Cyclotomic q–Schur algebras

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1 Introduction

Recently, we [**9**] and, independently, Du and Scott [**11**], defined an analogue of the q–Schur algebra [**6, 7**] for an Iwahori–Hecke algebra of type **B**. In this paper we study an analogue of the q–Schur algebra for an arbitrary Ariki–Koike algebra.

The Ariki–Koike algebra $\mathscr{H}$ is a cyclotomic algebra of type $G(r,1,n)$ [**2**], and it becomes the Iwahori–Hecke algebra of type **A** or **B** when $r = 1$ or 2 respectively. By working over a ring R which contains a primitive rth root of unity, and by specializing the parameters appropriately, the Ariki–Koike algebra turns into the group algebra $R(C_r \wr \mathfrak{S}_n)$ of the wreath product of the cyclic group C_r of order r with the symmetric group $\mathfrak{S}_n$ of degree n.

For each multicomposition λ of n, we construct a right ideal M^λ of $\mathscr{H}$ (see Definition 3.8). The cyclotomic q–Schur algebra is then defined to be $\mathscr{S} = \operatorname{End}_{\mathscr{H}}(\oplus_\lambda M^\lambda)$. (Under the specialization above where $\mathscr{H} \cong R(C_r \wr \mathfrak{S}_n)$, the module M^λ becomes a module induced from a subgroup of the form $(C_r \times \cdots \times C_r) \rtimes \mathfrak{S}_\lambda$.)

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In this paper we construct a cellular basis for the cyclotomic q–Schur algebra. As a consequence we obtain a Weyl module W^λ for each multipartition λ of n. We show that W^λ has simple head F^λ and that the set $\{F^\lambda\}$, as λ ranges over the multipartitions of n, is a complete set of non–isomorphic irreducible $\mathscr{S}$–modules. Using the cellular structure of $\mathscr{S}$, it is now easy to see that the cyclotomic q–Schur algebra is quasi–hereditary.

In order to prove these results about the cyclotomic q–Schur algebra, we need to examine the ideals M^λ in some detail. Using the cellular structure of the Ariki–Koike algebra $\mathscr{H}$ (cf. [**10, 13**]), we obtain a basis of M^λ and a special series of submodules of M^λ, known as a Specht series. From the Specht series of M^λ we construct the cellular basis of $\mathscr{S}$.

2 The Ariki–Koike algebra

Throughout this paper, r and n will be fixed positive integers with $r \geq 1$ and $n \geq 1$.

Let R be a commutative ring with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. The **Ariki–Koike algebra** $\mathscr{H}$ is the associative unital R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ subject to the following relations

$$\begin{aligned} (T_0 - Q_1)\cdots(T_0 - Q_r) &= 0 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\ (T_i + 1)(T_i - q) &= 0 && \text{for } 1 \leq i \leq n-1 \\ T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i && \text{for } 1 \leq i \leq n-2 \\ T_iT_j &= T_jT_i && \text{for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

Suppose that $\pi \in R$ is a primitive rth root of 1 and that $q = 1$ and $Q_k = \pi^k$ for $k = 1, 2, \dots, r$. Then it follows from the definition that $\mathscr{H}$ is isomorphic to $R(C_r \wr \mathfrak{S}_n)$.

Let $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \dots, n\})$ be the symmetric group on $\{1, 2, \dots, n\}$ and let $s_i = (i, i+1)$ for $1 \leq i < n$. Then $s_1, s_2, \dots, s_{n-1}$ are the standard Coxeter generators of $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then a word $w = s_{i_1} \dots s_{i_k}$ for w is a **reduced expression** for w if k is minimal; in this case we say that w has **length** k and write $\ell(w) = k$. Given a reduced expression $s_{i_1} \dots s_{i_k}$ for $w \in \mathfrak{S}_n$, let $T_w = T_{i_1} \dots T_{i_k}$; the relations in $\mathscr{H}$ ensure that T_w is independent of the choice of reduced expression. We denote by $\mathscr{H}(\mathfrak{S}_n)$ the subalgebra of $\mathscr{H}$ generated by $T_1, T_2, \dots, T_{n-1}$. Then $\mathscr{H}(\mathfrak{S}_n)$ has basis $\{\, T_w \,|\, w \in \mathfrak{S}_n \,\}$, and it is isomorphic to an Iwahori–Hecke algebra of type **A**.

Define elements $L_m = q^{1-m}T_{m-1} \dots T_1T_0T_1 \dots T_{m-1}$ for $m = 1, 2, \dots, n$; these are analogues of the q–Murphy operators of the Iwahori–Hecke algebras of type **A** [**5, 15**]. An easy calculation using the relations in $\mathscr{H}$ (cf. [**5**, (2.1), (2.2)]) shows that we have the following results.

(2.1) Suppose that $1 \leq i \leq n-1$ and $1 \leq m \leq n$. Then

(i) L_i and L_m commute.

(ii) T_i and L_m commute if $i \neq m-1, m$.

(iii) T_i commutes with L_iL_{i+1} and $L_i + L_{i+1}$.

(iv) If $a \in R$ and $i \neq m$ then T_i commutes with $(L_1 - a)(L_2 - a)\dots(L_m - a)$.

Using the elements T_w and L_m defined above, Ariki and Koike gave a basis for $\mathscr{H}$ as follows.

(2.2) Theorem (Ariki–Koike [1, (3.10)]) *The algebra $\mathscr{H}$ is a free R–module with basis*

$$\{\, L_1^{c_1} L_2^{c_2} \dots L_n^{c_n} T_w \mid w \in \mathfrak{S}_n \quad and \quad 0 \le c_m \le r-1 \quad for \quad m = 1, 2, \dots, n \,\}.$$

In particular, $\mathscr{H}$ is free of rank $r^n n!$

(2.3) Let $*$ be the R–linear antiautomorphism of $\mathscr{H}$ determined by $T_i^* = T_i$ for all i with $0 \le i \le n-1$. Then $T_w^* = T_{w^{-1}}$ and $L_m^* = L_m$ for all $w \in \mathfrak{S}_n$ and $m = 1, 2, \dots, n$.

We therefore have the following result.

(2.4) $\{\, T_w L_1^{c_1} L_c^{c_2} \dots L_n^{c_n} \mid w \in \mathfrak{S}_n \quad \text{and} \quad 0 \le c_m \le r-1 \quad \text{for} \quad m = 1, 2, \dots, n \,\}$ is a basis of $\mathscr{H}$.

3 A cellular basis of $\mathscr{H}$

In their paper [**12**], which introduced the concept of cellular algebras, Graham and Lehrer gave a cellular basis of $\mathscr{H}$, using the Kazhdan–Lusztig basis of $\mathscr{H}(\mathfrak{S}_n)$. We require a different cellular basis, namely one similar to the basis of $\mathscr{H}(\mathfrak{S}_n)$ introduced by Murphy [**16**]. Although the construction of the cellular basis of $\mathscr{H}$ in this section is similar to that in [**10, 13**], we are obliged to keep track of new information concerned with the cellular basis (see Corollary 3.24 below).

Consider the R–submodule of $\mathscr{H}$ which is spanned by

$$\{\, L_1^{c_1} L_2^{c_2} \dots L_n^{c_n} \mid 0 \le c_i \le r-1 \quad \text{for} \quad 1 \le i \le n \,\}.$$

When $q = 1$, this is a subalgebra of $\mathscr{H}$, but one of the main difficulties of working with the Ariki–Koike algebra is that it is not a subalgebra in general. (To see this, consider L_2^2 when $r = 2$.) We shall need certain elements $u_{\mathbf{a}}^+$ of this R–submodule of $\mathscr{H}$, and we introduce these now.

(3.1) Definition *Suppose that $\mathbf{a} = (a_1, \dots, a_r)$ is an r–tuple of integers a_i such that $0 \le a_i \le n$ for all i. Let $u_{\mathbf{a}}^+ = u_{\mathbf{a},1} u_{\mathbf{a},2} \dots u_{\mathbf{a},r}$ where*

$$u_{\mathbf{a},k} = \prod_{m=1}^{a_k} (L_m - Q_k) \quad for \quad 1 \le k \le r.$$

(3.2) Remarks (i) Suppose that every a_k is non–zero. Then $(L_1 - Q_k)$ is a factor of each $u_{\mathbf{a},k}$; so $u_\mathbf{a}^+$ has a factor

$$\prod_{k=1}^{r}(L_1 - Q_k) = \prod_{k=1}^{r}(T_0 - Q_k) = 0.$$

Therefore, $u_\mathbf{a}^+$ is zero in this case.

(ii) Rearranging the order of $a_1, a_2, \dots, a_r$ amounts just to reordering the parameters $Q_1, Q_2, \dots, Q_r$. For example, if we define $u_\mathbf{a}^- = u^+_{(a_r, a_{r-1}, \dots, a_1)}$ then $u_\mathbf{a}^-$ is obtained from $u_\mathbf{a}^+$ by replacing Q_k by Q_{r-k+1} for $1 \le k \le r$.

(iii) In practice, we shall use $u_\mathbf{a}^+$ only for r–tuples $\mathbf{a} = (a_1, a_2, \dots, a_r)$ such that $0 = a_1 \le a_2 \le \dots \le a_r \le n$. Our last two remarks show that there is no loss in doing this, and that we could equally well work with the elements $u_\mathbf{a}^-$ defined in Remark (ii).

(3.3) Example Suppose that $r = 4$, $n \ge 5$ and $\mathbf{a} = (0, 2, 4, 5)$. Then

$$\begin{aligned} u_\mathbf{a}^+ \;=\; & (L_1 - Q_2)(L_2 - Q_2) \\ & \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \\ & \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)(L_5 - Q_4). \end{aligned}$$

Our first lemma relates $u_\mathbf{a}^+$ and $u_\mathbf{b}^+$ when $\mathbf{b}$ is obtained from $\mathbf{a}$ by increasing a single part by one.

(3.4) Lemma *Let* $\mathbf{a} = (a_1, a_2, \dots, a_r)$ *and assume that* $1 \le k \le r$ *and* $a_k + 1 \le n$. *Let* $\mathbf{b} = (a_1, \dots, a_{k-1}, a_k + 1, a_{k+1}, \dots, a_r)$. *Then, for some* $h_1, h_2 \in \mathscr{H}(\mathfrak{S}_n)$, *we have* $u_\mathbf{a}^+ T_{a_k} T_{a_k - 1} \dots T_1 T_0 = u_\mathbf{a}^+ h_1 + u_\mathbf{b}^+ h_2$.

PROOF: The definition of $u_\mathbf{b}^+$ gives $u_\mathbf{b}^+ = u_\mathbf{a}^+(L_{a_k+1} - Q_k)$. Hence,

$$u_\mathbf{a}^+ T_{a_k} T_{a_k-1} \dots T_1 T_0 T_1 \dots T_{a_{k-1}} T_{a_k} = q^{a_k} u_\mathbf{a}^+ L_{a_k+1} = q^{a_k} Q_k u_\mathbf{a}^+ + q^{a_k} u_\mathbf{b}^+.$$

The desired result follows by postmultiplying by $T_{a_k}^{-1} \dots T_1^{-1}$. □

We next turn our attention to the subalgebra $\mathscr{H}(\mathfrak{S}_n)$ of $\mathscr{H}$. Here we shall need the notation and combinatorics of multipartitions.

A **composition** $\alpha = (\alpha_1, \alpha_2, \dots)$ is a finite sequence of non–negative integers; we denote by $|\alpha|$ the sum of this sequence. A **multicomposition** of n is an ordered r–tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{k=1}^{r} |\lambda^{(k)}| = n$. We call $\lambda^{(k)}$ the kth **component** of λ. A **partition** is a composition whose parts are non–increasing; a multicomposition is a **multipartition** if all its components are partitions.

For each composition $\alpha = (\alpha_1, \alpha_2, \dots)$ with $|\alpha| = m$ we have a Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots$ of $\mathfrak{S}_m$. Similarly, to each multicomposition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of n we associate the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \mathfrak{S}_{\lambda^{(2)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ of $\mathfrak{S}_n$.

We are now in a position to define certain key elements m_λ of $\mathscr{H}$. The element m_λ depends upon the Young subgroup $\mathfrak{S}_\lambda$ and also involves one of the elements $u_\mathbf{a}^+$ defined above.

(3.5) Definition *Suppose that λ is a multicomposition of n and define $\mathbf{a} = (a_1, a_2, \dots, a_r)$ by $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$. Let $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$ and set $m_\lambda = u_{\mathbf{a}}^+ x_\lambda$.*

(3.6) Example Suppose that $\lambda = ((2,1),(1^2),(2))$. Then $\mathfrak{S}_\lambda = \mathfrak{S}_2 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_2$, $\mathbf{a} = (0,3,5)$, and

$$\begin{aligned} m_\lambda &= (L_1 - Q_2)(L_2 - Q_2)(L_3 - Q_2) \\ &\quad \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3)(L_5 - Q_3) \\ &\quad \times (1 + T_1)(1 + T_6). \end{aligned}$$

(3.7) Remark If $\alpha = (|\lambda^{(1)}|, |\lambda^{(2)}|, \dots, |\lambda^{(r)}|)$ then all of the elements in $\mathscr{H}(\mathfrak{S}_\alpha)$ commute with $u_{\mathbf{a}}^+$ by (2.1)(iv). In particular, $m_\lambda = u_{\mathbf{a}}^+ x_\lambda = x_\lambda u_{\mathbf{a}}^+$. Hence, $m_\lambda^* = m_\lambda$, where $*$ is the antiautomorphism of (2.3).

The $\mathscr{H}$–modules which will be our main concern in this paper are the right ideals generated by the m_λ, as λ varies over the multicompositions of n. The cyclotomic q–Schur algebra will be built from endomorphisms between such right ideals.

(3.8) Definition *Suppose that λ is a multicomposition of n. Let $M^\lambda = m_\lambda \mathscr{H}$.*

We leave the proof of the following remarks to the reader.

(3.9) Remarks (i) If the multicomposition μ is obtained from λ by reordering the parts in each component then $M^\mu \cong M^\lambda$.

(ii) Suppose that $q = 1$ and that $Q_k = \pi^k$, for $k = 1, 2, \dots, r$, where π is a primitive rth root of unity in R. Then $\mathscr{H} \cong R(C_r \wr \mathfrak{S}_n)$. Let $\alpha_k = |\lambda^{(k)}|$ for $1 \le k \le r$. Then M^λ is induced from a module U for the subgroup $(C_r^{\alpha_1} \times C_r^{\alpha_2} \times \cdots \times C_r^{\alpha_r}) \rtimes \mathfrak{S}_\lambda$. The restriction of U to the subgroup $C_r^{\alpha_1} \times \cdots \times C_r^{\alpha_r}$ has the form $U_1^{\otimes \alpha_1} \otimes \cdots \otimes U_r^{\otimes \alpha_r}$ where U_k has rank k for $1 \le k \le r$. The restriction of U to $\mathfrak{S}_\lambda$ is the trivial module.

We shall construct a basis of M^λ, and study $\mathscr{H}$–homomorphisms between the various modules M^λ. To this end, we introduce λ–tableaux.

The diagram of a composition $\alpha = (\alpha_1, \alpha_2, \dots)$ is $\{ (i,j) \mid 1 \le i \text{ and } 1 \le j \le \alpha_i \}$, which we regard as an array of nodes, or boxes, in the plane. The **diagram** of a multicomposition is the ordered r–tuple of the diagrams of its components.

Let λ be a multicomposition of n. A λ**–tableau** $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ is obtained from the diagram of λ by replacing each node by one of the integers $1, 2, \dots, n$, allowing no repeats. We call the tableaux $\mathfrak{t}^{(k)}$ the **components** of $\mathfrak{t}$.

(3.10) Definition (i) *A λ–tableau is* **row standard** *if the entries in each row of each component increase from left to right.*

(ii) *A λ–tableau $\mathfrak{t}$ is* **standard** *if λ is a multipartition of n, $\mathfrak{t}$ is row standard and the entries in each column of each component of $\mathfrak{t}$ increase from top to bottom.*

(iii) *If λ is a multipartition of n, then let* $\operatorname{Std}(\lambda)$ *be the set of standard λ–tableaux.*

Note, particularly, that while row standard λ–tableaux are defined for all multicompositions λ, there exist standard λ–tableaux only if λ is a multipartition of n.

We require partial orders on the set of multicompositions and on the set of row standard tableaux.

If $\mathfrak{t}$ is a row standard λ–tableau and $1 \leq m \leq n$, then the entries $1, 2, \ldots, m$ in $\mathfrak{t}$ occupy the diagram of a multicomposition; let $\mathfrak{t} \downarrow m$ denote this multicomposition. For example, $\mathfrak{t} \downarrow n = \lambda$. We use this notation in our next definition.

(3.11) Definition *Suppose that* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ *and* $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ *are multicompositions of* n.

(i) *We say that* λ **dominates** μ, *and write* $\lambda \trianglerighteq \mu$, *if*

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda_i^{(k)} \geq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{i=1}^{j} \mu_i^{(k)}$$

for all k *and* j *with* $1 \leq k \leq r$ *and* $j \geq 0$. *If* $\lambda \trianglerighteq \mu$ *and* $\lambda \neq \mu$ *then we write* $\lambda \triangleright \mu$.

(ii) *Suppose that* $\mathfrak{s}$ *is a row standard* λ*–tableau and that* $\mathfrak{t}$ *is a row standard* μ*–tableau. We say that* $\mathfrak{s}$ **dominates** $\mathfrak{t}$, *and write* $\mathfrak{s} \trianglerighteq \mathfrak{t}$ *if* $\mathfrak{s} \downarrow m \trianglerighteq \mathfrak{t} \downarrow m$ *for all* m *with* $1 \leq m \leq n$. *If* $\mathfrak{s} \trianglerighteq \mathfrak{t}$ *and* $\mathfrak{s} \neq \mathfrak{t}$ *then we write* $\mathfrak{s} \triangleright \mathfrak{t}$.

For example, if $n = r = 2$, then the multipartitions of 2 are ordered by $((2), (0)) \triangleright ((1^2), (0)) \triangleright ((1), (1)) \triangleright ((0), (2)) \triangleright ((0), (1^2))$.

Note that if $\mathfrak{s}$ is a row standard λ–tableau and $\mathfrak{t}$ is a row standard μ tableau such that $\mathfrak{s} \trianglerighteq \mathfrak{t}$ then $\lambda \trianglerighteq \mu$.

Our next definition gives another relation between tableaux.

(3.12) Definition *Suppose that* $\mathfrak{s}$ *is a tableau and that* $1 \leq j \leq n$. *We write* $\operatorname{comp}_{\mathfrak{s}}(j) = k$ *if* j *appears in the* k*th component* $\mathfrak{s}^{(k)}$ *of* $\mathfrak{s}$.

Suppose that $\mathfrak{t}$ *is another tableau. Then* $\operatorname{comp}_{\mathfrak{s}} = \operatorname{comp}_{\mathfrak{t}}$ *if* $\operatorname{comp}_{\mathfrak{s}}(j) = \operatorname{comp}_{\mathfrak{t}}(j)$ *for all* j *with* $1 \leq j \leq n$. *We also write* $\operatorname{comp}_{\mathfrak{s}} \geq \operatorname{comp}_{\mathfrak{t}}$ *if* $\operatorname{comp}_{\mathfrak{s}}(j) \geq \operatorname{comp}_{\mathfrak{t}}(j)$ *for all* $1 \leq j \leq n$; *and* $\operatorname{comp}_{\mathfrak{s}} > \operatorname{comp}_{\mathfrak{t}}$ *if* $\operatorname{comp}_{\mathfrak{s}} \geq \operatorname{comp}_{\mathfrak{t}}$ *and* $\operatorname{comp}_{\mathfrak{s}} \neq \operatorname{comp}_{\mathfrak{t}}$.

(3.13) Example Let

$$\mathfrak{s} = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 3 & 8 & 9 \\ \hline 2 & 7 \\ \cline{1-2} \end{array} \right), \quad \mathfrak{t} = \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 7 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 6 & 8 \\ \hline 5 & 9 \\ \cline{1-2} \end{array} \right) \quad \text{and} \quad \mathfrak{u} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 7 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 9 \\ \hline 8 \\ \cline{1-1} \end{array} \right).$$

Then $\mathfrak{s}$ is row standard, and $\mathfrak{t}$ and $\mathfrak{u}$ are standard. We have $\mathfrak{u} \triangleright \mathfrak{s}$ and $\mathfrak{u} \triangleright \mathfrak{t}$ but $\mathfrak{s}$ and $\mathfrak{t}$ are incomparable in the dominance order. Also, $\operatorname{comp}_{\mathfrak{t}} > \operatorname{comp}_{\mathfrak{u}}$, but there are no other equations or inequalities between $\operatorname{comp}_{\mathfrak{s}}$, $\operatorname{comp}_{\mathfrak{t}}$ and $\operatorname{comp}_{\mathfrak{u}}$.

Let λ be a multicomposition of n. The symmetric group $\mathfrak{S}_n$ acts from the right on the set of λ–tableaux by permuting the entries in each tableau. Let $\mathfrak{t}^\lambda$ be the λ–tableau where $1, 2, \ldots, n$ appear in order along the rows of the first component, and then along the rows of the second component, and so on. The row stabilizer of $\mathfrak{t}^\lambda$ is the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$. For example, if $\lambda = ((3,2),(1^2),(3))$ then

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \; \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline \end{array}, \; \begin{array}{|c|c|c|}\hline 8 & 9 & 10 \\ \hline \end{array} \right).$$

For a row standard λ–tableau $\mathfrak{s}$, let $d(\mathfrak{s})$ be the element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$. Then $d(\mathfrak{s})$ is a distinguished right coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$ and we obtain, in this way, a correspondence between the set of row standard λ–tableaux and the set of right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

Recall the antiautomorphism $*$ of $\mathscr{H}$ from (2.3).

(3.14) Definition *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Let $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}$.*

Note that $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$. Also, $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$ (cf. Remark 3.7).

One of the aims of this section is to show that the elements $m_{\mathfrak{s}\mathfrak{t}}$, as $(\mathfrak{s},\mathfrak{t})$ varies over the ordered pairs of standard tableaux of the same shape, give a cellular basis of $\mathscr{H}$. We shall also establish useful properties of the right ideals M^λ. Initially, though, we concentrate upon the two–sided ideal generated by m_λ.

(3.15) Lemma *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Let $h \in \mathscr{H}(\mathfrak{S}_n)$. Then $m_{\mathfrak{s}\mathfrak{t}} h$ is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{v}}$ where each $\mathfrak{v}$ is a row standard λ–tableau.*

PROOF: Suppose that $w \in \mathfrak{S}_n$. Then there exist $y \in \mathfrak{S}_\lambda$ and a distinguished right coset representative d of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$ such that $w = yd$ and $\ell(w) = \ell(y) + \ell(d)$. Hence, $x_\lambda T_w = x_\lambda T_y T_d = q^{\ell(y)} x_\lambda T_d$. If $m_\lambda = u^+_{\mathbf{a}} x_\lambda$ then

$$m_{\mathfrak{s}\mathfrak{t}^\lambda} T_w = T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_w = q^{\ell(y)} T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_d = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{v}}$$

where the tableau $\mathfrak{v} = \mathfrak{t}^\lambda d$ is row standard.

Now, $m_{\mathfrak{s}\mathfrak{t}} h = m_{\mathfrak{s}\mathfrak{t}^\lambda} T_{d(\mathfrak{t})} h$, and this is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{t}^\lambda} T_w$ with $w \in \mathfrak{S}_n$. Therefore, the required result follows. □

In order to apply a result of Murphy in Proposition 3.18 below, we require a combinatorial lemma which concerns the dominance order on row standard tableaux. A preliminary definition sets the scene.

(3.16) Definition *We say that a tableau $\mathfrak{s}$ is of the* initial kind *(for λ) if* $\mathrm{comp}_{\mathfrak{s}} = \mathrm{comp}_{\mathfrak{t}^\lambda}$.

Note that a μ–tableau $\mathfrak{s}$ can be of the initial kind for λ even though $\mu \ne \lambda$.

(3.17) Lemma *Suppose that* $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ *is a multicomposition of* n*, and let* $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$*. Suppose that* w *is a distinguished right coset representative of* $\mathfrak{S}_\alpha$ *in* $\mathfrak{S}_n$ *and that* $\mathfrak{s}$ *is a row standard* λ*–tableau of the initial kind. Then the following hold.*

(i) *The tableau* $\mathfrak{s}w$ *is row standard.*
(ii) *If* $\mathfrak{s}$ *is standard then* $\mathfrak{s}w$ *is standard.*
(iii) *If* $\mathfrak{t}$ *is a row standard* λ*–tableau of the initial kind with* $\mathfrak{s} \rhd \mathfrak{t}$ *then* $\mathfrak{s}w \rhd \mathfrak{t}w$*.*

PROOF: The fact that w is a distinguished right coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ implies that whenever x and y are integers such that x and y are in the same component of $\mathfrak{s}$ and $x < y$, then $xw < yw$. Hence, (i) and (ii) are true.

Now assume that $\mathfrak{t}$ is a row standard λ–tableau of the initial kind, and that $\mathfrak{s} \rhd \mathfrak{t}$. Then $d(\mathfrak{s}) \rhd d(\mathfrak{t})$ in the Bruhat–Chevalley order $\rhd$ on $\mathfrak{S}_n$, by [**4**, (1.3)]. (Our notation for the order is such that $1 \unrhd v$ for all $v \in \mathfrak{S}_n$.) Since w is a distinguished right coset representative of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$, we also have $\ell(d(\mathfrak{t})w) = \ell(d(\mathfrak{t})) + \ell(w)$. The well–known "cancellation" property of the Bruhat–Chevalley order now lets us conclude that $d(\mathfrak{s})w \rhd d(\mathfrak{t})w$. But $\mathfrak{s}w$ and $\mathfrak{t}w$ are row standard, so $d(\mathfrak{s})w = d(\mathfrak{s}w)$ and $d(\mathfrak{t})w = d(\mathfrak{t}w)$. By applying [**4**, (1.3)] again, we deduce that $\mathfrak{s}w \rhd \mathfrak{t}w$. □

(3.18) Proposition *Suppose that* λ *is a multicomposition of* n *and that* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are row standard* λ*–tableaux. Then* $m_{\mathfrak{s}\mathfrak{t}}$ *is a linear combination of terms of the form* $m_{\mathfrak{u}\mathfrak{v}}$ *where* $\mathfrak{u}$ *and* $\mathfrak{v}$ *are standard* μ*–tableaux for some multipartition* μ *of* n*, and*

(i) $\mathfrak{u} \unrhd \mathfrak{s}$ *and* $\operatorname{comp}_\mathfrak{u} = \operatorname{comp}_\mathfrak{s}$*; and,*
(ii) $\mathfrak{v} \unrhd \mathfrak{t}$ *and* $\operatorname{comp}_\mathfrak{v} = \operatorname{comp}_\mathfrak{t}$.

PROOF: When $r = 1$, this is a Theorem of Murphy [**16**, (4.18)]. We deduce the general case from this.

Let $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$. We may write $\mathfrak{s} = \mathfrak{s}'w_1$ and $\mathfrak{t} = \mathfrak{t}'w_2$ where $\mathfrak{s}'$ and $\mathfrak{t}'$ are row standard λ–tableau of the initial kind, and w_1 and w_2 are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$.

Let $\mathbf{a} = (a_1, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$, as in the definition of m_λ. We have $m_{\mathfrak{s}\mathfrak{t}} = T^*_{w_1} m_{\mathfrak{s}'\mathfrak{t}'} T_{w_2}$ and $m_{\mathfrak{s}'\mathfrak{t}'} = T^*_{d(\mathfrak{s}')} u^+_\mathbf{a} x_\lambda T_{d(\mathfrak{t}')} = u^+_\mathbf{a} T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ since $d(\mathfrak{s}') \in \mathfrak{S}_\alpha$.

We may write $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ as a product of r commuting terms, one for each component of λ; say, $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')} = x_1 x_2 \dots x_r$, where the kth term x_k involves only elements T_w with $w \in \mathfrak{S}(\{a_k + 1, a_k + 2 \dots, a_{k+1}\})$. For example, $x_1 = T^*_{d(\mathfrak{s}'_1)} x_{\lambda^{(1)}} T_{d(\mathfrak{t}'_1)}$ where $\mathfrak{s}'_1$ is the first component of $\mathfrak{s}'$ and $\mathfrak{t}'_1$ is the first component of $\mathfrak{t}'$. By applying Murphy's result [**16**, (4.18)] to the Hecke algebra $\mathscr{H}(\mathfrak{S}_{\alpha_1})$ we may write x_1 as a linear combination of terms $T^*_{d(\mathfrak{u}'_1)} x_{\mu^{(1)}} T_{d(\mathfrak{v}'_1)}$ where $\mathfrak{u}'_1$ and $\mathfrak{v}'_1$ are standard $\mu^{(1)}$–tableaux for some partition $\mu^{(1)}$ of a_1, and $\mathfrak{u}'_1 \unrhd \mathfrak{s}'_1$ and $\mathfrak{v}'_1 \unrhd \mathfrak{t}'_1$.

We can apply the same technique for the other factors $x_2, \dots, x_r$, to conclude that $T^*_{d(\mathfrak{s}')} x_\lambda T_{d(\mathfrak{t}')}$ is a linear combination of terms of the form $T^*_{d(\mathfrak{u}')} x_\mu T_{d(\mathfrak{v}')}$ where $\mathfrak{u}'$ and $\mathfrak{v}'$ are standard μ–tableaux for some multipartition μ of n, and $\mathfrak{u}' \unrhd \mathfrak{s}'$ and $\mathfrak{v}' \unrhd \mathfrak{t}'$. Also, $\mathfrak{u}'$ and $\mathfrak{v}'$

are of the initial kind, and $|\mu^{(k)}| = |\lambda^{(k)}|$ for $1 \le k \le r$. Therefore, $m_{\mathfrak{s}'\mathfrak{t}'}$ is a linear combination of terms $m_{\mathfrak{u}'\mathfrak{v}'}$ where the sum runs over the same set of pairs $(\mathfrak{u}', \mathfrak{v}')$. Consequently, $m_{\mathfrak{st}}$ is a linear combination of terms $T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}'} T_{w_2}$.

Now, $T_{d(\mathfrak{v}')} T_{w_2} = T_{d(\mathfrak{v}')w_2} = T_{d(\mathfrak{v}'w_2)}$, so $T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}'} T_{w_2} = T^*_{w_1} m_{\mathfrak{u}'\mathfrak{v}}$ where $\mathfrak{v} = \mathfrak{v}'w_2$. By Lemma 3.17, since $\mathfrak{v}'$ is of the initial kind, $\mathfrak{v} = \mathfrak{v}'w_2$ is standard, and $\mathfrak{v}'w_2 \trianglerighteq \mathfrak{t}'w_2$; that is, $\mathfrak{v} \trianglerighteq \mathfrak{t}$. Moreover, $\operatorname{comp}_{\mathfrak{v}'w_2} = \operatorname{comp}_{\mathfrak{t}'w_2}$ since $\mathfrak{v}'$ and $\mathfrak{t}'$ are of the initial kind.

Similar remarks applied to $T^*_{w_1} T^*_{d(\mathfrak{u}')}$ now complete the proof of the Proposition. □

(3.19) Corollary *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. If $h \in \mathscr{H}(\mathfrak{S}_n)$ then $m_{\mathfrak{st}}h$ is a linear combination of terms of the form $m_{\mathfrak{uv}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n, and $\mu \trianglerighteq \lambda$, $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\operatorname{comp}_{\mathfrak{u}} = \operatorname{comp}_{\mathfrak{s}}$.*

PROOF: Combine Lemma 3.15 and Proposition 3.18, and recall that $\mathfrak{u} \trianglerighteq \mathfrak{s}$ implies that $\mu \trianglerighteq \lambda$. □

Corollary 3.19 provides the kind of information we need when we multiply $m_{\mathfrak{st}}$ by elements of $\mathscr{H}(\mathfrak{S}_n)$. More complicated is our next proposition, which shows what happens when we multiply $m_{\mathfrak{st}}$ by T_0. It is vital to the proposition that λ is a multipartition, not merely a multicomposition.

(3.20) Proposition *Suppose that λ is a multipartition of n, and that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableaux. Then $m_{\mathfrak{st}}T_0 = x_1 + x_2$ where*

(i) *x_1 is a linear combination of terms of the form $m_{\mathfrak{uv}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard λ–tableaux, with $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\operatorname{comp}_{\mathfrak{u}} = \operatorname{comp}_{\mathfrak{s}}$, and*

(ii) *x_2 is a linear combination of terms of the form $m_{\mathfrak{uv}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n, with $\mu \triangleright \lambda$ and $\operatorname{comp}_{\mathfrak{s}} > \operatorname{comp}_{\mathfrak{u}}$.*

PROOF: Let $\alpha = (|\lambda^{(1)}|, \dots, |\lambda^{(r)}|)$ and let $\mathbf{a} = (a_1, a_2, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$ for $1 \le k \le r$. Write $d(\mathfrak{t}) = yc$ with $y \in \mathfrak{S}_\alpha$ and c a distinguished right coset representative for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. Then the α–tableau $\mathfrak{t}^\alpha c$ is row standard. Assume that 1 is in row k of $\mathfrak{t}^\alpha c$; thus, $1 \le k \le r$.

Now, $c = (a_k, a_k+1)(a_k-1, a_k)\dots(1,2)c'$, where $\ell(c) = a_k + \ell(c')$ and c' fixes 1. Thus, $T_c = T_{a_k} T_{a_{k-1}} \dots T_1 T_{c'}$ and $T_{c'} T_0 = T_0 T_{c'}$. Let $\mathbf{b} = (a_1, \dots, a_k + 1, \dots, a_r)$. Then

$$u^+_{\mathbf{a}} T_c T_0 = u^+_{\mathbf{a}} T_{a_k} T_{a_{k-1}} \dots T_1 T_0 T_{c'} = u^+_{\mathbf{a}} h_1 + u^+_{\mathbf{b}} h_2$$

for some $h_1, h_2 \in \mathfrak{S}_n$ by Lemma 3.4. Since $y \in \mathfrak{S}_\alpha$ and $\mathfrak{t}^\lambda y$ is standard, y fixes $a_k + 1$; therefore, T_y commutes with $u^+_{\mathbf{b}}$. Hence, using Remark 3.7, we have

$$\begin{aligned} m_{\mathfrak{st}} T_0 &= T^*_{d(\mathfrak{s})} u^+_{\mathbf{a}} x_\lambda T_{d(\mathfrak{t})} T_0 = T^*_{d(\mathfrak{s})} x_\lambda u^+_{\mathbf{a}} T_y T_c T_0 = T^*_{d(\mathfrak{s})} x_\lambda T_y u^+_{\mathbf{a}} T_c T_0 \\ &= T^*_{d(\mathfrak{s})} x_\lambda T_y (u^+_{\mathbf{a}} h_1 + u^+_{\mathbf{b}} h_2) = T^*_{d(\mathfrak{s})} x_\lambda (u^+_{\mathbf{a}} T_y h_1 + u^+_{\mathbf{b}} T_y h_2). \end{aligned}$$

The first term, $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathfrak{a}}T_y h_1$, is a linear combination of terms of the required form by Corollary 3.19. If $k = 1$ then $u^+_{\mathfrak{b}} = 0$ by Remark 3.2(i), and the proof is complete in this case. Assume therefore that $k \geq 2$. We turn out attention to the term $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathfrak{b}}T_y h_2$. We now need to digress in order to prove that $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathfrak{b}}T_y h_2$ has the required form.

Define the multicomposition $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$ of n by $\nu^{(i)} = \lambda^{(i)}$ for $i \neq k-1, k$, $\nu^{(k-1)} = (\lambda^{(k-1)}_1, \lambda^{(k-1)}_2, \dots, 1)$ and $\nu^{(k)} = (\lambda^{(k)}_1 - 1, \lambda^{(k)}_2, \dots)$. (Thus we reduce the first part of the kth component of λ by 1, and adjoin a new part of size 1 to the end of the $(k-1)$th component of λ.) Note that $\nu \rhd \lambda$.

Let $l = \lambda^{(k)}_1$ be the first part of $\lambda^{(k)}$. The entries in the first row of the kth component of $\mathfrak{t}^\lambda$ are then $a_k + 1, a_k + 2, \dots, a_k + l$. Let G_1 be the symmetric group on these numbers, and let G_2 be the symmetric group on $a_k + 2, \dots, a_k + l$. Let $c_1, c_2, \dots, c_l$ be the distinguished right coset representatives for G_2 in G_1, ordered in terms of increasing length. Then the tableaux $\mathfrak{t}^\nu c_i$ are row standard, and they agree with $\mathfrak{t}^\nu$ except on the last row of the $(k-1)$th component and the first row of the kth component. The last row of the $(k-1)$th component of $\mathfrak{t}^\nu c_i$ contains the single entry $a_k + i$.

We are given that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$ is a row standard λ–tableau. Let $\mathfrak{u}_i = \mathfrak{t}^\nu c_i d(\mathfrak{s})$ for $1 \leq i \leq l$. Each $\mathfrak{u}_i$ agrees with $\mathfrak{s}$ except on the last row of the $(k-1)$th component of $\mathfrak{u}_i$ and on the first row of the kth component of $\mathfrak{u}_i$. Furthermore, since $(a_k+1)d(\mathfrak{s}) < \cdots < (a_k+l)d(\mathfrak{s})$, it follows that each $\mathfrak{u}_i$ is row standard.

We also have $\text{comp}_{\mathfrak{u}_i}(j) = \text{comp}_{\mathfrak{s}}(j)$ if $j \notin \{(a_k+1)d(\mathfrak{s}), \dots, (a_k+l)d(\mathfrak{s})\}$ and, for $j \in \{(a_k+1)d(\mathfrak{s}), \dots, (a_k+l)d(\mathfrak{s})\}$, we have $\text{comp}_{\mathfrak{u}_i}(j) = k-1$ or k and $\text{comp}_{\mathfrak{s}}(j) = k$. Therefore, $\text{comp}_{\mathfrak{s}} > \text{comp}_{\mathfrak{u}_i}$.

Now x_λ has a factor

$$\sum_{w \in G_1} T_w = \sum_{i=1}^{l} T^*_{c_i} \sum_{w \in G_2} T_w.$$

Hence, $x_\lambda = \sum_{i=1}^{l} T^*_{c_i} x_\nu$. Note that $m_\nu = x_\nu u^+_{\mathfrak{b}}$. Thus,

$$T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathfrak{b}} = T^*_{d(\mathfrak{s})} \sum_{i=1}^{l} T^*_{c_i} x_\nu u^+_{\mathfrak{b}} = \sum_{i=1}^{l} T^*_{c_i d(\mathfrak{s})} x_\nu u^+_{\mathfrak{b}} = \sum_{i=1}^{l} m_{\mathfrak{u}_i \mathfrak{t}^\nu}.$$

Hence, by Corollary 3.19, $T^*_{d(\mathfrak{s})}x_\lambda u^+_{\mathfrak{b}}T_y h_2$ is a linear combination of the required form.

This completes the proof of the Proposition. □

(3.21) Definition *Suppose that λ is a multicomposition of n.*

(i) *Let N^λ be the R–module spanned by*

$$\left\{ m_{\mathfrak{s}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \mu\text{–tableaux for some} \\ \text{multipartition } \mu \text{ of } n \text{ with } \mu \unrhd \lambda \end{array} \right\}.$$

(ii) *Let $\overline{N^\lambda}$ be the R–module spanned by*

$$\left\{ m_{\mathfrak{st}} \;\middle|\; \begin{array}{c} \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \mu\text{–tableaux for some} \\ \text{multipartition } \mu \text{ of } n \text{ with } \mu \rhd \lambda \end{array} \right\}.$$

We now apply Propositions 3.18 and 3.20 to obtain a sequence of useful results.

(3.22) Proposition *Suppose that λ is a multicomposition of n. Then N^λ and $\overline{N^\lambda}$ are two–sided ideals of $\mathscr{H}$.*

PROOF: Corollary 3.19 shows that N^λ is closed under postmultiplication by $T_1, T_2, \dots, T_{n-1}$ and Proposition 3.20 shows that N^λ is also closed under postmultiplication by T_0. Because $\mathscr{H}$ is generated by $T_0, T_1, \dots, T_{n-1}$, we deduce that N^λ is a right ideal of $\mathscr{H}$. Since $m_{\mathfrak{st}}^* = m_{\mathfrak{ts}}$, by applying the antiautomorphism $*$ we see that N^λ is also a left ideal of $\mathscr{H}$.

Finally, $\overline{N^\lambda}$ is a two sided ideal of $\mathscr{H}$ since $\overline{N^\lambda} = \sum_{\mu \rhd \lambda} N^\mu$. □

A proof very similar to that of Proposition 3.22 gives our next result.

(3.23) Proposition *Suppose that λ is a multicomposition of n and that $\mathfrak{s}$ is a row standard λ–tableau. Let $I_\mathfrak{s}$ be the R–module spanned by*

$$\left\{ m_{\mathfrak{uv}} \;\middle|\; \begin{array}{c} \mathfrak{u} \text{ and } \mathfrak{v} \text{ are standard } \mu\text{–tableaux for some multipartition} \\ \mu \text{ of } n \text{ with } \mu \unrhd \lambda \text{ and } \operatorname{comp}_\mathfrak{s} \geq \operatorname{comp}_\mathfrak{u} \end{array} \right\}.$$

Then $I_\mathfrak{s}$ is a right ideal of $\mathscr{H}$.

Recall that $M^\lambda = m_\lambda \mathscr{H}$.

(3.24) Corollary *Suppose that λ is a multicomposition of n. Then every element of M^λ is a linear combination of terms of the form $m_{\mathfrak{uv}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableaux for some multipartition μ of n with $\mu \unrhd \lambda$ and $\operatorname{comp}_{\mathfrak{t}^\lambda} \geq \operatorname{comp}_\mathfrak{u}$.*

PROOF: Since $m_\lambda = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$, Proposition 3.18 shows that $m_\lambda \in I_{\mathfrak{t}^\lambda}$ (note that λ may not be a multipartition). Therefore, $M^\lambda \subseteq I_{\mathfrak{t}^\lambda}$ and the desired result follows by Proposition 3.23. □

(3.25) Proposition *Suppose that λ is a multipartition of n and that $\mathfrak{t}$ is a standard λ–tableau. Let $h \in \mathscr{H}$. Then for every standard λ–tableau $\mathfrak{v}$ there exists $r_\mathfrak{v} \in R$ such that, for all standard λ–tableau $\mathfrak{s}$, we have*

$$m_{\mathfrak{st}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_\mathfrak{v} m_{\mathfrak{sv}} \bmod \overline{N^\lambda}.$$

PROOF: Let U be the R–module spanned by $\{\, m_{\mathfrak{t}^\lambda\mathfrak{t}'} \mid \mathfrak{t}' \text{ is a standard } \lambda\text{–tableau}\,\}$. Note that $\mathfrak{t}^\lambda$ is the unique λ–tableau $\mathfrak{u}$ such that $\mathfrak{u} \trianglerighteq \mathfrak{t}^\lambda$. Therefore, $U + \overline{N^\lambda}$ is closed under postmultiplication by $T_1, T_2, \dots, T_{n-1}$ by Proposition 3.18, and it is closed under postmultiplication by T_0 by Proposition 3.20. Hence $U + \overline{N^\lambda}$ is a right ideal of $\mathscr{H}$. Thus, for each $\mathfrak{v} \in \operatorname{Std}(\lambda)$ there exists $r_\mathfrak{v} \in R$ such that

$$m_{\mathfrak{t}^\lambda\mathfrak{t}} h \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\lambda)} r_\mathfrak{v} m_{\mathfrak{t}^\lambda\mathfrak{v}} \bmod \overline{N^\lambda}.$$

Multiply this congruence on the left by $T^*_{d(\mathfrak{s})}$, and use the fact that $\overline{N^\lambda}$ is an ideal, to obtain the congruence of the proposition. □

(3.26) Theorem ([**13**, (1.7)]) *The algebra $\mathscr{H}$ is a free R–module with basis*

$$\mathscr{M} = \left\{\, m_{\mathfrak{s}\mathfrak{t}} \;\middle|\; \begin{array}{c}\mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \lambda\text{–tableaux for some}\\ \text{multipartition } \lambda \text{ of } n\end{array} \right\}.$$

Moreover, $\mathscr{M}$ is a cellular basis of $\mathscr{H}$.

PROOF: Since $m_\lambda = 1$ when $\lambda = ((0), \dots, (0), (1^n))$, Proposition 3.25 shows that $\mathscr{M}$ spans $\mathscr{H}$. By Theorem 2.2, $\mathscr{H}$ is free of rank $r^n n!$ Since this is also the cardinality of $\mathscr{M}$, we deduce that $\mathscr{M}$ is a basis of $\mathscr{H}$. Finally, the properties that $\mathscr{M}$ must satisfy in order to be a cellular basis of $\mathscr{H}$ (as given in [**12**, (1.1)]), are covered by Proposition 3.25 and the fact that $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$. □

(3.27) Definition *We call $\mathscr{M}$ the* standard basis *of $\mathscr{H}$.*

We can now apply Graham and Lehrer's theory of cellular algebras [**12**] to describe the representation theory of $\mathscr{H}$.

(3.28) Definition *Suppose that λ is a multipartition of n. Let $z_\lambda = (\overline{N^\lambda} + m_\lambda)/\overline{N^\lambda}$. The* Specht module *$S^\lambda$ is the submodule of $\mathscr{H}/\overline{N^\lambda}$ given by $S^\lambda = z_\lambda \mathscr{H}$.*

As in [**12**], or directly from Proposition 3.25 and Theorem 3.26, S^λ is a free R–module with basis $\{\, z_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \text{ a standard } \lambda\text{–tableau}\,\}$.

Define a bilinear form $\langle\ ,\ \rangle$ on the Specht module S^λ by

$$m_\lambda T_{d(\mathfrak{s})} T^*_{d(\mathfrak{t})} m_\lambda \equiv \langle z_\lambda T_{d(\mathfrak{s})}, z_\lambda T_{d(\mathfrak{t})}\rangle m_\lambda \bmod \overline{N^\lambda}.$$

By Proposition 3.25 (and the version of Proposition 3.25 obtained by applying the antiautomorphism $*$), the bilinear form is well defined (cf. [**12**]). Moreover, the bilinear form is symmetric and $\langle uh, v\rangle = \langle u, vh^*\rangle$ for all $u, v \in S^\lambda$ and all $h \in \mathscr{H}$ [**12**, (2.4)]. Consequently, $\operatorname{rad} S^\lambda$, the radical of the bilinear form, is an $\mathscr{H}$–module.

(3.29) Definition *Suppose that λ is a multipartition of n. Let $D^\lambda = S^\lambda/\operatorname{rad} S^\lambda$.*

(3.30) Theorem *Suppose that R is a field. Then the non–zero $\mathscr{H}$ modules in*

$$\{\, D^\lambda \mid \lambda \text{ a multipartition of } n \,\}$$

form a complete set of non–isomorphic irreducible $\mathscr{H}$–modules. Moreover, each irreducible module D^λ is absolutely irreducible.

PROOF: Since $\mathscr{M}$ is a cellular basis of $\mathscr{H}$ the Theorem follows from the general theory of cellular algebras [**12**, (3.4)]. □

The theory in [**12**] also shows that if $D^\mu \neq 0$ and D^μ is a composition factor of S^λ then $\lambda \trianglerighteq \mu$.

When $r = 1$ the partitions μ for which $D^\mu \neq 0$ have been classified in [**4**]. When $r > 1$, if $q = 1$, or if the parameters Q_k are powers of q, the multipartitions μ for which $D^\mu \neq 0$ are classified by the results of [**12, 13, 14**]; in type **B**, see also [**8, 10**].

4 A Specht series for M^μ

In the last section, we used various elements m_μ of $\mathscr{H}$ to construct a cellular basis of $\mathscr{H}$. We saw that the right ideal M^μ generated by m_μ is an analogue of an induced module; indeed, if $r = 1$ and $q = 1$ then M^μ is the permutation module of $\mathfrak{S}_n$ on the Young subgroup $\mathfrak{S}_\mu$. We shall use the right ideals M^μ of $\mathscr{H}$ to construct the cyclotomic q–Schur algebra, but, before that, we study the individual modules M^μ in more detail. In particular, we shall give a semistandard basis of M^μ and construct a Specht series for M^μ.

We are going to define a new kind of λ–tableau. This will have entries which are ordered pairs (i, k) where i is a positive integer and $1 \le k \le r$. We denote such tableaux by capital letters.

(4.1) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. A λ–tableau S has* type *μ if its entries are ordered pairs (i, k), as above, and for all m and k the number of times (i, k) is an entry in S is $\mu_i^{(k)}$.*

Next, we introduce a function which converts standard λ–tableaux (or, indeed any tableau whose entries are $1, 2, \ldots, n$) into a λ–tableau of type μ.

(4.2) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathfrak{s}$ be a standard λ–tableau. Define $\mu(\mathfrak{s})$ to be the λ–tableau obtained from $\mathfrak{s}$ by replacing each entry m in $\mathfrak{s}$ by (i, k) if m is in row i of the kth component of $\mathfrak{t}^\mu$.*

In our examples, we shall always write i_k for the ordered pair (i, k).

(4.3) Example Suppose that $\mu = ((3,1),(2),(2,1^2))$. Then

$$t^\mu = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1} 10\\\cline{1-1}\end{array} \right) \quad\text{and}\quad \mu(t^\mu) = \left(\begin{array}{|c|c|c|}\hline 1_1&1_1&1_1\\\hline 2_1\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1_2&1_2\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1_3&1_3\\\hline 2_3\\\cline{1-1} 3_3\\\cline{1-1}\end{array} \right).$$

Suppose that

$$\mathfrak{s}_1 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 6&10\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1}\end{array} \right), \qquad \mathfrak{s}_2 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 5&10\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1}\end{array} \right)$$

and

$$\mathfrak{s}_3 = \left(\begin{array}{|c|c|c|c|}\hline 1&2&5&9\\\hline 7\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 3&10\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 4&6\\\hline 8\\\cline{1-1}\end{array} \right).$$

Then

$$\mu(\mathfrak{s}_1) = \mu(\mathfrak{s}_2) = \left(\begin{array}{|c|c|c|c|}\hline 1_1&1_1&1_1&1_2\\\hline 2_1\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1_2&3_3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1_3&1_3\\\hline 2_3\\\cline{1-1}\end{array} \right)$$

and

$$\mu(\mathfrak{s}_3) = \left(\begin{array}{|c|c|c|c|}\hline 1_1&1_1&1_2&2_3\\\hline 1_3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1_1&3_3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 2_1&1_2\\\hline 1_3\\\cline{1-1}\end{array} \right).$$

Given two pairs (i_1,k_1) and (i_2,k_2), we write $(i_1,k_1) < (i_2,k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$. Note that if $\mathfrak{s}$ is a standard λ–tableau, then $\mu(\mathfrak{s})$ is a λ–tableau of type μ whose entries are weakly increasing along rows and down columns. We next single out some of these tableaux $\mu(\mathfrak{s})$.

(4.4) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathrm{S} = (\mathrm{S}^{(1)},\dots,\mathrm{S}^{(r)})$ be a λ–tableau of type μ. Then S is* semistandard *if*

(i) *the entries in each row of each component $\mathrm{S}^{(k)}$ of S are non–decreasing; and*
(ii) *the entries in each column of each component $\mathrm{S}^{(k)}$ of S are strictly increasing; and*
(iii) *for each k with $1 \le k \le r$, no entry in $\mathrm{S}^{(k)}$ has the form (i,l) with $l < k$.*

Let $\mathcal{T}_0(\lambda,\mu)$ denote the set of semistandard λ–tableaux of type μ.

For example, $\mu(\mathfrak{s}_1)$ in Example 4.3 is semistandard, but $\mu(\mathfrak{s}_3)$ is not, since the third part of Definition 4.4 is not satisfied.

The point of the third condition in Definition 4.4 is the following result (cf. Corollary 3.24).

(4.5) Suppose that $\mathfrak{s}$ is a standard λ–tableau. Then $\mu(\mathfrak{s})$ satisfies Definition 4.4(iii) if and only if $\mathrm{comp}_{t^\mu} \ge \mathrm{comp}_{\mathfrak{s}}$

If λ is a multipartition and $\mathrm{S} \in \mathcal{T}_0(\lambda,\mu)$, then it is clear that there exists a standard λ–tableau $\mathfrak{s}$ with $\mathrm{S} = \mu(\mathfrak{s})$. In order to say more about this, we introduce another definition.

(4.6) Definition *If $\mathfrak{s}$ is a standard λ–tableau and $1 \le m \le n$ then let* $\operatorname{row}_{\mathfrak{s}}(m) = (i,k)$ *if m belongs to row i of the kth component of $\mathfrak{s}$.*

The definition of $\operatorname{row}_{\mathfrak{s}}$ allows us to recover $\mathfrak{s}$ from the function $\operatorname{row}_{\mathfrak{s}}$

Suppose that $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and let $\mathfrak{s}$ be a standard λ–tableau such that $\mu(\mathfrak{s}) = \mathtt{S}$. If there exists an integer i such that $\operatorname{row}_{\mathfrak{s}}(i) \ne \operatorname{row}_{\mathfrak{s}}(i+1)$ and i and $i+1$ are in the same row of $\mathfrak{t}^\mu$ then the tableau $\mathfrak{s}_1 = \mathfrak{s}(i,i+1)$ is standard and $\mu(\mathfrak{s}_1) = \mathtt{S}$. Also, $\mathfrak{s} \rhd \mathfrak{s}_1$ if $\operatorname{row}_{\mathfrak{s}}(i) < \operatorname{row}_{\mathfrak{s}}(i+1)$, and $\mathfrak{s}_1 \rhd \mathfrak{s}$ if $\operatorname{row}_{\mathfrak{s}}(i) > \operatorname{row}_{\mathfrak{s}}(i+1)$. Hence we have shown the following.

(4.7) Suppose that $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$. Then there exist standard λ–tableaux $\operatorname{first}(\mathtt{S})$ and $\operatorname{last}(\mathtt{S})$ such that

(i) $\mu(\operatorname{first}(\mathtt{S})) = \mu(\operatorname{last}(\mathtt{S})) = \mathtt{S}$; and,

(ii) if $\mathfrak{s}$ is any standard λ–tableau such that $\mu(\mathfrak{s}) = \mathtt{S}$ then $\operatorname{first}(\mathtt{S}) \trianglerighteq \mathfrak{s} \trianglerighteq \operatorname{last}(\mathtt{S})$.

(4.8) Definition *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then $m_{\mathtt{S}\mathfrak{t}}$ is the element of $\mathscr{H}$ given by*

$$m_{\mathtt{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathtt{S}}} m_{\mathfrak{s}\mathfrak{t}}.$$

Note that $m_{\mathtt{S}\mathfrak{t}}$ is a sum of standard basis elements $m_{\mathfrak{s}\mathfrak{t}}$ where $\operatorname{first}(\mathtt{S}) \trianglerighteq \mathfrak{s} \trianglerighteq \operatorname{last}(\mathtt{S})$.

(4.9) Example Suppose that $\lambda = ((4,3),(2,1),(2,1))$, $\mu = ((3^2,1),(1^2),(2,1^2))$ and

$$\mathtt{S} = \left(\begin{array}{|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 \\ \hline 2_1 & 2_1 & 3_1 \\ \cline{1-3}\end{array},\ \begin{array}{|c|c|}\hline 1_2 & 3_3 \\ \hline 2_2 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1}\end{array} \right).$$

Then $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and for any $\mathfrak{t} \in \operatorname{Std}(\lambda)$ we have $m_{\mathtt{S}\mathfrak{t}} = m_{\mathfrak{s}_1\mathfrak{t}} + m_{\mathfrak{s}_2\mathfrak{t}} + m_{\mathfrak{s}_3\mathfrak{t}}$ where

$$\mathfrak{s}_1 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3}\end{array},\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1}\end{array} \right),\quad \mathfrak{s}_2 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 \\ \cline{1-3}\end{array},\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1}\end{array} \right)$$

and

$$\mathfrak{s}_3 = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3}\end{array},\ \begin{array}{|c|c|}\hline 8 & 13 \\ \hline 9 \\ \cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 10 & 11 \\ \hline 12 \\ \cline{1-1}\end{array} \right).$$

Here $\mathfrak{s}_1 = \operatorname{first}(\mathtt{S})$ and $\mathfrak{s}_3 = \operatorname{last}(\mathtt{S})$.

(4.10) Lemma *Suppose that λ is a multipartition of n and that μ is a multicomposition of n. Let $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then $m_{\mathtt{S}\mathfrak{t}} \in M^\mu$.*

PROOF: Let $\mathbf{a} = (a_1,\dots,a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$, and $\mathbf{b} = (b_1,\dots,b_r)$ where $b_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$.

Let $\mathfrak{s}_1 = \text{first}(\mathrm{S})$ and let $d = d(\mathfrak{s}_1)$. Then, as in [4],

$$\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathrm{S}}} x_\lambda T_{d(\mathfrak{s})} = \sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w = hT_d x_\mu,$$

where $h = \sum T_v$, the sum running over certain elements $v \in \mathfrak{S}_\lambda$. Since $h \in \mathscr{H}(\mathfrak{S}_\lambda)$, we have $h^* u_{\mathbf{b}}^+ = u_{\mathbf{b}}^+ h^*$; so we obtain

$$m_{\mathrm{St}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathrm{S}}} m_{\mathfrak{s}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathrm{S}}} T_{d(\mathfrak{s})}^* x_\lambda u_{\mathbf{b}}^+ T_{d(\mathfrak{t})} = x_\mu T_d^* u_{\mathbf{b}}^+ h^* T_{d(\mathfrak{t})}.$$

Therefore, since $m_\mu = x_\mu u_{\mathbf{a}}^+$, it is sufficient to prove that $T_d^* u_{\mathbf{b}}^+ \in u_{\mathbf{a}}^+ \mathscr{H}$.

Assume that $1 \le k \le r$. Recall that $\mathfrak{s}_1 = \text{first}(\mathrm{S})$, $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$ and $b_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$ and note that $a_k \le b_k$ since S is semistandard. Define $\mathfrak{t}_k$ to be the standard λ–tableau such that $1, 2, \dots, a_k$ occupy the same positions in $\mathfrak{s}_1$ and $\mathfrak{t}_k$; $b_k + 1, \dots, n$ occupy the same positions in $\mathfrak{t}^\lambda$ and $\mathfrak{t}_k$; and that for $a_k + 1 \le i < j \le b_k$ we have $\text{row}_{\mathfrak{t}_k}(i) \le \text{row}_{\mathfrak{t}_k}(j)$ — thus the numbers $a_k+1, \dots, b_k$ are inserted into $\mathfrak{t}_k$ in "row order". The fact that S is semistandard ensures that $\mathfrak{t}_k$ is well defined. Note that $\mathfrak{t}_1 = \mathfrak{t}^\lambda$ and $\mathfrak{t}_r = \mathfrak{s}_1$.

Now define $w_k \in \mathfrak{S}_n$ inductively by requiring that $\mathfrak{t}_k = \mathfrak{t}^\lambda w_1 w_2 ... w_k$. Then $w_1 = 1$ and $d = w_1 w_2 ... w_r$; moreover, $\ell(d) = \ell(w_1) + ... + \ell(w_r)$. By construction, the element w_k fixes each of the numbers $1, 2, \dots, a_{k-1}$ and it also fixes each of $b_k + 1, b_k + 2, \dots, n$.

For $1 \le k \le r$, let $u_{\mathbf{a},k}$ and $u_{\mathbf{b},k}$ be as defined in Definition 3.1, and let $v_k = \prod_{m=a_k+1}^{b_k} (L_m - Q_k)$. Then $u_{\mathbf{b},k} = u_{\mathbf{a},k} v_k$.

Now, T_{w_k} commutes with $u_{\mathbf{a},l}$ for $l < k$ (since w_k fixes $1, 2, \dots, a_{k-1}$), and T_{w_k} commutes with $u_{\mathbf{b},l}$ for $l \ge k$ (since w_k fixes $b_k + 1, b_k + 2, \dots, n$); see (2.1)(iv). Therefore,

$$\begin{aligned}
T_d^* u_{\mathbf{b}}^+ &= T_{w_r}^* \dots T_{w_1}^* u_{\mathbf{b},1} u_{\mathbf{b},2} \dots u_{\mathbf{b},r} \\
&= T_{w_r}^* \dots T_{w_2}^* u_{\mathbf{b},1} u_{\mathbf{b},2} \dots u_{\mathbf{b},r} T_{w_1}^* \\
&= u_{\mathbf{a},1} T_{w_r}^* \dots T_{w_2}^* u_{\mathbf{b},2} \dots u_{\mathbf{b},r} v_1 T_{w_1}^* \\
&= u_{\mathbf{a},1} u_{\mathbf{a},2} T_{w_r}^* \dots T_{w_3}^* u_{\mathbf{b},3} \dots u_{\mathbf{b},r} v_2 T_{w_2}^* v_1 T_{w_1}^* \\
&\;\;\vdots \\
&= u_{\mathbf{a},1} \dots u_{\mathbf{a},r} v_r T_{w_r}^* v_{r-1} T_{w_{r-1}}^* \dots v_2 T_{w_2}^* v_1 T_{w_1}^*.
\end{aligned}$$

Thus, $T_d^* u_{\mathbf{b}}^+ \in u_{\mathbf{a}}^+ \mathscr{H}$, as required. □

Having shown that $m_{\mathrm{St}} \in M^\mu$, our next aim is to prove that the set of elements of the form m_{St} give a basis of M^μ.

Let $\mathbf{a} = (a_1, \dots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$ as in the previous proof; then $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Since $u_{\mathbf{a}}^+$ and x_μ commute, we have $M^\mu = m_\mu \mathscr{H} \subseteq u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$. We shall show that we have equality here.

We have seen that $\mathscr{H}$ has a standard basis which consists of elements $m_{\mathfrak{s}\mathfrak{t}}$ with $\mathfrak{s}$ and $\mathfrak{t}$ standard λ–tableaux for some multipartition λ of n. Suppose that $h \in \mathscr{H}$ and let $h =$

$\sum_{\mathfrak{s},\mathfrak{t}} r_{\mathfrak{st}} m_{\mathfrak{st}}$, with $r_{\mathfrak{st}} \in R$, be the unique expression for h in terms of the standard basis. We say that $m_{\mathfrak{st}}$ is involved in h if $r_{\mathfrak{st}} \neq 0$.

(4.11) Lemma *Suppose that μ is a multipartition of n and that $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Let $h \in x_\mu \mathscr{H} \cap u_{\mathbf{a}}^+ \mathscr{H}$ and suppose that $(\mathfrak{s}, \mathfrak{t})$ is a pair of standard tableaux of the same shape such that*

(i) *$m_{\mathfrak{st}}$ is involved in h; and,*

(ii) *if $(\mathfrak{u}, \mathfrak{v})$ is a pair of standard tableaux of the same shape such that $\mathfrak{s} \trianglerighteq \mathfrak{u}$, $\mathfrak{t} \trianglerighteq \mathfrak{v}$ and $(\mathfrak{s}, \mathfrak{t}) \neq (\mathfrak{u}, \mathfrak{v})$ then $m_{\mathfrak{uv}}$ is not involved in h.*

Let $\mathsf{S} = \mu(\mathfrak{s})$. Then S is semistandard and $\mathfrak{s} = \mathrm{last}(\mathsf{S})$.

PROOF: Since $\mathfrak{s}$ is standard, the entries in S are non–decreasing down rows and columns.

Suppose that i and $i+1$ belong to the same row of $\mathfrak{t}^\mu$. Then $T_i x_\mu = q x_\mu$, so $T_i h = qh$. Therefore, by [**16**, (4.19)], i and $i+1$ do not belong to the same column of $\mathfrak{s}$. Hence the entries in S are strictly increasing down columns.

Since $h \in u_{\mathbf{a}}^+ \mathscr{H}$ we have $\mathrm{comp}_{\mathfrak{t}^\mu} \geq \mathrm{comp}_{\mathfrak{s}}$ by Corollary 3.24 (applied to the multicomposition $((1^{\alpha_1}), (1^{\alpha_2}), \dots, (1^{\alpha_r}))$, where $\alpha = (|\mu^{(1)}|, \dots, |\mu^{(r)}|)$). Hence, by (4.5), S is semistandard.

Finally, suppose that $\mathfrak{s} \neq \mathrm{last}(\mathsf{S})$. Then there exist integers i and $i+1$ in the same row of $\mathfrak{t}^\mu$ such that the tableau $\mathfrak{s}' = \mathfrak{s}(i, i+1)$ is standard and $\mathfrak{s} \triangleright \mathfrak{s}'$. As in [**16**, (4.19)], $m_{\mathfrak{s}'\mathfrak{t}}$ is involved in $T_i h = qh$, contradicting part (ii) of our hypothesis. Therefore, $\mathfrak{s} = \mathrm{last}(\mathsf{S})$. □

(4.12) Corollary *Suppose that μ is a multicomposition of n and $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Then $u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ is spanned by*

$$\left\{ m_{\mathsf{S}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}_0(\lambda, \mu) \text{ and } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some} \\ \text{multipartition } \lambda \text{ of } n \end{array} \right\}.$$

PROOF: Note that $m_{\mathsf{S}\mathfrak{t}} \in M^\mu$ by Lemma 4.10.

For each multipartition λ of n let

$$\mathcal{T}_l(\lambda, \mu) = \{\, \mathfrak{s} \in \mathrm{Std}(\lambda) \mid \mu(\mathfrak{s}) = \mathrm{last}(\mathsf{S}) \text{ for some } \mathsf{S} \in \mathcal{T}_0(\lambda, \mu) \,\}.$$

By Lemma 4.11 every non–zero element of $u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ involves a standard basis element $m_{\mathfrak{st}}$ where $\mathfrak{s} \in \mathcal{T}_l(\lambda, \mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$ for some multipartition λ.

Now suppose that $h \in u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$ and write h in terms of the standard basis; say, $h = \sum r_{\mathfrak{uv}} m_{\mathfrak{uv}}$ with $r_{\mathfrak{uv}} \in R$. Let

$$h' = h - \sum_\lambda \sum_{\mathfrak{s} \in \mathcal{T}_l(\lambda,\mu)} \sum_{\mathfrak{t} \in \mathrm{Std}(\lambda)} r_{\mathfrak{st}} m_{\mu(\mathfrak{s})\mathfrak{t}}.$$

Then $h' \in u_{\mathbf{a}}^+ \mathscr{H} \cap x_\mu \mathscr{H}$, but h' does not involve any term $m_{\mathfrak{st}}$ for any $\mathfrak{s} \in \mathcal{T}_l(\lambda, \mu)$. Therefore, $h' = 0$, and we have obtained an expression for h as a linear combination of the required form. □

(4.13) Corollary *Suppose that μ is a multicomposition of n and $m_\mu = u_{\mathbf{a}}^+ x_\mu$. Then $M^\mu = u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H}$.*

PROOF: We have that $M^\mu \subseteq u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H}$. The inclusion $u_{\mathbf{a}}^+\mathscr{H} \cap x_\mu\mathscr{H} \subseteq M^\mu$ follows from Corollary 4.12 and Lemma 4.10 □

The next theorem for Iwahori–Hecke algebras of type **A** (that is, the case $r = 1$), is due to Murphy [**16**]; for Iwahori–Hecke algebras of type **B** (that is, the case $r = 2$), the result is due to Du and Scott [**11**]

(4.14) Theorem *Suppose that μ is a multicomposition of n. Then M^μ is free as an R–module with basis*

$$\left\{ m_{\mathfrak{S}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathfrak{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}.$$

PROOF: By Lemma 4.10 each element $m_{\mathfrak{S}\mathfrak{t}}$ belongs to M^μ. Since distinct elements $m_{\mathfrak{S}\mathfrak{t}}$ involve distinct standard basis elements $m_{\mathfrak{s}\mathfrak{t}}$, the elements $m_{\mathfrak{S}\mathfrak{t}}$ are linearly independent. Finally, Corollaries 4.12 and 4.13 show that the elements $m_{\mathfrak{S}\mathfrak{t}}$ span M^μ. □

(4.15) Corollary *Suppose that μ is a multicomposition of n. Then there exists a filtration of M^μ,*

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$$

such that for each i with $1 \le i \le k$ there exists a multipartition λ_i of n with $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, if λ is a multipartition of n, then the number of factors S^{λ_i} isomorphic to S^λ is equal to the number of semistandard λ–tableaux of type μ.

PROOF: Choose an ordering $\mathfrak{S}_1 > \mathfrak{S}_2 > \cdots > \mathfrak{S}_k$ on the set of semistandard tableaux of type μ such that $j > i$ if $\lambda_i \rhd \lambda_j$ where $\mathfrak{S}_i \in \mathcal{T}_0(\lambda_i,\mu)$ and $\mathfrak{S}_j \in \mathcal{T}_0(\lambda_j,\mu)$. Let M_i be the R–submodule of M^μ with basis $\{\, m_{\mathfrak{S}_j\mathfrak{t}} \mid j \ge i \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda_j)\,\}$. Then

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0,$$

and, by Proposition 3.25 and Theorem 4.14, each M_i is a right ideal of $\mathscr{H}$.

Suppose that $1 \le i \le k$. Then $M_i \cap \overline{N^{\lambda_i}} \subseteq M_{i+1}$. Hence, there is a well defined $\mathscr{H}$–homomorphism θ from S^{λ_i} onto M_i/M_{i+1} such that $\theta(z_{\lambda_i}) = m_{\mathfrak{S}_i\mathfrak{t}^{\lambda_i}} + M_{i+1}$. Since both S^{λ_i} and M_i/M_{i+1} have rank equal to the number of standard λ_i–tableaux, θ is an isomorphism. The Corollary now follows. □

5 The double annihilator of m_μ

The purpose of this section is to compute the double annihilator of the element m_μ of $\mathscr{H}$ for any multicomposition μ of n. This will enable us to calculate a basis for $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ in the next section.

(5.1) Definition *Suppose S is a subset of $\mathscr{H}$ and define* $\mathrm{r}(S) = \{\, h \in \mathscr{H} \mid Sh = 0 \,\}$ *and* $\mathrm{l}(S) = \{\, h \in \mathscr{H} \mid hS = 0 \,\}$. *The* double annihilator *of S is* $\mathrm{lr}(S) = \mathrm{l}(\mathrm{r}(S))$.

It is elementary that $\mathrm{lr}(S)$ contains the left ideal of $\mathscr{H}$ generated by S. If $\mathscr{H}$ is a quasi–Frobenius algebra, then $\mathrm{lr}(S)$ is equal to the left ideal of $\mathscr{H}$ generated by S [**3**, (61.2)]. Although $\mathscr{H}$ is quasi–Frobenius for $r \le 2$ (see, for example, [**4**, section 2]), we do not know whether it is quasi–Frobenius for $r > 2$. If $\mathscr{H}$ is quasi–Frobenius for all r, then the main result of this section, namely Theorem 5.16, is immediate.

A connection between double annihilators and homomorphisms is provided by the following easy lemma.

(5.2) Lemma *Suppose that $m \in \mathscr{H}$ and that $\mathrm{lr}(m) = \mathscr{H}m$. Let I be a right ideal of $\mathscr{H}$.*

(i) *For all $\varphi \in \operatorname{Hom}_{\mathscr{H}}(m\mathscr{H}, I)$ there exists $h_\varphi \in \mathscr{H}$ such that $\varphi(m) = h_\varphi m$.*

(ii) $\operatorname{Hom}_{\mathscr{H}}(m\mathscr{H}, I) \cong \mathscr{H}m \cap I$.

PROOF: For all $x \in \mathrm{r}(m)$ we have $\varphi(m)x = \varphi(mx) = 0$. Therefore, $\varphi(m) \in \mathrm{lr}(m) = \mathscr{H}m$; so $\varphi(m) = h_\varphi m$ for some $h_\varphi \in \mathscr{H}$. The proof of part (ii) is now straightforward. □

The main result of this section is that $\mathrm{lr}(m_\mu) = \mathscr{H}m_\mu$. Because the Iwahori–Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ is quasi–Frobenius, our first step is easy.

(5.3) Lemma *Suppose that μ is a multicomposition of n. Then $\mathrm{lr}(x_\mu) = \mathscr{H}x_\mu$.*

PROOF: Since $\mathscr{H}(\mathfrak{S}_n)$ is a quasi–Frobenius algebra, we have $\mathrm{lr}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n) = \mathscr{H}(\mathfrak{S}_n)x_\mu$. Now suppose that $z \in \mathrm{lr}(x_\mu)$. By Theorem 2.2, we may write

$$z = \sum_{\mathbf{c}\in\mathscr{C}} L_1^{c_1}, \dots L_n^{c_n} h_{\mathbf{c}}$$

where $\mathscr{C} = \{\, \mathbf{c} = (c_1, \dots, c_r) \mid 0 \le c_j < r \text{ for } 1 \le j \le n \,\}$ and $h_{\mathbf{c}} \in \mathscr{H}(\mathfrak{S}_n)$. Then, for all $y \in \mathrm{r}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n)$, we have

$$0 = zy = \sum_{\mathbf{c}\in\mathscr{C}} L_1^{c_1}, \dots L_n^{c_n} h_{\mathbf{c}} y.$$

Therefore, by Theorem 2.2, we have $h_{\mathbf{c}}y = 0$ for all $\mathbf{c} \in \mathscr{C}$. Thus, $h_{\mathbf{c}} \in \mathrm{lr}(x_\mu) \cap \mathscr{H}(\mathfrak{S}_n) = \mathscr{H}(\mathfrak{S}_n)x_\mu$, and so $z \in \mathscr{H}x_\mu$. □

For the remainder of this section we fix a multicomposition μ and define $\mathbf{a} = (a_1, \dots, a_r)$ by $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$; so $0 = a_1 \le a_2 \le \dots \le a_r \le n$. Our main task in this section is to prove that $\mathrm{lr}(u_{\mathbf{a}}^+) = \mathscr{H}u_{\mathbf{a}}^+$. To achieve this goal, we use the next result.

(5.4) Lemma *Assume that I is a left ideal of $\mathscr{H}$ and suppose that S is a subset of* $\mathrm{r}(I)$ *such that* $\mathrm{l}(S) \subseteq I$. *Then* $\mathrm{lr}(I) = I$.

PROOF: By assumption $S \subseteq \mathrm{r}(I)$ and $\mathrm{l}(S) \subseteq I$. Also, $I \subseteq \mathrm{lr}(I)$. Therefore, $\mathrm{lr}(I) \subseteq \mathrm{l}(S) \subseteq I \subseteq \mathrm{lr}(I)$; so $\mathrm{lr}(I) = I$ as required. □

Our definition of $u_{\mathbf{a}}^+$ expresses $u_{\mathbf{a}}^+$ as a double product

$$u_{\mathbf{a}}^+ = \prod_{k=1}^{r} \prod_{m=1}^{a_k} (L_m - Q_k).$$

We now introduce notation which allows us to reverse the order of the multiplication here.

(5.5) Definition *Suppose that $1 \leq i \leq n$.*
(i) *Let $\gamma_i = k$ if k is maximal such that $a_k < i$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.*
(ii) *Let* $v_i = \prod_{k=\gamma_i+1}^{r} (L_i - Q_k)$.

Then $u_{\mathbf{a}}^+ = v_1 v_2 \ldots v_n$ and all of these factors commute.

(5.6) Definition *Suppose that $1 \leq i \leq n$. Let* $y_i = T_{i-1} \ldots T_2 T_1 \prod_{k=1}^{\gamma_i} (L_1 - Q_k)$.

(5.7) Example Suppose that $r = 4$, $n = 5$ and $\mathbf{a} = (0, 2, 4, 5)$. Then

$$\begin{aligned} u_{\mathbf{a}}^+ = {} & (L_1 - Q_2)(L_2 - Q_2) \\ & \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \\ & \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)(L_5 - Q_4). \end{aligned}$$

Also $\gamma = (1, 1, 2, 2, 3)$ and, for $1 \leq i \leq n$, the product of the factors in the ith column of the above array is v_i. We have

$$\begin{aligned} y_1 &= (L_1 - Q_1) \\ y_2 &= T_1(L_1 - Q_1) \\ y_3 &= T_2T_1(L_1 - Q_1)(L_1 - Q_2) \\ y_4 &= T_3T_2T_1(L_1 - Q_1)(L_1 - Q_2) \\ y_5 &= T_4T_3T_2T_1(L_1 - Q_1)(L_1 - Q_2)(L_1 - Q_3). \end{aligned}$$

(5.8) Lemma *Suppose that $1 \leq i \leq n$. Then $v_1 v_2 \ldots v_i y_i = 0$.*

PROOF: Assume that k satisfies $\gamma_i+1 \le k \le r$. Then $v_1v_2\dots v_i$ has a factor $(L_1 - Q_k)(L_2-Q_k)\dots(L_i-Q_k)$. By (2.1)(iv), this factor commutes with $T_{i-1}\dots T_2T_1$. Hence, $v_1v_2\dots v_iT_{i-1}\dots T_2T_1$ has a right factor $\prod_{k=\gamma_i+1}^r(L_1-Q_k)$. However,

$$\prod_{k=1}^r(L_1-Q_k)=\prod_{k=1}^r(T_0-Q_k)=0;$$

so it follows that $v_1v_2\dots v_iy_i=0$ as claimed. □

Since $u_{\mathbf{a}}^+=v_1v_2\dots v_n$ we have the following Corollary.

(5.9) Corollary *Suppose that $1\le i\le n$. Then $u_{\mathbf{a}}^+y_i=0$.*

It turns out that $y_1,y_2,\dots,y_n$ generate $\mathrm{r}(u_{\mathbf{a}}^+)$. More importantly, together with Lemma 5.4 the elements y_i will allow us to prove that $\mathrm{lr}(u_{\mathbf{a}}^+)=\mathscr{H}u_{\mathbf{a}}^+$. For our proof, we require the technical Lemma 5.13 below; for this result we need some preparation.

(5.10) Lemma *Suppose that $1\le i<n$ and that $a,b\in\{0,1,\dots,r-1\}$.*

(i) *If $a\le b$ then* $L_i^aL_{i+1}^bT_i=T_iL_i^bL_{i+1}^a+(q-1)\sum_{c=1}^{b-a}L_i^{b-c}L_{i+1}^{a+c}$.

(ii) *If $a>b$ then* $L_i^aL_{i+1}^bT_i=qT_i^{-1}L_i^bL_{i+1}^a-(q-1)\sum_{c=1}^{a-b-1}L_i^{b+c}L_{i+1}^{a-c}$.

PROOF: (i) Assume that $a\le b$. If $b=a$ then T_i commutes with $L_i^aL_{i+1}^a$ by (2.1)(iii), so the result is correct in this case. Now suppose that $b>a$. Since $L_{i+1}T_i=T_iL_i+(q-1)L_{i+1}$ we have

$$L_i^aL_{i+1}^bT_i=L_i^aL_{i+1}^{b-1}(T_iL_i+(q-1)L_{i+1}),$$

which gives the required result by induction on $b-a$.

(ii) Either argue similarly, or apply the antiautomorphism $*$ to the result of part (i) and rearrange, interchanging a and b. □

We next generalize Lemma 5.10 as follows.

(5.11) Lemma *Suppose that $1\le i\le n$ and $b_j\in\{0,1,\dots,r-1\}$ for $j=1,2,\dots,n$. Then there exists an integer b and $\epsilon_1,\dots,\epsilon_{i-1}\in\{\pm1\}$ such that*

$$L_1^{b_1}L_2^{b_2}\dots L_i^{b_i}T_{i-1}\dots T_2T_1=x_1+x_2,$$

where $x_1=q^bT_{i-1}^{\epsilon_{i-1}}\dots T_2^{\epsilon_2}T_1^{\epsilon_1}L_1^{b_i}L_2^{b_1}L_3^{b_2}\dots L_i^{b_{i-1}}$ and x_2 is a linear combination of terms of the form $T_wL_1^{c_1}L_2^{c_2}\dots L_i^{c_i}$ where

(i) *$w\in\mathfrak{S}_n$; and*

(ii) $c_1, \ldots, c_i \in \{0, 1, \ldots, r-1\}$ *with* $c_1 + c_2 + \cdots + c_i = b_1 + b_2 + \cdots + b_i$*; and*

(iii) *either* $\prod_{j=1}^{i}(c_j + 1) < \prod_{j=1}^{i}(b_j + 1)$, *or* $c_1, c_2, \ldots, c_i$ *is a permutation of* $b_1, b_2, \ldots, b_i$ *and* $c_1 < b_i$.

PROOF: The result is true when $i = 1$. (In this case, $x_1 = L_1^{b_1}$ and $x_2 = 0$.) Assume, inductively, that $i < n$ and that the result as stated is true. By Lemma 5.10, there exists $b' \in \{0, 1\}$ and $\epsilon_i = \pm 1$ such that

$$L_i^{b_i} L_{i+1}^{b_{i+1}} T_i = q^{b'} T_i^{\epsilon_i} L_i^{b_{i+1}} L_{i+1}^{b_i} \pm (q-1) \sum L_i^{d_i} L_{i+1}^{d_{i+1}}$$

where $d_i, d_{i+1} \in \{0, 1, \ldots, r-1\}$ and $d_i + d_{i+1} = b_i + b_{i+1}$ and either $(d_i + 1)(d_{i+1} + 1) < (b_i + 1)(b_{i+1} + 1)$ or $d_i = b_i < b_{i+1}$. Using this result, and (2.1)(ii), we obtain

$$\begin{aligned} L_1^{b_1} L_2^{b_2} \ldots L_i^{b_i} T_{i-1} \ldots T_2 T_1 &= q^{b'} T_i^{\epsilon_i} L_1^{b_1} \ldots L_{i-1}^{b_{i-1}} L_i^{b_{i+1}} T_{i-1} \ldots T_2 T_1 L_{i+1}^{b_i} \\ &\pm (q-1) \sum L_1^{b_1} \ldots L_{i-1}^{b_{i-1}} L_i^{d_i} T_{i-1} \ldots T_2 T_1 L_{i+1}^{d_{i+1}}, \end{aligned}$$

where b', d_i and d_{i+1} are as above. The result now follows from our inductive hypothesis.

□

For the statement and proof of our next Lemma, we need some more notation.

(5.12) Notation Suppose that $0 \le i \le n$. Let V_i be the R–module with basis

$$\left\{ T_w L_1^{b_1} L_2^{b_2} \ldots L_n^{b_n} v_1 v_2 \ldots v_i \;\middle|\; \begin{array}{l} w \in \mathfrak{S}_n,\ 0 \le b_j < \gamma_j, \text{ for } 1 \le j \le i, \\ \text{and } 0 \le b_j < r, \text{ for } i+1 \le j \le n \end{array} \right\}.$$

Result (2.4) shows the elements in these sets are indeed linearly independent. Note also that $V_0 = \mathscr{H}$ and $V_n = \mathscr{H} u_{\mathbf{a}}^{+}$.

We shall use (2.4) extensively in the proof of Lemma 5.13.

(5.13) Lemma *Suppose that* $1 \le i \le n$ *and that* $z \in V_{i-1}$ *and* $z y_i = 0$. *Then* $z \in V_i$.

PROOF: Since $z \in V_{i-1}$ we may write z as a linear combination of linearly independent terms

$$T_w L_1^{b_1} \ldots L_{i-1}^{b_{i-1}} f(L_i) L_{i+1}^{b_{i+1}} \ldots L_n^{b_n} v_1 v_2 \ldots v_{i-1}$$

where $w \in \mathfrak{S}_n$ and $0 \le b_j < \gamma_j$ for $1 \le j \le i-1$ and $0 \le b_j < r$ for $i+1 \le j \le n$, and $f(X)$ is a polynomial in $R[X]$ of degree less than r. Write

$$f(X) = g(X) \prod_{k=\gamma_i+1}^{r} (X - Q_k) + h(X),$$

where $g(X)$ and $h(X)$ are polynomials in $R[X]$ such that $\deg g(X) < \gamma_i$ and $\deg h(X) < r-\gamma_i$. We may apply the "Euclidean algorithm" in this way, even though $R[X]$ need not be a Euclidean domain, because the divisor is a monic polynomial.

Note that $\prod_{k=\gamma_i+1}^{r}(L_i - Q_k) = v_i$. As a consequence, we have written z as $z_1 + z_2$ where $z_1 \in V_i$ and z_2 is a linear combination of terms

$$T_w L_1^{b_1} \dots L_{i-1}^{b_{i-1}} h(L_i) L_{i+1}^{b_{i+1}} \dots L_n^{b_n} v_1 v_2 \dots v_{i-1}.$$

We shall prove that $z_2 = 0$.

Note that $z_1 y_i = 0$, by Lemma 5.8, since $z_1 \in V_i$. Therefore, because $zy_i = 0$ by assumption, it follows that $z_2 y_i = 0$.

For each n–tuple $\mathbf{c} = (c_1, \dots, c_n)$ with $0 \le c_i < r$, for all i with $1 \le i \le n$, let $U_{\mathbf{c}} = \mathscr{H}(\mathfrak{S}_n) L_1^{c_1}, \dots L_n^{c_n}$. Then we may write

$$z_2 = \sum_{\mathbf{c}\in\mathscr{C}} z_{\mathbf{c}} \quad \text{where} \quad z_{\mathbf{c}} \in U_{\mathbf{c}}$$

and $\mathscr{C} = \{\, \mathbf{c} = (c_1, \dots, c_n) \mid 0 \le c_j < r \text{ for } 1 \le j \le n \text{ and } c_i < r - \gamma_i \,\}$.

Assume that $z_2 \ne 0$. Among the $\mathbf{c}$ such that $z_{\mathbf{c}} \ne 0$ choose $\mathbf{d} = (d_1, \dots, d_n)$ such that

(5.14) (i) $d_1 + d_2 + \dots + d_n$ is maximal; and,
(ii) $(d_1+1)(d_2+1)\dots(d_n+1)$ is maximal subject to (i); and,
(iii) d_i is maximal subject to (i) and (ii).

Then $z_{\mathbf{d}} = h_{\mathbf{d}} L_1^{d_1}, \dots L_n^{d_n}$ for some non–zero element $h_{\mathbf{d}}$ of $\mathscr{H}(\mathfrak{S}_n)$.

Now, $T_{i-1} \dots T_2 T_1$ commutes with each of the elements $L_{i+1}, L_{i+2}, \dots, L_n$. Therefore, by Lemma 5.11 there exists an invertible element $h \in \mathscr{H}(\mathfrak{S}_n)$ (namely, $h = q^b T_{i-1}^{\epsilon_{i-1}} \dots T_2^{\epsilon_2} T_1^{\epsilon_1}$) such that

$$z_2 T_{i-1} \dots T_2 T_1 = h_{\mathbf{d}} h L_1^{d_i} L_2^{d_1} L_3^{d_2} \dots L_i^{d_{i-1}} L_{i+1}^{d_{i+1}} \dots L_n^{d_n} + \text{ terms in the sets } U_{\mathbf{e}},$$

where $\mathbf{e} = (e_1, \dots, e_n)$ and $e_1 + \dots + e_n \le d_1 + \dots + d_n$, and either
(i) $e_1 + \dots + e_n < d_1 + \dots + d_n$ (by Lemma 5.11(ii) and (5.14)(i)); or,
(ii) $(e_1+1)\dots(e_n+1) < (d_1+1)\dots(d_n+1)$ (using Lemma 5.11(iii) and (5.14)(ii)); or,
(iii) $e_1, \dots, e_n$ is a permutation of $d_1, \dots, d_n$ and $e_1 < d_i$ (using Lemma 5.11(iii) and (5.14)(iii)); or,
(iv) $e_1, \dots, e_n$ is a permutation of $d_1, \dots, d_n$ but $\mathbf{e} \ne (d_i, d_1, d_2, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$ (consider the term x_1 in Lemma 5.11).

In particular, no $\mathbf{e}$ on the right hand side is equal to $(d_i, d_1, d_2, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$.

Next we postmultiply our expression for $z_2 T_{i-1} \dots T_2 T_1$ by $\prod_{m=1}^{\gamma_i}(L_1 - Q_m)$ to obtain $z_2 y_i$. Note that if $v_{\mathbf{e}} \in U_{\mathbf{e}}$ and k is any positive integer then $v_{\mathbf{e}} L_1^k$ is a linear combination of terms $v_{\mathbf{f}} \in U_{\mathbf{f}}$ where $\mathbf{f} = (f_1, e_2, \dots, e_n)$ and $f_1 < r$. Thus, if $e_1 + \dots + e_n \le d_1 + \dots + d_n$ and $e_1 + k \ge r$ then $v_{\mathbf{e}} L_1^k$ is a linear combination of terms $v_{\mathbf{f}}$ with $f_1 + e_2 + \dots + e_n < d_1 + \dots + d_n + k$. Therefore,

$$z_2 y_i = h_{\mathbf{d}} h L_1^{d_i+\gamma_i} L_2^{d_1} L_3^{d_2} \dots L_i^{d_{i-1}} L_{i+1}^{d_{i+1}} \dots L_n^{d_n} + \text{ terms in the sets } U_{\mathbf{f}}$$

where $\mathbf{f} \neq (d_i + \gamma_i, d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)$.

Note that $d_i + \gamma_i < r$ since $\mathbf{d} \in \mathscr{C}$. Therefore, since $h_\mathbf{d} \neq 0$ and h is invertible, $z_2 y_i \neq 0$ giving a contradiction. Hence, our assumption that $z_2 \neq 0$ is false. Thus, $z = z_1 \in V_i$ as required. □

(5.15) Corollary *We have* $\mathrm{lr}(u_\mathbf{a}^+) = \mathscr{H} u_\mathbf{a}^+$.

PROOF: Let $I = \mathscr{H} u_\mathbf{a}^+$ and $S = \{y_1, y_1, \ldots, y_n\}$. Then Corollary 5.9 shows that $S \subseteq \mathrm{r}(I)$.

Suppose that $z \in \mathscr{H}$ and $z y_i = 0$ for all i with $1 \leq i \leq n$. Since $V_0 = \mathscr{H}$ and $V_n = \mathscr{H} u_\mathbf{a}^+ = I$, it follows from Lemma 5.13 that $z \in I$. Thus, $\mathrm{l}(S) \subseteq I$. Lemma 5.4 now implies that $\mathrm{lr}(I) = I$; that is, $\mathrm{lr}(u_\mathbf{a}^+) = \mathscr{H} u_\mathbf{a}^+$. □

Finally, we can prove the main result of this section.

(5.16) Theorem *Suppose that μ is a multicomposition of n. Then* $\mathrm{lr}(m_\mu) = \mathscr{H} m_\mu$.

PROOF: Write $m_\mu = u_\mathbf{a}^+ x_\mu$ as in Definition 3.5. By applying the definitions, Corollary 5.15 and Lemma 5.3, we obtain

$$\mathscr{H} m_\mu \subseteq \mathrm{lr}(m_\mu) \subseteq \mathrm{lr}(u_\mathbf{a}^+) \cap \mathrm{lr}(x_\mu) = \mathscr{H} u_\mathbf{a}^+ \cap \mathscr{H} x_\mu.$$

However, $(u_\mathbf{a}^+ \mathscr{H} \cap x_\mu \mathscr{H})^* = (m_\mu \mathscr{H})^*$ by Corollary 4.13. Hence, we have equality throughout and $\mathrm{lr}(m_\mu) = \mathscr{H} m_\mu$ as claimed. □

(5.17) Corollary *Suppose that μ and ν are multicompositions of n. Then*

(i) *For every element φ in $\mathrm{Hom}_\mathscr{H}(M^\nu, M^\mu)$ there exists $h_\varphi \in \mathscr{H}$ such that $\varphi(m_\nu) = h_\varphi m_\nu$; in particular, $\varphi(m_\nu) \in M^{\nu *} \cap M^\mu$.*

(ii) $\mathrm{Hom}_\mathscr{H}(M^\nu, M^\mu) \cong M^{\nu *} \cap M^\mu$.

PROOF: By Theorem 5.16 we may apply Lemma 5.2(i) with $m = m_\nu$ to obtain that $\varphi(m_\nu) = h_\varphi m_\nu$ for some $h_\varphi \in \mathscr{H}$. Thus, $\varphi(m_\nu) \in M^{\nu *}$. It is clear that $\varphi(m_\nu) \in M^\mu$. Part (ii) follows from Lemma 5.2(ii), where an explicit isomorphism is given by $\varphi \longmapsto \varphi(m_\nu)$. □

6 The cyclotomic q–Schur algebra

In order to make our results as general as possible, let Λ_r be a poset ideal in the set of all multicompositions of n. Thus, if $\mu \in \Lambda_r$ and $\nu \rhd \mu$ then $\nu \in \Lambda_r$. We also let Λ_r^+ be the set of multipartitions in Λ_r.

(6.1) Definition *The* cyclotomic q–Schur algebra *is the endomorphism algebra*

$$\mathscr{S} = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda_r} M^\mu\Big).$$

Thus, $\mathscr{S} \cong \bigoplus_{\mu,\nu\in\Lambda_r} \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$.

Suppose that $\varphi \in \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$. Then $\varphi(m_\nu h) = \varphi(m_\nu)h$ for all $h \in \mathscr{H}$; thus φ is completely determined by $\varphi(m_\nu)$. Moreover, $\varphi(m_\nu) \in M^{\nu*} \cap M^\mu$ by Corollary 5.17. These remarks motivate us to construct a basis of $M^{\nu*} \cap M^\mu$.

(6.2) Definition *Suppose that μ and ν are multicompositions of n and that λ is a multipartition of n. Assume that $\mathrm{S} \in \mathcal{T}_0(\lambda,\mu)$ and that $\mathrm{T} \in \mathcal{T}_0(\lambda,\nu)$. Let*

$$m_{\mathrm{ST}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$$

where the sum is over all $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\lambda)$ with $\mu(\mathfrak{s}) = \mathrm{S}$ and $\nu(\mathfrak{t}) = \mathrm{T}$.

Note that $m_{\mathrm{ST}}^* = m_{\mathrm{TS}}$.

(6.3) Proposition *Suppose that μ and ν are multicompositions of n. Then*

$$\Big\{\, m_{\mathrm{ST}} \;\Big|\; \begin{array}{c}\mathrm{S} \in \mathcal{T}_0(\lambda,\mu) \text{ and } \mathrm{T} \in \mathcal{T}_0(\lambda,\nu) \text{ for some}\\ \text{multipartition } \lambda \text{ of } n\end{array} \Big\}$$

is a basis of $M^{\nu*} \cap M^\mu$.

PROOF: Since

$$m_{\mathrm{ST}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathrm{S}}} (m_{\mathrm{T}\mathfrak{s}})^* = \sum_{\substack{\mathfrak{t}\in\operatorname{Std}(\lambda)\\ \nu(\mathfrak{t})=\mathrm{T}}} m_{\mathrm{S}\mathfrak{t}},$$

Lemma 4.10 shows that $m_{\mathrm{ST}} \in M^{\nu*} \cap M^\mu$. Moreover, the elements m_{ST} are linearly independent since they involve distinct elements $m_{\mathfrak{s}\mathfrak{t}}$ of the standard basis of $\mathscr{H}$.

Now suppose that $h \in M^{\nu*} \cap M^\mu$. Since $h \in \mathscr{H}$, we may express h in terms of the standard basis $\mathscr{M}$; say

$$h = \sum_{m_{\mathfrak{s}\mathfrak{t}}\in\mathscr{M}} r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$$

where $r_{\mathfrak{s}\mathfrak{t}} \in R$. Since $h \in M^\mu$, we have $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}}$ if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$, by Theorem 4.14. Similarly, since $h \in M^{\nu*}$, we have $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}\mathfrak{t}'}$ if $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$. Thus, if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$ and $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$ then $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}'}$. Furthermore, $r_{\mathfrak{s}\mathfrak{t}} = 0$ unless both $\mu(\mathfrak{s})$ and $\nu(\mathfrak{t})$ are semistandard. Therefore, h is a linear combination of elements m_{ST}. This completes the proof of the Proposition. □

Proposition 6.3 shows that, in our next Definition, φ_{ST} is a well defined $\mathscr{H}$–homomorphism from M^ν into M^μ.

(6.4) Definition *Suppose that $\mu,\nu\in\Lambda_r$ and $\lambda\in\Lambda_r^+$. Assume that $\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T}\in\mathcal{T}_0(\lambda,\nu)$. Define $\varphi_{\mathtt{ST}}\in\operatorname{Hom}_{\mathscr{H}}(M^\nu,M^\mu)$ by*

$$\varphi_{\mathtt{ST}}(m_\nu h)=m_{\mathtt{ST}}h$$

for all $h\in\mathscr{H}$. Extend $\varphi_{\mathtt{ST}}$ to an element of the cyclotomic q–Schur algebra $\mathscr{S}$ by defining $\varphi_{\mathtt{ST}}$ to be zero on M^κ when $\nu\neq\kappa\in\Lambda_r$.

(6.5) Definition *Suppose that $\lambda\in\Lambda_r^+$. Let $\overline{\mathscr{S}_\lambda}$ be the R–submodule of $\mathscr{S}$ spanned by*

$$\left\{\varphi_{\mathtt{UV}}\;\middle|\;\begin{array}{c}\mathtt{U}\in\mathcal{T}_0(\alpha,\mu),\ \mathtt{V}\in\mathcal{T}_0(\alpha,\nu)\text{ for some}\\ \mu,\nu\in\Lambda_r\text{ and }\alpha\in\Lambda_r^+\text{ with }\alpha\rhd\lambda\end{array}\right\}.$$

This brings us to one of the main results of our paper. The work in Sections 4 and 5 was aimed at proving part (i) of the Theorem.

(6.6) Theorem (The Semistandard Basis Theorem)

(i) *The cyclotomic q–Schur algebra $\mathscr{S}$ is free as an R–module with basis*

$$\left\{\varphi_{\mathtt{ST}}\;\middle|\;\begin{array}{c}\mathtt{S}\in\mathcal{T}_0(\lambda,\mu),\ \mathtt{T}\in\mathcal{T}_0(\lambda,\nu)\text{ for some}\\ \mu,\nu\in\Lambda_r\text{ and }\lambda\in\Lambda_r^+\end{array}\right\}.$$

(ii) *Suppose that $\mu,\nu\in\Lambda_r$ and $\lambda\in\Lambda_r^+$ and let $\varphi\in\mathscr{S}$. Then, for every $\kappa\in\Lambda_r$ and every $\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)$ there exists $r_{\mathtt{T}'}\in R$ such that for all $\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)$, we have*

$$\varphi_{\mathtt{ST}}\varphi\equiv\sum_{\kappa\in\Lambda_r}\sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)}r_{\mathtt{T}'}\varphi_{\mathtt{ST}'}\bmod\overline{\mathscr{S}_\lambda}.$$

PROOF: (i) By Corollary 5.17(ii), $\operatorname{Hom}_{\mathscr{H}}(M^\nu,M^\mu)\cong M^{\nu*}\cap M^\mu$, where an isomorphism is given by $\varphi\longmapsto\varphi(m_\nu)$. Hence, by Proposition 6.3 and Definition 6.4, the set $\{\varphi_{\mathtt{ST}}\}$ is a basis of $\mathscr{S}$.

(ii) It is sufficient to consider the case where $\varphi\in\operatorname{Hom}_{\mathscr{H}}(M^\kappa,M^\nu)$ for some $\kappa\in\Lambda_r$. Suppose that $\varphi(m_\kappa)=m_\nu h$ where $h\in\mathscr{H}$. Then, for all $\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T}\in\mathcal{T}_0(\lambda,\nu)$, we have

$$\varphi_{\mathtt{ST}}\varphi(m_\kappa)=m_{\mathtt{ST}}h\in M^{\kappa*}\cap M^\mu.$$

By Proposition 6.3, $m_{\mathtt{ST}}h=\sum r_{\mathtt{UV}}m_{\mathtt{UV}}$, where $r_{\mathtt{UV}}\in R$ and the sum is over $\mathtt{U}\in\mathcal{T}_0(\alpha,\mu)$ and $\mathtt{V}\in\mathcal{T}_0(\alpha,\kappa)$ for some $\alpha\in\Lambda_r^+$. By applying Proposition 3.25 we deduce that

$$m_{\mathtt{ST}}h=\sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)}r_{\mathtt{T}'}m_{\mathtt{ST}'}+\sum_{\mathtt{U}',\mathtt{V}'}r_{\mathtt{U}'\mathtt{V}'}m_{\mathtt{U}'\mathtt{V}'}$$

where $r_{\mathtt{T}'},r_{\mathtt{U}'\mathtt{V}'}\in R$ and the second sum is over $\mathtt{U}'\in\mathcal{T}_0(\alpha,\mu)$ and $\mathtt{V}'\in\mathcal{T}_0(\alpha,\nu)$ for some $\alpha\in\Lambda_r^+$ with $\alpha\rhd\lambda$. Therefore,

$$\varphi_{\mathtt{ST}}\varphi\equiv\sum_{\mathtt{T}'\in\mathcal{T}_0(\lambda,\kappa)}r_{\mathtt{T}'}\varphi_{\mathtt{ST}'}\bmod\overline{\mathscr{S}_\lambda}.$$

This completes the proof. □

(6.7) Definition *We call the basis $\{\varphi_{\mathrm{ST}}\}$ the* semistandard basis *of $\mathscr{S}$.*

(6.8) Remark If $d \geq n$ then Λ_r^+ consists of all multipartitions of n. It is straightforward to prove that if $d > n$, then the algebra $\mathscr{S}$ is Morita equivalent to the algebra we obtain by taking $d = n$.

In order to show that the semistandard basis of $\mathscr{S}$ is a cellular basis we need an appropriate antiautomorphism $*$ for $\mathscr{S}$.

(6.9) Proposition *Let $*: \mathscr{S} \longrightarrow \mathscr{S}$ be the unique R–linear map such that $\varphi_{\mathrm{ST}}^* = \varphi_{\mathrm{TS}}$ for all elements φ_{ST} in the semistandard basis. Then $*$ is an antiautomorphism of $\mathscr{S}$.*

PROOF: Assume that $\mathrm{S} \in \mathcal{T}_0(\lambda, \mu)$ and that $\mathrm{T} \in \mathcal{T}_0(\lambda, \nu)$ for some $\mu, \nu \in \Lambda_r$ and $\lambda \in \Lambda_r^+$. Then

$$\varphi_{\mathrm{ST}}(m_\nu) = m_{\mathrm{ST}} = (m_{\mathrm{TS}})^* = (\varphi_{\mathrm{TS}}(m_\mu))^*.$$

The proposition now follows from the next general lemma.

(6.10) Lemma *Suppose that $\mathscr{H}$ is an algebra with an antiautomorphism $*$. Let $\{\, m_\mu \mid \mu \in \Lambda\,\}$ be a set of elements of $\mathscr{H}$ such that $m_\mu^* = m_\mu$ for all $\mu \in \Lambda$ and let*

$$S = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu \in \Lambda} m_\mu \mathscr{H}\Big).$$

Assume that, for all $\mu, \nu \in \Lambda$, every $\mathscr{H}$-homomorphism from $m_\nu \mathscr{H}$ to $m_\mu \mathscr{H}$ is given by left multiplication by an element of $\mathscr{H}$. Then the following hold.

(i) *For each $\varphi \in \operatorname{Hom}_{\mathscr{H}}(m_\nu \mathscr{H}, m_\mu \mathscr{H})$ there is a unique $\varphi^* \in \operatorname{Hom}_{\mathscr{H}}(m_\mu \mathscr{H}, m_\nu \mathscr{H})$ such that $\varphi^*(m_\mu) = (\varphi(m_\nu))^*$.*

(ii) *The map $*$ is an antiautomorphism of $\mathscr{S}$.*

PROOF: (i) Suppose that $\varphi \in \operatorname{Hom}_{\mathscr{H}}(m_\nu \mathscr{H}, m_\mu \mathscr{H})$. Then $\varphi(m_\nu) = x_1 m_\nu = m_\mu y$ for some $x_1, y \in \mathscr{H}$. Define $\varphi^* \in \operatorname{Hom}_{\mathscr{H}}(m_\mu \mathscr{H}, m_\nu \mathscr{H})$ by $\varphi^*(m_\mu h) = y^* m_\mu h$ for all $h \in \mathscr{H}$. Then φ^* is a well–defined $\mathscr{H}$–homomorphism; also, it maps into $m_\nu \mathscr{H}$, since $y^* m_\mu = m_\nu x_1^*$. Since $\varphi^*(m_\mu) = (\varphi(m_\nu))^*$, the proof of (i) is complete.

(ii) Suppose that $\psi \in \operatorname{Hom}_{\mathscr{H}}(m_\mu \mathscr{H}, m_\lambda \mathscr{H})$ for some $\lambda \in \Lambda$. Then $\psi(m_\mu) = x_2 m_\mu$ for some $x_2 \in \mathscr{H}$. We have,

$$\begin{aligned}(\psi\varphi)^*(m_\lambda) &= (\psi\varphi(m_\nu))^* &&= (x_2 x_1 m_\nu)^* &&= m_\nu x_1^* x_2^* \\ &= \varphi^*(m_\mu) x_2^* &&= \varphi^*(m_\mu x_2^*) &&= \varphi^* \psi^*(m_\lambda).\end{aligned}$$

Therefore, $(\psi\varphi)^* = \varphi^*\psi^*$, and it follows that $*$ is an antiautomorphism of $\mathscr{S}$. □

(6.11) Corollary *The R–module $\overline{\mathscr{S}_\lambda}$ is a two–sided ideal of $\mathscr{S}$.*

PROOF: Theorem 6.6(ii) shows that $\overline{\mathscr{S}_\lambda}$ is a right ideal. By applying the antiautomorphism $*$, we deduce that it is also a left ideal. □

(6.12) Theorem *The semistandard basis of $\mathscr{S}$ is a cellular basis.*

PROOF: This follows at once from Theorem 6.6, Corollary 6.11 and Proposition 6.9. □

We now apply the theory of cellular algebras to the representation theory of $\mathscr{S}$, just as we treated $\mathscr{H}$ in Section 3.

Suppose that λ is a multipartition of n. Let $\mathtt{T}^\lambda = \lambda(\mathtt{t}^\lambda)$ (see Definition 4.2). Then $\mathtt{T}^\lambda$ is the unique semistandard λ–tableau of type λ (cf. Example 4.3). Define $\varphi_\lambda = \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$; then φ_λ is the identity map on M^λ.

(6.13) Definition *Suppose that $\lambda \in \Lambda_r^+$, The* Weyl module *W^λ is the submodule of $\mathscr{S}/\overline{\mathscr{S}_\lambda}$ given by $W^\lambda = \mathscr{S}(\varphi_\lambda + \overline{\mathscr{S}_\lambda})$.*

If $\mathtt{S}$ is a semistandard λ–tableau let $\varphi_\mathtt{S} = \varphi_{\mathtt{S}\mathtt{T}^\lambda}(\varphi_\lambda + \overline{\mathscr{S}_\lambda}) = \varphi_{\mathtt{S}\mathtt{T}^\lambda} + \overline{\mathscr{S}_\lambda}$. Then, from Theorem 6.12, we obtain the following result.

(6.14) Corollary *The Weyl module W^λ is a free R–module with basis*

$$\{\, \varphi_\mathtt{S} \mid \mathtt{S} \in \mathcal{T}_0(\lambda,\mu) \text{ for some } \mu \in \Lambda_r \,\}.$$

Define a bilinear form $\langle\ ,\ \rangle$ on W^λ by requiring that

$$\varphi_{\mathtt{T}^\lambda\mathtt{S}}\varphi_{\mathtt{T}\mathtt{T}^\lambda} \equiv \langle \varphi_\mathtt{S}, \varphi_\mathtt{T}\rangle \varphi_\lambda \bmod \overline{\mathscr{S}_\lambda}$$

for all semistandard λ–tableaux $\mathtt{S}$ and $\mathtt{T}$. This bilinear form is well–defined and symmetric, and also satisfies $\langle su, v\rangle = \langle u, s^*v\rangle$ for all $u, v \in W^\lambda$ and all $s \in \mathscr{S}$ [**12**, (2.4)]. Consequently, $\operatorname{rad} W^\lambda = \{\, u \in W^\lambda \mid \langle u, v\rangle = 0 \text{ for all } v \in W^\lambda \,\}$ is a submodule of W^λ.

(6.15) Definition *Suppose that $\lambda \in \Lambda_r^+$. Let $F^\lambda = W^\lambda / \operatorname{rad} W^\lambda$.*

(6.16) Theorem *Suppose that R is a field. Then*

$$\{\, F^\lambda \mid \lambda \in \Lambda_r^+ \,\}$$

is a complete set of non–isomorphic irreducible $\mathscr{S}$–modules. Moreover, each F^λ is absolutely irreducible.

PROOF: Let $\lambda \in \Lambda_r^+$. From the definition of the bilinear form on W^λ, we have

$$\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda}\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda} \equiv \langle\varphi_{\mathrm{T}^\lambda}, \varphi_{\mathrm{T}^\lambda}\rangle\varphi_\lambda \bmod \overline{\mathscr{S}_\lambda}.$$

However, $\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda}\varphi_{\mathrm{T}^\lambda\mathrm{T}^\lambda} = \varphi_\lambda$ is the identity on M^λ; so $\langle\varphi_{\mathrm{T}^\lambda}, \varphi_{\mathrm{T}^\lambda}\rangle = 1$. Consequently, F^λ is non–zero. The Theorem now follows from [**12**, (3.4)]. □

If $\lambda, \mu \in \Lambda_r^+$, let $d_{\lambda\mu}$ denote the composition multiplicity of F^μ as a composition factor of W^λ. Then

$$(d_{\lambda\mu})_{\lambda,\mu\in\Lambda_r^+}$$

is the decomposition matrix of $\mathscr{S}$. The theory of cellular algebras [**12**, (3.6)] yields the following.

(6.17) Corollary *The decomposition matrix of $\mathscr{S}$ is unitriangular. That is, for $\lambda, \mu \in \Lambda_r^+$, we have $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.*

Finally, Theorem 6.16 combined with [**12**, (3.10)], gives us our last result.

(6.18) Corollary *The cyclotomic q–Schur algebra is quasi–hereditary.*

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