Unitary Kloosterman Sums and Gelfand-Graev Representation of GL_2

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Introduction

In this note, we obtain relations between two types of Kloosterman sums and corresponding ones for finite field extensions, using L-series and Euler product expansions. The methods used were first applied by A. Weil in his proof of the Davenport-Hasse relation for Gauss sums ([8], Chapter 11), and for other exponential sums in ([13], Appendix V). Generalized Kloosterman sums have also been considered in the context of l-adic sheaves over certain algebraic varieties by Deligne [6].

The relations proved in §1,2 are closely connected with the representation theory over the complex field $\mathbb{C}$ of the connected reductive algebraic group $G = GL_2$ defined over a finite field $\mathsf{F} = \mathbf{F}_q$, with Frobenius map σ, so that $G^\sigma = GL_2(\mathsf{F})$. In [5], a norm map Δ was defined, as a homomorphism of algebras from the Hecke algebra $\mathcal{H}_m$ of a Gelfand-Graev representation of G^{σ^m} to a corresponding Hecke algebra $\mathcal{H}$ of G^σ. The norm map Δ is characterized by intertwining relations involving the two classes of σ-stable maximal tori in G. The identities for Kloosterman sums give these intertwining relations, and an explicit formula for the norm map Δ, for certain basis elements of $\mathcal{H}_m$.

The relations for Kloosterman sums can also be used to calculate Gauss sums for certain irreducible representations of classical groups defined over finite fields, extending formulas obtained for representations of GL_n by Kondo [9] and Macdonald [11], based on the Davenport-Hasse relations for Gauss sums.

Notation $\mathsf{F} = \mathsf{F}_1 = \mathbf{F}_q$ denotes a finite field with q elements, and $\mathsf{F}_m = \mathbf{F}_{q^m}$ the extension field of degree m of F, contained in a fixed algebraic closure $\overline{\mathsf{F}}$ of F. For a field K, we denote by $K^\times$ the multiplicative group $K - \{0\}$ of K. If m divides n, the norm map $N_{n,m} : \mathsf{F}_n \to \mathsf{F}_m$ is defined by $N_{n,m}(\alpha) = \alpha^{\frac{1-q^n}{1-q^m}}$ and the trace map $T_{n,m} : \mathsf{F}_n \to \mathsf{F}_m$ is defined by $T_{n,m}(\alpha) = \sum_{i=0}^{d-1} \alpha^{q^{mi}}$ where $d = \frac{n}{m}$. Let $C_m \subset \mathsf{F}_{2m}^\times$ be the kernel of the norm map $N_{2m,m}$, so that $C_m = \{\alpha \in \mathsf{F}_{2m} : \alpha^{q^m+1} = 1\}$. In particular, $C = C_1$ is the subgroup of $\mathsf{F}_2^\times$ consisting of elements of norm 1.

We fix a nontrivial additive character χ of F throughout this paper, and let $\chi_m = \chi \circ Tr_{m,1}$ be the canonical lift of χ to F_m. Similarly, for a multiplicative character π (resp. φ) of $\mathsf{F}^\times$(resp. C), $\pi_m = \pi \circ N_{m,1}$ (resp. $\varphi_m = \varphi \circ N_{2m,2}$) denotes the

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canonical lift of π (resp. φ) to $\mathsf{F}_m^\times$ (resp. C_m). We shall also use the convention that $\pi(0) = \varphi(0) = 0$.

1. Generalized Kloosterman Sums and Finite Field Extensions

Let π be a multiplicative character of $\mathsf{F}^\times$. A generalized Kloosterman sum is defined as the complex valued function on $\mathsf{F}^\times$ given by

$$a \mapsto K(\chi, \pi, a) = \sum_{st=a} \chi(s+t)\pi(s), \; a \in \mathsf{F}^\times$$

where $s, t \in \mathsf{F}^\times$. The corresponding generalized Kloosterman sum for F_m is the function on $\mathsf{F}^\times$ defined by

$$a \mapsto K(\chi_m, \pi_m, a) = \sum_{\alpha\beta=a} \chi_m(\alpha+\beta)\pi_m(\alpha), \; a \in \mathsf{F}^\times$$

where $\alpha, \beta \in \mathsf{F}_m^\times$. If $a = 1$, we shall simply write $K(\chi, \pi)$ and $K(\chi_m, \pi_m)$ to denote $K(\chi, \pi, 1)$ and $K(\chi_m, \pi_m, 1)$ respectively.

We shall obtain a relation between $K(\chi_m, \pi_m, a)$ and $K(\chi, \pi, a)$, by a method similar to the proof of the Davenport-Hasse relation for Gauss sums ([8], Chapter 11, §4).

Let Φ be the set of monic polynomials in $\mathsf{F}[x]$ with nonzero constant term and let Φ_n be the set of polynomials in Φ of degree n. We begin by defining the complex valued function $\xi(f)$ for

$$f(x) = x^n - c_1x^{n-1} + \cdots + (-1)^n c_n \in \Phi_n$$

by the formula

$$\xi(f) = \begin{cases} 1 & \text{if } n = 0, \\ \chi(c_1 + ac_{n-1}c_n^{-1})\pi(c_n) & \text{if } n > 0. \end{cases}$$

It is easily verified that the function $\xi(f)$ has the following properties:

$$\begin{aligned} \xi(fg) &= \xi(f)\xi(g) \text{ for all } f, g \in \Phi, \\ \sum_{f\in\Phi_1} \xi(f) &= K(\chi, \pi, a), \text{ and} \\ \sum_{f\in\Phi_2} \xi(f) &= G(\chi, \pi)G(\chi, \overline{\pi})\pi(a) = q\pi(-a) \end{aligned}$$

where $G(\chi, \pi) = \sum_{t\in\mathsf{F}} \chi(t)\pi(t)$ is a Gauss sum, π is nontrivial, and $\overline{\pi}(t) = \pi(t^{-1})$. It is easily shown that if π is trivial, one has $\sum_{f\in\Phi_2} \xi(f) = q$. Finally, if $d > 2$

$$\sum_{f\in\Phi_d} \xi(f) = 0.$$

Following the proof of the Davenport-Hasse relation, one has

$$K(\chi_m, \pi_m, a) = \sum_f \deg f\, \xi(f)^{\frac{m}{\deg f}},$$

where the sum is taken over all irreducible polynomials in Φ whose degree divide m. Continuing as in ([8], p. 165), one obtains

$$\frac{uz + 2vz^2}{1 + uz + vz^2} = \sum_f \left(\sum_{r=1}^{\infty} \deg f\, \xi(f)^r z^{r \deg f} \right),$$

where the sum is taken over irreducible polynomials in Φ and

$$u = K(\chi, \pi, a), \;\; v = q\pi(-a)$$

for all multiplicative characters π. It follows that $K(\chi_m, \pi_m, a)$ is the coefficient of z^m on the left hand side. Therefore we have the following theorem.

Theorem 1. *Let π be an arbitrary character of $\mathsf{F}^\times$ and let ω_1 and ω_2 be the complex numbers defined by the equation*

$$(1 - \omega_1 z)(1 - \omega_2 z) = 1 + K(\chi, \pi, a)z + q\pi(-a)z^2.$$

Then for all positive integers m, we have

$$-\omega_1^m - \omega_2^m = K(\chi_m, \pi_m, a).$$

Remark. This result, in case π is trivial, is due to Carlitz [1].

2. Unitary Kloosterman Sums and Finite Field Extensions

2.1. Let φ be a character of C. The sum

$$J(\chi, \varphi) = \sum_{\alpha \in C} \chi(\alpha + \alpha^{-1})\varphi(\alpha)$$

is called a unitary Kloosterman sum. More generally, this definition can be extended to give a complex valued function on $\mathsf{F}^\times$ defined by

$$a \mapsto J(\chi, \varphi', a) = \sum_{\alpha \in \mathsf{F}_2, N_{2,1}(\alpha) = a} \chi_2(\alpha)\varphi'(\alpha),$$

for a multiplicative character φ' of $\mathsf{F}_2^\times$, where $\chi_2 = \chi \circ Tr_{2,1}$. Then $J(\chi, \varphi', 1) = J(\chi, \varphi' \mid_C)$. The sum $J(\chi, \varphi', a)$ appears in the calculation of cuspidal characters of

$GL_2(\mathsf{F})$ ([12], [2]).

2.2. Before establishing a relation between a unitary Kloosterman sum and the corresponding sum for a finite field extension, we consider the unitary Kloosterman sum $J(\chi,\varphi)$ with $\varphi^2 = 1$.

Firstly, letting $1_{\mathsf{F}^\times}$ and 1_C denote the trivial characters of $\mathsf{F}^\times$ and C, respectively, one has

$$J(\chi, 1_C, a) = -K(\chi, 1_{\mathsf{F}^\times}, a),$$

for $a \in \mathsf{F}^\times$, by Chang's lemma ([3], Lemma 1.2).

Now assume that q is odd and let π_0 be the quadratic character of $\mathsf{F}^\times$. For $s \in \mathsf{F}$, let χ^s be the additive character of F defined by $\chi^s(a) = \chi(sa), a \in \mathsf{F}$. Then one also has

$$K(\chi^s, \pi_0) = \pi_0(s)G(\chi,\pi_0)(\chi(2s) + \chi(-2s))$$

and in particular

$$K(\chi,\pi_0) = G(\chi,\pi_0)(\chi(2) + \chi(-2))$$

by a theorem of Evans [7] (cf. also [10], Exercise 5.85).

A similar result holds for a unitary Kloosterman sum. Let φ_0 be the quadratic character of C.

Proposition. *For all $s \in \mathsf{F}$, we have*

$$J(\chi^s, \varphi_0) = \pi_0(s)G(\chi,\pi_0)(\chi(-2s) - \chi(2s)),$$

and in particular

$$J(\chi, \varphi_0) = G(\chi,\pi_0)(\chi(-2) - \chi(2)).$$

Proof. Let $t = 2s$, then as in ([7], Theorem 2.6) it suffices to show that for all $a \in \mathsf{F}$

$$\begin{aligned} q^{-1}\sum_t J(\chi^{\frac{t}{2}}, \varphi_0)\chi(-at) \\ = q^{-1}\pi_0(2)G(\chi,\pi_0)\sum_t \pi_0(t)(\chi(t(-1-a)) - \chi(t(1-a)). \end{aligned}$$

The calculation in [loc.cit.] also holds in this case, and the left hand side of the above equation becomes $0, \varphi_0(a)$ or $2\varphi_0(a+\sqrt{a^2-1})$ according as $\pi_0(a^2-1) = 1, 0$ or -1, respectively, and the right hand side is $\pi_0(2a+2) - \pi_0(2a-2)$. Therefore the proof is reduced to showing that if $\pi_0(a^2-1) = -1$, then

$$\pi_0(2a+2) = \varphi_0(a+\sqrt{a^2-1}).$$

Now take the $(q-1)/2$-th power of the following equation:

$$a+\sqrt{a^2-1} = \left(1 + \frac{\sqrt{a^2-1}}{a+1}\right)^2 \frac{a+1}{2}.$$

Then the conclusion follows, since $\varphi_0(a+\sqrt{a^2-1}) = (a+\sqrt{a^2-1})^{(q+1)/2}$,

$$\pi_0(2a+2) = (\frac{a+1}{2})^{(q-1)/2} \quad \text{and}$$

$$(a+\sqrt{a^2-1})^{-1} = \left(1+\frac{\sqrt{a^2-1}}{a+1}\right)^{q-1} = a-\sqrt{a^2-1}.$$

Remark. The sums $K(\chi^s,\pi)$ and $J(\chi^s,\varphi)$ have also appeared in ([4], Theorem 5.2), in connection with the representation theory of SL_2.

2.3. Now we shall obtain a relation between a unitary Kloosterman sum and a corresponding one for a finite field extension.

For a polynomial

$$f(x) = x^n - c_1x^{n-1} + \cdots + (-1)^n c_n \in \mathsf{F}_2[x],$$

with $c_n \neq 0$, define

$$f^*(x) = (-1)^n \overline{f}(x^{-1})x^n\overline{c_n}^{-1},$$

where $\overline{c} = c^q$ is the algebraic conjugate of the element $c \in \mathsf{F}_2$ by the nontrivial element of the Galois group $c \mapsto \overline{c}$, and $\overline{f}(x)$ is the polynomial $\sum_{i=0}^n (-1)^{n-i}\overline{c}_{n-i}x^i$. It is easily verified that

$$f^{**}(x) = f(x),$$
$$(f(x)g(x))^* = f^*(x)g^*(x),$$

and that

$$f^*(x) = f(x)$$

if and only if

$$c_i = \overline{c}_{n-i}\overline{c_n}^{-1}, \ 0 \leq i \leq [\frac{n}{2}],$$

and

$$c_n \in C.$$

Let Ψ be the set of monic polynomials $f(x) \in \mathsf{F}_2[x]$ with nonzero constant term, satisfying $f^*(x) = f(x)$, and let Ψ_n be the set of polynomials in Ψ of degree n. Introduce the complex valued function $\eta(f)$, for $f \in \Psi$, by setting

$$\eta(f) = \begin{cases} 1, & \text{if } n = \deg f = 0, \\ \chi(c_1 + c_{n-1}c_n^{-1})\varphi(c_n), & \text{if } n = \deg f > 0, \end{cases}$$

noting that $c_1 + c_{n-1}c_n^{-1} \in \mathsf{F}$ if $f \in \Psi$. Thus as in §1, it follows that

$$\eta(fg) = \eta(f)\eta(g), \text{ for } f, g \in \Psi.$$

The L-function associated with the set of polynomials Ψ is the series

$$L_\Psi(z) = \sum_{n\geq 0}\left(\sum_{f\in\Psi_n}\eta(f)\right)z^n.$$

The multiplicative property of η implies that $L_\Psi(z)$ has the Euler product expansion

$$L_\Psi(z) = \prod_f (1-\eta(f)z^{\deg f})^{-1} \prod_{\{g,g^*\}} (1-\eta(gg^*)z^{2\deg g})^{-1}, \tag{2.3.1}$$

where $\{f\}$ and $\{g\}$ are irreducible polynomials in $\mathsf{F}_2[x]$ such that $f = f^*$ and $g \neq g^*$, respectively.

We now proceed to calculate the coefficients of $L_\Psi(z)$. We have immediately

$$\begin{aligned}
\sum_{f\in\Psi_1} \eta(f) &= J(\chi,\varphi),\\
\sum_{f\in\Psi_2} \eta(f) &= \begin{cases} G(\chi_2,\varphi'), & \text{if } \varphi \neq 1_C,\\ q, & \text{if } \varphi = 1_C \end{cases}\\
\sum_{f\in\Psi_n} \eta(f) &= 0, \text{ if } n > 2,
\end{aligned}$$

where $\chi_2 = \chi \circ Tr_{2,1}$ as before and $\varphi'(\alpha) = \varphi(\alpha^{1-q})$ for $\alpha \in \mathsf{F}_2^\times$. By Stickelberger's Theorem ([10], Theorem 5.16) we have $G(\chi_2,\varphi') = q\varphi(-1)$ in case φ is nontrivial. If $\varphi = 1_C$, the formula follows from the definition of $\eta(f)$. At this point we have

$$L_\Psi(z) = 1 + J(\chi,\varphi)z + \varphi(-1)qz^2.$$

Define the complex numbers ω_1 and ω_2 by

$$L_\Psi(z) = (1-\omega_1 z)(1-\omega_2 z).$$

Now let

$$z\frac{d}{dz}\log L_\Psi(z) = \sum_{m\geq 1} L_m z^m.$$

By the preceding discussion it follows that L_m is the coefficient of z^m in

$$\frac{uz+2vz^2}{1+uz+vz^2},$$

where $u = J(\chi,\varphi)$, $v = \varphi(-1)q$, and can also be expressed in the form

$$L_m = -\omega_1^m - \omega_2^m. \tag{2.3.2}$$

The desired relation will follow from

2.4. Lemma. *For $m \geq 1$,*

$$L_m = \begin{cases} \sum_{\gamma\in\mathsf{F}_m^\times} \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})), & \text{if } m \text{ is even},\\ \sum_{\gamma\in C_m} \chi(Tr_{2m,2}(\gamma+\gamma^{-1}))\varphi(N_{2m,2}(\gamma)), & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Before starting a proof of this lemma, we prove the following result, which plays a key role in the proof.

(2.4.1) *Let* $f(x) \in \mathsf{F}_2[x]$ *be irreducible and* $f^*(x) = f(x)$. *Then* $d = \deg f$ *is an odd integer. Moreover if* γ *is a root of* f, *then* $\gamma^{q^d+1} = 1$.

To prove this assertion, let γ be a root of $f(x)$. Then the condition $f^*(x) = f(x)$ implies that $f(\gamma^{-q}) = 0$ so that $\gamma = (\gamma^{-q})^{q^{2i}}$ for some i, $0 \le i < d = \deg f$, since f is irreducible. Thus γ is contained in $\mathsf{F}_{2(2i+1)}$ and hence $\mathsf{F}_{2d} \subseteq \mathsf{F}_{2(2i+1)}$, because γ generates the field F_{2d}. Therefore d divides $2i+1$ which is possible only when $d = 2i+1$.

Now by the Euler product expansion (2.3.1) and the definition of L_m, we have

$$L_m = \sum_{\substack{f=f^* \\ \deg f|m}} \deg f\, \eta(f)^{\frac{m}{\deg f}} + \sum_{\substack{\{g,g^*\} \\ 2\deg g|m}} 2\deg g\, \eta(gg^*)^{\frac{m}{2\deg g}}. \tag{2.4.2}$$

Assume that m is even. If $f \in \mathsf{F}_2[x]$ is an irreducible polynomial satisfying $f = f^*$ and $d|m$, where $d = \deg f$, then $2d$ divides m, since d is odd by (2.4.1). So if γ is a root of $f(x) = \sum_{i=0}^{d}(-1)^{d-i}c_{d-i}x^i$, then $\gamma \in \mathsf{F}_m$ and $c_1 = Tr_{2d,2}(\gamma)$, $c_{d-1}c_d^{-1} = Tr_{2d,2}(\gamma^{-1})$ and $c_d = N_{2d,2}(\gamma) \in C$. Therefore we have

$$\begin{aligned}
\eta(f)^{\frac{m}{d}} &= \chi(\frac{m}{d}Tr_{2d,2}(\gamma+\gamma^{-1}))\varphi(N_{2d,2}(\gamma^{\frac{m}{d}})) \\
&= \chi(Tr_{2m,2}(\gamma+\gamma^{-1}))\varphi(N_{2m,2}(\gamma)) \\
&= \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})),
\end{aligned}$$

where to get the third equation we have used the fact that $\gamma \in \mathsf{F}_m$ and γ is conjugate to γ^{-q} over F_2.

Similarly if $g \in \mathsf{F}_2[x]$ is an irreducible polynomial satisfying $g \neq g^*$ and $2\deg g|m$, and if γ is a root of g or g^*, we have

$$\eta(gg^*)^{\frac{m}{2\deg g}} = \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})).$$

Since such f and g have distinct roots, we have by (2.4.2)

$$\begin{aligned}
L_m &= \sum_{\substack{f=f^* \\ \deg f|m}} \sum_{\substack{\gamma\in\mathsf{F}_m \\ f(\gamma)=0}} \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})) \\
&\quad + \sum_{\substack{\{g,g^*\} \\ 2\deg g|m}} \sum_{\substack{\gamma\in\mathsf{F}_m \\ g(\gamma)g^*(\gamma)=0}} \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})) \\
&= \sum_{\gamma\in\mathsf{F}_m^\times} \chi(Tr_{m,1}(\gamma+\gamma^{-1}))\varphi(N_{m,2}(\gamma^{1-q})).
\end{aligned}$$

When m is odd, the second sum in the right hand side of (2.4.2) vanishes. Moreover if $f \in \Psi_d$ is irreducible with $d|m$ and if γ is a root of f, then $\gamma^{q^d+1} = 1$ by (2.4.1) and hence $\gamma \in C_m$, since both m and d are odd and $d|m$. The rest of the proof for this case holds as in the case when m is even.

2.5. When m is even, define the multiplicative character φ_m of $\mathsf{F}_m^\times$ by

$$\varphi_m(\gamma) = \varphi(N_{m,2}(\gamma^{1-q})) = \varphi(\gamma^{\frac{1-q^m}{1+q}}),$$

and $\chi_m = \chi \circ Tr_{m,1}$. Then, as introduced in §1, the generalized Kloosterman sum associated with χ_m and φ_m is

$$K(\chi_m, \varphi_m) = \sum_{\gamma \in \mathsf{F}_m^\times} \chi_m(\gamma + \gamma^{-1})\varphi_m(\gamma),$$

and is equal to L_m by the proof of Lemma 2.4. When m is odd, define the character φ_m of C_m by $\varphi_m(\gamma) = \varphi(N_{2m,2}(\gamma))$, noting that $N_{2m,2}(\gamma) \in C$ if $\gamma \in C_m$, because m is odd, $\gamma^{q^m} = \gamma^{-1}$, and $N_{2m,2}(\gamma) = \gamma^{\frac{1+q^m}{1+q}}$. We also have, for $\gamma \in C_m$, $Tr_{2m,2}(\gamma + \gamma^{-1}) = Tr_{m,1}(\gamma + \gamma^{-1})$. In this case, L_m is equal to the unitary Kloosterman sum

$$J(\chi_m, \varphi_m) = \sum_{\gamma \in C_m} \chi_m(\gamma + \gamma^{-1})\varphi_m(\gamma).$$

Combining these remarks with (2.3.2) and (2.4), we have:

Theorem 2. *Let ω_1 and ω_2 be the complex numbers determined by*

$$(1 - \omega_1 z)(1 - \omega_2 z) = 1 + J(\chi, \varphi)z + \varphi(-1)qz^2.$$

Then for every positive integer m, we have

$$-\omega_1^m - \omega_2^m = \begin{cases} K(\chi_m, \varphi_m), & \text{if } m \text{ is even}, \\ J(\chi_m, \varphi_m), & \text{if } m \text{ is odd}. \end{cases}$$

2.6. Let φ' be a multiplicative character of F_2 and a be a fixed element in $\mathsf{F}^\times$. We shall extend the results obtained in (2.5) to the unitary Kloosterman sum introduced in (2.1).

We start by defining an involutive mapping $*a$ on the set of monic polynomials over F_2 with nonzero constant term. For $f(x) = \sum_{i=0}^n (-1)^{n-i} c_{n-i} x^i \in \mathsf{F}_2[x]$ with $c_0 = 1$ and $c_n \neq 0$, define the polynomial f^{*a} by

$$f^{*a}(x) = \overline{f}(\frac{a}{x})x^n(-1)^n \overline{c}_n^{-1}.$$

Then it can be easily checked that $(f^{*a})^{*a} = f$ and that $f^{*a} = f$ if and only if

$$\begin{cases} c_i = \overline{c}_{n-i}\overline{c_n}^{-1}a^i, & 0 < i \leq [\frac{n}{2}], \\ N_{2,1}(c_n) = a^n. \end{cases}$$

Let $\Psi^a = \{f(x) \in \mathsf{F}_2[x] : f(x)$ is monic, $f(0) \neq 0$ and $f^{*a} = f\}$ and Ψ_n^a be the subset of Ψ^a consisting of polynomials of degree n. Defining a complex valued function $\eta_a(f)$ on Ψ^a for

$$f(x) = x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n \in \Psi_n^a$$

by the formula

$$\eta_a(f) = \begin{cases} 1, & \text{if } n = \deg f = 0, \\ \chi(c_1 + a c_{n-1} c_n^{-1})\varphi'(c_n), & \text{if } n = \deg f > 0, \end{cases}$$

we can proceed with the same manner as in (2.3) and (2.4) and, omitting the details, we can prove the following theorem.

Theorem 3. *Let φ' be a character of $\mathsf{F}_2^\times$ and ω_i $(i = 1, 2)$ be the complex numbers determined by*

$$(1 - \omega_1 z)(1 - \omega_2 z) = 1 + J(\chi, \varphi', a)z + \varphi'(-a)qz^2.$$

Then for every positive integer m, we have

$$-\omega_1^m - \omega_2^m = \begin{cases} \varphi'(a^{\frac{m}{2}})K(\chi_m, {\varphi'_m}^{1-q}, a), & \text{if } m \text{ is even}, \\ J(\chi_m, \varphi'_{2m}, a), & \text{if } m \text{ is odd}, \end{cases}$$

where $\chi_m = \chi \circ Tr_{m,1}, \varphi'_{2m} = \varphi' \circ N_{2m,2}$ and $\varphi'_m = \varphi' \circ N_{m,2}$.

Remark. A proof of the Davenport-Hasse relation based on étale cohomology has been given by Deligne ([6], p. 177). The same approach can be applied to the relations considered in Theorem 1-3. For example, Theorem 3, in case φ' is trivial, will follow from ([6], Exemple 3.7, p. 193) by taking the Frobenius map defined on the projective line $\mathbf{P}^1$ by $[x : y] \mapsto [y^q : ax^q]$, for $a \in \mathsf{F}^\times$.

3. Connections with the Gelfand-Graev representation of GL_2

The fact that the formula for $K(\chi_m, \pi_m)$ in terms of $K(\chi, \pi)$, and for $J(\chi_m, \varphi_m)$ or $K(\chi_m, \pi_m)$ in terms of $J(\chi, \varphi)$, are both obtained as the coefficient of z^m in the expansion of the rational function

$$\frac{uz + 2vz^2}{1 + uz + vz^2}$$

is explained by the norm map Δ from the Hecke algebra $\mathcal{H}_m$ of a Gelfand-Graev representation of $GL_2(\mathsf{F}_m)$ to the Hecke algebra $\mathcal{H}$ of a Gelfand-Graev representation of $GL_2(\mathsf{F})$ ([5]).

To make this connection, let G be the connected reductive group GL_2, with its usual BN-pair, $\mathbf{F}_q$-structure, and Frobenius map σ, so that $G^\sigma = GL_2(\mathsf{F})$ and $G^{\sigma^m} = GL_2(\mathsf{F}_m)$, for a fixed $m \geq 1$. The definition of Δ involves representatives of the two

G^σ-classes of σ-stable maximal tori in G, and their norm maps. The classes of tori are parametrized by the elements $\{1, w\}$ of the Weyl group of G. The split torus T_1 consists of the diagonal matrices in G, with the norm map $N_1 = N_{T_1^{\sigma^m}/T_1^\sigma}$ from $T_1^{\sigma^m}$ to T_1^σ, given by

$$N_1 = 1 + \sigma + \cdots + \sigma^{m-1}.$$

For the other class of tori, we have

$$T_w^\sigma \cong T^{\sigma \mathrm{ad}\,\dot{w}} \cong \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix} : \alpha \in \mathsf{F}_2^\times \right\},$$

with $\dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathrm{ad}\,\dot{w}(t) = \dot{w}t\dot{w}^{-1}$, for $t \in T_1$. The norm map N_w from $T_w^{\sigma^m}$ to T_w^σ, after these identifications, is given by

$$N_w = 1 + \sigma\mathrm{ad}\dot{w} + \cdots + (\sigma\mathrm{ad}\dot{w})^{m-1}.$$

Let χ be a nontrivial additive character of F, and $\Gamma = \mathrm{ind}_{U_0^\sigma}^{G^\sigma}\psi$ the Gelfand-Graev character defined by the linear charcter ψ of U_0^σ, with

$$\psi \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \chi(a),\ a \in \mathsf{F}.$$

Let $\mathcal{H}$ be the Hecke algebra associated with Γ, as in ([4], §5). For fixed $m \geq 1$, $\chi_m = \chi \circ Tr_{m,1}$ is a nontrivial additive character of F_m, and defines a Gelfand-Graev character Γ_m of G^{σ^m}, whose Hecke algebra is denoted by $\mathcal{H}_m$.

For an σ-stable maximal torus T of G, let $f_T : \mathcal{H} \to \mathbb{C}T^\sigma$, and $f_T^{(m)} : \mathcal{H}_m \to \mathbb{C}T^{\sigma^m}$ be the homomorphisms defined in ([4], Theorem 4.2). By Theorem 1 and Theorem 2 of [5], there exists a surjective homomorphism of algebras

$$\Delta : \mathcal{H}_m \to \mathcal{H},$$

which is characterized as the unique linear map such that

$$\tilde{N}_1 \circ f_{T_1}^{(m)} = f_{T_1} \circ \Delta$$

and

$$\tilde{N}_w \circ f_{T_w}^{(m)} = f_{T_w} \circ \Delta,$$

with $\tilde{N}_1 : \mathbb{C}T^{\sigma^m} \to \mathbb{C}T^\sigma$ the extension of $N_1 : T^{\sigma^m} \to T^\sigma$, etc. We shall prove that these two intertwining formulas correspond exactly to the identities for $K(\chi_m, \pi_m)$ in terms of $K(\chi, \pi)$ and $J(\chi_m, \varphi_m)$ or $K(\chi_m, \varphi_m)$ in terms of $J(\chi, \varphi)$, in the calculation of Δ at the standard basis element of $\mathcal{H}_m$ corresponding to $\dot{w} \in G^{\sigma^m}$.

3.1. Lemma. *Let $c_{\dot{w}}$ be the standard basis element of $\mathcal{H}$ indexed by $\dot{w}$ (as in [4], §5, for the case of SL_2). Then $f_{T_1}(c_{\dot{w}}) \in \mathbb{C}T_1^\sigma$, and can be identified with the element in the group algebra of $\mathsf{F}^\times$ whose coefficient at $a \in \mathsf{F}^\times$ is $\chi(a + a^{-1})$. The image $f_{T_w}(c_{\dot{w}})$*

of $c_{\dot w}$ in $\mathbb{C}T^{\sigma\mathrm{ad}\dot w}$ can be identified with the element of the group algebra of C whose coefficient at $\alpha \in C$ is $-\chi(\alpha+\alpha^{-1})$.

The proof of the Lemma is straightforward, using the formula

$$f_T(c_{\dot w})(t) = \\ < Q_T^G, \Gamma > \mathrm{ind}\dot w |U_0^\sigma|^{-1} |C_G(t)^{0\sigma}|^{-1} \sum_{\substack{g\in G^\sigma, u \in U_0^\sigma \\ (gu\dot w g^{-1})_{ss}=t}} \psi(u^{-1}) Q_T^{C_G(t)^0}((gu\dot w g^{-1})_{uni}),$$

for $t \in T^\sigma$, as in the proof of Theorem 5.2 of [4].

By the Lemma, the image of the standard basis element $c_{\dot w,m}$ of $\mathcal{H}_m$ corresponding to $\dot w$, under the homomorphism $f_{T_1}^{(m)} : \mathcal{H}_m \to \mathbb{C}T_1^{\sigma^m}$, is the element of the group algebra of $\mathsf{F}_m^\times$ whose coefficient at $\alpha \in \mathsf{F}_m^\times$ is $\chi_m(\alpha+\alpha^{-1})$. Noting that $T_w^{\sigma^m} \cong T_1^{(\sigma\mathrm{ad}\dot w)^m} \cong T_1^{\sigma^m \mathrm{ad}\dot w}$ if m is odd, and $T_w^{\sigma^m} \cong T_1^{\sigma^m}$ if m is even, the image of $c_{\dot w,m}$ by the homomorphism f_{T_w} can be identified with the element of the group algebra of C_m whose coefficient at $\alpha \in C_m$ is $-\chi_m(\alpha+\alpha^{-1})$ for $\alpha \in C_m$ if m is odd, and with the element of the group algebra of $\mathsf{F}_m^\times$ whose coefficient at $\alpha \in \mathsf{F}_m^\times$ is $\chi_m(\alpha+\alpha^{-1})$, if m is even.

The irreducible representations of the group algebra of $\mathsf{F}^\times$ are the extensions $\tilde\pi$ of multiplicative characters π of $\mathsf{F}^\times$. For each such representation, we have

$$\tilde\pi \circ f_{T_1}(c_{\dot w}) = \sum_{a\in \mathsf{F}^\times} \pi(a)\chi(a+a^{-1}) = K(\chi,\pi)$$

and, as is easily verified,

$$\tilde\pi \circ \tilde N_1 \circ f_{T_1}^{(m)}(c_{\dot w,m}) = K(\chi_m, \pi_m),$$

where $\tilde N_1$ denotes the extension of the norm map $N_1 : T^{\sigma^m} \to T^\sigma$ to a homomorphism of group algebras.

Similarly, the irreducible representations of the group algebra of C are the extensions $\tilde\varphi$ of characters φ of C, and one has

$$\tilde\varphi \circ f_{T_w}(c_{\dot w,m}) = -\sum_{\alpha\in C} \varphi(\alpha)\chi(\alpha+\alpha^{-1}) = -J(\chi,\varphi).$$

Using the twisted norm map $N_w = 1+\sigma\mathrm{ad}\dot w+\cdots+(\sigma\mathrm{ad}\dot w)^{m-1}$ from $T_1^{(\sigma\mathrm{ad}\dot w)^m}$ to $T_1^{\sigma\mathrm{ad}\dot w}$, one has

$$\tilde\varphi \circ \tilde N_w \circ f_{T_w}^{(m)}(c_{\dot w,m}) = -J(\chi_m,\varphi_m)$$

if m is odd, and when m is even,

$$\tilde\varphi \circ \tilde N_w \circ f_{T_w}^{(m)}(c_{\dot w,m}) = K(\chi_m,\varphi_m),$$

where $J(\chi_m,\varphi_m)$ and $K(\chi_m,\varphi_m)$ are defined as in §2.5.

Let $P_m(u,v)$ be the coefficient of z^m in the expansion of the rational function

$$\frac{uz+2vz^2}{1+uz+vz^2},$$

with u, v viewed as indeterminates over $\mathbb{C}(z)$. The formulas for $K(\chi_m, \pi_m)$ in terms of $K(\chi, \pi)$, and for $J(\chi_m, \varphi_m)$ or $K(\chi_m, \varphi_m)$ in terms of $J(\chi, \varphi)$ are both given in terms of the polynomial $P_m(u, v)$, by Theorems 1 and 2. It is shown here that the value of the norm map $\Delta : \mathcal{H}_m \to \mathcal{H}$ at the standard basis element $c_{\dot{w},m}$ is also given by the polynomial $P_m(u, v)$. Other values of Δ can be obtained from the result (see the remarks following the Proposition). The polynomial $(-1)^{m-1}P_m(u, v)$, called the *Dickson polynomial* in [10], is given by

$$P_m(u,v) = \sum_{j=0}^{[\frac{m}{2}]} (-1)^{m-j-1} \frac{m}{m-j} \binom{m-j}{j} u^{m-2j} v^j$$

([10] p. 228 and p. 355).

Let $P_m(c_{\dot{w}}, qc_{-I})$ be the element of $\mathcal{H}$ obtained by substituting for u and v the standard basis elements $c_{\dot{w}}$ and c_{-I}, where I is the identity matrix in G. It is easily verified that $\tilde{\pi} \circ f_{T_1}(c_{-I}) = \pi(-1)$, for each multiplicative character π of $\mathsf{F}^\times$, and $\tilde{\varphi} \circ f_{T_w}(c_{-I}) = \varphi(-1)$ for an arbitrary character φ of C.

We now have

Proposition. *The element $P_m(c_{\dot{w}}, qc_{-I}) \in \mathcal{H}$ satisfies the intertwining relations*

$$\tilde{N}_1 \circ f_{T_1}^{(m)}(c_{\dot{w},m}) = f_{T_1}(P_m(c_{\dot{w}}, qc_{-I}))$$

and

$$\tilde{N}_w \circ f_{T_w}^{(m)}(c_{\dot{w},m}) = f_{T_w}(P_m(c_{\dot{w}}, qc_{-I})).$$

It follows that

$$\Delta(c_{\dot{w},m}) = P_m(c_{\dot{w}}, qc_{-I}).$$

Proof. By Theorem 1, it follows that, for each character π of $\mathsf{F}^\times$,

$$K(\chi_m, \pi_m) = P_m(K(\chi, \pi), q\pi(-1)).$$

By the remarks preceding the statement of the proposition, this formula implies that

$$\begin{aligned} \tilde{\pi} \circ \tilde{N}_1 \circ f_{T_1}^{(m)}(c_{\dot{w},m}) &= P_m(\tilde{\pi} \circ f_{T_1}(c_{\dot{w}}), q\tilde{\pi} \circ f_{T_1}(c_{-I})) \\ &= \tilde{\pi}(P_m(f_{T_1}(c_{\dot{w}}), qf_{T_1}(c_{-I})) \\ &= \tilde{\pi} \circ f_{T_1}(P_m(c_{\dot{w}}, qc_{-I})) \end{aligned}$$

using the facts that $\tilde{\pi}$ and f_{T_1} are homomorphisms of algebras. As the maps $\{\tilde{\pi}\}$ are a complete set of irreducible representations of the group algebra of $\mathsf{F}^\times$, it follows that

$$\tilde{N}_1 \circ f_{T_1}^{(m)}(c_{\dot{w},m}) = f_{T_1}(P_m(c_{\dot{w}}, qc_{-I})),$$

proving the first relation.

Now let φ be a character of C, and let m be odd. Then, by Theorem 2,

$$-J(\chi_m, \varphi_m) = P_m(-J(\chi, \varphi), q\varphi(-1)).$$

This implies

$$\tilde{\varphi} \circ \tilde{N}_w \circ f_{T_w}^{(m)}(c_{\dot{w},m}) = P_m(\tilde{\varphi} \circ f_{T_w}(c_{\dot{w}}), q\tilde{\varphi} \circ f_{T_w}(c_{-I}))$$

and the relation

$$\tilde{N}_w \circ f_{T_w}^{(m)}(c_{\dot{w},m}) = f_{T_w}(P_m(c_{\dot{w}}, qc_{-I}))$$

follows as in the first case.

If m is even, Theorem 2 implies that

$$K(\chi_m, \varphi_m) = P_m(-J(\chi, \varphi), q\varphi(-1)),$$

and hence

$$\tilde{\varphi} \circ \tilde{N}_w \circ f_{T_w}^{(m)}(c_{\dot{w},m}) = P_m(\tilde{\varphi} \circ f_{T_w}(c_{\dot{w}}), q\tilde{\varphi} \circ f_{T_w}(c_{-I})).$$

The relation

$$\tilde{N}_w \circ f_{T_w}^{(m)}(c_{\dot{w},m}) = f_{T_w}(P_m(c_{\dot{w}}, qc_{-I}))$$

follows as before.

The fact that $\Delta(c_{\dot{w},m}) = P_m(c_{\dot{w}}, qc_{-I})$ now follows from Theorem 1 of [5].

The preceding result also gives the value of Δ at standard basis elements $c_{t\dot{w},m}$, with t in the center of $GL_2(\mathsf{F}_m)$. If $t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathsf{F}_m^\times$, then $f_{T_1}^{(m)}(c_{t\dot{w},m})$ can be identified with the element of the group algebra of $\mathsf{F}_m^\times$ whose coefficient at $\alpha \in \mathsf{F}_m^\times$ is $\chi_m(\alpha\lambda^{-1} + \alpha^{-1}\lambda)$. For a character π of $\mathsf{F}^\times$, one has

$$\tilde{\pi} \circ \tilde{N}_1 \circ f_{T_1}^{(m)}(c_{t\dot{w},m}) = \sum_{a\in \mathsf{F}^\times} \Big(\sum_{\substack{\alpha \in \mathsf{F}_m^\times \\ N\alpha = a}} \chi_m(\alpha\lambda^{-1} + \alpha^{-1}\lambda) \Big) \pi(a) = K(\chi_m, \pi_m)\pi_m(\lambda).$$

A similar result holds for the torus T_w, using the formula for $f_{T_w}^{(m)}(c_{t\dot{w},m})$ from ([2], §2). By the Proposition above, and the facts that $f_{T_1}(c_s)$ and $f_{T_w}(c_s)$ can both be identified with the element s in the group algebra of the torus for a central element $s \in GL_2(\mathsf{F})$, it follows that

$$\Delta(c_{t\dot{w},m}) = c_{N_1(t)}\Delta(c_{\dot{w},m}) = c_{N_1(t)}P_m(c_{\dot{w}}, qc_{-I}).$$

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