CHARACTER SHEAVES AND GENERALIZED GELFAND–GRAEV CHARACTERS

MEINOLF GECK

1. INTRODUCTION

Kawanaka [10] has associated with every unipotent class of a finite reductive group a so-called generalized Gelfand-Graev character. These characters are deeply related with the geometry of unipotent classes of the underlying algebraic group G. Lusztig [17] has expressed the generalized Gelfand-Graev characters in terms of intersection cohomology complexes of closures of unipotent classes with coefficients in various local systems. The resulting formulas for their character values contain as unknown quantities certain fourth roots of unity which relate the characteristic functions of those intersection cohomology complexes with their Fourier transforms.

It is one purpose of this paper to determine these fourth roots of unity explicitly, under the assumption that the center of G is connected and that the characteristic of the field over which G is defined is large enough. After some preparations in Section 2, this will be achieved in Theorem 3.6 and Theorem 3.8. Note that if the center is not connected, the problem seems to be more subtle; for the case of ordinary Gelfand-Graev characters associated with regular unipotent elements, see [5]. The assumption on p comes from [17] where it is also necessary.

Our main tools are formulas relating the values of a class function on unipotent elements with the scalar products between that class function and the various generalized Gelfand-Graev characters, see Corollary 2.6. These results are formally deduced from Lusztig's results in [17]. We then derive in Corollary 3.2 an integrality condition which imposes sufficiently strong restrictions on the possible values of the unknown roots of unity. In order to apply this integrality condition we need some results from [12, Chap. 13] about the classification of the irreducible characters of our finite reductive group and relations with unipotent classes in G.

Lusztig's theory of character sheaves [14] provides the framework for the determination of character values of our finite reductive group. The explicit formulas for the values of generalized Gelfand-Graev characters allow us to revise a number of results concerning character sheaves with non-zero restriction to the unipotent variety of G in a more conceptual framework; see Theorem 4.5, Remark 5.4 and Proposition 5.5. Results of this kind were first obtained by Lusztig in [15], where groups of type B_n and exceptional groups were considered in a thoroughly explicit manner. (See [2] for similar

results about the other types of classical groups.) One of our motivations was to find a conceptual approach to the remarks in [15, (1.6)], assuming that the characteristic is large; see Theorem 4.5. It was mentioned to me by Lusztig that some additional hypothesis as in Theorem 4.5 is actually needed so that these remarks hold.

In order to prove these results, we use again integrality conditions coming from ordinary irreducible characters and scalar product relations with generalized Gelfand-Graev characters. In order to make this work, we also need a weak form of Shoji's results [21] on the relation between irreducible characters and characteristic functions of character sheaves, see Theorem 7.1.

Applications of these results on character sheaves can be found in [6], for example, where they were used to prove the existence of basic sets of 2-modular Brauer characters for finite classical groups over a field of odd characteristic.

Basic assumptions. G will always be a connected reductive group over an algebraic closure k of the finite field $\mathbb{F}_q$ where q is a power of some prime p. Whenever G is defined over $\mathbb{F}_q$ we let $F \colon G \to G$ be the corresponding Frobenius map. The finite group of fixed points will then be denoted by G^F. We assume throughout that p is good for G, so that the generalized Gelfand-Graev characters are defined (see [10]). We will have to make use of the main results of [17], which are only proved under the assumption that p and q are large enough. We have collected the results that we shall need in the form of axioms in (2.4) below. The assumption that the characteristic is large will only be used in reference to these axioms.

2. SOME BASIS RELATIONS

Let G be a connected reductive group defined over $\mathbb{F}_q$ with corresponding Frobenius map F. The purpose of this section is to introduce our basic notation and establish some relations between the values of a class functions on unipotent elements and the scalar products of this class function with the various generalized Gelfand-Graev characters of G^F; formulas of this kind are due to Lusztig [17, (9.11)].

(**2.1**) All of our characters and class functions will have values in an algebraic closure of $\mathbb{Q}_l$, where l is prime not dividing q. If f, f' are two class functions on G^F we denote by

$$\langle f, f' \rangle := \frac{1}{|G^F|} \sum_{g \in G^F} f(g)\overline{f'(g)}$$

their usual hermitian product, where $x \mapsto \bar{x}$ is a field automorphism which maps roots of unity to their inverses.

We denote by G_{uni} the set of unipotent elements in G. For each element $g \in G$ we let (g) denote the G-conjugacy class of g. There is a canonical partial order on the set of unipotent classes of G: if C, C' are two such

classes we write $C \leq C'$ if C is contained in the Zariski closure of C'. We write $C < C'$ if $C \leq C'$ but $C \neq C'$.

For each unipotent class C of G we choose once and for all a representative $u \in C$ and let $A(u)$ be the group of components of $C_G(u)$. If C is F-stable we tacitly assume that $u \in C^F$ so that F induces an action on $A(u)$. Let $\mathcal{N}_G$ be the set of pairs (C, ψ) where C is a unipotent class and ψ is an irreducible character of $A(u)$ (for the chosen $u \in C$). If $i \in \mathcal{N}_G$ we also write $i = (C_i, \psi_i)$.

(2.2) The map F acts naturally on $\mathcal{N}_G$. With each F-stable pair $i = (C, \psi) \in \mathcal{N}_G^F$ we can associate a class function Y_i as follows. Since ψ is invariant under the action of F, we can extend it to a character $\tilde{\psi}$ of the semidirect product of $A(u)$ with the cyclic group of automorphisms generated by F. For each $y \in A(u)$, we denote by u_y an element in C^F obtained by twisting the given representative u with y. We let

(a)
$$Y_i(g) = \begin{cases} \tilde{\psi}(Fy) & \text{if } g = u_y \text{ for some } y \in A(u), \\ 0 & \text{otherwise.} \end{cases}$$

The set of functions $\{Y_i \mid i \in \mathcal{N}_G^F\}$ forms a basis of the space of class functions of G^F with support on C^F. (Note that it depends on the choice of extensions $\tilde{\psi}$; thus, it is only well-defined up to non-zero scalar multiples.)

By the generalized Springer correspondence, we can partition the set $\mathcal{N}_G$ into blocks, and with each block we can associate a Levi subgroup L of some parabolic subgroup of G and a pair $i' \in \mathcal{N}_L$ which is cuspidal (see [13] and [17, Section 4]). We define

(b)
$$b_i = \frac{1}{2}(\dim G - \dim C - \dim Z(L)^\circ)$$

where $Z(L)$ denotes the center of L. We will see later that b_i is an integer if the center of G is connected, see (3.3).

(2.3) Let C be an F-stable unipotent class in G and $u = u_1, \ldots, u_d$ be representatives for the G^F-conjugacy classes contained in C^F. Let $\Gamma_{u_1}, \ldots, \Gamma_{u_d}$ be the corresponding generalized Gelfand-Graev characters. We denote the order of $A(u)$ by a and that of $A(u_r)^F$ by a_r, for $1 \leq r \leq d$. As in [17, (7.5)] we define:

(a)
$$\Gamma_i = \sum_{r=1}^d \frac{a}{a_r} Y_i(u_r) \Gamma_{u_r} \quad \text{for } i \in \mathcal{N}_G^F \text{ with } C_i = C.$$

Note that the functions Y_i with $C_i = C$ satisfy orthogonality relations:

(b)
$$\sum_{r=1}^d \frac{a}{a_r} Y_i(u_r) \overline{Y_{i'}(u_r)} = a\delta_{ii'} \quad \text{and} \quad \sum_i Y_i(u_r) \overline{Y_i(u_{r'})} = a_r \delta_{rr'}.$$

Thus, we also have

(c)
$$\Gamma_{u_r} = \frac{1}{a} \sum_i \overline{Y_i(u_r)} \Gamma_i \quad \text{for all } 1 \leq r \leq d,$$

where the sum is over all $i \in \mathcal{N}_G^F$ with $C_i = C$.

(2.4) We keep the notation of (2.3), and fix a pair $i = (C, \psi) \in \mathcal{N}_G^F$. Lusztig [17, Theorem 7.3] expresses each Γ_i as a linear combination of Y_j's, for various $j \in \mathcal{N}_G^F$. The advantage of using Γ_i instead of Γ_u is that the coefficients in these expressions are independent of the normalization of the functions Y_j. These coefficients involve a certain 4-th root of unity ζ_i' which, for our purposes, is conveniently characterized as follows:

(a) $$\langle D_G(\Gamma_i), Y_{i'} \rangle = a\zeta_i' q^{-b_i} \delta_{i,i'} \quad \text{if } C_i = C = C_{i'},$$

where D_G denotes Alvis-Curtis-Kawanaka duality. This is deduced in [6, Lemma 3.5] from the formulas in [17, (6.12), (7.3), (8.6)]. We have $\zeta_i' = \delta \zeta^{-1}$ where δ, ζ are defined in [17, (8.4) and (7.2)]. Since δ and ζ only depend on the block of $\mathcal{N}_G$ containing i, the same holds for ζ_i' as well. The definition of these numbers shows that

(b) we have $\zeta_i' = 1$ if the Levi subgroup L associated with the block of $\mathcal{N}_G$ containing i is just a maximal torus.

We shall not need the exact form of all the coefficients in the expression of Γ_i as a linear combination of Y_j's but only the following additional basic relation:

(c) $$\text{if } D_G(\Gamma_i)(g) \neq 0 \text{ for some } g \in G^F \text{ then } (g) \leq C_i.$$

This is proved by combining [17, (6.13) and (8.6)]. Note that these results are proved in [17] under the assumption that q is a sufficiently large power of a sufficiently large prime. We shall take these properties here as axioms on which the subsequent arguments are built.

Lemma 2.5 (Cf. [17], (9.11)). *Let $i = (C, \psi) \in \mathcal{N}_G^F$ and f be any class function on G^F satisfying the condition:*

(*) $$f(g) = 0 \text{ for all unipotent } g \in G^F \text{ such that } C < (g)$$

Then, with the notation in (2.3), we have

$$\sum_{r=1}^d \frac{a}{a_r} f(u_r) \overline{Y_i(u_r)} = \zeta_i' q^{b_i} \langle f, D_G(\Gamma_i) \rangle.$$

Proof. By condition (*) and property (2.4)(c) we see that the scalar product on the right hand side only depends on the restriction of f to C^F. Clearly, the same also holds for the left hand side. Thus, it will be sufficient to prove the lemma in the special case where we take $f = Y_{i'}$ with $i' \in \mathcal{N}_G^F$, $C_{i'} = C$. In this case, the left hand sides evaluates to:

$$\sum_{r=1}^d \frac{a}{a_r} Y_{i'}(u_r) \overline{Y_i(u_r)} = a\delta_{ii'},$$

by the orthogonality relations in (2.3)(b). On the other hand, the right hand side evaluates to:

$$\zeta_i' q^{b_i} \langle Y_{i'}, D_G(\Gamma_i) \rangle = \zeta_i' q^{b_i} \overline{\langle D_G(\Gamma_i), Y_{i'} \rangle} = \zeta_i' q^{b_i} a \bar{\zeta}_i' q^{-b_i} \delta_{ii'} = a \delta_{ii'},$$

by using the basic relation in (2.4)(a). In both cases we obtain the same result. Since the functions $Y_{i'}$ form a basis of the space of class functions with support on C^F, the proof is complete. $\square$

Corollary 2.6. *With the assumptions of Lemma 2.5 we have*

$$f(u_r) = \frac{1}{a} \sum_i \zeta_i' q^{b_i} Y_i(u_r) \langle f, D_G(\Gamma_i) \rangle \quad for \ 1 \le r \le d,$$

where the sum is over all $i \in \mathcal{N}_G^F$ with $C_i = C$.

Proof. We multiply both sides of the equation in Lemma 2.5 by $Y_i(u_{r'})$ and sum over all $i \in \mathcal{N}_G^F$ with $C_i = C$. Using the orthogonality relations in (2.3)(b) the left hand side of that equation evaluates to:

$$\sum_i Y_i(u_{r'}) \left(\sum_{r=1}^{d} \frac{a}{a_r} f(u_r) \overline{Y_i(u_r)} \right) =$$

$$\sum_{r=1}^{d} \frac{a}{a_r} f(u_r) \left(\sum_i Y_i(u_{r'}) \overline{Y_i(u_r)} \right) = \sum_{r=1}^{d} \frac{a}{a_r} f(u_r) a_r \delta_{rr'} = a f(u_{r'}).$$

The right hand side just gives a times the desired expression. This completes the proof. $\square$

(2.7) Let ρ be an irreducible character of G^F. Then there is a canonical unipotent class in G so that condition (*) in Lemma 2.5 is satisfied. This canonical class is determined as follows. For each F-stable unipotent class C we define

(a) $$\text{AV}(C, \rho) := \sum_{r=1}^{d} \frac{a}{a_r} \rho(u_r) \qquad (\text{notation of (2.3)}).$$

Following [17] we say that C is the unipotent support of ρ if $\text{AV}(C, \rho) \neq 0$ and if C is the unique class of maximal possible dimension with this property. We shall denote the unipotent support of an irreducible character ρ by C_ρ. By [17], [7], [9] the unipotent support always exists, without any restriction on p or q. The properties in (2.4) formally imply that

(b)   if $\rho(g) \neq 0$ for some $g \in C^F$ then $\dim C < \dim C_\rho$ or $C = C_\rho$,

where C is any F-stable unipotent class in G. Indeed, assume that $\rho(g) \neq 0$ for some $g \in C^F$ and that C has maximal possible dimension with this property. Then condition (*) in Lemma 2.5 is satisfied. So we can apply Corollary 2.6 and conclude that there exists some $i \in \mathcal{N}_G^F$ with $C_i = C$ and $\langle \rho, D_G(\Gamma_i) \rangle \neq 0$. Using the defining formula (2.3)(a) we see that there exists some $r \in \{1, \ldots, d\}$ such that $\langle \rho, D_G(\Gamma_{u_r}) \rangle \neq 0$. Now we take the

pair $(C, 1) \in \mathcal{N}_G$ (where 1 stands for the trivial character) and normalize the corresponding function $Y_{(C,1)}$ so that it has constant value 1 on C^F. Then $\Gamma_{(C,1)}$ is a sum of generalized Gelfand-Graev characters in which Γ_{u_r} appears with non-zero multiplicity. Since $\pm D_G(\rho) \in \mathrm{Irr}(G^F)$ and $\langle D_G(\rho), \Gamma_{u_r} \rangle = \langle \rho, D_G(\Gamma_{u_r}) \rangle \neq 0$, we also get that $\langle \rho, D_G(\Gamma_{(C,1)}) \rangle \neq 0$. Since this scalar product is a non-zero scalar multiple of $\mathrm{AV}(C, \rho)$, we are done. (This is the argument in the last part of the proof of [17, Theorem 11.2].)

(**2.8**) There is in fact an explicit formula for the average value of an irreducible character ρ on its unipotent support C_ρ. The Levi subgroup associated with the block of $\mathcal{N}_G$ containing the pair $(C_\rho, 1)$ is just a maximal torus of G (see, for example, [7, (3.2)]), and hence we have $b_{(C_\rho,1)} = \dim \mathcal{B}_u$, where $\mathcal{B}_u$ denotes the variety of Borel subgroups containing $u \in C_\rho$ (see the dimension formula in [4, (5.10.1)]). By (2.4)(b), we have $\zeta'_{(C_\rho,1)} = 1$ in this case. Therefore, Lemma 2.5 implies (see also [17, (9.11)]) that

(a) $$\mathrm{AV}(C_\rho, \rho) = q^{\dim \mathcal{B}_u} \langle D_G(\rho), \Gamma_{(C_\rho,1)} \rangle.$$

By [9, Theorem 3.7] we can evaluate the scalar product on the right hand side and obtain the formula:

(b) $$\mathrm{AV}(C_\rho, \rho) = q^{\dim \mathcal{B}_u} \langle D_G(\rho), \Gamma_{(C_\rho,1)} \rangle = \pm \frac{a}{n_\rho} q^{\dim \mathcal{B}_u},$$

where the sign is such that $\pm D_G(\rho) \in \mathrm{Irr}(G^F)$ and n_ρ denotes the generic denominator of ρ, see [9, Section 3B] and (6.1) below. Since p is a good prime, n_ρ is uniquely determined by the condition that $n_\rho \rho(1) = \pm q^{\dim \mathcal{B}_u} N$ where N is an integer satisfying $N \equiv 1 \bmod q$, see [12, (4.26.3)] and note that n_ρ is divisible by bad primes only.

The above results show that the formula in Corollary 2.6 expresses the values of ρ on its unipotent support C_ρ as a linear combination of the scalar products of the dual of ρ with the various Gelfand-Graev characters associated with C_ρ, where the coefficients are $|A(u)|^{-1}$ times algebraic integers. Since we know that the result is a character value, and hence an algebraic integer, we therefore obtain a divisibility condition which imposes restrictions on the coefficients of that linear combination. We shall make this explicit in the next section.

3. Determination of the 4-th roots of unity ζ'_i

Throughout this section we assume that the center of G is connected and that $G/Z(G)$ is simple. Recall also that the properties in (2.4) are assumed to hold; in particular, p is a good prime.

If C is an F-stable unipotent class C in G, we let $u_1, \dots, u_d$ be representatives for the G^F-classes contained in C^F, and $a = |A(u)|$ (for $u \in C$), $a_r = |A(u_r)^F|$, as in (2.3). Recall also that $\rho \mapsto C_\rho$ denotes the map which associates with each irreducible character its unipotent support. In this section we determine explicitly the 4-th roots of unity ζ'_i occuring in the formula (2.4)(a). The main idea in our argument is to analyze the terms in

the formula in Corollary 2.6 when applied to an irreducible character satisfying a certain extremality condition. The condition that we need and the existence of a corresponding character ρ is contained in the following result, which is merely a restatement of the remarks in [12, (13.3)].

Proposition 3.1. *Let C be an F-stable unipotent class. Then there exists an irreducible character ρ with unipotent support C and such that $n_\rho = |A(u)|$ ($u \in C$). For each character ρ satisfying these conditions, there exists an $r_0 \in \{1, \ldots, d\}$ such that*

$$a = a_{r_0} \quad and \quad \langle \rho', \Gamma_i \rangle = \overline{Y_i(u_{r_0})},$$

for all $i \in \mathcal{N}_G^F$ with $C_i = C$, and where ρ' is the irreducible character such that $\rho' = \pm D_G(\rho)$. We have $\langle \rho', \Gamma_{u_{r_0}} \rangle = 1$ and $\langle \rho', \Gamma_{u_r} \rangle = 0$ for all $r \neq r_0$.

Proof. Suppose first that ρ has unipotent support C and satisfies $n_\rho = |A(u)|$. The formulas in (2.8)(b) then show that

$$\langle \rho', \sum_{r=1}^{d} \frac{a}{a_r} \Gamma_{u_r} \rangle = \langle \rho', \Gamma_{(C,1)} \rangle = \pm 1.$$

Note that the sign must be $+$ since ρ' and all Γ_{u_r} are actual characters. (Note also that a_r divides a.) The above equation therefore implies that there exists an index r_0 such that $\langle \rho', \Gamma_{u_{r_0}} \rangle = 1$ and $\langle \rho', \Gamma_{u_r} \rangle = 0$ for all $r \neq r_0$; moreover, we have $a = a_{r_0}$. Using the defining equation (2.3)(a) we conclude that $\langle \rho', \Gamma_i \rangle = \overline{Y_i(u_{r_0})}$ for all $i \in \mathcal{N}_G^F$ with $C = C_i$. Hence we are done.

It remains to show that there exists an irreducible character satisfying the above conditions. By [17, Theorem 11.2] (for large p) and [9, Theorem 3.7] (for arbitrary p, q) the map $\rho \mapsto C_\rho$ is the composition of the map $\rho \mapsto \rho' = \pm D_G(\rho)$ with the map ξ from irreducible characters to unipotent classes defined in [12, (13.4)]. By [12, (13.4.3)] there exists some $\rho \in \mathrm{Irr}(G^F)$ such that $\xi(\rho) = C$ and $n_\rho = a$. By [1, (5.5)], we have $n_\rho = n_{\rho'}$. Hence ρ' is a character as desired. This completes the proof. $\square$

Corollary 3.2. *Let C, ρ, r_0 as in Proposition 3.1. Then*

$$\pm \rho(u_{r_0}) = \frac{1}{a} \sum_i \zeta_i' q^{b_i} \psi_i(1)^2,$$

where the sum is over all $i \in \mathcal{N}_G^F$ with $C_i = C$. In particular, the expression on the right hand side of the above formula is an algebraic integer.

Proof. Since C is the unipotent support of ρ, condition (*) in Lemma 2.5 is satisfied (see the remarks in (2.7)). Thus, the formula in Corollary 2.6

yields that

$$
\begin{aligned}
\rho(u_{r_0}) &= \frac{1}{a} \sum_i \zeta_i' q^{b_i} Y_i(u_{r_0}) \langle \rho, D_G(\Gamma_i) \rangle \\
&= \pm \frac{1}{a} \sum_i \zeta_i' q^{b_i} Y_i(u_{r_0}) \langle \rho', \Gamma_i \rangle \\
&= \pm \frac{1}{a} \sum_i \zeta_i' q^{b_i} Y_i(u_{r_0}) \overline{Y_i(u_{r_0})}
\end{aligned}
$$

where the last equality comes from Proposition 3.1. Now, by [3] and [19], we can always find a representative $u \in C^F$ such that F acts trivially on $A(u)$. Hence, if $A(u)$ is abelian then $Y_i(u_{r_0})$ is a root of unity and $Y_i(u_{r_0})\overline{Y_i(u_{r_0})} = 1 = \psi_i(1) = \psi_i(1)^2$. If $A(u)$ is not abelian, then the condition $a = a_{r_0}$ forces that u_{r_0} corresponds to a central element in $A(u)$. So, again, we get that $Y_i(u_{r_0})\overline{Y_i(u_{r_0})} = \psi_i(1)^2$. This completes the proof.   □

The assumption that the center of G is connected was already used in the proof of Proposition 3.1 where we explicitly refer to the classification of irreducible characters of G^F in [12]. We will now describe other consequences of that assumption; recall also that p is a good prime for G.

(**3.3**) If a block of $\mathcal{N}_G$ is a singleton set then the unique pair in that block is called a cuspidal pair (see [13]). The Levi subgroup corresponding to such a pair is just G. The classification of cuspidal pairs can be reduced to the case where G is simple modulo its center (see [13, (10.1)]). From the list in the introduction of [13] we then see that

  (a) *The set $\mathcal{N}_G$ contains at most one cuspidal pair, and if such a pair exists then rank $G/Z(G)$ is even.*

We have already mentioned in (2.4) that ζ_i' only depends on the block of $\mathcal{N}_G$ to which i belongs. More precisely, we have:

  (b) *Let L be a Levi subgroup in G and assume that $i_0 \in \mathcal{N}_L$ is cuspidal. If $i \in \mathcal{N}_G$ lies in a block to which L, i_0 are associated via the generalized Springer correspondence then $\zeta_i' = \zeta_{i_0}'$.*

Finally, there is one extreme case in which we know ζ_i'. This is the case when i lies in a block for which the associated Levi subgroup L is a maximal torus in G. (These pairs are called uniform pairs in $\mathcal{N}_G$.) By (2.4)(b), we have $\zeta_i' = 1$ in this case.

These properties together with the divisibility condition in Corollary 3.2 will provide the frame for an inductive argument for the determination of ζ_i'.

Remark 3.4. Assume that $G/Z(G)$ is simple of type A_n. Then all pairs in $\mathcal{N}_G$ are uniform (see the table in the introduction of [13]) and hence we have $\zeta_i' = 1$ for all $i \in \mathcal{N}_G^F$ by (2.4)(b).

(**3.5**) Assume that $G/Z(G)$ is simple of type G_2, F_4 or E_8. By the table in the introduction of [13] these are the only groups of exceptional type which

have a cuspidal pair in $\mathcal{N}_G$. In each of these cases the unique cuspidal pair in $\mathcal{N}_G$ has the form $i_0 = (C, sgn)$ where C is the unique unipotent class such that $A(u)$ is isomorphic to $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$, respectively, and sgn denotes the sign character. Moreover, all other elements in $\mathcal{N}_G$ are uniform. Hence, by (2.4)(b), we have $\zeta_i' = 1$ for all pairs $i \in \mathcal{N}_G$ except possibly for the unique cuspidal pair.

Theorem 3.6. *Let $i_0 = (C, sgn)$ be the unique cuspidal pair in $\mathcal{N}_G$ where $G/Z(G)$ is simple of type G_2, F_4 or E_8 in good characteristic. Let $\varepsilon \in \{\pm 1\}$ such that $q \equiv \varepsilon \bmod 3$. Then*

$$\zeta_{i_0}' = \varepsilon^{(\mathrm{rank}\ G - \dim Z(G))/2}.$$

Proof. Since G is the direct product of its derived subgroup and $Z(G)$, we may assume that $Z(G) = 1$. By Corollary 3.2, the expression

$$\frac{1}{|A(u)|} \sum_i \zeta_i' q^{b_i} \psi_i(1)^2 \qquad (u \in C)$$

is an algebraic integer, where the sum is over all $i \in \mathcal{N}_G^F$ such that $C_i = C$. By [3], we can choose a representative $u \in C^F$ so that F acts trivially on $A(u)$.

As remarked in (3.5), all pairs except the cuspidal one are uniform. Moreover, for a uniform pair $i \in \mathcal{N}_G$ with $C_i = C$ we have $\zeta_i' = 1$, and $b_i = \dim \mathcal{B}_u$ (using the formula in [4, (5.10.1)]). Since, furthermore, the sum of the squares of all character degrees of $A(u)$ gives just the order of this group, we conclude that the above expression can be written in the form:

$$\frac{1}{|A(u)|} \left(\zeta_{i_0}' q^{b_{i_0}} + (|A(u)| - 1) q^{\dim \mathcal{B}_u} \right) = q^{\dim \mathcal{B}_u} + \frac{1}{|A(u)|} \left(\zeta_{i_0}' q^{b_{i_0}} - q^{\dim \mathcal{B}_u} \right).$$

Since this is an algebraic integer, we deduce that

$$(1) \qquad\qquad |A(u)| \quad \text{divides} \quad \zeta_{i_0}' q^{b_{i_0}} - q^{\dim \mathcal{B}_u}$$

in the ring of algebraic integers. We multiply the term on the right hand with the similar term where we have replaced the sign $-$ by the sign $+$. Since ζ_i' is a 4-th root of unity, the resulting product is an integer and hence

$$(2) \qquad\qquad |A(u)| \quad \text{divides} \quad (\zeta_{i_0}')^2 q^{2b_{i_0}} - q^{2\dim \mathcal{B}_u} \quad \text{in } \mathbb{Z}.$$

Since the exponents of q are even integers, this implies $(\zeta_{i_0}')^2 \equiv 1 \bmod 3$ and hence that $\zeta_{i_0}' = \pm 1$. The tables in [4, (13.1)] give the following information:

Type	$|A(u)|$	$\dim \mathcal{B}_u$	b_{i_0}
G_2	6	1	2
F_4	24	4	6
E_8	120	16	20

In type G_2, the condition (1) yields that $3 \mid \zeta_{i_0}' q^2 - q$. Hence $\zeta_{i_0}' = \varepsilon$ in this case. Since rank $G = 2$, this yields the desired result. In type F_4 and E_8,

the condition (1) yields $3 \mid \zeta_{i_0}' q^6 - q^4$ and $3 \mid \zeta_{i_0}' q^{20} - q^{16}$, respectively. Hence we must have $\zeta_{i_0}' = 1$ in these cases. Since furthermore rank G is divisible by 4, this also yields the desired result.   □

(**3.7**) Assume that $G/Z(G)$ is simple of type B_n, C_n or D_n, where $n \geq 2$. The following table gives the conditions for the existence of a cuspidal pair $i_0 = (C, \psi) \in \mathcal{N}_G$, the sizes of the Jordan blocks of $u \in C$ in the natural matrix representation of such a group, and the exponent $a(u)$ such that $A(u)$ has order $2^{a(u)}$. (This information is extracted from [13] and the tables in [4, p.399].) Note that, in any case, there exists at most one cuspidal pair in $\mathcal{N}_G$.

Type	condition	Jordan blocks	$a(u)$
B_n	$n = 2t(t+1)$	$1, 3, 5, \ldots$	$2t$
C_n	$n = 2t(4t+1)$	$2, 4, 6, \ldots$	$4t - 1$
	or $n = 2t(4t-1)$	$2, 4, 6, \ldots$	$4t - 2$
D_n	$n = 8t^2$	$1, 3, 5, \ldots$	$4t - 2$

where $t \geq 1$ in each case. Note that $A(u)$ is always an elementary abelian group of order $2^{a(u)}$.

Theorem 3.8. *Let $p \neq 2$ and $i_0 = (C, \psi)$ be the unique cuspidal pair in $\mathcal{N}_G$ where $G/Z(G)$ is simple of type B_n, C_n or D_n (for suitable $n \geq 0$). Let $\varepsilon \in \{\pm 1\}$ such that $q \equiv \varepsilon \bmod 4$. Then we have*

$$\zeta_{i_0}' = \varepsilon^{(\operatorname{rank} G - \dim Z(G))/2}.$$

Proof. We proceed by induction on rank $G/Z(G)$. If this is zero then G is a torus and $\zeta_{i_0}' = 1$ by (2.4)(b). So we are done in this case. Now let rank $G/Z(G) > 0$. If L is a Levi subgroup of some parabolic subgroup of G then $Z(L)$ is also connected; moreover, if $\mathcal{N}_L$ contains a cuspidal pair then $L/Z(L)$ is simple. (For otherwise $L/Z(L)$ would have a non-trivial component of type A which is impossible by Remark 3.4.) By induction, the root of unity corresponding to the unique cuspidal pair of such a Levi subgroup L is given by the above formula. The strategy is now the same as in Theorem 3.6, with the difference that now we have to take into account various blocks and their associated Levi subgroups. By [19], a representative $u \in C^F$ can be chosen so that F acts trivially on $A(u)$.

Let $i \in \mathcal{N}_G^F$ with $C_i = C$ and denote by L_i the Levi subgroup associated with the block containing i. To abbreviate notation we let $e_i := (\operatorname{rank} L_i - \dim Z(L_i))/2$. Note that this is an integer, by (3.3)(a). By induction and (3.3)(b) we have $\zeta_i' = \varepsilon^{e_i}$ if $L \neq G$, and we must show that $\zeta_{i_0}' = \varepsilon^{e_{i_0}}$. Since rank $G = \operatorname{rank} L_i$ we can write b_i in the form

$$
\begin{aligned}
b_i &= (\dim G - \dim C - \dim Z(L_i))/2 \\
&= (2N + \operatorname{rank} G - \dim C - \dim Z(L_i))/2 \\
&= (2N - \dim C)/2 + e_i
\end{aligned}
$$

where N denotes the number of positive roots of G. Since $\dim C$ is always even (see the formula [4, (5.10.1)]) we can extract the integral power $q^{N-\dim C/2}$ from each term q^{b_i} and are left with q^{e_i}. For the cuspidal pair i_0 we obtain $e_{i_0} = (\text{rank } G - \dim Z(G))/2$. Putting things together, Corollary 3.2 now yields the condition that

$$|A(u)| \quad \text{divides} \quad \zeta'_{i_0} q^{e_{i_0}} + \sum_i \zeta'_i q^{e_i} = \zeta'_{i_0} q^{e_{i_0}} + \sum_i (\varepsilon q)^{e_i}$$

in the ring of algebraic integers, where the sum is over all $i \in \mathcal{N}_G^F$ such that $C_i = C$ and $i \neq i_0$; note also that $A(u)$ is abelian and so $\psi_i(1) = 1$ for all i. The table in (3.7) shows that $a(u) \geq 2$ in all cases. This means that 4 divides the order of $A(u)$. We will now just consider the above expression modulo 4. Since $\varepsilon q \equiv 1 \bmod 4$ we conclude that

(3) $\qquad\qquad 4 \quad \text{divides} \quad \zeta'_{i_0} q^{e_{i_0}} - 1$

in the ring of algebraic integers. As in the proof of Theorem 3.6 we deduce from this a divisibility condition in the ring of rational integers:

(4) $\qquad\qquad 4 \quad \text{divides} \quad (\zeta'_{i_0})^2 q^{2e_{i_0}} - 1.$

Since e_{i_0} is an integer, we have $q^{2e_{i_0}} \equiv 1 \bmod 4$. So (4) forces $\zeta'_{i_0} = \pm 1$. Now we can again return to (3) and deduce that $\zeta'_{i_0} \equiv q^{e_{i_0}} \bmod 4$, as desired. This completes the proof. $\qquad\qquad\square$

4. Character sheaves and unipotent classes

In this and the following sections we will revise some known results on the restriction of character sheaves to their unipotent support. These results were obtained in a case-by-case manner in [15], where in fact only groups of type B_n and exceptional groups were considered in detail. Note, however, that we have to assume that p is large enough whenever we use results on generalized Gelfand-Graev characters.

Throughout we will keep the assumptions from Section 3. In particular, p is a good prime for G, the center of G is connected, and $G/Z(G)$ is simple. Let T be a fixed maximal torus in G and $W = N_G(T)/T$ the Weyl group. Let G^* be a dual group and $T^* \subseteq G^*$ a dual maximal torus. We can naturally identify W with the Weyl group of G^* with respect to T^*.

(4.1) Let $\hat{G}$ be the set of character sheaves on G. These are certain G-equivariant perverse sheaves in the derived category of constructible $\bar{\mathbb{Q}}_l$-sheaves on G. We briefly summarize the main results about the classification of character sheaves from [14]. First, by [14, Cor. 11.4], there is a canonical surjective map from $\hat{G}$ to the W-orbits on T^*. Let $s \in T^*$ and $\hat{G}_s$ be the set of character sheaves in the fiber over (s) of this map. Let W_s be the Weyl group of $C_{G^*}(s)$ (with respect to T^*), identified with a subgroup of W. By [14, Theorem 23.1] we have a bijection

(a) $\qquad \hat{G}_s \longleftrightarrow \coprod_{\mathcal{F}} \mathcal{M}(\mathcal{G}_{\mathcal{F}}),$ together with injections $\quad \mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}_{\mathcal{F}}),$

where $\mathcal{F}$ runs over the families of irreducible characters of W_s, $\mathcal{G}_\mathcal{F}$ is a finite group, and $\mathcal{M}(\mathcal{G}_\mathcal{F})$ is the set of $\mathcal{G}_\mathcal{F}$-conjugacy classes of pairs (x, σ) where $x \in \mathcal{G}_\mathcal{F}$ and σ is an irreducible character of the centralizer of x in $\mathcal{G}_\mathcal{F}$. Note that W_s is a finite Weyl group, since the center of G is connected; for the definition of families, see [12, Chap. 4]. There is a non-degenerate, hermitian pairing $\{\, ,\, \}$ on $\mathcal{M}(\mathcal{G}_\mathcal{F})$, defined by the formula in [12, (4.14.3)]. For example, we have

(b) $\qquad \{(x, \sigma), (y, \tau)\} = \dfrac{1}{|C_{\mathcal{G}_\mathcal{F}}(y)|} \sigma(y)\overline{\tau(x)}$ if x lies in $Z(\mathcal{G}_\mathcal{F})$.

Then, the bijection in (a) satisfies an additional condition as follows: for each $E \in \mathcal{F}$ let $\mathcal{R}_s(E)$ be the rational linear combination of perverse sheaves on G defined in [14, (14.10)]. Then the multiplicity of $A \in \hat{G}_s$ in $\mathcal{R}_s(E)$ is given by

(c) $\qquad (A : \mathcal{R}_s(E)) = \begin{cases} \{x_A, x_E\} & \text{if } A \leftrightarrow x_A \in \mathcal{M}(\mathcal{G}_\mathcal{F}), \\ 0 & \text{otherwise,} \end{cases}$

where $x_E \in \mathcal{M}(\mathcal{G}_\mathcal{F})$ corresponds to E under the embedding in (a).

(4.2) We will be mainly interested in the restriction of character sheaves to G_{uni}. This can be described as follows.

Let C be a unipotent class in G, and fix $u \in C$. Then the isomorphism classes of irreducible G-invariant $\bar{\mathbb{Q}}_l$-local systems on C are in bijection with the irreducible characters of $A(u)$. Thus, we can identify a pair $(C, \mathcal{E})$, for $\mathcal{E}$ a local system as above, with the corresponding pair $(C, \psi) \in \mathcal{N}_G$. This should not lead to any confusion.

Given a pair $i = (C, \mathcal{E}) \in \mathcal{N}_G$ we denote by $\mathrm{IC}(\bar{C}, \mathcal{E})$ the corresponding intersection cohomology complex on the closure of C. Recall that, by the generalized Springer correspondence, we can associate with each $i \in \mathcal{N}_G$ a Levi subgroup L; we shall denote $d_i = \dim C_i + \dim Z(L)$. Then, by [15, (2.6)(e)] and [13, (6.5)], the restriction of a character sheaf $A \in \hat{G}$ to G_{uni} can be expressed uniquely as follows:

(a) $\qquad A|_{G_{\mathrm{uni}}} = \sum_{i \in \mathcal{N}_G} m_{A,i} A_i$ where $A_i := \mathrm{IC}(\bar{C}_i, \mathcal{E}_i)[d_i]$,

where the $m_{A,i}$ are certain non-negative integers. If the restriction of A to G_{uni} is non-zero then $m_{A,i} \neq 0$ for some i. Our aim will be to describe those of these non-zero coefficients for which C_i has maximal possible dimension.

(4.3) Let $s \in T^*$ and $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family. We wish to associate with the pair $(s, \mathcal{F})$ a unipotent class in G. This is done as follows, see [12, (13.3)] and [17, (10.5)]: let $E_0 \in \mathcal{F}$ be the unique special character. Then there exists a unique $E_0' \in \mathrm{Irr}(W)$ such that $b(E_0') = b(E_0)$ and

$\qquad \mathrm{Ind}_{W_s}^W (E_0) = E_0' + $ sum of characters E' with $b(E') > b(E_0)$,

where, for any irreducible character E of a finite Weyl group, we denote by $b(E)$ the smallest $m \geq 0$ such that E is a constituent of the m-th symmetric

power of the natural reflection representation. Via the Springer correspondence, E'_0 corresponds to a pair $i = (C, \psi) \in \mathcal{N}_G$. Then C is the desired unipotent class. We have in fact $\psi = 1$, see the remarks in [12, (13.3)], and [17, Theorem 10.7(iii)] for a general proof. Moreover, we have

(a) $$\dim \mathcal{B}_u = b(E_0) \qquad (u \in C).$$

Now let $\hat{G}_{s,\mathcal{F}}$ be the subset of $\hat{G}_s$ corresponding to pairs in $\mathcal{M}(\mathcal{G}_\mathcal{F})$ under the bijection (4.1)(a). Then the class C associated with $(s, \mathcal{F})$ as above has the following properties:

$\star_1$ if $A \in \hat{G}_{s,\mathcal{F}}$ and $A|_{\{g\}} \neq 0$ for some $g \in G_{\mathrm{uni}}$ then g lies in C or in a class of strictly smaller dimension than C;

$\star_2$ there exists some $A \in \hat{G}_{s,\mathcal{F}}$ such that $A|_C \neq 0$.

This is proved in [17, Theorem 10.7] under the assumption that p is large. Note that the conditions $\star_1$ and $\star_2$ are slightly different from those in Lusztig's paper in that they only take into account the restriction of character sheaves to G_{uni}; see also [16, (4.6)] and the remarks in [6, (5.5)] for a short discussion of this point. This discussion also shows that the assumption that p is large is only needed in reference to properties of generalized Gelfand-Graev characters which in turn are established as formal consequences of those described in (2.4).

(4.4) We define a function $d \colon \mathrm{Irr}(W) \to \mathbb{N}_0$ as follows. Let $E \in \mathrm{Irr}(W)$. By the Springer correspondence, E corresponds to a pair $i_E = (C_E, \psi_E) \in \mathcal{N}_G$; we then let $d(E) := \dim \mathcal{B}_u$ where $u \in C_E$.

Now consider an element $s \in T^*$ and a family $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$. Let E_0 be the unique special character in $\mathcal{F}$ and $E'_0 \in \mathrm{Irr}(W)$ as in (4.3); we have $d(E'_0) = b(E_0)$. We say that $(s, \mathcal{F})$ is good if the following two conditions are satisfied:

(a) We have $\mathcal{G}_\mathcal{F} \cong A(u)$ $(u \in C)$ where C is the unipotent class attached to the pair $(s, \mathcal{F})$ as in (4.3).

(b) We have $\mathrm{Ind}^W_{W_s}(E_0) = E'_0 +$ sum of characters E' with $d(E') > b(E_0)$.

An example illustrating these conditions is given in (6.4) below. It is likely that (a) implies (b). In type A_n this is clear since $d(E) = b(E)$ for all $E \in \mathrm{Irr}(W)$. If G is of exceptional type, this can be checked explicitly using induce/restrict matrices for the characters of W (computed, for example, with the CHEVIE system [8]) and the tables in [4, (13.3)] giving the Springer correspondence for exceptional types. For type B_n, an even more general statement is given in [15, (4.10)]. To settle the question for types C_n and D_n, it seems to be necessary to use an explicit combinatorial description of the induction of characters of the corresponding Weyl groups. This will be discussed elsewhere.

With this notation we can now state:

Theorem 4.5 (Cf. [15], (1.6)). *Let $s \in T^*$ and $\mathcal{F}$ be a family in $\mathrm{Irr}(W_s)$. Let C be the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). Assume that*

$(s, \mathcal{F})$ *satisfies the conditions (a) and (b) in (4.4). Then, for every $A \in \hat{G}_{s,\mathcal{F}}$, the set*

$$\{i \in \mathcal{N}_G \mid C_i = C \text{ and } m_{A,i} \neq 0\}$$

is either empty or contains precisely one element, in which case the corresponding coefficient $m_{A,i}$ equals 1. Conversely, for each $i \in \mathcal{N}_G$ with $C_i = C$ there exists a unique $A \in \hat{G}_{s,\mathcal{F}}$ such that $m_{A,i} \neq 0$.

Note that (4.3) shows that if $A \in \hat{G}_{s,\mathcal{F}}$ and C' is any unipotent class in G such that $A|_{C'} \neq 0$ then $\dim C' \leq \dim C$, with equality only for $C' = C$. The above theorem states that the map $A \mapsto A|_C$ defines a bijection between the set of character sheaves in $\hat{G}_{s,\mathcal{F}}$ with non-zero restriction to C and the set of $i \in \mathcal{N}_G$ with $C_i = C$.

Example 4.6 (Cf. [16], (4.7)(a)). Let G be of type E_8, $s = 1$ and $\mathcal{F} \subseteq \mathrm{Irr}(W)$ be the unique family with $\mathcal{G}_{\mathcal{F}} \cong \mathfrak{S}_5$. Let C be the corresponding unipotent class as in (4.3). Then the hypotheses of Theorem 4.5 are satisfied. Let $A \in \hat{G}_{s,\mathcal{F}}$ correspond to the pair $(x, \sigma) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$ under the bijection (4.1)(a). Then $A|_C = 0$ if $x \neq 1$. On the other hand, if $x = 1$, then $A|_C$ is up to shift the local system on C corresponding to the character σ of $A(u) \cong \mathcal{G}_{\mathcal{F}}$.

In order to prove Theorem 4.5 we will choose a suitable $\mathbb{F}_q$-rational structure on G, and then use results about character values and extremality conditions of a somehow similar kind as in Section 3. Note that we are interested in the coefficients $m_{A,i}$ defined by the decomposition in (4.2)(a), and these are independent of any $\mathbb{F}_q$-rational structure. Such a kind of argument is inspired from the proof of [17, Theorem 10.7]. In this section we start to set up the necessary general machinary. In Section 5 we will prove a relation which, for given i, bounds the number of possible A such that $m_{A,i} \neq 0$, see (5.7). A kind of complementary relation will be established in Section 6, but on the level on ordinary characters of G^F, see (6.5). The link between these two relations is given by Shoji [21], and this will be used in Section 7 to complete the proof of Theorem 4.5, see (7.3).

(4.7) Let q be any power of p. Then we can choose an $\mathbb{F}_q$-rational structure on G with Frobenius map $F \colon G \to G$. The dual group G^* inherits an $\mathbb{F}_q$-rational structure whose Frobenius map we also denote by F. We choose F in such a way that the tori $T \subseteq G$ and $T^* \subseteq G^*$ are F-stable and split over $\mathbb{F}_q$, and that we have an F-stable Borel subgroup $B \subseteq G$ containing T. For brevity we shall say that a closed subgroup $L \subseteq G$ is a Levi subgroup in G if there is a parabolic subgroup $P \subseteq G$ containing B such that L is the unique Levi complement in P which contains T.

Now fix an element $s \in T^*$. We can and will choose q so that the following conditions are satisfied:

(a) We have $F(s) = s$. Note that, since G is of split type, this also implies that F acts trivially on W_s.

(b) We have $q \equiv 1 \bmod 4$ if G is of type C_n, and $q \equiv 1 \bmod 3$ if G is of type G_2 or E_8.

(c) If L is an F-stable Levi subgroup in G then all character sheaves in $\hat{L}_s$ are F-stable.

Indeed, start with any q such that $F(s) = s$. Then, replacing q by q^2 if necessary, we can certainly make sure that (b) holds. Finally, if L is as in (c), F induces a permutation on the finite set of character sheaves in $\hat{L}_s$. Hence, replacing q by a suitable power if necesarry we can also make sure that all these character sheaves are F-stable. Since there is only a finite number of possibilities of L, we can find a common q so that (c) holds.

Since G is of split type, each unipotent class C of G is F-stable (see, for example, [9, 4C]). Since $q \equiv 1 \bmod 3$ if G is of type E_8, we can always find a split representative $u \in C^F$, unique up to G^F-conjugacy, see [19] and [3]. This implies, in particular, that F acts trivially on $A(u)$. We denote by $u = u_1, \ldots, u_d$ representatives of the G^F-classes contained in C^F, as in (2.3). Then we have a canonical normalization of the functions Y_i with $C_i = C$ by the condition:

(d) $\qquad\qquad Y_i(u) = \psi_i(1) \quad$ for all $i \in \mathcal{N}_G$ with $C_i = C$,

where $u \in C^F$ is the chosen split element. Thus, the matrix of values $(Y_i(u_r))_{i,r}$ (for $i \in \mathcal{N}_G$ with $C_i = C$, and $1 \le r \le d$) is just the character table of the finite group $A(u)$.

The other congruence conditions in (b) make sure that the 4-th roots of unity ζ_i' are always equal to 1, see (3.3)–(3.8).

(4.8) If A is any F-stable character sheaf on G, we can choose an isomorphism $\varphi\colon F^*A \to A$ and obtain a corresponding characteristic function $\chi_{A,\varphi}$ which is in fact a class function on G^F (see [14, (8.4)]). We always choose such an isomorphism as in [14, (25.1)]; such a choice is unique up to scalar multiples of absolute value 1. The corresponding characteristic function $\chi_{A,\varphi}$ then has norm 1.

Our first task is to specify a definite choice for an isomorphism $\varphi\colon F^*A \to A$ in the case where A has non-zero restriction to G_{uni}.

(4.9) Let A be any character sheaf on G with non-zero restriction to G_{uni}. Then A is a component of an induced complex $\mathrm{ind}_L^G(A_0)$ where L is a Levi subgroup in G (cf. the convention in (4.7)) and A_0 is a cuspidal character sheaf on L (in the sense of [14, Def. 3.10]); moreover, the restriction of A_0 to L_{uni} is non-zero, see [15, (2.9)]. Note that L is like G, i.e., $Z(L)$ is connected and $L/Z(L)$ is simple. This will allow us to argue by induction on $\dim G$ in several proofs below.

Now assume that A_0 is a cuspidal character sheaf on G. By [14, Prop. 3.12] we then have

(a) $\qquad\qquad A_0 = \mathrm{IC}(\bar{C}_0 Z(G), \mathcal{E}_0 \boxtimes \mathcal{L})[\dim C_0 + \dim Z(G)]$

where $i_0 := (C_0, \mathcal{E}_0) \in \mathcal{N}_G$ is a cuspidal pair and $\mathcal{L}$ is an irreducible $\bar{\mathbb{Q}}_l$-local local system on $Z(G)$. Thus, we have:

(b) $m_{A_0, i_0} = 1$ and $m_{A_0, i} = 0$ for $i_0 \neq i \in \mathcal{N}_G$.

Note that, moreover, A_0 is clean (see [14, (23.1)]). So the restriction of A_0 to G_{uni} is zero on $G_{\text{uni}} \setminus C_0$ and just $\mathcal{E}_0$ on C_0 (up to shift). Conversely, we also have

(c) $m_{A, i_0} = 0$ for all non-cuspidal $A \in \hat{G}$.

This follows from the fact that if A is a component of a complex induced from a Levi subgroup L as above, then $m_{A, i} = 0$ unless i lies in a block of $\mathcal{N}_G$ with associated Levi subgroup conjugate to L, see [15, (2.6)] and [13, (6.5)].

Proposition 4.10. *With the assumptions of (4.7), let $\mathcal{F} \subseteq \text{Irr}(W_s)$ be a family and C the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). Then, for any $A \in \hat{G}_{s, \mathcal{F}}$, there exists an isomorphism $\varphi \colon F^* A \to A$ as in (4.8) such that*

$$\chi_{A, \varphi}(u_r) = (-1)^{\text{rank } G} \sum_i q^{b_i} m_{A, i} Y_i(u_r) \quad \text{for } 1 \leq r \leq d,$$

where the sum is over all $i \in \mathcal{N}_G$ with $C = C_i$.

Proof. The proof of this is contained in [15, Sections 2 and 3]. We briefly recall the main ingredients. We may assume that the restriction of A to G_{uni} is non-zero. Then A is a component of an induced complex $\text{ind}_L^G(A_0)$ where L and A_0 are as in (4.9). Since A is F-stable, we can assume that L and A_0 are also F-stable, see [15, (3.2)]. By [14, Prop. 4.8(b)] we can also assume that $A_0 \in \hat{L}_s$. Now the conditions in (4.7) are all valid for L as well.

There is a canonical choice for an ismorphism $\varphi' \colon F^* A_0 \to A_0$. Indeed, as in (4.9)(b), there is a unique cuspidal pair $i' = (C', \mathcal{E}') \in \hat{L}$ such that $m_{A_0, i'} \neq 0$. Then φ' is determined by the condition that the restriction of $\chi_{A_0, \varphi'}$ to L_{uni}^F is given by $q^{(1/2)(\dim L/Z(L) - \dim C')} Y_{i'}'$ (see [15, (3.2)]). Note that this also implies that $\chi_{A_0, \varphi'}$ has norm 1.

For $i \in \mathcal{N}_G$ with $m_{A, i} \neq 0$ let A_i be as in (4.2)(a). The choice of φ' uniquely determines isomorphisms $\varphi \colon F^* A \to A$ and $\varphi_i \colon F^* A_i \to A_i$. The corresponding characteristic functions all have norm 1. By [15, (3.2)(a)], they are also compatible with the decomposition (4.2)(a), and so we have

$$\chi_{A, \varphi} = \sum_{i \in \mathcal{N}_G} m_{A, i} \chi_{A_i, \varphi_i} \quad \text{on } G_{\text{uni}}^F.$$

Finally, the isomorphisms φ_i have the property that

$$\chi_{A_i, \varphi_i}(u_r) = (-1)^{\text{rank } G} q^{b_i} Y_i(u_r) \quad \text{for } 1 \leq r \leq d,$$

see [15, (3.4)(a)]. Putting things together, the proof is complete. $\square$

Corollary 4.11. *For any $A \in \hat{G}_{s,\mathcal{F}}$ and any $i \in \mathcal{N}_G$ with $C_i = C$ we have*

$$m_{A,i} = \frac{1}{|A(u)|}(-1)^{\operatorname{rank} G} \langle \chi_{A,\varphi}, D_G(\Gamma_i) \rangle \qquad (u \in C).$$

Proof. By (2.4)(c) and (4.3)($\star_1$), it suffices to know the restriction of $\chi_{A,\varphi}$ to C^F in order to evaluate the scalar product on the right hand side. It remains to combine Proposition 4.10 and the orthogonality relations in (2.4)(a). Note that $\zeta_i' = 1$ by the choice of q in (4.7). $\qquad\square$

The above result shows that the unknown coefficients $m_{A,i}$ can be expressed in terms of scalar products of characteristic functions of character sheaves in $\hat{G}_{s,\mathcal{F}}$ with the generalized Gelfand-Graev characters associated with C. In the next section, these characteristic functions will be linked with certain irreducible characters of G^F.

5. Principal series characters

We keep the notation and basic assumptions of the previous section. Let us also fix an element $s \in T^*$ and a corresponding $\mathbb{F}_q$-rational structure on G with Frobenius map F as in (4.7). Recall that each character sheaf $A \in \hat{G}_s$ is F-stable. For any such A, we assume chosen an isomorphism $\varphi \colon F^*A \to A$ as in (4.8).

The following result gives a link between these characteristic functions and certain principal series characters of G^F. At the end of this section, we will use this link to obtain a non-trivial property of the coefficients $m_{A,i}$ that we are trying to compute.

(5.1) By duality, the element $s \in T^*$ corresponds to a character $\theta_s \in \operatorname{Irr}(T^F)$. We can lift this character to B^F using the natural map $B^F \to T^F$, and induce the resulting character to G^F. The irreducible constituents of this induced character are then parametrized by the irreducible characters of W_s; let ρ_E be the component corresponding to $E \in \operatorname{Irr}(W_s)$. We then have the following formula (see [14, (14.4)] and [15, (3.6)]):

(a) $$\rho_E = (-1)^{\dim G} \sum_{A \in \hat{G}_s} \xi_A(A : \mathcal{R}_s(E))\chi_{A,\varphi},$$

where ξ_A is an algebraic number of absolute value 1.

Let $\mathcal{F} \in \operatorname{Irr}(W_s)$ be a family and C be the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). Then we have:

(b) if $E \in \mathcal{F}$, the class C is the unipotent support of ρ_E.

Indeed, (4.3) and the formula (a) show that the unipotent support of ρ_E is either C or a class of strictly smaller dimension than C. Assume, if possible, that it has smaller dimension. Then the p-part in the degree of ρ would be strictly bigger than $q^{\dim \mathcal{B}_u}$ $(u \in C)$, see the equation defining n_ρ in (2.8). On the other hand, [12, (4.26.3)] shows that the p-part of ρ_E is also given by

$q^{b(E_0)}$ which, by (4.3)(a), equals $q^{\dim \mathcal{B}_u}$. Hence the assumption was wrong, and C is the unipotent support of ρ_E.

Our main tool to obtain information about the coefficients $m_{A,i}$ is the following formula:

Proposition 5.2. *Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family and C the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). Assume that $|\mathcal{G}_{\mathcal{F}}| = |A(u)|$ $(u \in C)$. Then there exists some $u_0 \in C^F$ such that $A(u_0)^F = A(u)$ and*

$$\overline{Y_i(u_0)} = \sum_{A \in \hat{G}_{s,\mathcal{F}}} [\mathcal{G}_{\mathcal{F}} : C_{\mathcal{G}_{\mathcal{F}}}(x_A)] \sigma_A(1) \xi_A m_{A,i} \quad \text{for all } i \in \mathcal{N}_G \text{ with } C_i = C,$$

where $(x_A, \sigma_A) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$ corresponds to A under the bijection (4.1)(a).

Proof. Let $E_0 \in \mathcal{F}$ be the unique special character. By [12, (4.14.2)], the generic denominator of ρ_{E_0} is given by $|\mathcal{G}_{\mathcal{F}}|$, see also the defining relation in [12, (4.26.3)]. Hence our assumption implies that $n_{\rho_{E_0}} = |A(u)|$. By Proposition 3.1, there exists some $u_0 \in C^F$ such that $A(u_0)^F = A(u)$ and

$$\langle D_G(\rho_{E_0}), \Gamma_i \rangle = \overline{Y_i(u_0)} \quad \text{for all } i \in \mathcal{N}_G^F \text{ with } C_i = C.$$

(Note that $D_G(\rho_E) \in \mathrm{Irr}(G^F)$ for all $E \in \mathrm{Irr}(W_s)$ since these are principal series characters.)

By [12, (4.14.2)], E_0 corresponds to the pair $(1, 1) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$ under the embedding (4.1)(a). The definition of the pairing $\{\,,\,\}$ therefore shows that

$$(A : \mathcal{R}_s(E_0)) = \frac{1}{|C_{\mathcal{G}_{\mathcal{F}}}(x_A)|} \sigma_A(1) 1_{\mathcal{G}_{\mathcal{F}}}(x_A) = \frac{1}{|C_{\mathcal{G}_{\mathcal{F}}}(x_A)|} \sigma_A(1).$$

Putting things together, we compute that

$$\begin{aligned}
\overline{Y_i(u_{r_0})} &= \langle D_G(\rho_{E_0}), \Gamma_i \rangle = \langle \rho_{E_0}, D_G(\Gamma_i) \rangle \\
&= (-1)^{\dim G} \sum_A \xi_A (A : R_s(E_0)) \langle \chi_{A,\varphi}, D_G(\Gamma_i) \rangle \quad \text{by (5.1)(a)} \\
&= (-1)^{\dim G + \mathrm{rank} G} \sum_A \xi_A (A : R_s(E_0)) |A(u)| m_{A,i} \quad \text{by Cor. 4.11}
\end{aligned}$$

Inserting the above expression for $(A : R_s(E_0))$, the proof is complete. $\square$

In order to make an efficient use of this formula we need to know the coefficients ξ_A for those A which have non-zero restriction to G_{uni}. These results were known before, see Remark 5.6 below. We shall use a similar inductive argument as in the proof of Theorem 3.8 to determine these coefficients. For this purpose, we first need the following preliminary result.

Proposition 5.3. *Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family and assume that $\hat{G}_{s,\mathcal{F}}$ contains a cuspidal character sheaf A. Let C be the unipotent class attached to $(s, \mathcal{F})$ as in (4.3).*

  (a) *We have $\mathcal{G}_{\mathcal{F}} \cong A(u)$ $(u \in C)$.*
  (b) *If the restriction of A to G_{uni} is non-zero then $A|_C \neq 0$.*

Proof. First note that the above statements do not refer to any $\mathbb{F}_q$-rational structure on G. Thus, we can and will replace the chosen q by a suitable power below so that the conditions in (4.7) hold for any given semisimple element in T^* that we encounter in the course of our argument.

Considering the adjoint quotient $G \to G_{\mathrm{ad}}$ and the reduction arguments in [14, (17.9), (17.10)] we see that we may assume, without loss of generality, that $Z(G) = \{1\}$.

In order to prove (a) we have to get an overview of the possible pairs $(s, \mathcal{F})$ so that $\hat{G}_{s,\mathcal{F}}$ contains a cuspidal character sheaf. By [14, (17.12)], the element s must be isolated, i.e., W_s has the same rank as W. Moreover, by [14, (17.13)], the family $\mathcal{F}$ must be a cuspidal family (which is then unique) in the sense of [12, (8.1)]. This already reduces drastically the possibilities.

If G is of type A_n then G does not have any cuspidal character sheaves at all unless $n = 0$ in which case there is nothing to prove.

Assume now that G is of exceptional type. By checking the possibilities for the existence of a cuspidal family in W_s we see that $W_s = W$ and $\mathcal{F}$ must be the unique cuspidal family, see [14, Sections 20, 21]. Using the description of the groups $\mathcal{G}_{\mathcal{F}}$ in [12, Chap. 4], the definition of C in (4.3) and the tables for the Springer correspondence in [4, (13.3)] we can check by inspection that (a) holds.

Now let G be of classical type B_n, C_n or D_n, for some $n \geq 2$. From [14, Section 23] (see also [12, (9.9.1)]) we see that the set $\hat{G}_{s,\mathcal{F}}$ contains a cuspidal character sheaf if and only if W_s has the same rank as W, each factor of W_s has even rank, and $\mathcal{F}$ is a cuspidal family (which is unique). Using the description of cuspidal families in [12, (8.1)] and that of the corresponding groups $\mathcal{G}_{\mathcal{F}}$ in [12, Chap. 4], we find the following possibilities:

Type	W_s	$|\mathcal{G}_{\mathcal{F}}|$
B_n	$C_{r(r+1)} \times C_{t(t+1)}$	2^{r+t}
C_n	$D_{4r^2} \times B_{t(t+1)}$	$2^{2r+t-\delta(r)}$
D_n	$D_{4r^2} \times D_{4t^2}$	$2^{2r+2t-1-\delta(rt)}$

where, in each case, the rank of W_s equals the rank of W; moreover, for any integer m, we write $\delta(m) = 1$ if $m \neq 0$ and $\delta(m) = 0$ if $m = 0$.

The map $(s, \mathcal{F}) \mapsto C$ can be explicitly computed using combinatorial descriptions for the Springer correspondence (see [4, (13.3)]) and the induction of characters. The results are given in [18, (1.10)–(1.12)]:

In type B_n, the class C corresponding to a pair $(s, \mathcal{F})$ as in the above table has Jordan blocks of type $(J_1 + J_3 + \ldots + J_{2u-1}) + (J_1 + J_3 + \ldots + J_{2v-1})$ where $u = r + t + 1$ and $v = |r - t|$. (Here, J_i denotes a Jordan block of size i.)

In type C_n, the class C has Jordan blocks of type $(J_2 + J_4 + \ldots + J_{2u}) + (J_2 + J_4 + \ldots + J_{2v})$ where $u = 2r + t$ and $v = 2r - t - 1$ if $2r > t$ or $v = t - 2r$ if $2r \leq t$.

In type D_n, the class C has Jordan blocks of type $(J_1 + J_3 + \ldots + J_{2u-1}) + (J_1 + J_3 + \ldots + J_{2v-1})$ where $u = 2r + 2t$ and $v = |2r - 2t|$.

Using the description of the groups $A(u)$ in [4, p.399] we can check that, in each case, $|A(u)|$ is given by the number in the right most column of the above table. Thus, (a) is proved.

Now consider (b). Let $A \in \hat{G}_{s,\mathcal{F}}$ be a cuspidal character sheaf with non-zero restriction to G_{uni}. Then $m_{A,i_0} \neq 0$ where $i_0 = (C_0, \mathcal{E}_0) \in \mathcal{N}_G$ is the unique cuspidal pair, see (4.9)(b). The problem is to show that $C = C_0$.

If G is of type A_n there is nothing to prove.

As far as exceptional groups are concerned, we only need to consider G of type G_2, F_4 or E_8 (cf. (3.5)). We have already seen in (a) that $W_s = W$ and $\mathcal{F}$ must be the unique cuspidal family. Using the definition of C in (4.3) we see that C is the unique class such that $A(u)$ ($u \in C$) has size 6,24,120, respectively. Comparing with the list in (3.5), we can conclude that $C_{i_0} = C$, and hence the proof is complete in this case.

It remains to consider groups of classical type B_n, C_n or D_n, for some $n \geq 2$. The possibilities for the cuspidal pair $i_0 \in \mathcal{N}_G$ are listed in (3.8). Comparing with the lists in (a) above we see that there exists some pair $(s_0, \mathcal{F}_0)$ such that C_0 is the class attached to $(s_0, \mathcal{F}_0)$ as in (4.3) and such that $\hat{G}_{s_0,\mathcal{F}_0}$ contains a cuspidal character sheaf. By (a) the assumptions of Proposition 5.2 are satisfied. So we conclude that

$$0 \neq \overline{Y_{i_0}(u_0)} = \sum_{A'} \xi_{A'} m_{A',i_0} \qquad \text{(sum over all } A' \in \hat{G}_{s_0,\mathcal{F}_0}\text{)}.$$

(We may have to replace q by some suitable power so that the conditions in (4.7) hold for s_0 as well.) Hence there exists some $A_0 \in \hat{G}_{s_0,\mathcal{F}_0}$ such that $m_{A_0,i_0} \neq 0$. By (4.9), A_0 must be cuspidal and $m_{A_0,i_0} \neq 0$.

Thus, we have two cuspidal character sheaves, A and A_0, which have non-zero restriction to C_0. By [14, (23.2)], there exists at most one cuspidal character sheaf with non-zero restriction to G_{uni}. So we conclude that $A = A_0$ and hence that $C = C_0$. This completes the proof. $\qquad \square$

Remark 5.4. From the above proof we can also find the type of W_s so that $\hat{G}_{s,\mathcal{F}}$ (where $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ is the unique cuspidal family) contains a cuspidal character sheaf with non-zero restriction to G_{uni}; for classical groups we obtain:

Type of G	condition on n	type of W_s
B_n	$n = 2t(t+1)$	$C_{t(t+1)} \times C_{t(t+1)}$
C_n	$n = 2t(4t \pm 1)$	$D_{4t^2} \times B_{4t^2 \pm 2t}$
D_n	$n = 8t^2$	$D_{4t^2} \times D_{4t^2}$

where $t \geq 1$ in each case. This coincides with the remarks in [15, (7.11)]. The assignment $(s, \mathcal{F}) \mapsto C$, for *all* pairs $(s, \mathcal{F})$ such that $\hat{G}_{s,\mathcal{F}}$ contains a cuspidal character sheaf, is determined in [18], using a refinement of property $(\star_1)$ in (4.3), see [17, Theorem 10.7]. Since we are only interested in character sheaves with non-zero restriction to G_{uni} we could avoid using that refinement in the above proof.

Proposition 5.5. *Let $A \in \hat{G}_s$ such that the restriction of A to G_{uni} is non-zero. Then $\xi_A = 1$, where ξ_A is the algebraic number appearing in the formula (5.1)(a) and where the isomorphism $\varphi\colon F^*A \to A$ is chosen as in Proposition 4.10.*

Proof. We proceed by induction on rank $G/Z(G)$. If this is zero then G is a torus and there is nothing to prove. Now let rank $G/Z(G) > 0$.

As in the proof of Proposition 4.10, A is a component of an induced complex $\mathrm{ind}_L^G(A_0)$ where $A_0 \in \hat{L}_s$ is cuspidal and the restriction of A_0 to L_{uni} is non-zero. The conditions in (4.7) hold in L as well. In particular, ξ_{A_0} is defined. By [15, (3.5)(a)], we have $\xi_A = \xi_{A_0}$. So we are done by induction if $L \neq G$. Hence we can assume that $L = G$ and $A = A_0$.

Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be the family such that $A_0 \in \hat{G}_{s,\mathcal{F}}$, and let C be the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). Let $i_0 \in \mathcal{N}_G$ be the cuspidal pair as in (4.9)(b). By Proposition 5.3, we have $C_{i_0} = C$ and $|\mathcal{G}_{\mathcal{F}}| = |A(u)|$.

We can apply the formula in Corollary 5.2. The classification of cuspidal character sheaves in [14, Sections 20,21,23] shows that all character sheaves in $\hat{G}_s$ not equal to A_0 are non-cuspidal. Hence, by induction, we have $\xi_A = 1$ for all $A \in \hat{G}_s$, $A \neq A_0$. So the formula in (5.2) now reads:

$$\overline{Y_i(u_0)} = \xi_{A_0}[\mathcal{G}_{\mathcal{F}} : C_{\mathcal{G}_{\mathcal{F}}}(x_{A_0})]\sigma_{A_0}(1)m_{A_0,i} + \sum_A [\mathcal{G}_{\mathcal{F}} : C_{\mathcal{G}_{\mathcal{F}}}(x_A)]\sigma_A(1)m_{A,i},$$

for all $i \in \mathcal{N}_G$ with $C_i = C$.

Take any $i \in \mathcal{N}_G$ with $C_i = C$ and $i \neq i_0$. Then $m_{A_0,i} = 0$ (see (4.9)(b)) and hence the above formula shows that $\overline{Y_i(u_0)} \geq 0$. If $A(u)$ is abelian then the matrix of all values $Y_i(u_r)$ is the character table of $A(u)$. Since $|A(u)| > 2$ (see the list of possibilities in (3.7)) the condition that $\overline{Y_i(u_0)} \geq 0$ for all $i \neq i_0$ can only be satisfied if u_0 corresponds to the trivial element in $A(u)$, i.e., if u_0 is split. If the group $A(u)$ is non-abelian, it is isomorphic to $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$ and the condition that $A(u_0)^F = A(u)$ forces that u_0 is split.

Now we can take $i = i_0$ in the above formula. We have just seen that u_0 is a split elements. So the left hand side of that formula gives $\overline{Y_{i_0}(u_0)} = \psi_{i_0}(1)$. If $A(u)$ is abelian then this must be 1; if $A(u)$ is not abelian, we are in type G_2, F_4 ot E_8 and the remarks in (3.5) show that, again, $\psi_{i_0}(1) = 1$. On the other hand, by (4.9)(c), we have $m_{A,i_0} = 0$ for all non-cuspidal character sheaves $A \in \hat{G}_s$. So the summation on the righ hand side contains only one non-zero term, namely that corresponding to A_0. Hence we deduce that $1 = \psi_{i_0}(1) = \xi_{A_0}[\mathcal{G}_{\mathcal{F}} : C_{\mathcal{G}_{\mathcal{F}}}(x_{A_0})]\sigma_{A_0}(1)m_{A_0,i_0}$. This forces $\xi_{A_0} = 1$, and also $\sigma_{A_0}(1) = 1$. $\qquad\square$

Remark 5.6. A slightly different proof for type B_n was first given by Lusztig in [15, Prop. 5.3(c)]. See the remarks in [15, (7.11)] for the other classical types, and [15, (8.9)] for type E_8. Note, however, that we have to assume that p is large enough while in [15], only congruence conditions like those in (4.7) are needed.

(5.7) Recall that we are given an element $s \in T^*$, a family $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$, and the corresponding unipotent class C as in (4.3). Assume now that

$$|\mathcal{G}_\mathcal{F}| = |A(u)| \qquad (u \in C).$$

We have chosen an $\mathbb{F}_q$-rational structure on G with Frobenius map F as in (4.7). In the proof of Proposition 5.5 we used the formula in Corollary 5.2 to conclude that $\xi_A = 1$. Now we can return to that formula and try to obtain information about $m_{A,i}$. Indeed, we now have

$$\overline{Y_i(u_0)} = \sum_A [\mathcal{G}_\mathcal{F} : C_{\mathcal{G}_\mathcal{F}}(x_A)]\sigma_A(1)m_{A,i}, \quad \text{for all } i \in \mathcal{N}_G \text{ with } C_i = C,$$

where the sum is over all $A \in \hat{G}_{s,\mathcal{F}}$. The fact that the right hand side is always positive forces that u_0 is a split element. Hence the left hand side is always equal to $\psi_i(1)$, where $\psi_i \in \mathrm{Irr}(A(u))$. Thus, we conclude that

(a) $$\psi_i(1) = \sum_A [\mathcal{G}_\mathcal{F} : C_{\mathcal{G}_\mathcal{F}}(x_A)]\sigma_A(1)m_{A,i},$$

for all $i \in \mathcal{N}_G$ with $C_i = C$. Since the left hand side is non-zero, we see that there exists at least some $A \in \hat{G}_{s,\mathcal{F}}$ with $m_{A,i} \neq 0$, and the number of such A is bounded above by $\psi_i(1)$.

6. Multiplicities of irreducible characters

The basic assumptions of Section 4 remain in force. We have fixed an element $s \in T^*$ and chosen a suitable $\mathbb{F}_q$-rational structure with corresponding Frobenius map F as in (4.7). We will now go one step further and not only consider principal series characters but all irreducible characters of G^F.

(6.1) Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family. Then the pair $(s, \mathcal{F})$ also defines a subset $\mathcal{E}_{s,\mathcal{F}} \subseteq \mathrm{Irr}(G^F)$, see [12, (8.4.4) and (6.17)]. Moreover, since F acts trivially on W_s, [12, Main Theorem 4.23] shows that we have again a bijection

(a) $$\mathcal{E}_{s,\mathcal{F}} \longleftrightarrow \mathcal{M}(\mathcal{G}_\mathcal{F}), \quad \rho \leftrightarrow \bar{x}_\rho$$

such that the following holds. For each $E \in \mathrm{Irr}(W_s)$ let $R_s(E)$ be the corresponding almost character, defined in [12, (3.7)] as a certain rational linear combination of Deligne-Lusztig generalized characters of G^F. (Note that it only depends on E and not on a certain extension of E since F acts trivially on W_s.) Then the multiplicity of $\rho \in \mathcal{E}_{s,\mathcal{F}}$ in $R_s(E)$ is given by

(b) $$\langle \rho, R_s(E) \rangle = \begin{cases} \Delta(\bar{x}_\rho)\{\bar{x}_\rho, x_E\} & \text{if } E \leftrightarrow x_E \in \mathcal{M}(\mathcal{G}_\mathcal{F}), \\ 0 & \text{otherwise,} \end{cases}$$

where $x_E \in \mathcal{M}(\mathcal{G}_\mathcal{F})$ corresponds to E under the embedding in (4.1)(a), and $\Delta\colon \mathcal{M}(\mathcal{G}_\mathcal{F}) \to \{\pm 1\}$ is as defined in [12, (4.14)]. Note that Δ is identically 1 except if W_s has a component of type E_7 or E_8 and the family contains a character of degree 512 or 4096 in type E_7 or E_8, respectively. We can now

also give the formal definition of the generic denominator of ρ which already appeared in (2.8); we have, cf. [12, (4.26.3)]:

(c) $\qquad n_\rho^{-1} = \{\bar{x}_\rho, (1,1)\} = \dfrac{\sigma(1)}{|C_{\mathcal{G}_{\mathcal{F}}}(x)|} \quad$ if $\bar{x}_\rho = (x, \sigma) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$.

We will also need to know, for each $\rho \in \mathcal{E}_{s,\mathcal{F}}$, the sign so that $\pm D_G(\rho) \in \mathrm{Irr}(G^F)$. By [12, (6.8) and (6.20)] we have

(d) $\qquad \Delta(\bar{x}_\rho) D_G(\rho) \in \mathrm{Irr}(G^F) \quad$ for $\rho \in \mathcal{E}_{s,\mathcal{F}}$.

(6.2) Let $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ be a family and $E_0 \in \mathcal{F}$ the unique special character. We wish to determine the restriction of $R_s(E_0)$ to C^F, where C is as in (4.3).

Recall that $R_s(E_0)$ is defined as a certain linear combination of Deligne-Lusztig generalized characters. The restriction of such a Deligne-Lusztig generalized character to G_{uni} is given in terms of the corresponding Green function. For $w \in W$ let $T_w \subseteq G$ denote an F-stable maximal torus obtained by twisting the given split torus T with w, and let $Q_T \colon G_{\mathrm{uni}}^F \to \mathbb{Z}$ the corresponding Green function. For $E \in \mathrm{Irr}(W)$ let

$$ Q_E = \frac{1}{|W|} \sum_{w \in W} \mathrm{Trace}(w, E) Q_{T_w}. $$

Then, using the exact definition of almost characters in [12, (3.7)] we obtain the following relation:

$$ R_s(E_0)|_{G_{\mathrm{uni}}^F} = \sum_{E \in \mathrm{Irr}(W)} c(E_0, E) Q_E, \quad c(E_0, E) := \langle \mathrm{Ind}_{W_s}^W(E_0), E \rangle. $$

Via the Springer correspondence, each irreducible character of W corresponds to a pair in $\mathcal{N}_G$; we denote this correspondence by $E \mapsto i_E = (C_E, \psi_E)$. Let $E \in \mathrm{Irr}(W)$ such that $c(E_0, E) \neq 0$. Then [17, Cor. 10.9] shows that $d(E) \geq b(E_0)$ with equality only for $C_E = C$; recall the definition of $d(E)$ in (4.4). Since we have chosen q so that split elements always exist, we have by [20]:

(a) $\qquad Q_E = q^{b_{i_E}} Y_{i_E} +$ combination of Y_i with $\dim C_i < \dim C_E$.

Putting things together we see that

(b) $\qquad R_s(E_0)|_{C^F} = q^{\dim \mathcal{B}_u} \sum_E c(E_0, E) Y_{i_E}$

where the sum is over all $E \in \mathrm{Irr}(W)$ with $d(E) = b(E_0)$.

Proposition 6.3. *Assume that $(s, \mathcal{F})$ is good, cf. (4.4). Recall that E_0 is the unique special character in the family $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$ and that C is the unipotent class associated with $(s, \mathcal{F})$ as in (4.3). Then we have*

$$ R_s(E_0)|_{C^F} = q^{\dim \mathcal{B}_u} Y_{(C,1)}. $$

Proof. Combine (6.2)(b) and condition (b) in (4.4). Note that the character E_0' corresponds to the pair $(C, 1) \in \mathcal{N}_G$, as already remarked in (4.3). $\qquad \square$

The following example shows the necessity of the assumption that $A(u) \cong \mathcal{G}_{\mathcal{F}}$.

Example 6.4. Let G be of type C_3 and $s \in T^*$ such that $C_{G^*}(s)$ is of type $A_1 \times D_2$. Then every irreducible character of W_s is special and each family consists of just one element. Let $\mathcal{F} = \{\text{sign} \otimes \text{id}\}$. The unipotent class C attached to $(s, \mathcal{F})$ is uniquely determined by the requirement that $\dim \mathcal{B}_u = 1$. We compute that

$$\text{Ind}_{W_s}^{W}(\text{sign} \otimes \text{id}) = \phi_{(2,1)} + \phi_{(-,21)} + \phi_{(-,3)}.$$

(Recall that the irreducible characters of W are parametrized by double partitions of 3.) Using the algorithms in [4, §13.3] we can explicitly compute the Springer correspondence for type C_3. This shows that $\phi_{(2,1)}$ and $\phi_{(-,3)}$ correspond to pairs $i \in \mathcal{N}_G$ with $C_i = C$. Thus, Proposition 6.3 does not hold.

(6.5) We keep the assumptions of Proposition 6.3. Let $u_1, \ldots, u_d$ be representatives of the G^F-classes contained in C^F, where $u = u_1$ is a split element as in (4.7). Let us fix an index $r \in \{1, \ldots, d\}$ and consider the corresponding generalized Gelfand-Graev character Γ_{u_r}. Then we have:

(a) $$\langle D_G(R_s(E_0)), \Gamma_{u_r} \rangle = \langle R_s(E_0), D_G(\Gamma_{u_r}) \rangle = 1.$$

Indeed, using (2.3)(c), this is equivalent to showing that, for any $i \in \mathcal{N}_G$ with $C_i = C$, we have

$$\langle R_s(E_0), D_G(\Gamma_i) \rangle = \begin{cases} |A(u)| & \text{if } i = (C, 1), \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove this, first note that $R_s(E_0)$ satisfies condition (*) in Lemma 2.5. Hence we just need to consider the restriction of $R_s(E_0)$ to C^F in order to evaluate the above scalar product. By Proposition 6.3 this restriction is just a multiple of $Y_{(C,1)}$. It remains to use the orthogonality relations in (2.3) and the fact that $b_{(C,1)} = \dim \mathcal{B}_u$. Thus, (a) is proved.

On the other hand, we can write the left hand side of (a) as a linear combination of irreducible characters of G^F, using (6.1). The special character E_0 corresponds to the pair $(1,1) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$. The property (4.1)(a) of the pairing $\{\,,\,\}$ shows that $\{(y, \tau), (1,1)\} = \tau(1)/|C_{\mathcal{G}_{\mathcal{F}}}(y)|$ for any $(y, \tau) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$. So (6.1)(b) yields that

(b) $$R_s(E_0) = \sum_{\rho} \frac{\tau_\rho(1)}{|C_{\mathcal{G}_{\mathcal{F}}}(y_\rho)|} \Delta(\bar{x}_\rho) \rho,$$

where ρ runs over the characters in $\mathcal{E}_{s,\mathcal{F}}$ and $(y_\rho, \sigma_\rho) \in \mathcal{M}(\mathcal{G}_{\mathcal{F}})$ corresponds to ρ under the bijection (6.1)(a). The inverse of the coefficient of ρ in the above formula in fact equals the generic denominator n_ρ of ρ, see (6.1)(c). For each $\rho \in \mathcal{E}_{s,\mathcal{F}}$ let $\rho' := \Delta(\bar{x}_\rho) D_G(\rho) \in \text{Irr}(G^F)$, see (6.1)(d). Putting

things together we conclude that

(c) $$\sum_{\rho} \frac{1}{n_\rho} \langle \rho', \Gamma_{u_r} \rangle = 1,$$

where, as before, ρ runs over the characters in $\mathcal{E}_{s,\mathcal{F}}$.

Proposition 6.6. *Assume that the hypotheses of Theorem 4.5 are satisfied and that $A(u)$ is abelian, of order d say. Write $A(u) = \{u_t \mid 1 \le t \le d\}$ and $\mathcal{E}_{s,\mathcal{F}} = \{\rho_{rr'} \mid 1 \le r, r' \le d\}$. Then this labelling can be chosen in such a way that*

$$\langle \rho_{rr'}, D_G(\Gamma_{u_t}) \rangle = \begin{cases} \pm 1 & \text{if } r = t \\ 0 & \text{otherwise} \end{cases}$$

where $1 \le r, r', t \le d$.

Note that since $A(u)$ is abelian of order d we have $|\mathcal{E}_{s,\mathcal{F}}| = d^2$.

Proof. Since $A(u)$ is abelian we have $n_\rho = |\mathcal{G}_\mathcal{F}| = d$ for all $\rho \in \mathcal{E}_{s,\mathcal{F}}$. This follows from the definition of n_ρ in (6.1)(c) and the property (4.1)(a) of the pairing $\{\,,\,\}$. For $\rho \in \mathcal{E}_{s,\mathcal{F}}$ let $\rho' = \Delta(\bar{x}_\rho) D_G(\rho) \in \operatorname{Irr}(G^F)$ as in (6.1)(d). We can apply Proposition 3.1 and conclude that

  (a) for each $\rho \in \mathcal{E}_{s,\mathcal{F}}$ there exists precisely one t such that $\langle \rho', \Gamma_{u_t} \rangle \ne 0$, and this non-zero multiplicity equals 1.

Now fix $t \in \{1, \ldots, d\}$. Since $n_\rho = d$ for all $\rho \in \mathcal{E}_{s,\mathcal{F}}$, equation (6.5)(c) simplifies to: $\sum_\rho \langle \rho', \Gamma_{u_r} \rangle = d$. By (a), all non-zero terms in this sum are equal to 1. So we conclude that

  (b) for each t there exist precisely d characters $\rho \in \mathcal{E}_{s,\mathcal{F}}$ such that ρ' occurs with non-zero multiplicity in Γ_{u_t}.

Combining (a) and (b) yields the desired result. □

Proposition 6.7. *Assume that the hypotheses of Theorem 4.5 are satisfied and that $A(u) \cong \mathfrak{S}_3$. Then we have the following matrix of multiplicities between the irreducible characters in $\mathcal{E}_{s,\mathcal{F}}$ and the duals of the generalized Gelfand-Graev characters associated with C (where . stands for 0):*

n_ρ	6	6	3	3	3	3	2	2
$D_G(\Gamma_{u_1})$	1	1	2	.	.	.	.	.
$D_G(\Gamma_{u_2})$	.	.	.	1	1	1	.	.
$D_G(\Gamma_{u_3})$	.	.	.	.	.	.	1	1

Note that since $A(u) \cong \mathfrak{S}_3$, there are 3 representatives u_1, u_2, u_3. We arrange notation so that $a_1 = 6$, $a_2 = 3$ and $a_3 = 2$, where $a_r = |A(u_r)^F|$ as in (2.3).

Proof. The formula (6.1)(c) shows that n_ρ only depends on $\mathcal{G}_\mathcal{F}$ and the pair $\bar{x}_\rho \in \mathcal{M}(\mathcal{G}_\mathcal{F})$. Thus, we see that $\mathcal{E}_{s,\mathcal{F}}$ always contains 2 characters with $n_\rho = 6$, 4 characters with $n_\rho = 3$ and 2 characters with $n_\rho = 2$. Furthermore, we have $\Delta(\bar{x}_\rho) = 1$ for all $\rho \in \mathcal{E}_{s,\mathcal{F}}$, and so $D_G(\rho) \in \operatorname{Irr}(G^F)$.

Now let $\rho \in \mathcal{E}_{s,\mathcal{F}}$ and consider equation (2.8)(b). Having fixed the above notation, this equation yields that

(a) $\qquad \langle D_G(\rho), \Gamma_{u_1} \rangle + 2\langle D_G(\rho), \Gamma_{u_2} \rangle + 3\langle D_G(\rho), \Gamma_{u_3} \rangle = \dfrac{6}{n_\rho}.$

We first apply this to the 2 characters with $n_\rho = 6$. Then we are just in the situation of Proposition 3.1 and we conclude that the duals of these two characters must occur with multiplicity 1 in Γ_{u_1}, and they do not occur in $\Gamma_{u_2}, \Gamma_{u_3}$.

Let $m \geq 0$ be the sum of the multiplicities of characters $D_G(\rho)$ with $n_\rho = 3$ in Γ_{u_1}, and $n \geq 0$ the corresponding number for characters with $n_\rho = 2$. Then equation (6.5)(c) (with $r = 1$) yields that

$$\frac{1}{6} + \frac{1}{6} + \frac{m}{3} + \frac{n}{2} = 1.$$

This forces that $n = 0$ and $m = 2$. In particular, the dual of a character with $n_\rho = 2$ cannot occur in Γ_{u_1}. Let ρ be such a character. We have just seen that its dual only occurs in Γ_{u_2} and Γ_{u_3}. Hence equation (a) yields that

$$2\langle D_G(\rho), \Gamma_{u_2} \rangle + 3\langle D_G(\rho), \Gamma_{u_3} \rangle = \frac{6}{2} = 3.$$

This forces that $D_G(\rho)$ occurs with multiplicity 1 in Γ_{u_3} and does not occur in Γ_{u_3}. Now equation (6.5)(c) (with $r = 3$) shows that Γ_{u_3} contains no other characters than the two $D_G(\rho)$'s with $n_\rho = 2$.

It remains to consider the characters with $n_\rho = 3$. We have just seen that they cannot occur in Γ_{u_3}. Equation (a) yields that

$$\langle D_G(\rho), \Gamma_{u_1} \rangle + 2\langle D_G(\rho), \Gamma_{u_2} \rangle = \frac{6}{3} = 2.$$

This shows that if $D_G(\rho)$ with $n_\rho = 3$ occurs in Γ_{u_2} then it occurs with multiplicity 1 and does not occur in Γ_{u_1}. Conversely, if it occurs in Γ_{u_1} then it occurs with multiplicity 2 and does not occur in Γ_{u_2}. Equation (6.5)(c) (with $r = 2$) then shows that 3 such characters must occur in Γ_{u_2} and 1 such character in Γ_{u_2}. This completes the proof. $\qquad \square$

(6.8) Assume that the hypotheses of Theorem 4.5 are satisfied and that $A(u) \cong \mathfrak{S}_4$ or $A(u) \cong \mathfrak{S}_5$. Then G must have type F_4 or E_8, respectively, and in either case, $s \in T^*$ must be central. So we can in fact assume that $s = 1$. All pairs in $\mathcal{N}_G$ except one are uniform, and the exceptional pair is cuspidal (cf. (3.5)); it corresponds to the sign character of $A(u)$. The decomposition of Y_{i_0} into irreducible characters in this case was found by Kawanaka in [10, Theorem 4.2.2]; see also [15, (8.6), (8.12)]. The decomposition of Y_i for a non-uniform pair i is given by first writing it as a linear combination of Green functions (by the algorithm in [20]) and then using the multiplicity formula in [12, (4.23)]. Combining this with the knowledge of the generalized Gelfand-Graev characters from Section 3 it is thus possible

to compute explicitly the matrix of scalar products $\langle \rho, D_G(\Gamma_u) \rangle$, for $\rho \in \mathcal{E}_{s,\mathcal{F}}$ and $u \in C^F$. We omit the details since we will not need this here.

7. Proof of Theorem 4.4

The aim of this final section is to complete the proof of Theorem 4.5. Recall that we are given a semisimple element $s \in T^*$ and a family $\mathcal{F} \subseteq \mathrm{Irr}(W_s)$. Let C be the unipotent class attached to $(s, \mathcal{F})$ as in (4.3). We choose an $\mathbb{F}_q$-rational structure with corresponding Frobenius map $F \colon G \to G$ as in (4.7). Recall also the notation $u_1, \dots, u_d$ for representatives of the G^F-classes contained in C^F, where $u = u_1$ is a split element.

In order to apply the results from Section 6 to our problem of determining the coefficients $m_{A,i}$, we need a link between the ordinary characters of G^F and the characteristic functions of character sheaves. This link is provided by Shoji [21] who proved Lusztig's conjecture about the relation between almost characters of G^F and characteristic functions of F-stable character sheaves. We do not need this result in its full strength; for example, we consider only series of characters and character sheaves which correspond to an element $s \in T^*$ which is F-fixed. In fact we shall only need the following weak form of Shoji's results:

Theorem 7.1 (Shoji [21]). *In the setup of (4.7), let us choose for any $A \in \hat{G}_s$ an isomorphism $\varphi \colon F^* A \to A$ as in (4.8). Then every character in $\mathcal{E}_{s,\mathcal{F}}$ can be written as a linear combination of characteristic functions $\chi_{A,\varphi}$, for various $A \in \hat{G}_{s,\mathcal{F}}$. Conversely, every such $\chi_{A,\varphi}$ is a linear combination of the characters in $\mathcal{E}_{s,\mathcal{F}}$.*

Lemma 7.2. *Assume that the hypotheses of Theorem 4.5 are satisfied. Consider the matrix of all coefficients $m_{A,i}$ where $A \in \hat{G}_{s,\mathcal{F}}$ and $i \in \mathcal{N}_G$ with $C_i = C$. Then this matrix has full rank.*

Proof. Assume first that $A(u)$ is abelian or isomorphic to $\mathfrak{S}_3$. Then, by Prop. 6.6 and Prop. 6.7, the matrix of all scalar products $\langle \rho, D_G(\Gamma_{u_r}) \rangle$, where $\rho \in \mathcal{E}_{s,\mathcal{F}}$ and $1 \leq r \leq d$, has full rank.

Now consider the matrix of all values $\rho(u_r)$ where $\rho \in \mathcal{E}_{s,\mathcal{F}}$ and $1 \leq r \leq d$. Using Theorem 7.1 and (4.3) we see that the condition (*) in Lemma 2.5 is satisfied. Using the relations in Corollary 2.6 and (2.3) we can therefore express the matrix $(\langle \rho, D_G(\Gamma_{u_r}) \rangle)$ as a product where one factor is the matrix $(\rho(u_r))$. It follows that the latter also has full rank.

Next we consider the matrix of all values $\chi_{A,\varphi}(u_r)$ where $A \in \hat{G}_{s,\mathcal{F}}$ and $1 \leq r \leq d$. Using Theorem 7.1 we deduce that also this matrix has full rank.

Finally, consider for each $A \in \hat{G}_{s,\mathcal{F}}$ the equation in Proposition 4.10 expressing the values of $\chi_{A,\varphi}$ on $u_1, \dots, u_d$ in terms of our unknown coefficients $m_{A,i}$. Since the matrix of values $Y_i(u_r)$ (where $1 \leq r \leq d$ and $i \in \mathcal{N}_G$ with $C_i = C$) is invertible, we finally conclude that the matrix that we are interested in, namely that of all coefficients $m_{A,i}$ where $A \in \hat{G}_{s,\mathcal{F}}$ and $i \in \mathcal{N}_G$ with $C_i = C$, has full rank, and we are done.

It remains to consider the case where $A(u) \cong \mathfrak{S}_4$ or $A(u) \cong \mathfrak{S}_5$. Then G is of type F_4 or E_8, respectively, and s must be central. Take any $E \in \mathcal{F}$. Then, by definition (note that s is central), the restriction of $R_s(E)$ to G_{uni}^F equals Q_E, with Q_E as in (6.2). Hence we have

$$R_s(E)|_{C^F} = Q_E|_{C^F} = q^{\dim \mathcal{B}_u} Y_{i_E}|_{C^F},$$

where $E \mapsto i_E$ denotes the Springer correspondence, see (6.2)(a). From the tables in [4, p.428–433] we see that all pairs $i \in \mathcal{N}_G$ with $C_i = C$ except one arise in this way. The exceptional pair is just the cuspidal pair $i_0 \in \mathcal{N}_G$, see (3.5). Let $A_0 \in \hat{G}_{s,\mathcal{F}}$ be the cuspidal character sheaf such that $m_{A_0,i_0} \neq 0$, which exists by [14, (21.1)(a) and (21.3)(a)]. We can now conclude that the matrix of values of the functions $\{\chi_{A_0,\varphi_0}; R_s(E), E \in \mathcal{F}\}$ on $u_1, \dots, u_d$ is invertible. Each of these functions can be written as a linear combination of characters in $\mathcal{E}_{s,\mathcal{F}}$, by (6.1) and Theorem 7.1. Hence the matrix of values $\rho(u_r)$, where $\rho \in \mathcal{E}_{s,\mathcal{F}}$ and $1 \leq r \leq d$, has full rank. We can then proceed as before to complete the proof. $\square$

(7.3) Assume that the hypotheses of Theorem 4.5 are satisfied. Consider the matrix of all coefficients $m_{A,i}$ where $A \in \hat{G}_{s,\mathcal{F}}$ and $i \in \mathcal{N}_G$ with $C_i = C$. We want to show that each row and each column of this matrix contains exactly one non-zero entry, and this non-zero entry should be 1. Our main tool to do this is the formula in (5.7)(a).

We write $\text{Irr}(A(u)) = \{\psi_r \mid r = 1, 2, \dots, d\}$ and choose the labelling so that $\psi_1(1) \leq \psi_2(1) \leq \dots \leq \psi_d(1)$. For each pair $(x, \sigma) \in \mathcal{M}(\mathcal{G}_\mathcal{F})$ let $e(x, \sigma) := [\mathcal{G}_\mathcal{F} : C_{\mathcal{G}_\mathcal{F}}(x)]\sigma(1)$. We then write $\mathcal{M}(\mathcal{G}_\mathcal{F}) = \{(x_t, \sigma_t) \mid t = 1, 2, \dots\}$ so that $e(x_1, \sigma_1) \leq e(x_2, \sigma_2) \leq \dots$. The main (and maybe strange) observation is now that the first d numbers in this list are exactly the same as $\psi_1(1), \psi_2(1), \dots, \psi_d(1)$, i.e., we have

(a) $\psi_1(1) = e(x_1, \sigma_1), \dots, \psi_d(1) = e(x_d, \sigma_d).$

(Note that $\mathcal{M}(\mathcal{G}_\mathcal{F})$ contains at least d elements.) Indeed, if $A(u)$ is abelian this is trivial. The only remaining cases are when $A(u)$ is isomorphic to $\mathfrak{S}_3$, $\mathfrak{S}_4$ or $\mathfrak{S}_5$. In each of these cases, the observation is easily checked by inspection.

If $i = (C, \psi_r)$ and if $A \in \hat{G}_{s,\mathcal{F}}$ corresponds to $(x_t, \sigma_t) \in \mathcal{M}(\mathcal{G}_\mathcal{F})$ under the bijection (4.1)(a), we simply write $m_{t,r}$ instead of $m_{A,i}$. The formula in (5.7)(a) now reads:

(b) $\psi_r(1) = e(x_1, \sigma_1)m_{1,r} + e(x_2, \sigma_2)m_{2,r} + \dots$

In particular, this implies that

(c) $\psi_r(1) \geq e(x_t, \sigma_t)m_{t,r} \quad$ for all t.

With this notation, Theorem 4.5 is equivalent to the statement that the above ordering of $\mathcal{M}(\mathcal{G}_\mathcal{F})$ can be chosen so that

$$m_{t,r} = \begin{cases} 1 & \text{if } r = t, \\ 0 & \text{if } r \neq t. \end{cases}$$

To prove this statement we proceed by induction on r. We certainly have $\psi_1(1) = 1$, by the above ordering of $\mathrm{Irr}(A(u))$. So the left hand side in (b) is just 1. Hence there is precisely one t_1 such that $m_{t_1,r} \neq 0$, and we have in fact $m_{t_1,r} = 1$. We are certainly allowed to reorder the elements in $\mathcal{M}(\mathcal{G}_\mathcal{F})$ so that $t_1 = 1$. Hence the case $r = 1$ is all right. Now let $r > 1$ and assume that we have reordered the set $\mathcal{M}(\mathcal{G}_\mathcal{F})$ so that the first $r - 1$ columns have non-zero entries only in the first $r - 1$ rows. Since our matrix has full rank, there must exist some $t \geq r$ with $m_{t,r} \neq 0$. Since $t \geq r$ we have $e(x_t, \sigma_t) \geq e(x_r, \sigma_r) = \psi_r(1)$, where the equality comes from (a). Since $m_{t,r} \neq 0$, the inequality (c) forces that $e(x_r, \sigma_r) = e(x_t, \sigma_t)$. Hence we may interchange the r-th and t-th row, so that now $m_{r,r} \neq 0$. Having done this reordering, it remains to look once more at equation (b); combining it with (a) it now implies $\psi_r(1) \geq e(x_r, \sigma_r) m_{r,r} = \psi_r(1) m_{r,r}$. Hence we must have equality which means that $m_{r,r} = 1$ and $m_{t,r} = 0$ for all $t \neq r$. This completes the proof of Theorem 4.5.

REFERENCES

[1] D. ALVIS, The duality operation in the character ring of a finite Chevalley group, Bull. Amer. Math. Soc. **1** (1979), 907–911.

[2] A.M. AUBERT, Some properties of character sheaves, preprint ENS, Paris (1996).

[3] W.M. BEYNON AND N. SPALTENSTEIN, Green functions of finite Chevalley groups of type E_n ($n = 6, 7, 8$), J. of Algebra **88** (1984), 584–614.

[4] R.W. CARTER, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley, New York (1985).

[5] F. DIGNE, G.I. LEHRER AND J. MICHEL, On the Gelfand-Graev characters of reductive groups with disconnected centre, preprint (1996).

[6] M. GECK, Basic sets of Brauer characters of finite groups of Lie type, III. Manuscripta Math. **85** (1994), 195–216.

[7] M. GECK, On the average values of the irreducible characters of finite groups of Lie type on geometric unipotent classes, Documenta Math. **1** (1996), 293–317.

[8] M. GECK, G. HISS, F. LÜBECK, G. MALLE, AND G. PFEIFFER, CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, AAECC **7** (1996), 175–210.

[9] M. GECK AND G. MALLE, On the existence of a unipotent support for the irreducible characters of finite groups of Lie type, submitted to Trans. AMS.

[10] N. KAWANAKA, Generalized Gelfand-Graev representations of exceptional algebraic groups I, Invent. Math. **84** (1986), 575–616.

[11] N. KAWANAKA, Shintani lifting and Gelfand-Graev representations, Proc. Symp. Pure Math. **47** (1987), 147–163, Amer. Math. Soc.

[12] G. LUSZTIG, Characters of reductive groups over a finite field, Annals Math. Studies **107**, Princeton University Press (1984).

[13] ――――, Intersection cohomology complexes on a reductive group, Invent. Math. **75**, 205–272 (1984).

[14] _____, Character sheaves, Advances in Math. **56**, 193–237 (1985); II, **57**, 226–265 (1985); III, **57**, 266–315 (1985); IV, **59**, 1–63 (1986); V, **61**, 103–155 (1986).

[15] _____, On the character values of finite Chevalley groups at unipotent elements, J. of Algebra, **104**, 146–194 (1986).

[16] _____, Introduction to character sheaves, Proc. Symp. Pure Math. **47** (1987), 165–179, Amer. Math. Soc.

[17] _____, A unipotent support for irreducible representations, Advances in Math. **94**, 139–179 (1992).

[18] _____, Remarks on computing irreducible characters, J. Amer. Math. Soc. **5**, 971–986 (1992).

[19] T. SHOJI, On the Green polynomials of classical groups, Invent. Math. **74** (1983), 237–267.

[20] _____, Green functions of reductive groups over a finite field, Proc. Symp. Pure Math. **47** (1987), 289–302, Amer. Math. Soc.

[21] _____, Character sheaves and almost characters of reductive groups, Advances in Math. **111** (1995), 244–313; II, **111** (1995), 314–354.

UFR DE MATHÉMATIQUES AND UMR 9994 DU CNRS, UNIVERSITÉ PARIS 7, 2 PLACE JUSSIEU, F–75251 PARIS.