{s_a}\left(\sum_{w\in\mathcal{D}} \mathcal{Z}'\pi_a T_{t_{a+1}}T_w\right) \\ &= \sum_{w\in\mathcal{D}} \mathcal{Z}'\pi_{a-1}[T_{s_a}(q^{a-1} + T_{t_a})T_{t_{a+1}}]T_w. \end{aligned}$$

Now, it is equivalent to prove $M \cap \pi_a\mathcal{H}' = \{0\}$.

Suppose $h \in M \cap \pi_a\mathcal{H}'$ and write, for some elements $\alpha_w \in \mathcal{Z}'$

$$(*)\qquad \begin{aligned} h &= \sum_{w\in\mathcal{D}} \alpha_w\pi_{a-1}[T_{s_a}(q^{a-1} + T_{t_a})T_{t_{a+1}}]T_w \\ &= \sum_{w\in\mathcal{D}} \alpha_w\pi_{a-1}[q^aT_{t_a s_a} + q^{a-1}(q-1)T_{t_{a+1}} + qT_{t_a s_a t_a} + (q-1)T_{t_a t_{a+1}}]T_w. \end{aligned}$$

(Note $T_{t_a}T_{t_{a+1}} = T_{t_{a+1}}T_{t_a}$.) On the other hand, we have by (4.1.1)

$$(**)\qquad \begin{aligned} h &= \sum_{w\in\mathcal{D}} \beta_w\pi_a T_w + \sum_{w\in\mathcal{D}} \beta_{t_{a+1}w}\pi_a T_{t_{a+1}}T_w \\ &= \sum_{w\in\mathcal{D}} [\beta_w\pi_{a-1}(q^{a-1} + T_{t_a}) + \beta_{t_{a+1}w}\pi_{a-1}(q^{a-1}T_{t_{a+1}} + T_{t_a t_{a+1}})]T_w, \end{aligned}$$

for some $\beta_y \in \mathcal{Z}'$. Using the basis for $\pi_{a-1}\mathcal{H}'$ with $i = 2$ in (4.1.1), we have $\beta_w = 0$ for all such w by equating the coefficients of $\pi_{a-1}T_w$ in both (*) and (**). (Note that any term $\pi_{a-1}T_{t_a s_a t_a}T_w$ in (*) is a linear combination of basis elements of the form $\pi_{a-1}T_{t_a t_{a+1}}T_y$, where $y = w$ or $s_a w \in \mathcal{D}$, cf. (2.2.7).) Thus, equating the coefficients of $\pi_{a-1}T_{t_a}T_w$, we obtain that

$$\beta_w = \begin{cases} q^a\alpha_{s_a w} + q^a(q-1)\alpha_w, & \text{if } s_a w < w \\ q^{a+1}\alpha_{s_a w}, & \text{if } s_a w > w. \end{cases}$$

Since $\beta_w = 0$ for all $w \in \mathcal{D}_{\langle t_1,\cdots,t_{a+1}\rangle}$, the relation above implies that $\alpha_w = 0$ for all w. Therefore, $h = 0$, and the result is proved. □

Let $O_{t_i} = T_{s_{i-1}}^{-1}\cdots T_{s_1}^{-1}T_{s_0}T_{s_1}^{-1}\cdots T_{s_{i-1}}^{-1}$ and $\hat{O}_{t_i} = T_{s_{i-1}}^{-1}\cdots T_{s_1}^{-1}T_{s_1}^{-1}\cdots T_{s_{i-1}}^{-1}$. We now have the following characterization of permutation modules.

(4.1.3) Corollary. *For any bicomposition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of r with $a = |\lambda^{(1)}|$, we have*

$$x_\lambda\mathcal{H}' = x_{\bar\lambda}\mathcal{H}' \cap \pi_a\mathcal{H}'$$
$$= \{h \in \mathcal{H}' \mid T_s h = qh, \forall s \in W_{\bar\lambda} \cap S, O_{t_i}h = q_0\hat{O}_{t_i}h, 1 \le i \le a\}.$$

Proof. Clearly, we have $x_\lambda\mathcal{H}' \subseteq x_{\bar\lambda}\mathcal{H}' \cap \pi_a\mathcal{H}'$ since $x_\lambda = x_{\bar\lambda}\pi_a = \pi_a x_{\bar\lambda}$. Suppose $h = \pi_a h_1 \in x_{\bar\lambda}\mathcal{H}'$ and write, for some $\xi_w \in \mathcal{Z}'$ with $w \in \mathcal{D} := \mathcal{D}_{\langle t_1,\cdots,t_a\rangle}$, $h = \sum_{w\in\mathcal{D}}\xi_w\pi_a T_w$. Then we have $T_s h = qh$ where $s = s_i \in W_{\bar\lambda}$ for which we have also $sw \in \mathcal{D}$ whenever $w \in \mathcal{D}$ (see (2.2.7)). Equating the coefficients of $\pi_a T_w$ in $T_s h$ and qh, we obtain $\xi_w = \xi_{sw}$. Consequently, we have $\xi_d = \xi_{wd}$ for any $w \in W_{\bar\lambda}$ and $d \in \mathcal{D}_\lambda$. Therefore, we have $h_1 = x_{\bar\lambda}h_2$ with $h_2 = \sum_{d\in\mathcal{D}_\lambda}\xi_d T_d$, and hence, $h = x_\lambda h_2 \in x_\lambda\mathcal{H}'$, proving the first equality.

Now, by (4.1.2), we have $\pi_a\mathcal{H}' = \{h \in \mathcal{H}' \mid O_{t_i}h = q_0\hat{O}_{t_i}h, 1 \le i \le a\}$, while $x_{\bar\lambda} = \{h \in \mathcal{H}' \mid T_s h = qh, \forall s \in W_{\bar\lambda} \cap S\}$ (see, for example, [DPS1; (2.1.2)]). So the second equality follows from the first one. □

4.2 Bases for $\mathrm{Hom}_{\mathcal{H}'}(\mathcal{T}'_\mu, \mathcal{T}'_\lambda)$. Let λ, μ be bicompositions of r. Fix $d \in \mathcal{D}_{\lambda\mu}$ and let $d = u\hat{d}v$ be a right distinguished decomposition of d as in §2.3. Thus, $\hat{d} \in \mathcal{D}_{\hat\lambda\hat\mu}$, $u \in \bar{\mathcal{D}}_{\lambda^{(1)}}$ and $v^{-1} \in \bar{\mathcal{D}}_{\mu^{(1)}}$ (see (2.3.1)). Also, we have $\ell(d) = \ell(u) + \ell(\hat{d}) + \ell(v)$ and $\hat{d}v \in \mathcal{D}_{\lambda\mu}$. As $\hat{d}$ is a distinguished representative of a double coset of parabolic subgroups, the subgroup $W_{\hat\lambda}^{\hat d} \cap W_{\hat\mu} = W_{\hat\nu}$ is parabolic, where $\hat\nu$ identifies with a tight bicomposition. Clearly, $C_{\hat\nu}$ is a subgroup of $C_\mu = C_{\hat\mu}$, and we have a decomposition $C_\mu = C_{\hat\nu} \times C_{\mu\backslash\hat\nu}$. Accordingly, we may write $\pi_\mu = \pi_{\hat\nu}\pi_{\mu\backslash\hat\nu}$ where $\pi_{\mu\backslash\hat\nu}$ is in the span of $\{T_w \mid w \in C_{\mu\backslash\hat\nu}\}$ and $\pi_\mu, \pi_{\hat\nu}$ are given by (3.2.1). (The element $\pi_{\mu\backslash\hat\nu}$ is unique in $\mathcal{H}$.) Note that this decomposition is different from the one given in (2.3.2), though it uses a similar notation scheme. The relation is $v^{-1}C_{\hat\nu}v = C_{\lambda d\cap\mu}$, and $v^{-1}C_{\mu\backslash\hat\nu}v = C_{\mu\backslash\lambda d\cap\mu}$. On the other hand, if $t_j \in C_\lambda$ and $\hat{d}^{-1}t_j\hat{d} \in C_\mu$ then $\hat{d}^{-1}t_j\hat{d} = t_j$ by the additivity of lengths, and of n_0 (see (2.1.1b)) for $t_j\hat{d} = \hat{d}\hat{d}^{-1}t_j\hat{d}$. So we have $C_{\hat\nu} = Z_{C_\lambda\cap C_\mu}(\hat{d})$, the centralizer of $\hat{d}$ in $C_\lambda \cap C_\mu$. Thus we may also write $C_\lambda = C_{\hat\nu} \times C_{\lambda\backslash\hat\nu}$ and $\pi_\lambda = \pi_{\hat\nu}\pi_{\lambda\backslash\hat\nu}$.

Let $d = u_0\hat{d}v_0$ be the left distinguished decomposition of d. By definition, $u_0 \in \mathcal{D}_{\hat\tau}^{-1} \cap W_{\hat\lambda}$, where $\hat\tau = \hat\lambda \cap \hat{d}\hat\mu$. Also, $u_0 \in \bar{\mathcal{D}}_{\lambda^{(1)}}^{-1} \subseteq \mathcal{D}_\lambda^{-1}$. Consider the subgroup $W_1 = W_{\hat\lambda} \cap W_\mu^{(\hat{d}v_0)^{-1}}$ of $W_{\hat\tau}$ and the subgroup $W_{\nu^1} = W_\lambda^{u_0} \cap W_\mu^{(\hat{d}v_0)^{-1}} = W_\lambda^{u_0} \cap W_1$ of W_1, and put $W_\lambda \cap W_\mu^{d^{-1}} = W_{\nu^0}$. Clearly, W_{ν^1} is conjugate to W_{ν^0}, and $W_{\nu^1} = u_0^{-1}W_{\nu^0}u_0 = C_{\nu^0}^{u_0}\bar{W}_{\nu^0}^{u_0}$ has top part $\bar{W}_{\nu^1} = W_{\bar\nu^1} = \bar{W}_{\nu^0}^{u_0}$ and bottom part $C_{\nu^0}^{u_0} = C_{\hat\tau} = C_{\hat\nu}$. We have the following.

(4.2.1) Lemma. *Keep the notation introduced above.*

(a) $T_{\hat d}\pi_{\hat\nu} = \pi_{\hat\nu}T_{\hat d}$.

(b) *$\bar{W}_{\nu^1}$ is parabolic, and hence, W_{ν^1} is quasi-parabolic (in the restricted sense (2.2.1)). Moreover, we have $x_{\bar\nu^0}T_{u_0} = T_{u_0}x_{\bar\nu^1}$.*

(c) $x_{\bar\nu^1}\pi_{\lambda\backslash\hat\nu} = \pi_{\lambda\backslash\hat\nu}x_{\bar\nu^1}$.

Proof. We have seen that $t_i = \hat{d}^{-1}t_i\hat{d}$ for any $t_i \in C_{\hat\nu}$. So (a) is obvious. Since $\ell(u_0y) = \ell(u_0) + \ell(y)$ for all $y \in W_1$, and $\bar{W}_{\nu^1} = u_0^{-1}\bar{W}_{\nu^0}u_0$, we have, for all

$x \in \bar{W}_{\nu^0}$, $y = u_0^{-1}xu_0 \in \bar{W}_{\nu^1}$, $xu_0 = u_0y$ and $\ell(x) + \ell(u_0) = \ell(u_0) + \ell(y)$. In particular, $x \in S$ if and only if $y \in S$. Since $\bar{W}_{\nu^0}$ is parabolic (see (2.2.8)), so is $\bar{W}_{\nu^1}$. On the other hand, W_{ν^1} is a subgroup of $W_{\hat{\tau}}$ with the same bottom $C_{\hat{\tau}} = C_{\hat{\nu}}$. So it is quasi-parabolic. The last assertion in (b) follows easily. The assertion (c) follows, calculating in $\mathcal{H}$, from the fact that $C_{\nu^1} = C_{\hat{\nu}}$ and the fact that, if $sw = ws$ with $s \in S$, then $T_sT_w = T_wT_s$. $\square$

With the notation above, let $h_{\bar{\lambda}d\cap\bar{\mu}} = T_{\mathcal{D}_{\bar{\lambda}d\cap\bar{\mu}}\cap W_{\bar{\mu}}}$ and $h_{\bar{\lambda}\cap d\bar{\mu}} = T_{\mathcal{D}_{\bar{\lambda}\cap d\bar{\mu}}\cap W_{\bar{\lambda}}}$. Then $x_{\bar{\lambda}} = h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}x_{\bar{\nu}^0}$ and $T_{W_{\bar{\lambda}}dW_{\bar{\mu}}} = T_{W_{\bar{\lambda}}}T_dh_{\bar{\lambda}d\cap\bar{\mu}} = h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}T_dT_{W_{\bar{\mu}}}$. Here ι is the $\mathcal{Z}'$-linear anti-involution on $\mathcal{H}'$ given by $T_w^{\iota} = T_{w^{-1}}$ which clearly fixes the elements π_λ and $x_{\bar{\lambda}}$, for any λ.

(4.2.2) Proposition. *Maintain the notation introduced above. For any $d \in \mathcal{D}_{\lambda\mu}$ with right distinguished decomposition $d = u\hat{d}v$, let*

(4.2.2.1)
$$X_{W_\lambda dW_\mu} = x_\lambda T_{u\hat{d}}\pi_{\mu\backslash\hat{\nu}}T_vh_{\bar{\lambda}d\cap\bar{\mu}}.$$

(a) If $d = u_0\hat{d}v_0$ is a left distinguished decomposition for d, then

(4.2.2.2)
$$X_{W_\lambda dW_\mu} = x_\lambda T_{u_0\hat{d}}\pi_{\mu\backslash\hat{\nu}}T_{v_0}h_{\bar{\lambda}d\cap\bar{\mu}}.$$

(b) $X_{W_\lambda dW_\mu} = (X_{W_\mu d^{-1}W_\lambda})^{\iota}$.

(c) For $d \in \mathcal{D}_{\underline{\lambda\mu}}$ and $s \in \bar{W}_{\bar{\lambda}} \cap S$, we have $X_{W_{\underline{\lambda}}dW_{\underline{\mu}}} = \pi_\lambda T_{u\hat{d}}\pi_{\mu\backslash\hat{\nu}}T_v$ and $sd \in \mathcal{D}_{\underline{\lambda\mu}}$

$$T_sX_{W_{\underline{\lambda}}dW_{\underline{\mu}}} = \begin{cases} X_{W_{\underline{\lambda}}sdW_{\underline{\mu}}} & \text{if } sd > d, \\ qX_{W_{\underline{\lambda}}sdW_{\underline{\mu}}} + (q-1)X_{W_{\underline{\lambda}}dW_{\underline{\mu}}} & \text{if } sd < d. \end{cases}$$

A similar result holds for $s \in \bar{W}_{\bar{\mu}} \cap S$ and $X_{W_{\underline{\lambda}}dW_{\underline{\mu}}}T_s$.

(d) Let $\mathcal{H}^{\lambda\mu}_{\mathcal{Z}'}$ be the submodule of $\mathcal{H}'$ generated by all $X_{W_\lambda dW_\mu}$, $d \in \mathcal{D}_{\lambda\mu}$. Then it is a free $\mathcal{Z}'$-module of rank $\#\mathcal{D}_{\lambda\mu}$.

Proof. The statement (a) is easy. (Write $v_0 = xv$ where $x \in W_{\hat{\nu}}$ (see the proof of (2.3.1)), and $u = u_0x'$ where $x' \in W_{\hat{\tau}} = \hat{d}W_{\hat{\nu}}\hat{d}^{-1}$. Then $\hat{d}x' = x'\hat{d}$ with length additivity on both sides, since $\hat{d} \in \mathcal{D}_{\hat{\lambda}\hat{\mu}}$.) From (4.2.1) above, we have by (a)

$$\begin{aligned}
X_{W_\lambda dW_\mu} &= x_{\bar{\lambda}}T_{u_0}\pi_\lambda T_{\hat{d}}\pi_{\mu\backslash\hat{\nu}}T_{v_0}h_{\bar{\lambda}d\cap\bar{\mu}} \\
&= x_{\bar{\lambda}}T_{u_0}\pi_{\lambda\backslash\hat{\nu}}T_{\hat{d}v_0}h_{\bar{\lambda}d\cap\bar{\mu}}\pi_\mu \quad \text{by (4.2.1a)} \\
&= h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}x_{\bar{\nu}^0}T_{u_0}\pi_{\lambda\backslash\hat{\nu}}T_{\hat{d}v_0}h_{\bar{\lambda}d\cap\bar{\mu}}\pi_\mu \\
&= h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}T_{u_0}x_{\bar{\nu}^1}\pi_{\lambda\backslash\hat{\nu}}T_{\hat{d}v_0}h_{\bar{\lambda}d\cap\bar{\mu}}\pi_\mu \quad \text{by (4.2.1b)} \\
&= h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}T_{u_0}\pi_{\lambda\backslash\hat{\nu}}x_{\bar{\nu}^1}T_{\hat{d}v_0}h_{\bar{\lambda}d\cap\bar{\mu}}\pi_\mu \quad \text{by (4.2.1c)} \\
&= h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}T_{u_0}\pi_{\lambda\backslash\hat{\nu}}T_{u_0}^{-1}x_{\bar{\nu}^0}T_dh_{\bar{\lambda}d\cap\bar{\mu}}\pi_\mu \quad \text{by (4.2.1b) again} \\
&= h^{\iota}_{\bar{\lambda}\cap d\bar{\mu}}T_{u_0}\pi_{\lambda\backslash\hat{\nu}}T_{\hat{d}v_0}x_\mu \\
&= (X_{W_\mu d^{-1}W_\lambda})^{\iota},
\end{aligned}$$

proving (b). The first assertion in (c) is from the definition, since $x_{\underline{\lambda}} = \pi_\lambda$, and the rest follows easily, since $sd \in \mathcal{D}_{\underline{\lambda\mu}}$ by (2.2.7).

Observe from (3.2.1) and the definition of $\pi_{\mu\backslash\hat{\nu}}$ that

$$\pi_{\mu\backslash\hat{\nu}} = \begin{cases} q^m T_1 + \sum_{1\neq y\in C_{\mu\backslash\hat{\nu}}} q^{m_y} T_y, & \text{if } C_{\mu\backslash\hat{\nu}} \neq \{1\} \\ 1, & \text{otherwise.} \end{cases}$$

Note $n_0(u_0\hat{d}yv_0x) = n_0(\hat{d}y) > n_0(\hat{d}) = n_0(d)$ for every such y and $x \in W_{\bar{\mu}}$, by (2.1.1b,c). Write $y = d'y'$ with $y' \in W_{\bar{\mu}}$ and d' (left) distinguished with respect to $W_{\bar{\mu}}$ (cf. (2.2.2)). Then $u_0\hat{d}d' \in \mathcal{D}_{\lambda\bar{\mu}}$ since $u_0\hat{d} \in \mathcal{D}_{\bar{\mu}}^{-1}$. Thus, in (4.2.2.2), every term in $T_{u_0\hat{d}}\pi_{\lambda\backslash\hat{\nu}}T_{v_0}h_{\bar{\lambda}d\cap\bar{\mu}}$ of the form $T_{u_0\hat{d}}T_yT_{v_0}T_x = T_{u_0\hat{d}d'}T_{y'}T_{v_0}T_x$ where $x \in W_{\bar{\mu}}$ is a linear combination of T_w's with $w \in W_{\bar{\lambda}}u_0\hat{d}d'W_{\bar{\mu}}$. Since $X_{W_\lambda dW_{\bar{\mu}}} = x_\lambda T_d h_{\bar{\lambda}d\cap\bar{\mu}}$, we have by (3.1.1b) and (3.2.2)

$$X_{W_\lambda dW_\mu} = q^m X_{W_\lambda dW_{\bar{\mu}}} + \sum_{\substack{d'\in\mathcal{D}_\lambda \\ n_0(d')>n_0(d)}} \xi_{d'}x_\lambda T_{d'},$$

with $\xi_{d'} \in \mathcal{Z}'$. By (3.1.2), the set $\{X_{W_\lambda dW_{\bar{\mu}}} \mid d \in \mathcal{D}_{\lambda\mu}\}$ is linearly independent. An argument by induction on $n_0(d)$ proves (d). □

For $d \in \mathcal{D}_{\lambda\mu}$, let $\hat{\nu}(d) = \hat{\nu}(\hat{d}) = \hat{\lambda}\hat{d} \cap \hat{\mu}$, $\bar{\nu}(d) = \bar{\lambda}d \cap \bar{\mu}$ be as before.

(4.2.3) Lemma. *Maintain the notation introduced above, and let $d = u\hat{d}v \in \mathcal{D}_{\lambda\mu}$ be the left or right distinguished decomposition of d.*

(a) W is a disjoint union $W = \cup_{d\in\mathcal{D}_{\lambda\mu}} W_{\hat{\lambda}}u\hat{d}C_{\mu\backslash\hat{\nu}(d)}v(\mathcal{D}_{\bar{\nu}(d)} \cap W_{\bar{\mu}})$.

(b) If $c \in C_{\mu\backslash\hat{\nu}(d)}$ and $c \neq 1$, then $W_{\hat{\lambda}}\hat{d}\bar{W}_{\hat{\mu}} \cap W_{\hat{\lambda}}\hat{d}c\bar{W}_{\hat{\mu}} = \emptyset$.

Proof. (a) follows from (2.3.3) and the relation $vC_{\mu\backslash\lambda d\cap\mu}v^{-1} = C_{\mu\backslash\hat{\nu}(d)}$. To prove (b), note $wc \in \mathcal{D}_{\hat{\nu}}$ for some $w \in \bar{W}_{\hat{\nu}}$, as follows using (2.1.1) and (2.2.6.1). We have $\hat{d}w = w'\hat{d}$ where $w' \in \hat{d}\bar{W}_{\hat{\nu}}\hat{d}^{-1} \subseteq \bar{W}$. Thus, by [C; (2.7.5)] and (2.1.1), $n_0(\hat{d}c) > n_0(\hat{d})$, minimum for its coset $W_{\hat{\lambda}}\hat{d}wc = W_{\hat{\lambda}}\hat{d}c$, or for its double coset $W_{\hat{\lambda}}\hat{d}c\bar{W}_{\hat{\mu}}$. □

Put

$$\varepsilon(d) = \begin{cases} 0, & \text{if } C_{\mu\backslash\hat{\nu}(d)} = \{1\} \\ 1, & \text{otherwise.} \end{cases}$$

(4.2.4) Lemma. *Keep the notation introduced above, and assume $\lambda = \underline{\lambda}$ and $\mu = \underline{\mu}$. For any $d \in \mathcal{D}_{\lambda\mu}$, let $d = u\hat{d}v$ be a left distinguished decomposition and write $C_{\mu\backslash\hat{\nu}(d)} = \langle t_{a(d)+1}, \cdots, t_{a(d)+i(d)}\rangle$ where $a(d), i(d)$ are non-negative integers, and $i(d) \geq 1$ if $C_{\mu\backslash\hat{\nu}(d)} \neq \{1\}$. Then the set*

$$\mathcal{B}_\lambda = \{\pi_\lambda T_{u\hat{d}}T_{t_{a(d)+1}}^{\varepsilon(d)\varepsilon_1} \cdots T_{t_{a(d)+i(d)}}^{\varepsilon(d)\varepsilon_{i(d)}} T_v \mid d = u\hat{d}v \in \mathcal{D}_{\lambda\mu}, \varepsilon_j = 0, 1\ \forall j\}$$

forms a basis for $\pi_\lambda\mathcal{H}'$.

Moreover, if M_λ is the free $\mathcal{Z}'$-submodule of $\pi_\lambda\mathcal{H}'$ spanned by the elements in $\mathcal{B}_\lambda$ with $\varepsilon_j = 1$ for some j, then $\pi_\lambda\mathcal{H}' = \mathcal{H}^{\lambda\mu}_{\mathcal{Z}'} \oplus M_\lambda$. Each element of M_λ is a $\mathcal{Z}'$-linear combination of elements T_x with $x \in W_{\hat\lambda}\hat{d}c\bar{W}_{\hat\mu}$ as in (4.2.3b), with $c \neq 1, c \in C_{\mu\backslash\hat\nu(\hat d)}$ and $\hat d \in \mathcal{D}_{\hat\lambda\hat\mu}$.

Proof. The left distinguished decomposition $d = u\hat{d}v$ implies that $u\hat d \in \mathcal{D}_{\bar\mu}^{-1}$. Applying (3.1.1b) to the product $T_{t_{a(d)+1}}^{\varepsilon(d)\varepsilon_1}\cdots T_{t_{a(d)+i(d)}}^{\varepsilon(d)\varepsilon_{i(d)}}T_v$, we see that

$$T_{u\hat d}T_{t_{a(d)+1}}^{\varepsilon(d)\varepsilon_1}\cdots T_{t_{a(d)+i(d)}}^{\varepsilon(d)\varepsilon_{i(d)}}T_v = q^aT_{u\hat d d'w'v} + \sum_{w\in\bar W_{\hat\mu}, w'v<w}\xi_wT_{u\hat dd'w}, \tag{4.2.4.1}$$

where ${t_{a(d)+1}}^{\varepsilon(d)\varepsilon_1}\cdots {t_{a(d)+i(d)}}^{\varepsilon(d)\varepsilon_{i(d)}} = c = d'w'$ as in (2.2.2) with d' a distinguished left coset representative for $\bar W$. As argued in the proof of (4.2.3b), $n_0(\hat dc)$ is minimal for $W_{\hat\lambda}\hat dc\bar W_{\hat\mu}$. Thus, $\hat dc \in \mathcal{D}_\lambda$ by (2.2.6.1). It follows that $u\hat dd'w'v$ and all $u\hat dd'w$ in the sum belong to $\mathcal{D}_\lambda = \mathcal{D}_{\underline{\lambda}}$, by (2.2.7). Also, $u\hat dd'w'v < u\hat dd'w$, since $u\hat d \in \mathcal{D}_{\bar\mu}^{-1}$ and $w'v < w$. By (4.2.3a) and (3.1.2), the set $\mathcal{B}_\lambda$ is linearly independent, and spans, over any field k which is a $\mathcal{Z}'$-algebra, a subspace with the same dimension as $\pi_\lambda\mathcal{H}'_k$. Now, applying [CPS1; (3.3.1)] as in the proof of (4.1.1) proves the first assertion.

Observe that $X_{W_{\underline\lambda}dW_{\underline\mu}} = \pi_\lambda T_{u\hat d}\pi_{\mu\backslash\hat\nu(d)}T_v$, and, if $C_{\mu\backslash\hat\nu(d)} = \{1\}$, then $X_{W_{\underline\lambda}dW_{\underline\mu}} = \pi_\lambda T_d$. Now, the direct sum assertion follows, using the definition of M_λ. The final assertion is obtained by rewriting the elements $u\hat dd'w'v$, $u\hat dd'w$, and using the length additivity for the products from C_μ with $\mathcal{D}_{\hat\mu}$ □

(4.2.5) Proposition. *For any bicomposition λ,μ of r, the $\mathcal{Z}'$-module $x_\lambda\mathcal{H}' \cap \mathcal{H}'x_\mu$ is free and $\mathcal{H}^{\lambda\mu}_{\mathcal{Z}'} = x_\lambda\mathcal{H}' \cap \mathcal{H}'x_\mu$. When $\mathcal{Z}' = \mathcal{Z}$, we have $\mathcal{H}^{\lambda\mu} = \mathcal{H}^{\lambda\mu}_K \cap \mathcal{H}$.*

Proof. The inclusion $\mathcal{H}^{\lambda\mu}_{\mathcal{Z}'} \subseteq x_\lambda\mathcal{H}' \cap \mathcal{H}'x_\mu$ follows from (4.2.2b). Conversely, by (4.1.3), we have $x_\lambda\mathcal{H}' \cap \mathcal{H}'x_\mu = \pi_\lambda\mathcal{H}' \cap \mathcal{H}'\pi_\mu \cap x_{\bar\lambda}\mathcal{H}' \cap \mathcal{H}'x_{\bar\mu}$. It suffices to prove the following: (1) $\pi_\lambda\mathcal{H}' \cap \mathcal{H}'\pi_\mu = \mathcal{H}^{\lambda\mu}_{\mathcal{Z}'}$; (2) $\mathcal{H}^{\lambda\mu}_{\mathcal{Z}'} \cap x_{\bar\lambda}\mathcal{H}' \cap \mathcal{H}'x_{\bar\mu} \subseteq \mathcal{H}^{\lambda\mu}_{\mathcal{Z}'}$.

To see (1), it suffices to prove $M_{\underline\lambda} \cap \mathcal{H}'\pi_\mu = \{0\}$ where $M_{\underline\lambda}$ is the complement of $\mathcal{H}^{\lambda\mu}_{\mathcal{Z}'}$ in $\pi_\lambda\mathcal{H}'$ as given in (4.2.4). This can be seen as follows. Suppose $h \in \mathcal{H}'\pi_\mu$. By (4.2.4), $\mathcal{H}'\pi_\mu$ has a basis $\mathcal{B}'_\mu$ (reversing the roles of $\underline\lambda$ and $\underline\mu$ in (4.2.4)). By looking at the constant term in π_μ, there is a term T_x in h with $x \in W_{\bar\lambda}d$ for some $d = u\hat dv \in \mathcal{D}_{\underline{\lambda\mu}}$. However, by the last assertion in (4.2.4) and (4.2.3b), the elements in $M_{\underline\lambda}$ do not have such a term, proving (1).

We now prove (2). Pick $h \in \mathcal{H}^{\lambda\mu}_{\mathcal{Z}'} \cap x_{\bar\lambda}\mathcal{H}' \cap \mathcal{H}'x_{\bar\mu}$ and write $h = \sum_{d\in\mathcal{D}_{\underline{\lambda\mu}}}\xi_dX_{W_{\underline\lambda}dW_{\underline\mu}}$. Then, for any $s \in \bar W_{\bar\lambda} \cap S$, $s' \in \bar W_{\bar\mu} \cap S$, $T_sh = qh = hT_{s'}$. By (4.2.2c) and equating the coefficients, one sees easily $\xi_d = \xi_{sd} = \xi_{ds'}$ for all $s \in \bar W_{\bar\lambda} \cap S$, $s' \in \bar W_{\bar\mu} \cap S$ and $d \in \mathcal{D}_{\underline{\lambda\mu}}$. Consequently, $\xi_d = \xi_{wd}$ for all $d \in \mathcal{D}_{\underline{\lambda\mu}}$, $w \in \bar W_{\bar\lambda}$ and $\xi_d = \xi_{dy}$ for all

$d \in \mathcal{D}_{\underline{\lambda}\mu}$, $y \in \bar{W}_{\bar{\mu}}$. Therefore,

$$\begin{aligned} h = & x_{\bar{\lambda}} \sum_{d \in \mathcal{D}_{\lambda\underline{\mu}}} \xi_d X_{W_{\underline{\lambda}} d W_{\underline{\mu}}} = \sum_{d \in \mathcal{D}_{\lambda\underline{\mu}}} \xi_d X_{W_\lambda d W_{\underline{\mu}}} \\ = & \sum_{d \in \mathcal{D}_{\lambda\mu}} \xi_d X_{W_\lambda d W_{\underline{\mu}}} h_{\bar{\lambda} d \cap \bar{\mu}} = \sum_{d \in \mathcal{D}_{\lambda\mu}} \xi_d X_{W_\lambda d W_\mu} \in \mathcal{H}_{\mathcal{Z}'}^{\lambda\mu} \end{aligned}$$

where $h_{\bar{\lambda} d \cap \bar{\mu}}$ is as in (4.2.2), as required.

The final assertion now follows from (4.1.3). □

The basis for $\mathcal{H}_{\mathcal{Z}'}^{\lambda\mu}$ gives actually a basis for the intertwining module between $\mathcal{T}'_\mu$ and $\mathcal{T}'_\lambda$. To this end, we define for $d \in \mathcal{D}_{\lambda\mu}$ a function $\varphi^d_{\lambda\mu} : \mathcal{T}'_\mu \to \mathcal{T}'_\lambda$ by setting $\varphi^d_{\lambda\mu}(x_\mu h) = X_{W_\lambda d W_\mu} h \in x_\lambda \mathcal{H}' = \mathcal{T}'_\lambda$ for all $h \in \mathcal{H}'$. Note that $\varphi^d_{\lambda\mu}$ is well-defined, since $X_{W_\lambda d W_\mu} = h_\lambda x_\mu$ for some $h_\lambda \in \mathcal{H}'$, and $\varphi^d_{\lambda\mu}$ is a homomorphism of right $\mathcal{H}'$-modules. Here is the answer to the first and third questions raised at the end of §3.3.

(4.2.6) Theorem. *The $\mathcal{Z}'$-module $\mathrm{Hom}_{\mathcal{H}'}(\mathcal{T}'_\mu, \mathcal{T}'_\lambda)$ is free with basis $\{\varphi^d_{\lambda\mu} \mid d \in \mathcal{D}_{\lambda\mu}\}$. In particular, the q-Schur^2 algebra $\mathcal{S}^2_q(n, r, \mathcal{Z}')$ is a free $\mathcal{Z}'$-module. Moreover, base change induces isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{T}_\mu, \mathcal{T}_\lambda)_{\mathcal{Z}'} \cong \mathrm{Hom}_{\mathcal{H}'}(\mathcal{T}'_\mu, \mathcal{T}'_\lambda) \quad \textit{and} \quad \mathcal{S}^2_q(n, r)_{\mathcal{Z}'} \cong \mathcal{S}^2_q(n, r, \mathcal{Z}').$$

Proof. As noted, the mappings $\varphi^d_{\lambda\mu}$ are $\mathcal{H}'$-linear, and they clearly form a linearly independent set. It remains to check the spanning condition. Let φ be an element in the Hom set. Then $\varphi(x_\mu) \in x_\lambda \mathcal{H}' \cap \mathcal{H}' x_\mu$ by (4.1.3). By (4.2.5), we have $\varphi(x_\mu) \in \mathcal{H}^{\lambda\mu}_{\mathcal{Z}'}$. So $\varphi(x_\mu) = \sum_{d \in \mathcal{D}_{\lambda\mu}} \xi_d X_{W_\lambda d W_\mu}$ for some $\xi_d \in \mathcal{Z}'$. Consequently, we have $\varphi = \sum_{d \in \mathcal{D}_{\lambda\mu}} \xi_d \varphi^d_{\lambda\mu}$, proving the first assertion. Noting $\mathcal{S}^2_q(n, r, \mathcal{Z}') \cong \bigoplus_{\lambda, \mu \in \Pi(n,r)} \mathrm{Hom}_{\mathcal{H}'}(\mathcal{T}'_\lambda, \mathcal{T}'_\mu)$, we have immediately the second assertion. The last assertion follows from the fact that the definition of $\varphi^d_{\lambda\mu}$ is the same for any coefficient ring $\mathcal{Z}'$. □

We remark that the first assertion in the theorem is really a version of the Frobenius reciprocity and Mackey decomposition theorem for quasi-parabolic subgroups, though we couldn't state it in terms of induction from subalgebras. (The induced *modules* have q-analogs, but not the subgroups or subgroup algebras involved.)

We remark also that, as the anti-automorphism ι takes a right ideal of $\mathcal{H}'$ to a left ideal, the elements $(\varphi^d_{\lambda\mu})^\iota$ defined by setting $(\varphi^d_{\lambda\mu})^\iota(hx_\mu) = (\varphi^d_{\lambda\mu}(x_\mu h^\iota))^\iota$ for any $h \in \mathcal{H}'$ form a basis for $\mathrm{Hom}_{\mathcal{H}'}(\mathcal{H}' x_\mu, \mathcal{H}' x_\lambda)$.

4.3 Self duality of permutation modules. For any right $\mathcal{H}'$-module M, let M^ι be the left $\mathcal{H}'$-module obtained by shifting the right action on M to the left via ι. That is, we have $h * m = mh^\iota$ for all $h \in \mathcal{H}'$, $m \in M$. In particular, if M happens to be a right ideal of $\mathcal{H}'$, then M^ι is isomorphic to the left ideal M^ι, the image of the right ideal M under ι. The $\mathcal{Z}'$-dual $M^* = \mathrm{Hom}_{\mathcal{Z}'}(M, \mathcal{Z}')$ of M is a left $\mathcal{H}'$-module via the action $(hf)(m) = f(mh)$ where $h \in \mathcal{H}'$, $f \in M^*$ and $m \in M$.

We say that M is *self dual* if $M^* \cong M^\iota$. Note that we have $(M^*)^\iota \cong (M^\iota)^*$. Define $M^\star = (M^*)^\iota$. Note that a 'star' and not an 'asterisk' is used for this same-side dual, though the left/right context is also a guide as to which dual is being used. Now, self-duality is equivalent to $M^\star \cong M$.

Let $\tau : \mathcal{H}' \to \mathcal{Z}'$ be the projection onto the identity element of $\mathcal{H}'$ (with respect to the usual basis of T_w's) and define a bilinear form $(\ ,\ ) : \mathcal{H}' \times \mathcal{H}' \to \mathcal{Z}'$ by $(h, h') = \tau(hh')$. This is symmetric and associative, and the map $h \mapsto (-, h)$ gives an $\mathcal{H}'$-module isomorphism $\mathcal{H}' \to (\mathcal{H}')^\star$. In particular, $\mathcal{H}'$ is self dual.

(4.3.1) Proposition. *The permutation modules $x_\lambda\mathcal{H}'$ are self dual.*

Proof. First, we assume $\mathcal{Z}' = \mathcal{Z}$. It is enough to prove that $(x_\lambda\mathcal{H})^* \cong \mathcal{H}x_\lambda$. Let $x_\lambda^2 = z_\lambda x_\lambda$. Then z_λ is a unit in $\mathcal{H}_K$ and commutes with x_λ (see [DJ4; (4.5)]). Define $\langle -, - \rangle : x_\lambda\mathcal{H} \times \mathcal{H}x_\lambda \to \mathcal{Z}$ by $\langle x_\lambda h, h'x_\lambda \rangle = \tau(x_\lambda hh'x_\lambda/z_\lambda) = \tau(x_\lambda hh') = (x_\lambda h, h')$. For $y \in \mathcal{D}_\lambda$, write

$$x_\lambda T_y = \sum_{x \in C_\lambda, w \in \bar{W}_\lambda y} q^{m_x} T_x T_w = q^a T_y + \sum_z \xi_z T_z,$$

with $a = m_1$, z ranging over $W_\lambda y$, and $\xi_z \in \mathcal{Z}'$. Since y has minimal length among the elements of $W_\lambda y$ by (2.2.3), we have by (3.1.1a) that $\ell(z) > \ell(y)$ whenever $\xi_z \neq 0$. Order the elements of W in the way $w_1, w_2, \cdots$ such that $\ell(w_i) \leq \ell(w_j) \Rightarrow i \leq j$. We have for $y, w^{-1} \in \mathcal{D}_\lambda$,

$$\langle x_\lambda T_y, T_w x_\lambda \rangle = (x_\lambda T_y, T_w) = \begin{cases} 0, & \text{if } \ell(w) \leq \ell(y), w \neq y \\ q^{m_1}, & \text{if } w = y. \end{cases}$$

Then the matrix of the form $\langle -, - \rangle$ with respect to the induced order on $\mathcal{D}_\lambda$ is upper triangular with invertible diagonal product, and hence is non-degenerate. The general case follows from base change. The proposition is proved. □

5. Bistandard bases and Specht series for permutation modules

The classical Young rule for symmetric groups [JK; p.89] gives the multiplicity of a Specht module in a permutation module in terms of the number of semi-standard tableaux. We shall show in this section that similar rule holds for the q-permutation modules and (twisted) Specht modules introduced in previous sections. This is in fact a character problem which will be treated in the first subsection. Our task then is to prove the existence of Specht filtrations. We shall achieve this by introducing the bistandard basis for $\mathcal{T}_\lambda$ with which we will get bistandard bases for q-Schur^2 algebras.

By (4.2.6), it is sufficient to look at the integral case $\mathcal{Z}' = \mathcal{Z}$. From now on we will keep this assumption. The reader can easily interpret our results for general $\mathcal{Z}'$; see (6.2.3c).

5.1 The multiplicities $m_{\lambda\mu}$. All representations in this subsection are complex representations.

Let $\mu \in \Pi_r^+$ be a bipartition and $\lambda \in \Pi_r$ a bicomposition. Let $\mathfrak{T}^{ss}(\mu, \lambda)$ be the set of all semi-standard μ-bitableaux of type λ (see (1.2.2)). To avoid too many indices we just write $\lambda = (\alpha, \beta)$ and $\mu = (\gamma, \delta)$. We assume $\gamma \vdash c$ and $\delta \vdash d$. Let ε_δ be the sign representation of C_δ, where $C_\delta = C_{[c+1,r]} = \langle t_{c+1}, \cdots, t_r\rangle$ is the complement of $C_\gamma := C_\mu = C_{[1,c]}$. We extend it to a linear representation of C, denoted again by ε_δ, such that it acts on C_μ trivially. The group W acts on ε_δ by $w\varepsilon_\delta(x) = \varepsilon_\delta(w^{-1}xw)$, for $w \in W, x \in C$. Clearly, the inertia group of ε_δ is $C\bar{W}_{(c,d)} = W_{((c,d),-)}$. Thus, ε_δ extends to a linear representation of $W_{((c,d),-)}$ by setting $\varepsilon_\delta(xw) = \varepsilon_\delta(x)$ for $x \in C$ and $w \in \bar{W}_{(c,d)}$. Let $\bar{S}_\gamma$ and $\bar{S}_\delta$ be the irreducible modules of $\mathfrak{S}_c$ and $\mathfrak{S}_d$, respectively, corresponding to γ and δ. We lift their *outer* tensor product $\bar{S}_\gamma \odot \bar{S}_\delta$ to an irreducible module of $W_{((c,d),-)}$ with C acting trivially, and form the (inner) tensor product $(\bar{S}_\gamma \odot \bar{S}_\delta) \otimes \varepsilon_\delta$. Note that this can also be viewed as an outer tensor product $\bar{S}_\gamma \odot \bar{S}^\varepsilon_\delta$ of $\bar{S}_\gamma$ and $\bar{S}^\varepsilon_\delta = \bar{S}_\delta \otimes \varepsilon_\delta$, lifted to a module of $W_{((c,d),-)}$. Then one sees easily that the induced $\mathbb{C}W$-module S_μ of $\bar{S}_\gamma \odot \bar{S}^\varepsilon_\delta$ is irreducible. Let T_λ be the complex permutation module of W on the right cosets of W_λ.

(5.1.1) Theorem. *The dimension of* $\mathrm{Hom}_{\mathbb{C}W}(S_\mu, T_\lambda)$ *is equal to the number* $\#\mathfrak{T}^{ss}(\mu,\lambda)$ *of semi-standard μ-bitableaux of type λ.*

Proof. Let $a = |\alpha|$ and $b = |\beta|$, and let $T_{(\alpha,-)}$ and T_β be the permutation modules of W_a and $\mathfrak{S}_b$ on the right cosets of $C_{(\alpha,-)}\mathfrak{S}_\alpha$ and $\mathfrak{S}_\beta$ respectively. Then

$$T_\lambda = 1_{W_\lambda} \uparrow^{W_{(a,b)}} \uparrow^{W}_{W_{(a,b)}} = (T_{(\alpha,-)} \odot T_\beta) \uparrow^{W}_{W_{(a,b)}}.$$

Let $\mathcal{P} = \mathcal{P}_b$ be the power set of $\{1, \cdots, b\}$. For any $J \in \mathcal{P}$, we have a corresponding linear representation ε_J of C_β, where $C_\beta = C_{[a+1,r]}$ is the complement of $C_\alpha := C_{[1,a]}$ defined similarly as above, and ε_J takes the sign representation on the copies of $C_2 \cong \langle t_j\rangle$ with indices in J and the trivial representation on the remaining copies. Thus, the regular representation of C_β is a direct sum of them: $\mathbb{C}C_\beta = \bigoplus_{J\in\mathcal{P}} \varepsilon_J$. This module extends to $C_\beta\mathfrak{S}_b \cong W_b$ by letting $\mathfrak{S}_b$ act by conjugation. So we have by transitivity of induction and a well-known tensor identity of Brauer

$$T_\lambda = (T_{(\alpha,-)} \odot T_\beta) \uparrow^{W}_{W_{(a,b)}} \cong (T_\alpha \odot (T_\beta \otimes \mathbb{C}C_\beta)) \uparrow^{W}_{W_{((a,b),-)}}.$$

By Frobenius reciprocity, we have

$$\mathrm{Hom}_{\mathbb{C}W}(S_\mu, T_\lambda) = \mathrm{Hom}_{\mathbb{C}W_{((c,d),-)}}(\bar{S}_\gamma \otimes \bar{S}^\varepsilon_\delta, (T_{(\alpha,-)} \odot (T_\beta \otimes \mathbb{C}C_\beta)) \uparrow^{W}_{W_{((a,b),-)}} \downarrow_{W_{((c,d),-)}}),$$

which is zero even on $\mathbb{C}C$ if $b < d$ as there is no ε_J equal to a conjugate of ε_δ in this case. (They all take -1 values on too few elements t_j.) Clearly, $\#\mathfrak{T}^{ss}(\mu,\lambda)$ equals zero, too, in this case.

We now assume $b \geq d$. Note that $a+b = r = c+d$. By the Mackey decomposition theorem, we have

$$\begin{aligned}
&(T_{(\alpha,-)} \odot (T_\beta \otimes \mathbb{C}C_\beta)) \uparrow^{W}_{W_{((a,b),-)}} \downarrow_{W_{((c,d),-)}} \\
&= \bigoplus_{d\in\bar{\mathcal{D}}_{(a,b),(c,d)}} (T_{(\alpha,-)} \odot (T_\beta \otimes \mathbb{C}C_\beta)) \downarrow_{W^d_{((a,b),-)} \cap W_{((c,d),-)}} \uparrow^{W_{((c,d),-)}} \\
&= (T_{(\alpha,-)} \odot (T_\beta \otimes \mathbb{C}C_\beta)) \downarrow_{W_{((a,b-d,d),-)}} \uparrow^{W_{((c,d),-)}} + \begin{pmatrix}\text{terms with no conjugate of} \\ \varepsilon_\delta \text{ as a subfactor on } \mathbb{C}C\end{pmatrix}
\end{aligned}$$

As above, we can ignore the terms on the right. Frobenius reciprocity can be applied to calculate on the left, though we prefer to keep the induction in evidence. We view $W_{((a,b-d,d),-)}$ as the direct product of $W_{(a,-)}$ with $W_{((b-d,d),-)}$. Note that any irreducible module for the latter group which contains $\varepsilon_\delta \downarrow C_\beta$ on C_β must be in the induction of $\varepsilon_\delta \downarrow C_\beta$ to $W_{((b-d,d),-)}$, so cannot contain any other irreducible representation on C_β. So we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{C}W}(S_\mu, T_\lambda)\\
&= \mathrm{Hom}_{\mathbb{C}W_{((c,d),-)}}(\bar{S}_\gamma \odot \bar{S}^\varepsilon_\delta, (T_{(\alpha,-)} \odot (T_\beta \otimes \mathbb{C}C_\beta)) \downarrow_{W_{((a,b-d,d),-)}} \uparrow^{W_{((c,d),-)}})\\
&= \mathrm{Hom}_{\mathbb{C}W_{((c,d),-)}}(\bar{S}_\gamma \odot \bar{S}^\varepsilon_\delta, (T_{(\alpha,-)} \odot (T_\beta \otimes \varepsilon_\delta)) \downarrow_{W_{((a,b-d,d),-)}} \uparrow^{W_{((c,d),-)}})\\
&= \mathrm{Hom}_{\mathbb{C}\bar{W}_{(c,d)}}(\bar{S}_\gamma \odot \bar{S}_\delta, (T_\alpha \odot T_\beta \downarrow_{\bar{W}_{(b-d,d)}}) \uparrow^{\bar{W}_{(c,d)}}).
\end{aligned}$$

However, by (1.2.4) and the Mackey decomposition theorem again, we have

$$T_\beta \downarrow_{\mathfrak{S}_{b-d}\times\mathfrak{S}_d} = \bigoplus_{(\beta^1,\beta^2)\in\Pi(\beta)} (1_{\beta^1} \odot 1_{\beta^2}) \downarrow_{\mathfrak{S}_{\beta^1}\times\mathfrak{S}_{\beta^2}} \uparrow \mathfrak{S}_{b-d} \times \mathfrak{S}_d.$$

Thus, we eventually have, using (1.2.4b)

$$\begin{aligned}
\dim \mathrm{Hom}_{\mathbb{C}W}(S_\mu, T_\lambda) &= \dim \bigoplus_{(\beta^1,\beta^2)\in\Pi(\beta)} \mathrm{Hom}_{\mathbb{C}\bar{W}_{(c,d)}}(\bar{S}_\gamma \odot \bar{S}_\delta, T_{\alpha\vee\beta^1} \odot T_{\beta^2})\\
&= \sum_{(\beta^1,\beta^2)\in\Pi(\beta)} \dim \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_c}(\bar{S}_\gamma, T_{\alpha\vee\beta^1}) \dim \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}(\bar{S}_\delta, T_{\beta^2})\\
&= \#\mathfrak{T}^{ss}(\mu,\lambda),
\end{aligned}$$

as required. □

As, under the specialization $\mathcal{Z} \to \mathbb{Q}$ such that $q_0 = q = 1$ in $\mathbb{Q}$, the Hecke algebra $\mathcal{H}_\mathbb{Q}$ becomes the group algebra, which is still semi-simple, we have immediately from Theorem 5.1.1 and [C; (10.11.2)] the following.

(5.1.2) Corollary. *(a) Let $m_{\lambda\mu}$ be the multiplicity defined in (3.4.3). Then $m_{\lambda\mu} = \#\mathfrak{T}^{ss}(\mu,\lambda)$.*

(b) For bicompositions λ and ν of r, we have

$$\#\mathcal{D}_{\lambda\nu} = \#\{(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \mathfrak{t} \in \mathfrak{T}^{ss}(\mu,\nu), \mu \in \Pi^+_r\}.$$

Proof. (a) follows from that fact that S_μ is isomorphic to the specialization of $S^\natural_\mu$ at $q_0 = q = 1$. (See [DJ4; §5] and [DJM; (4.22)].)

The statement (b) follows from (a) and the fact that

$$\mathrm{Hom}_{\mathcal{H}_K}(\mathcal{T}_{\lambda K}, \mathcal{T}_{\nu K}) \cong \bigoplus_{\mu\in\Pi^+_r} \mathrm{Hom}_{\mathcal{H}_K}(\mathcal{T}_{\lambda K}, S^\natural_{\mu K}) \otimes \mathrm{Hom}_{\mathcal{H}_K}(S^\natural_{\mu K}, \mathcal{T}_{\nu K}). \quad □$$

5.2 The bistandard bases for $\mathcal{T}_\lambda$ (and $\mathcal{T}'_\lambda$). Recall from §3.4 the bistandard basis $\{x_{\mathfrak{st}} \mid \mathfrak{s},\mathfrak{t} \in \mathbf{T}^s(\mu), \mu \in \Pi_r^+\}$ for $\mathcal{H}$. We are now ready to introduce bistandard bases for permutation modules $\mathcal{T}_\lambda$. Let λ be a bicomposition of r and μ a bipartition of r. Associated to each $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)$, there is a distinguished double coset representative $\delta(\mathfrak{s}) \in \mathcal{D}_{\bar\lambda\bar\mu} \cap \bar W$. Since $\mathcal{D}_{\bar\lambda\bar\mu} \cap \bar W = \mathcal{D}_{\lambda\mu} \cap \bar W$, we have $\delta(\mathfrak{s}) \in \mathcal{D}_{\lambda\mu}$ and therefore, it defines a basis element $\varphi_{\lambda\mu}^{\delta(\mathfrak{s})} \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{T}_\mu, \mathcal{T}_\lambda)$ (cf. (4.2.6)).

The following lemma is a generalization of a result in [M; §7] to the type B case.

(5.2.1) Lemma. *For $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)$ and any standard μ-bitableau $\mathfrak{t} \in \mathbf{T}^s(\mu)$, let $X^\mu_{\mathfrak{st}} = \varphi_{\lambda\mu}^{\delta(\mathfrak{s})}(x_{\mu\mathfrak{t}})$. Then the set $\mathfrak{X}_\lambda = \{X^\mu_{\mathfrak{st}} \mid \mu \in \Pi_r^+, \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \mathfrak{t} \in \mathbf{T}^s(\mu)\}$ forms a basis for $\mathcal{T}_\lambda$. Putting $X^\mu_{\mathfrak{ts}} = (X^\mu_{\mathfrak{st}})^\iota$, the set $\mathfrak{X}^\iota_\lambda = \{X^\mu_{\mathfrak{ts}} \mid \mu \in \Pi_r^+, \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \mathfrak{t} \in \mathbf{T}^s(\mu)\}$ is a basis for $\mathcal{H}x_\lambda$.*

Proof. By the definitions given above (3.4.2), and above (4.2.6), we have $X^\mu_{\mathfrak{st}} = \varphi_{\lambda\mu}^{\delta(\mathfrak{s})}(x_{\mu\mathfrak{t}}) = X_{W_\lambda\delta(\mathfrak{s})W_\mu}T_{\delta(\mathfrak{t})}$. Using (1.2.3a), we have $\delta(\mathfrak{s})^{-1}C_\lambda\delta(\mathfrak{s}) \subseteq C_\mu$. Recall from (1.1.1) and (1.2.3) that $\delta(\mathfrak{s}) \in \bar{\mathcal{D}}_{\bar\lambda\bar\mu} \subseteq \mathcal{D}_{\lambda\mu}$. So, putting $d = \delta(\mathfrak{s})$ and defining $\hat d$ as above (4.2.1), we have $C_\lambda = \hat d^{-1}C_\lambda\hat d \subseteq C_\mu$ and $\pi_{\hat\nu} = \pi_\lambda$ where $\hat\nu = \hat\lambda\hat d \cap \hat\mu$. In particular, $\pi_{\lambda\backslash\hat\nu} = T_1$. Therefore, using (4.2.2a,b) and (1.2.3c),

$$X^\mu_{\mathfrak{st}} = X_{W_\lambda\delta(\mathfrak{s})W_\mu}T_{\delta(\mathfrak{t})} = \sum_{x\in\mathcal{D}^{-1}_{\bar\lambda\cap\delta(\mathfrak{s})\bar\mu}\cap\bar W_\lambda} T_xT_{\delta(\mathfrak{s})}x_\mu T_{\delta(\mathfrak{t})} = \sum_{\mathfrak{s}\in\mathbf{T}_{\mathfrak{s}}} x^\mu_{\mathfrak{st}}. \tag{5.2.2}$$

(See above (3.4.1) for the definition of the basis elements $x^\mu_{\mathfrak{st}}$.) By (1.2.3c), these sums are all disjoint, and thus, linearly independent. They form a basis for $\mathcal{T}_{\lambda K}$ by (5.1.2) and a comparison on dimension. Finally, the result follows from (3.2.2d), since the $x^\mu_{\mathfrak{st}}$ form a basis for $\mathcal{H}$. □

(5.2.3) Theorem. *For any bicomposition λ of r, there is a submodule sequence of $\mathcal{T}_\lambda$:*

$$0 = \mathcal{T}^0_\lambda \subset \mathcal{T}^1_\lambda \subset \cdots \subset \mathcal{T}^m_\lambda = \mathcal{T}_\lambda$$

such that $\mathcal{T}^i_\lambda/\mathcal{T}^{i-1}_\lambda \cong S^\natural_{\mu_{(i)}}$ $(1 \le i \le m)$ for some $\mu_{(i)} \in \Pi_r^+$ and $\mu_{(m)} = \lambda''$.

Proof. Since $\mathcal{T}_\lambda$ is isomorphic to $\mathcal{T}_{\lambda''}$ (3.2.2c), we may assume that λ is a bipartition. We order Π_r^+ by $\mu_{(1)}, \mu_{(2)}, \cdots$ such that $\mu_{(i)} \trianglerighteq \mu_{(j)}$ implies $i < j$. Let $\mu = \mu_{(i)}$ be a bipartition of r and let $\mathcal{T}^i_\lambda$ be the $\mathcal{Z}$-submodule of $\mathcal{T}_\lambda$ generated by all $X^{\mu^j}_{\mathfrak{st}}$ with $j \le i$. Clearly, $\mathcal{T}^i_\lambda = 0$ unless $\mu \trianglerighteq \lambda$ and each $\mathcal{T}^i_\lambda$ is an $\mathcal{H}$-submodule of $\mathcal{T}$ by [DJM; (4.18)]. Moreover, $\mathcal{T}^i_\lambda/\mathcal{T}^{i-1}_\lambda$ is $\mathcal{Z}$-free with basis $\{X^\mu_{\mathfrak{st}} \mid \mathfrak{t} \in \mathbf{T}^s(\mu), \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)\}$. For fixed $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)\}$, the set $\{X^\mu_{\mathfrak{st}} \mid \mathfrak{t} \in \mathbf{T}^s(\mu)\}$ generates an $\mathcal{H}$-submodule of $\mathcal{T}^i_\lambda/\mathcal{T}^{i-1}_\lambda$, which is isomorphic to $S^\natural_\mu$, by [DJM; (4.11)] (or (3.4.1)), (3.4.2) and (5.2.2). Hence $\mathcal{T}^i_\lambda/\mathcal{T}^{i-1}_\lambda$ is isomorphic to a direct sum of $\#\mathfrak{T}^{ss}(\mu,\lambda)$ copies of $S^\natural_\mu$. Note that $S^\natural_\lambda$ appears at the top of $\mathcal{T}_\lambda$ with multiplicity 1. Therefore, $\mathcal{T}_\lambda$ has a submodule sequence with $S^\natural_\mu$ as sections. □

We call such a sequence for $\mathcal{T}_\lambda$ a (twisted) *Specht filtration* of $\mathcal{T}_\lambda$. We remark that similar terminology can be used for $\mathcal{T}'_\lambda$. (See (6.2.3c) below.)

6. Quasi-heredity of q-Schur^2 algebras

We are now in a position to prove that a q-Schur^2 algebra is quasi-hereditary. We shall construct a "codeterminant"-type basis which gives easily the required structure. (No actual determinants are involved.) This kind of basis was first introduced by J. A. Green [G; §7] for classical Schur algebras (see also [D1] for the GL_2 case). A proof of the quasi-heredity property can also be given using [DPS2] when q_0 is a fractional power of q; see (6.2.3b) below. Also, see [DR] for a more general construction of standard bases for any split quasi-hereditary algebra.

6.1 Bistandard bases for q-Schur^2 algebras. Recall from (4.2.6) the basis $\{\varphi^d_{\lambda\mu}\}$ for the algebra $\mathcal{S}^2_q(n,r)$. This is the natural basis possessed by a centralizer algebra of a permutation module. We are now ready to introduce a new basis for $\mathcal{S}^2_q(n,r)$ which reflects quite well the structure of its module categories $\mathcal{S}^2_q(n,r)$-mod and mod-$\mathcal{S}^2_q(n,r)$. Since it gives rise to bases (indexed by semi-standard bitableaux) for certain "standard" objects in both these categories, we will call it the *bistandard basis.*

(6.1.1) Theorem. *Let $\lambda,\nu \in \Pi(n,r)$ and $\mu\in\Pi^+(n,r)$. For any $\mathfrak{s}\in\mathfrak{T}^{ss}(\mu,\lambda)$ and $\mathfrak{t}\in\mathfrak{T}^{ss}(\mu,\nu)$, we define $\Phi_{\mathfrak{st}}=\Phi^\mu_{\mathfrak{st}}=\varphi^{\delta(\mathfrak{s})}_{\lambda\mu}\varphi^{\delta(\mathfrak{t})^{-1}}_{\mu\nu}$. Then the set*

$$\{\Phi_{\mathfrak{st}}\mid \mathfrak{s}\in\mathfrak{T}^{ss}(\mu,\lambda),\mathfrak{t}\in\mathfrak{T}^{ss}(\mu,\nu),\mu\in\Pi^+(n,r),\lambda,\nu\in\Pi(n,r)\}$$

forms a basis for $\mathcal{S}^2_q(n,r)$.

Proof. Fix λ,ν and put

$$X_{\mathfrak{st}}=X^\mu_{\mathfrak{st}}=\Phi^\mu_{\mathfrak{st}}(x_\nu)=\varphi^{\delta(\mathfrak{s})}_{\lambda\mu}(X_{W_\mu\delta(\mathfrak{t})^{-1}W_\nu})=\varphi^{\delta(\mathfrak{s})}_{\lambda\mu}(x_\mu T_{\delta(\mathfrak{t})^{-1}}h_{\bar\nu\cap\delta(\mathfrak{t})\bar\mu}).$$

Note that the last equality is obtained from (1.2.3a), as in the proof of (5.2.1), and that $\bar\nu\cap\delta(\mathfrak{t})\bar\mu=\bar\mu\delta(\mathfrak{t})^{-1}\cap\bar\nu$. From the argument given in (4.2.6), we see it suffices to show that the set

$$\{X^\mu_{\mathfrak{st}}\mid \mathfrak{s}\in\mathfrak{T}^{ss}(\mu,\lambda),\mathfrak{t}\in\mathfrak{T}^{ss}(\mu,\nu),\mu\in\Pi^+(n,r)\} \tag{6.1.2}$$

is a basis for $\mathcal{H}^{\lambda\nu}$. Let

$$h_1=\sum_{x\in\bar W_\lambda\cap\mathcal{D}_{\bar\lambda\cap\delta(\mathfrak{s})\bar\mu}}T_{x^{-1}}\quad\text{and}\quad h_2=\sum_{x\in\bar W_\nu\cap\mathcal{D}_{\bar\nu\cap\delta(\mathfrak{t})\bar\mu}}T_x=h_{\bar\nu\cap\delta(\mathfrak{t})\bar\mu}.$$

Then we have

$$X^\mu_{\mathfrak{st}}=\Phi_{\mathfrak{st}}(x_\nu)=h_1T_{\delta(\mathfrak{s})}x_\mu T_{\delta(\mathfrak{t})^{-1}}h_2=\sum_{\mathbf{u}\in\mathbf{T}_\mathfrak{s},\mathbf{v}\in\mathbf{T}_\mathfrak{t}}x_{\mathbf{uv}}, \tag{6.1.3}$$

by the above and the argument for the first part of (5.2.2), and (1.2.3c) for the last equality. Here $\mathbf{T}_\mathfrak{s}=\mathbf{T}_\mathfrak{s}(\mu,\lambda)$ and $\mathbf{T}_\mathfrak{t}=\mathbf{T}_\mathfrak{t}(\mu,\nu)$ as defined above (1.2.3), and the basis elements $x_{\mathbf{uv}}$ for $\mathcal{H}$ are discussed above (3.4.1). So the set in (6.1.2) is linearly independent and forms a basis for $\mathcal{H}_K^{\lambda\nu}$ by (5.1.2). Finally, by (4.2.5), we have $\mathcal{H}^{\lambda\nu} = \mathcal{H}_K^{\lambda\nu} \cap \mathcal{H}$ and, therefore, the set in (6.1.2) is a basis for $\mathcal{H}^{\lambda\nu}$. □

We remark that, if $\nu = (-, (1^r)) \in \Pi_r^+$ is the minimal element, then $\mathfrak{T}^{ss}(\mu,\nu) = \mathbf{T}^s(\mu)$. In this case, the basis given in (6.1.2) for $x_\lambda\mathcal{H}$ is the same as the basis $\mathfrak{X}_\lambda$ given in (5.2.1). Our notation is also consistent.

(6.1.4) Corollary. *For any $\lambda, \rho, \nu \in \Pi(n,r)$ and $x \in \mathcal{D}_{\lambda\rho}$, $y \in \mathcal{D}_{\rho\nu}$, the product $\varphi^x_{\lambda\rho}\varphi^y_{\rho\nu}$ is a linear combination of $\Phi^\tau_{\mathfrak{st}}$ where $\mathfrak{s} \in \mathfrak{T}^{ss}(\tau,\lambda)$, $\mathfrak{t} \in \mathfrak{T}^{ss}(\tau,\nu)$ and $\tau \in \Pi^+(n,r)$ with $\tau \trianglerighteq \rho''$. In particular, if $\mu \in \Pi^+(n,r)$, the submodule $\mathcal{S}^{\trianglerighteq\mu}$ (resp. $\mathcal{S}^{\triangleright\mu}$) generated by all $\Phi^\lambda_{\mathfrak{st}}$ with $\lambda \trianglerighteq \mu$ (resp. $\lambda \triangleright \mu$) and $\lambda \in \Pi^+(n,r)$ is a two-sided ideal of $\mathcal{S}^2_q(n,r)$.*

Proof. We have $\varphi^x_{\lambda\rho}\varphi^y_{\rho\nu}(x_\nu) = h_1x_\rho h_2 \in \mathcal{H}^{\lambda\nu}$ for some $h_i \in \mathcal{H}$. On the other hand, there exists $d \in \mathcal{D}_{\rho''\rho}$ such that $T_d^{-1}x_\rho T_d = x_{\rho''}$ (see the proof of (3.2.2c)). Therefore, we have $h_1x_\rho h_2 = h_1'x_{\rho''}h_2'' \in \mathcal{H}^{\lambda\nu}$, which, by [DJM; (4.8),(4.10)], is a linear combination of $x^\mu_{\mathfrak{st}}$ with $\mu \trianglerighteq \rho''$ and $\mathfrak{s},\mathfrak{t} \in \mathbf{T}^s(\mu)$. The first assertion follows. For the second, note any product $\varphi\Phi^\mu_{\mathfrak{st}}\psi$ with $\varphi,\psi \in \mathcal{S}^2_q(n,r)$ is a linear combination of products $\varphi^x_{\gamma\mu}\varphi^y_{\mu\omega}$ for suitable x, y, γ, ω, but the same μ. □

(6.1.5) Corollary. *For any $\varphi, \psi \in \mathcal{S}^2_q(n,r)$, we have*

$$\varphi\Phi^\mu_{\mathfrak{st}} \equiv \sum_{\mathfrak{u}} \xi_{\mathfrak{u}}\Phi^\mu_{\mathfrak{ut}} \operatorname{mod}(\mathcal{S}^{\triangleright\mu}) \quad \text{and} \quad \Phi^\mu_{\mathfrak{st}}\psi \equiv \sum_{\mathfrak{v}} \zeta_{\mathfrak{v}}\Phi^\mu_{\mathfrak{sv}} \operatorname{mod}(\mathcal{S}^{\triangleright\mu}),$$

where $\xi_{\mathfrak{u}}, \zeta_{\mathfrak{v}} \in \mathcal{Z}$ are independent of $\mathfrak{t}$, $\mathfrak{s}$ respectively.

Proof. This follows from (6.1.3) using (3.4.1). □

(6.1.6) Corollary. *Keep the notation introduced in (5.2.4). Let $\mathcal{T}^i_\lambda$ be the i-th filtration term defined there. Then $\mathcal{T}^i_\lambda \cap \mathcal{H}x_\nu$ is free with basis*

$$\{X^\mu_{\mathfrak{st}} \mid \mu = \mu_{(j)}, j \le i, \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \mathfrak{t} \in \mathfrak{T}^{ss}(\mu,\nu)\},$$

for any $\nu \in \Pi(n,r)$.

Proof. Recall from the proof of (5.2.3) that $\mathcal{T}^i_\lambda$ is spanned by all $X^{\mu_{(j)}}_{\mathfrak{s},\mathbf{t}}$ where $j \le i$, $\mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)$ and $\mathbf{t} \in \mathbf{T}^s(\mu)$, for $\mu = \mu_{(i)}$. By (6.1.3), $X^\mu_{\mathfrak{st}} = \sum_{\mathbf{u}\in\mathbf{T}_\mathfrak{s},\mathbf{v}\in\mathbf{T}_\mathfrak{t}} x_{\mathbf{uv}} = \sum_{\mathbf{v}\in\mathfrak{T}_\mathfrak{t}} X^\mu_{\mathfrak{s}\mathbf{v}} = \sum_{\mathbf{u}\in\mathbf{T}_\mathfrak{s}} X^\mu_{\mathbf{u}\mathfrak{t}}$, which is in $\mathcal{T}^i_\lambda \cap \mathcal{H}x_\nu$, by (5.2.1). Conversely, it is easy to see that any element in the intersection is a linear combination of the elements in the given set. □

6.2 The quasi-heredity of $\mathcal{S}^2_q(n,r)$. For simplicity, let $\mathcal{S} = \mathcal{S}^2_q(n,r)$. Fix a linear ordering for the set $\Pi^+(n,r) = \{\mu_{(1)} \ge \mu_{(2)} \ge \cdots \ge \mu_{(N)}\}$ which refines $\trianglelefteq$ (i.e., $\mu_{(i)} \trianglerighteq \mu_{(j)}$ implies $\mu_{(i)} \ge \mu_{(j)}$). Put $e_i = \sum_{j=1}^i \varphi_{\mu_{(j)}}$, $\mathcal{J}_i = \mathcal{S}e_i\mathcal{S}$ and $\mathcal{S}_i = \mathcal{S}/\mathcal{J}_i$, where $\varphi_\lambda = \varphi^1_{\lambda\lambda}$. We have the following main theorem of this paper.

(6.2.1) Theorem. *The q-Schur^2 algebra $\mathcal{S}^2_q(n,r)$ over $\mathcal{Z}$ is a (split) quasi-hereditary algebra with the "defining" sequence (or heredity chain)*

$$0 = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \subseteq \mathcal{J}_N = \mathcal{S}.$$

Therefore, for any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, $\mathcal{S}_q^2(n,r,\mathcal{Z}')$ is quasi-hereditary.

Proof. From [CPS1; §3] we have to show that, for each $1 \le i \le N$, $\mathcal{J}_i/\mathcal{J}_{i-1}$ is a heredity ideal of $\mathcal{S}_{i-1} = \mathcal{S}/\mathcal{J}_{i-1}$. To avoid using too many indices, we put $\mu = \mu_{(i)}$, $\mathcal{J} = \mathcal{J}(\mu) = \mathcal{J}_i/\mathcal{J}_{i-1}$ and $\overline{\mathcal{S}} = \mathcal{S}/\mathcal{J}_{i-1}$. We shall denote by $\overline{a}$ the image of $a \in \mathcal{S}$ in $\overline{\mathcal{S}}$.

From the theorem and corollary 6.1.4 above, we see that both $\overline{\mathcal{S}}$ and $\mathcal{J}$ are $\mathcal{Z}$-free and have bases $\{\overline{\Phi}_{\mathfrak{st}} \mid \mathfrak{s} \in \mathfrak{T}^{ss}(\rho,\lambda), \mathfrak{t} \in \mathfrak{T}^{ss}(\rho,\nu), \lambda,\nu \in \Pi(n,r), \rho \le \mu\}$ and $\{\overline{\Phi}_{\mathfrak{st}} \mid \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \mathfrak{t} \in \mathfrak{T}^{ss}(\mu,\nu), \lambda,\nu \in \Pi(n,r)\}$, respectively. Note that $\varphi_\mu = \Phi_{\mathfrak{s}_\mu\mathfrak{s}_\mu}$ where $\mathfrak{s}_\mu \in \mathfrak{T}^{ss}(\mu,\mu)$ is the unique μ-bitableau of type μ. So if $\overline{\Phi}_{\mathfrak{st}}\overline{\varphi}_\mu \ne 0$ then $\overline{\Phi}_{\mathfrak{st}}\overline{\varphi}_\mu = \overline{\Phi}_{\mathfrak{s}\mathfrak{s}_\mu}$. Therefore, we have $\mathcal{J} = \overline{\mathcal{S}}\overline{\varphi}_\mu\overline{\mathcal{S}}$ and, in particular, we have (1): $\mathcal{J}^2 = \mathcal{J}$. Let ${}^\mu\mathcal{J} = \overline{\varphi}_\mu\mathcal{J}$ and $\mathcal{J}^\mu = \mathcal{J}\overline{\varphi}_\mu$. Then the argument above shows also that ${}^\mu\mathcal{J}$ and $\mathcal{J}^\mu$ have bases $\{\overline{\Phi}_{\mathfrak{s}_\mu\mathfrak{t}} \mid \mathfrak{t} \in \mathfrak{T}^{ss}(\mu,\lambda), \lambda \in \Pi(n,r)\}$ and $\{\overline{\Phi}_{\mathfrak{s}\mathfrak{s}_\mu} \mid \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda), \lambda \in \Pi(n,r)\}$, respectively. Thus, we have (2): $\overline{\varphi}_\mu\overline{\mathcal{S}}\overline{\varphi}_\mu = \mathcal{J}^\mu \cap {}^\mu\mathcal{J} = \mathcal{Z}\overline{\varphi}_\mu$ and that the map $f : \mathcal{J}^\mu \otimes {}^\mu\mathcal{J} \to \mathcal{J}$ given by $f(\overline{\xi} \otimes \overline{\eta}) = \overline{\xi\eta}$ is bijective. Since ${}^\mu\mathcal{J} = \overline{\varphi}_\mu\mathcal{J}$ is a projective right $\overline{\mathcal{S}}$-module, it follows that (3): $\mathcal{J}$ is a projective right $\overline{\mathcal{S}}$-module. Altogether (1)–(3), we have proved that $\mathcal{J}$ is a heredity ideal.

The last assertion follows easily, since $\mathcal{S}_q^2(n,r,\mathcal{Z}') \cong \mathcal{S}_q^2(n,r)_{\mathcal{Z}'}$ by (4.2.6). □

From the theorem and proof above, we can easily describe the standard objects in the category of $\mathcal{S}_q^2(n,r)_k$-modules, where k is a field which is also a $\mathcal{Z}$-algebra. These modules are the counterparts of the q-Weyl modules for q-Schur algebras, and have the following "semi-standard" bases (cf. [DJ3; (8.1)]).

(6.2.2) Corollary. *Maintain the notation introduced above and put $\Delta(\mu_{(i)}) = \mathcal{S}\tilde{z}_{\mu_{(i)}}$ where $\tilde{z}_{\mu_{(i)}} = \varphi_{\mu_{(i)}} + \mathcal{J}_{i-1}$ for all i. Then, for any field k which is a $\mathcal{Z}$-algebra, the category $\mathcal{S}_q^2(n,r)_k$-mod of $\mathcal{S}_q^2(n,r)_k$-modules is a highest weight category* [CPS3], *and $\{\Delta(\mu)_k\}_{\mu\in\Pi^+(n,r)}$ is the set of standard objects in $\mathcal{S}_q^2(n,r)_k$-mod. Moreover, each $\Delta(\mu)$ is $\mathcal{Z}$-free with "semi-standard" basis*

$$\{\varphi_{\lambda\mu}^{\delta(\mathfrak{s})}\tilde{z}_\mu \mid \lambda \in \Pi(n,r), \mathfrak{s} \in \mathfrak{T}^{ss}(\mu,\lambda)\}.$$

Also, (6.2.2) guarantees formally that the $\mathcal{S}_q^2(n,r)$-module categories are integral highest weight categories in the sense of [DS]. We refer the reader to that paper for further discussion of the integral concept.

(6.2.3) Remark. (a) By (6.1.5), one checks easily that the bistandard basis for $\mathcal{S}_q^2(n,r)$ satisfies the axioms in [DR; (1.2.1)]. Thus, $\mathcal{S}_q^2(n,r)$ is a standardly based algebra in that sense. In fact, it is even cellular in the sense of [GL]. We leave the details to the reader. Now the result in [DR; (3.2.1)] gives a second proof of (6.2.1) by showing that this standardly based algebra is full.

(b) When q_0 is a rational power of q, the ring $\mathcal{Z}$ has Krull dim. 2. Thus the theory developed in [DPS1] applies. To get the quasi-heredity using that theory, it suffices to check the condition [DPS2; (1.7(4))] by using the filtration given in (5.2.4), result (6.1.6) and an argument similar to [DPS1; (2.3.7)]. We leave further details to the reader.

(c) As we have mentioned at the beginning of §5, we only consider the integral case $\mathcal{Z}' = \mathcal{Z}$ in §5.2 and §6. However, the definitions of bistandard bases for $\mathcal{H}$, $\mathcal{T}_\lambda$ and $\mathcal{S}_q^2(n,r)$ are the same for any coefficient ring $\mathcal{Z}'$ (using (4.2.6) in later cases). Therefore, all results in §5.2 and §6 remain true after base change.

6.3 Integral centralizer subalgebras. Let $\varphi_\lambda = \varphi^1_{\lambda\lambda}$ as before. These are idempotents, called *weight idempotents.* With these idempotents, we see that both $\hat{\mathcal{S}}_q^2(n,r)$ and $\bar{\mathcal{S}}_q^2(n,r)$ are centralizer "subalgebras" (without the same identity element, in general) of $\mathcal{S}_q^2(n,r)$ of the form $e\mathcal{S}_q^2(n,r)e$ for some idempotents e. In addition, the q-Schur algebra is also a centralizer subalgebra of the q-Schur^2 algebra: Recall from [DJ3] the q-Schur algebra $\mathcal{S}_q(n,r) = \mathrm{End}_{\bar{\mathcal{H}}}\left(\bigoplus_{\lambda\in\bar{\Pi}(n,r)} x_\lambda\bar{\mathcal{H}}\right)$, where $\bar{\mathcal{H}} = \mathcal{H}(\bar{W})$, and from (3.3.1) the coideal Ω_m of $\Pi(n,r)$.

(6.3.1) Proposition. *Let $e = \sum_{\lambda\in\Omega_0}\varphi_\lambda$. Then e is an idempotent of $\mathcal{S}_q^2(n,r)$ and $e\mathcal{S}_q^2(n,r)e$ is isomorphic to the q-Schur algebra $\mathcal{S}_q(n,r)$. Therefore, the restriction of the bistandard basis described in (6.1.1) gives the (bi)standard basis for a q-Schur algebra.*

Proof. Let $\mathcal{H}' = \mathcal{H}_{\mathcal{Z}'}$ where $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra. We note that, for any $\lambda\in\Omega_0$ (see (3.3.1)), $x_\lambda\mathcal{H}' = x_\lambda\bar{\mathcal{H}}' \cong x_{\bar{\lambda}}\bar{\mathcal{H}}'$ as $\bar{\mathcal{H}}'$-module, where $\bar{\mathcal{H}}' = \bar{\mathcal{H}}_{\mathcal{Z}'}$. Thus we have for $\lambda,\mu\in\Omega_0$

$$\mathrm{Hom}_{\mathcal{H}'}(x_\lambda\mathcal{H}', x_\mu\mathcal{H}') \subseteq \mathrm{Hom}_{\bar{\mathcal{H}}'}(x_\lambda\bar{\mathcal{H}}', x_\mu\bar{\mathcal{H}}') \cong \mathrm{Hom}_{\bar{\mathcal{H}}'}(x_{\bar{\lambda}}\bar{\mathcal{H}}', x_{\bar{\mu}}\bar{\mathcal{H}}').$$

Now, applying [CPS1; (3.3.1)] gives the required isomorphism. □

Note that the standard basis for a q-Schur algebra obtained in this way is the q-analogue of Green's codeterminant basis for a Schur algebra [G].

Before stating our final results on $\mathcal{S}_q^2(n,r)$, we observe a general theorem on integral centralizer algebras. Let $(\mathcal{O},K,k)$ be a local system[2] with $\mathcal{O}$ regular. Then $(\mathcal{O},K,k)$ determines a second local system $(\widehat{\mathcal{O}},\widehat{K},k)$ where the completion $\widehat{\mathcal{O}}$ is the completion $\varprojlim \mathcal{O}/\mathfrak{m}^i$ of $\mathcal{O}$ at the maximal ideal $\mathfrak{m}$. Let A be a finite $\mathcal{O}$-free algebra and $e\in A$ an idempotent such that eAe is $\mathcal{O}$-free. Put $B = eAe$.

(6.3.2) Theorem. *The decomposition matrix of B is part of the decomposition matrix of A. In particular, if A has a unitriangular decomposition matrix, so does B.*

Proof. Without loss of generality, we assume $\mathcal{O} = \widehat{\mathcal{O}}$. Thus we have a decomposition $e = \sum_i e_i$ over $\mathcal{O}$ such that each e_iA is a PIM and $e = \sum_j f_j$ in A_K with f_jA_K irreducible. Then

$$\begin{aligned}\dim \mathrm{Hom}_{A_K}(e_iA_K, f_jA_K) &= \dim(f_jA_Ke_i)\\ &= \dim(f_jB_Ke_i)\\ &= \dim \mathrm{Hom}_{B_K}(e_iB_K, f_jB_K),\end{aligned}$$

[2]A *local system* is a triple $(\mathcal{O},K,k)$ consisting of a commutative, Noetherian local domain $\mathcal{O}$ having fraction field K and residue field $k = \mathcal{O}/\mathfrak{m}$.

as desired. □

Applying this to the q-Schur^2 algebras, we have immediately the following.

(6.3.3) Corollary. *The decomposition matrix for each of the algebras $\mathcal{S}_q(n,r)$, $\hat{\mathcal{S}}_q^2(n,r)$ and $\bar{\mathcal{S}}_q^2(n,r)$ is unitriangular and is part of the decomposition matrix of $\mathcal{S}_q^2(n,r)$.*

(6.3.4) Remark. Because of work of Dipper and James, the Hecke endomorphism algebras $\hat{\mathcal{S}}_q^2(n,r)$ are important for the non-describing representation theory of finite groups of Lie type (see [DPS1]), and this has largely motivated our investigation. Several years ago (1993), we calculated that $\hat{\mathcal{S}}_q^2(2,2,\mathbb{F}_p)$ is quasi-hereditary when $p \neq 2$ and $q = q_0 = 2$. (More precisely, as remarked in [CPS4; p.111], we calculated that a Hecke endomorphism algebra associated to the finite group $\mathrm{Sp}(4,2)$ is quasi-hereditary over $\mathbb{F}_p$, and used a correspondence of Dipper and James described in [DPS1].) This result may now be viewed as a special case of our main theorem (6.2.1), since it turns out $\hat{\mathcal{S}}_q^2(2,2,\mathbb{F}_p)$ is Morita equivalent to $\mathcal{S}_q^2(2,2,\mathbb{F}_p)$. This may be seen by checking that the natural surjection $\pi_1\mathcal{H}' \to \pi_2\mathcal{H}'$, defined as left multiplication by $q+T_{t_2}$, is split in this case. The same argument works, whenever $q = q_0 \in \mathcal{Z}'$ and 2 (as well as q) is invertible in $\mathcal{Z}'$, showing $\hat{\mathcal{S}}_q^2(2,2,\mathcal{Z}')$ is quasi-hereditary in this generality.

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