QMW-97-2
NI97005
hep-th/9702067
February, 1997

Actions For $(2,1)$ Sigma-Models and Strings

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ABSTRACT

Effective actions are derived for (2,0) and (2,1) superstrings by studying the corresponding sigma-models. The geometry is a generalisation of Kahler geometry involving torsion and the field equations imply that the curvature with torsion is self-dual in four dimensions, or has $SU(n,m)$ holonomy in other dimensions. The Yang-Mills fields are self-dual in four dimensions and satisfy a form of the Uhlenbeck-Yau equation in higher dimensions. In four dimensions with Euclidean signature, there is a hyperkahler structure and the sigma-model has (4,1) supersymmetry, while for signature (2,2) there is a hypersymplectic structure consisting of a complex structure squaring to $-\mathbb{1}$ and two 'real structures' squaring to $\mathbb{1}$. The theory is invariant under a twisted form of the (4,1) superconformal algebra which includes an $SL(2,\mathbb{R})$ Kac-Moody algebra instead of an $SU(2)$ Kac-Moody algebra. Kahler and related geometries are generalised to ones involving real structures.

1. Introduction

Martinec and Kutasov [1-3] have argued that different vacua of the superstring with (2,1) world-sheet supersymmetry correspond to the 11-dimensional membrane, the type IIB string, the heterotic and the type I strings. This suggests that the (2,1) string could be useful in the search for the degrees of freedom appropriate for the description of the fundamental theory underpinning M-theory and superstring theory. (Another proposal is given by the matrix model [5].) It was shown in [6] that the usual string with (2,2) supersymmetry is a theory of self-dual gravity in a four-dimensional space-time with signature (2,2) governed by the Plebanski action, while in [7] it was argued that the (2,2) supersymmetric string based on twisted chiral multiplets is a theory of self-dual gravity with torsion, which turns out to be a free theory. The (2,1) and (2,0) strings [8] are again formulated in a four-dimensional space-time with signature (2,2), but now a null reduction must be imposed to obtain a space with signature (2,1) (which corresponds to a membrane world-volume [1]) or a space with signature (1,1) (which corresponds to a string world-sheet [1]). The (2,2)-dimensional theory before null-reduction is a theory of gravity with torsion coupled to Yang-Mills gauge fields. The purpose of this paper is to investigate further the target space geometry of (2,0) and (2,1) strings and sigma-models.

We give an effective action for the gravitational and anti-symmetric tensor degrees of freedom of the (2,1) string which was obtained independently by Martinec and Kutasov [3], who also proposed a generalisation to include the Yang-Mills fields and checked that this agrees with the S-matrix of (2,1) strings to quartic order in the fields. A sigma-model derivation of this action is given and generalised to give an effective action whose variation gives the conditions found in [10-13] for conformal invariance of general (2,0) and (2,1) sigma-models. The geometry is a generalisation of Kahler geometry with torsion [9] and the field equations imply that the curvature with torsion is self-dual in four dimensions, or has $SU(n,m)$ holonomy in other dimensions. In four dimensions with Euclidean signature, there

is a hyperkahler structure and the sigma-model has (4,1) supersymmetry, while for signature (2,2) there is a hypersymplectic structure [14,15] – instead of three complex structures squaring to $-\mathbb{1}$, there is a complex structure and two 'real structures' or 'locally product structures' squaring to $+\mathbb{1}$ – and the model is invariant under a twisted form of the (4,1) superconformal algebra which includes an $SL(2,\mathbb{R})$ Kac-Moody algebra instead of an $SU(2)$ Kac-Moody algebra. Kahler and related geometries are generalised to ones involving real structures. The Yang-Mills fields are self-dual in four dimensions and satisfy the Uhlenbeck-Yau equation $g^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = 0$ in higher dimensions, but where the metric $g_{\alpha\bar{\beta}}$ involves corrections dependent on the gauge-fields. The action is related to that of [16], and involves the Bott-Chern form [17,18].

The results regarding the amount of supersymmetry can be summarised in terms of the holonomy of a certain connection, which will have torsion in general. For Euclidean signature, the holonomy of a general D-dimensional manifold M is $O(D)$, and a (1,1) supersymmetric sigma-model can be defined on M. If $D = 2n$ and the holonomy $\mathcal{H}$ is in $U(n)$, there is a covariantly constant complex structure J, $J^2 = -\mathbb{1}$, and the (1,1) model in fact has (2,1) supersymmetry. If $\mathcal{H}$ is in $SU(n)$, then there is a covariantly constant spinor and so such a space preserves some space-time supersymmetry, and the space is a solution of string theory (or related to one by a certain deformation). In the case of compact M with vanishing torsion, these are the Calabi-Yau spaces. If $n = 2m$ and $\mathcal{H} \subseteq USp(m)$, (where $USp(m)$ is compact, with the convention that $USp(1) = SU(2)$) then there is a covariantly constant hyperkahler structure consisting of three complex structures I, J, K satisfying the quaternion algebra

$$\begin{aligned} &I^2 = J^2 = K^2 = -\mathbb{1}, \\ &IJ = -JI = K, JK = -KJ = I, KI = -IK = J \end{aligned} \tag{1.1}$$

and the (1,1) model has (4,1) supersymmetry.

These results [13] for Euclidean signatures are well-known, but they can be generalised to other signatures. For signature $(2n_1, 2n_2)$, if $\mathcal{H}$ is in $U(n_1, n_2)$ then

there is again a complex structure and (2,1) supersymmetry, and the generalised Calabi-Yau condition is $\mathcal{H} \subseteq SU(n_1, n_2)$. Consider now the case of signature (d, d), for which the name Kleinian geometry was suggested in [14]; the case $d = 2$ is relevant for $(2, p)$ strings. In general the holonomy is in $O(d, d)$, but if $d = 2n$ and $\mathcal{H} \subseteq U(n, n)$ there is a complex structure J leading to (2,1) supersymmetry and the Calabi-Yau-type condition is $\mathcal{H} \subseteq SU(n, n)$. If on the other hand $\mathcal{H} \subseteq GL(n, \mathbb{R})$, then there is a real structure S satisfying $S^2 = \mathbb{1}$ and there is an extra supersymmetry, but the right-handed superalgebra is of the form

$$\{Q^A, Q^B\} = \eta^{AB} P \tag{1.2}$$

where $A = 1, 2$ and $\eta^{AB} = diag(1, -1)$ and P is the right-moving momentum. If there is no torsion, then the metric is given in terms of a scalar potential analogous to the Kahler potential, while if there is torsion, both the metric and torsion are given in terms of a vector potential, analogous to the one in [9]. The condition $\mathcal{H} \subseteq SL(n, \mathbb{R})$ is the analogue of the Calabi-Yau condition; it implies Ricci-flatness if there is no torsion, or the generalisation of this that corresponds to the string field equations if there is torsion. Finally, if $n = 2m$ and $\mathcal{H} \subseteq Sp(m, \mathbb{R})$ (where $Sp(m, \mathbb{R})$ is non-compact, with $Sp(1) = SL(2, \mathbb{R})$), then there are three covariantly constant tensors J, S, T satisfying the pseudoquaternion algebra [14,15]

$$\begin{aligned} &J^2 = -\mathbb{1}, S^2 = T^2 = \mathbb{1} \\ &ST = -TS = -J, TJ = -JT = S, JS = -SJ = T \end{aligned} \tag{1.3}$$

J is a complex structure and S, T are real structures and the sigma-model again has a twisted $(4, 1)$ superconformal symmetry. The right-handed superalgebra is again of the form (1.2), where now $A = 1, 2, 3, 4$ and $\eta^{AB} = diag(1, 1, -1, -1)$. As $Sp(m, \mathbb{R})$ is a subgroup of both $SU(m, m)$ and $SL(2m, \mathbb{R})$, such spaces give string solutions. For $m = 1$, $SU(1, 1) = SL(2, \mathbb{R}) = Sp(1, \mathbb{R})$ and spaces with this holonomy have self-dual curvature (with torsion).

2. The (2,1) Supersymmetric Sigma-Model

The (1,1) supersymmetric sigma-model with metric g_{ij} and anti-symmetric tensor b_{ij} has the (1,1) superspace action [19]

$$S = \int d^2x d^2\theta \, [g_{ij} + b_{ij}] D_+\phi^i D_-\phi^j \tag{2.1}$$

It will be conformally invariant at one-loop if there is a function Φ such that

$$R^{(+)}_{ij} - 2\nabla_{(i}\nabla_{j)}\Phi - 2H_{ij}{}^k\nabla_k\Phi = 0 \tag{2.2}$$

where $R^{(+)}_{ij}$ is the Ricci tensor for a connection with torsion. We define the connections with torsion

$$\Gamma^{(\pm)i}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \pm H^i_{jk} \tag{2.3}$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ is the Christoffel connection and the torsion tensor is

$$H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]} \tag{2.4}$$

The curvature and Ricci tensors with torsion are

$$R^{(\pm)k}{}_{lij} = \partial_i\Gamma^{(\pm)k}_{jl} - \partial_j\Gamma^{(\pm)k}_{il} + \Gamma^{(\pm)k}_{im}\Gamma^{(\pm)m}_{jl} - \Gamma^{(\pm)k}_{jm}\Gamma^{(\pm)m}_{il}, \qquad R^{(\pm)}_{ij} = R^{(\pm)k}{}_{ikj} \tag{2.5}$$

The equation (2.2) can be obtained from varying the action

$$S = \int d^Dx \, e^{-2\Phi}\sqrt{|g|}\left(R - \frac{1}{3}H^2 + 4(\nabla\Phi)^2 \right) \tag{2.6}$$

The target space coordinates x^i are the lowest components of the superfields ϕ^i $(i = 1, ..., D)$.

The sigma model is invariant under (2,1) supersymmetry [9-13] if the target space is even dimensional ($D = 2n$) with a complex structure $J^i{}_j$ which is covariantly constant

$$\nabla^{(+)}_k J^i{}_j = 0 \tag{2.7}$$

with respect to the connection $\Gamma^{(+)}$ defined in (2.3), and with respect to which the metric is hermitian, so that $J_{ij} \equiv g_{ik} J^k{}_j$ is antisymmetric.

It is useful to introduce complex coordinates $z^\alpha, \bar{z}^{\bar{\beta}}$ in which the line element is $ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}$ and consider the Dolbeault cohomology. An N-form is decomposed into a set of (p,q) forms with p factors of dz and q factors of $d\bar{z}$ with $p + q = N$. The exterior derivative decomposes as $d = \partial + \bar{\partial}$ and it is useful to define $\hat{d} = i(\partial - \bar{\partial})$ and $\Delta = i\partial\bar{\partial} = \frac{1}{2}\hat{d}d$. As $\Delta^2 = 0$, Δ defines its own cohomology. Useful lemmas are (i) if $\partial U = 0$ and $\bar{\partial} U = 0$ for some (p,q) form U, then locally there is a $(p-1, q-1)$ form W such that $U = \Delta W$ (ii) if $\Delta U = 0$ for some for some (p,q) form U, then locally there is a $(p-1, q)$ form W and a $(p, q-1)$ form X such that $U = \bar{\partial} X + \partial W$.

The conditions for (2,1) supersymmetry imply that the (0,3) and (3,0) parts of the 3-form H vanish, and H is given in terms of the fundamental 2-form

$$J = \frac{1}{2} J_{ij} d\phi^i_\Lambda d\phi^j = i g_{\alpha\bar{\beta}} dz^\alpha_\Lambda d\bar{z}^{\bar{\beta}} \tag{2.8}$$

by

$$H = i(\partial - \bar{\partial}) J \tag{2.9}$$

Then the condition $dH = 0$ implies

$$i\partial\bar{\partial} J = 0 \tag{2.10}$$

so that locally there is a (1,0) form $k = k_\alpha dz^\alpha$ such that

$$J = i(\partial\bar{k} + \bar{\partial} k) \tag{2.11}$$

The metric and torsion potential are then given, in a suitable gauge, by

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha \\ b_{\alpha\bar{\beta}} &= \partial_\alpha \bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha \end{aligned} \tag{2.12}$$

If $k_\alpha = \partial_\alpha K$ for some K, then the torsion vanishes and the manifold is Kahler with Kahler potential K, but if $dk \neq 0$ then the space is a hermitian manifold of the type introduced in [9]. The metric and torsion are invariant under [25]

$$\delta k_\alpha = i\partial_\alpha \chi + \theta_\alpha \tag{2.13}$$

where χ is real and θ_α is holomorphic, $\partial_{\bar{\beta}}\theta_\alpha = 0$. It will be useful to define the vector

$$v^i = H_{jkl} J^{ij} J^{kl} \tag{2.14}$$

together with the $U(1)$ part of the curvature

$$C^{(+)}_{ij} = J^l{}_k R^{(+)k}{}_{lij} \tag{2.15}$$

and the $U(1)$ part of the connection (2.3)

$$\Gamma^{(+)}_i = J^k{}_j \Gamma^{(+)j}_{ik} = i(\Gamma^{(+)\alpha}_{i\alpha} - \Gamma^{+\bar{\alpha}}_{i\bar{\alpha}}) \tag{2.16}$$

In a complex coordinate system, (2.15) can be written as $C^{(+)}_{ij} = \partial_i \Gamma^{(+)}_j - \partial_j \Gamma^{(+)}_i$.

If the metric is Riemanian, then the holonomy of any metric connection (including $\Gamma^{(\pm)}$) is contained in $O(2n)$, while if it has signature $(2n_1, 2n_2)$ where $n_1 + n_2 = n$, it will be in $O(2n_1, 2n_2)$. The holonomy $\mathcal{H}(\Gamma^{(+)})$ of the connection $\Gamma^{(+)}$ is contained in $U(n_1, n_2)$. It will be contained in $SU(n_1, n_2)$ if in addition

$$C^{(+)}_{ij} = 0 \tag{2.17}$$

where the $U(1)$ part of the curvature is given by (2.15). As C_{ij} is a representative of the first Chern class, a necessary condition for this is the vanishing of the first Chern class.

It was shown in [10] that geometries for which

$$\Gamma^{(+)}_i = 0 \tag{2.18}$$

in some suitable choice of coordinate system will satisfy the one-loop conditions (2.2), provided the dilaton is chosen as

$$\Phi = -\log|det g_{\alpha\bar{\beta}}| \tag{2.19}$$

which implies

$$\partial_i \Phi = \frac{1}{2} v_i \tag{2.20}$$

Moreover, the one-loop dilaton field equation is also satisfied for compact manifolds, or for non-compact ones in which $\nabla\Phi$ falls off sufficiently fast [11-13]. This implies that $\mathcal{H}(\Gamma^{(+)}) \subseteq SU(n_1, n_2)$ and these geometries generalise the Kahler Ricci-flat or Calabi-Yau geometries, and reduce to these in the special case in which $H = 0$.

These are not the most general solutions of (2.2). In the special case in which $H = 0$ and the geometry is Kahler, the condition (2.2) becomes

$$R_{ij} = 2\nabla_i \nabla_j \Phi \tag{2.21}$$

which implies that either Φ is constant and the geometry is Kahler-Ricci-flat, or that $J^{ij}\nabla_j\Phi$ is a Killing vector, and the geometry is a generalised 'linear dilaton' vacuum of a type that has been analysed in [22]. If $H \neq 0$, then this generalises and the solutions are either of the type described above, or are ones in which (2.19) isn't satisfied and which have a Killing vector; this latter case will be discussed in [23] and here we will restrict ourselves to the case of $SU(n_1, n_2)$ holonomy with (2.18),(2.19) holding.

The equation (2.18) can be viewed as a field equation for the potential k_α. It can be obtained by varying the action

$$S = \int d^D x \sqrt{|det g_{\alpha\bar{\beta}}|} \tag{2.22}$$

where $g_{\alpha\bar{\beta}}$ is given in terms of k_α by (2.12). This action was obtained independently by Martinec and Kutasov [3,4]. It can be rewritten as

$$S = \int d^D x |det g_{ij}|^{1/4} \tag{2.23}$$

which is non-covariant, as the field equation (2.18) was obtained in a particular coordinate system. However, it is invariant under volume-preserving diffeomorphisms.

3. (4,1) Supersymmetry, Real Structures, Hypersymplectic Structures, and Kleinian Geometry

It was argued in [21] that (4,1) sigma-models are finite to all orders in perturbation theory. For Euclidean signature, the model (2.1) will have (4,1) supersymmetry if the complex dimension is even, $n = 2m$, and $\mathcal{H}(\Gamma^{(+)}) \subseteq USp(m)$ (with $USp(1) = SU(2)$). This implies that there are three complex structures I, J, K satisfying the quaternion algebra (1.1) and each satisfying (2.7):

$$\nabla^{(+)}_k I^i{}_j = \nabla^{(+)}_k J^i{}_j = \nabla^{(+)}_k K^i{}_j = 0 \tag{3.1}$$

The algebra (1.1) can be written as

$$J^a J^b = -\delta^{ab}\mathbf{1} + \epsilon^{abc} J^c \tag{3.2}$$

where $a = 1, 2, 3$ and $J^a = \{I, J, K\}$.

In particular, as $USp(1) = SU(2)$, it follows that in four dimensions, $D = 2n = 4$, a geometry satisfying (2.18) will be finite to all orders and so there are no corrections to the action (2.22) of higher order in the sigma-model coupling constant α'. The curvature is anti-self-dual, satisfying

$$R^{(+)}_{ijkl} = -\frac{1}{2}\epsilon_{ij}{}^{mn} R^{(+)}_{mnkl} \tag{3.3}$$

Consider now the case of non-Euclidean signature. For 4 dimensions with the Kleinian signature (2,2), the vanishing of the $U(1)$ part of the curvature implies that the curvature is again anti-self-dual, (3.3), and that the holonomy is $SU(1,1) = SL(2,\mathbb{R}) = Sp(1,\mathbb{R})$. There are no longer three complex structures but there are three covariantly constant tensors J, S, T satisfying the pseudoquaternion algebra [14,15] (1.3) or, equivalently,

$$S^a S^b = -\eta^{ab}\mathbb{I} + f^{ab}{}_c S^c \tag{3.4}$$

where $a = 1,2,3$, $S^1 = J, S^2 = S, S^3 = T$, $\eta^{ab} = diag(+1,-1,-1)$ is the $SL(2,\mathbb{R})$ Killing metric and $f^{ab}{}_c$ are the $SL(2,\mathbb{R})$ structure constants. Each of the S^a is covariantly constant with respect to the connection $\Gamma^{(+)}$

$$\nabla^{(+)}_k J^i{}_j = \nabla^{(+)}_k S^i{}_j = \nabla^{(+)}_k T^i{}_j = 0 \tag{3.5}$$

and each satisfies

$$S^a_{ij} = -S^a_{ji} \tag{3.6}$$

The complex structure is $J = S^1$ which squares to $-\delta^i{}_j$ while S^2, S^3 each square to $+\delta^i{}_j$. Each of the S^a is requireds to be integrable, so that the Nijenhuis-type tensor vanishes and there is a coordinate system in which the real or complex structure is constant.* However they are not simultaneously integrable in general i.e. for each S^a there is a coordinate system in which $(S^a)^i{}_j$ is constant, but there may not be one in which all three are simultaneously constant.

$\star$ The case of almost complex structures or almost real structures which are not integrable will not be considered here, although they do lead to more general models.

The S, T are each real structures [14,15], sometimes called locally product structures [19,24]. If the S^a_{ij} $(a = 2, 3)$ had been symmetric, the metric would have been a locally product metric and the space would have been a locally product space of the type discussed in [19]. The fact that they are anti-symmetric gives a different structure, however. Choosing adapted real coordinates $u^\alpha, v^{\tilde{\alpha}}$ $(\alpha = 1, 2; \tilde{\alpha} = 1, 2)$ in which S, say, takes the form

$$S^i{}_j = \begin{pmatrix} \delta^\alpha{}_\beta & 0 \\ 0 & -\delta^{\tilde{\alpha}}{}_{\tilde{\beta}} \end{pmatrix} \tag{3.7}$$

the condition (3.6) implies that the line element takes the form

$$ds^2 = 2g_{\alpha\tilde{\alpha}}(u, v)du^\alpha dv^{\tilde{\alpha}} \tag{3.8}$$

so that $\partial/\partial u^\alpha$ and $\partial/\partial v^{\tilde{\alpha}}$ are null vectors. In general, this coordinate system will be incompatible with the complex structure J.

Spaces of $SL(2, \mathbb{R})$ holonomy have two spinors that are covariantly constant with respect to $\Gamma^{(+)}$, ε^A $(A = 1, 2)$ and these can be used to construct three covariantly constant 2-forms $S^{(AB)} = \bar{\varepsilon}^A \gamma_{ij} \varepsilon^B$, which can be identified with the S^a; this gives the simplest way of obtaining the above results. The sigma-models with these target spaces do not have the usual (4,1) supersymmetry. They have three currents

$$j^a = \frac{1}{2} S^a_{ij} \psi^i \psi^j \tag{3.9}$$

generating an $SL(2, \mathbb{R})$ Kac-Moody algebra and four supercurrents

$$G^0 = g_{ij} \partial X^i \psi^j, \qquad G^a = S^a_{ij} \partial X^i \psi^j \tag{3.10}$$

The currents T, G^0, G^1, j^1 generate an $N = 2$ superconformal algebra (where T is the energy-momentum tensor). The full set of right-handed currents

$\{T, G^0, G^a, j^a\}$ generate a non-compact twisted form of the (small) $N = 4$ superconformal algebra. The global limit is a (4,1) superalgebra in which the four right-handed supercharges $Q^A = \{Q^0, Q^a\}$ (with $A = 0, 1, 2, 3$) satisfy

$$\{Q^A Q^B\} = \eta^{AB} P \tag{3.11}$$

where $\eta^{AB} = diag(1, 1, -1, -1)$ is the $O(2, 2)$ Killing metric and P is the right-moving momentum.

A similar structure obtains for spaces of signature $(2n, 2n)$ with holonomy $Sp(n, \mathbb{R})$ – the subgroup of $U(n, n)$ preserving an anti-symmetric matrix, or equivalently the subgroup of $O(2n, 2n)$ preserving three matrices S, T, J satisfying (1.3). Sigma-models with such target spaces will have twisted $(4, 1)$ supersymmetry. Spaces with one covariantly constant real structure S satisfying the conditions discussed above will have twisted (2,1) supersymmetry, with global limit given by (3.11) with $A = 1, 2$ and $\eta^{AB} = diag(1, -1)$. If the metric is to be invertible ($detg \neq 0$), this requires the metric to have signature (m, m) and the holonomy is then $\mathcal{H}(\Gamma^{(+)}) \subseteq GL(m, \mathbb{R})$.

Consider first the case in which there is no torsion, $H = 0$. Then the antisymmetric tensors J^a_{ij} or S^a_{ij} are closed, as a result of (3.1) or (3.5), and so each closed 2-form defines a symplectic structure. For Euclidean signature, the metric is Kahler with respect to each of the complex structures I, J, K and the space is hyperkahler. The $\{I, J, K\}$ constitute a hyperkahler structure. In complex coordinates adapted to any one of the complex structures, the metric is

$$ds^2 = 2g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}}, \qquad g_{\alpha\bar{\beta}} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}} K \tag{3.12}$$

for some locally defined Kahler potential K.

For signature $(2n, 2n)$, the $\{J, S, T\}$ constitute a hypersymplectic structure [15]. The metric is Kahler with respect to the complex structure J, while in

coordinates adapted to either of the real structures, S say, the metric takes the form

$$ds^2 = 2g_{\alpha\tilde{\beta}}(u,v)du^\alpha dv^{\tilde{\beta}}, \qquad g_{\alpha\tilde{\beta}} = \frac{\partial^2}{\partial u^\alpha \partial \bar{v}^{\tilde{\beta}}} K \tag{3.13}$$

for some locally defined potential K. In these coordinates, the symplectic structure is $S = g_{am} du^a_\wedge dv^m$.

If $H \neq 0$, then the 2-forms J^a or S^a are not closed, but I, J, K are Δ-closed. In the Euclidean case, one can choose complex coordinates adapted to any one of the three complex structures, and the formulae (2.8)-(2.12) then hold for each choice of complex structure. For the Kleinian signature $(2n, 2n)$, the complex structure J again leads to conditions (2.8)-(2.12). For the real structure S (or T) it is useful to introduce the adapted coordinates $u^\alpha, v^{\tilde{\beta}}$, and consider the analogue of Dolbeault cohomology. An N-form is decomposed into a set of (p,q) forms with p factors of du and q factors of dv with $p+q=N$. The exterior derivative decomposes as $d = \partial_u + \partial_v$ where $\partial_u : H^{(p,q)} \to H^{(p+1,q)}$ and $\partial_v : H^{(p,q)} \to H^{(p,q+1)}$. It is useful to define $\hat{d} = (\partial_u - \partial_v)$ and $\Delta = \partial_u \partial_v = \frac{1}{2}\hat{d}d$. Again $\Delta^2 = 0$, so that Δ defines its own cohomology. Then H is given in terms of the fundamental 2-form

$$S = \frac{1}{2} S_{ij} d\phi^i_\Lambda d\phi^j = g_{\alpha\tilde{\beta}} du^\alpha_\Lambda dv^{\tilde{\beta}} \tag{3.14}$$

by

$$H = (\partial_u - \partial_v) S \tag{3.15}$$

The condition $dH = 0$ then implies

$$\partial_u \partial_v S = 0 \tag{3.16}$$

so that locally there is a (1,0) form $k = k_\alpha du^\alpha$ and a (0,1) form $\tilde{k} = \tilde{k}_{\tilde{\beta}} dv^{\tilde{\beta}}$ such

that

$$S = \partial_u \tilde{k} + \partial_v k \tag{3.17}$$

The metric and torsion potential are then given, in a suitable gauge, by

$$\begin{aligned} g_{\alpha\tilde{\beta}} &= \partial_\alpha \tilde{k}_{\tilde{\beta}} + \partial_{\tilde{\beta}} k_\alpha \\ b_{\alpha\tilde{\beta}} &= \partial_\alpha \tilde{k}_{\tilde{\beta}} - \partial_{\tilde{\beta}} k_\alpha \end{aligned} \tag{3.18}$$

so that

$$H = \partial_u \partial_v (k + \tilde{k}) \tag{3.19}$$

If $k_\alpha = \partial_\alpha \kappa$ and $\tilde{k}_{\tilde{\beta}} = \partial_{\tilde{\beta}} \tilde{\kappa}$ for some locally defined potentials $\kappa, \tilde{\kappa}$, then the torsion vanishes and

$$S = \partial_u \partial_v (\tilde{\kappa} - \kappa) \tag{3.20}$$

so that (3.13) is satisfied with potential $K = \tilde{\kappa} - \kappa$.

The power-counting arguments of Howe and Papadopoulos [21] can be generalised to apply to models with this twisted $(4,1)$ supersymmetry, so that such models should again be finite. This is supported by the results of Martinec and Kutasov [3], who showed that the action (2.22) generates the correct S-matrix for part of the (2,1) string, confirming that this action receives no corrections in $D = 4$.

For target spaces of signature (m,m) with holonomy $GL(m,\mathbb{R})$ with one covariantly constant integrable real structure S satisfying (3.5),(3.6), the geometry is given in terms of a scalar potential by (3.13) if $H = 0$ or by a vector potential (3.18) if $H \neq 0$. The metric and torsion are preserved by the gauge transformations

$$\delta k_\alpha = \partial_\alpha \chi + \theta_\alpha, \qquad \delta \tilde{k}_{\tilde{\alpha}} = -\partial_{\tilde{\alpha}} \chi + \tilde{\theta}_{\tilde{\alpha}} \tag{3.21}$$

where $\partial_{\tilde{\alpha}} \theta_\alpha = \partial_\alpha \tilde{\theta}_{\tilde{\alpha}} = 0$. In analogy with (2.14),(2.15),(2.16), it will be useful to

define the vector

$$\tilde{v}_i = H_{ijk} S^{jk} \tag{3.22}$$

together with the $GL(1,\mathbb{R})$ part of the curvature

$$\tilde{C}^{(+)}_{ij} = S^l{}_k R^{(+)k}{}_{lij} \tag{3.23}$$

and the $GL(1)$ part of the connection (2.3)

$$\tilde{\Gamma}^{(+)}_i = S^k{}_j \Gamma^{(+)j}_{ik} = (\Gamma^{(+)\alpha}_{i\alpha} - \Gamma^{+\tilde{\alpha}}_{i\tilde{\alpha}}) \tag{3.24}$$

If $H = 0$, then the curvature 2-form is a (1,1) form and the only non-vanishing components of the curvature are $R_{\alpha\tilde{\beta}\gamma\tilde{\delta}}$. It follows that the Ricci tensor $R_{\alpha\tilde{\beta}}$ is proportional to is proportional to $\tilde{C}_{\alpha\tilde{\beta}}$ and is given by

$$R_{\alpha\tilde{\beta}} = \partial_\alpha \partial_{\tilde{\beta}} \log |det g_{\gamma\tilde{\delta}}| \tag{3.25}$$

with $R_{\alpha\beta} = 0$. Thus the Einstein equation $R_{ij} = 0$ is equivalent to demanding $SL(m,\mathbb{R})$ holonomy and gives, with a suitable choice of coordinates,

$$|det g_{\gamma\tilde{\delta}}| = 1 \tag{3.26}$$

which gives a Monge-Ampere equation for K on using (3.13).

If $H \neq 0$, the condition (2.18) of the complex case is replaced by

$$\tilde{\Gamma}^{(+)}_i = 0 \tag{3.27}$$

and this again implies that the one-loop field equation (2.2) is satisfied, provided the dilaton is chosen as (2.19). Furthermore, the condition (3.27) implies $\tilde{C}^{(+)}_{ij} = 0$ and so the holonomy is in $SL(m,\mathbb{R})$. The field equation (3.27) can again be obtained from the action (2.23), but where now the metric is given by (3.18) in terms of the potentials $k, \tilde{k}$ corresponding to the real structure S, and it is these that are varied to give the field equation (3.27).

It is remarkable how much of the geometry based on a complex structure J carries over to the case of a real structure S. Instead of using complex numbers, it is sometimes useful to use *double numbers* in this context, which are based on introducing a number e satisfying $e^2 = 1$ instead of the usual imaginary unit i satisfying $i^2 = -1$ [14].

4. The (2,0)-Supersymmetric Sigma-Model and the Bott-Chern Form

Consider now the (1,0) sigma-model. It consists of (1,0) scalar superfields ϕ^i taking values in the target space M and coupling to g_{ij} and b_{ij}, plus fermionic fields ψ^M which are sections of $S_+ \times V$ where S_+ is the world-sheet chiral spinor bundle and V is a vector bundle over M with structure group G; they couple to the connection A_i on V [9]. The (1,0) superspace action is [9]

$$S = \int d^2x d\theta \; (g_{ij} + b_{ij})\, \partial_- \phi^i D\phi^j + \psi^M (D\psi^M + A_i^M{}_N D\phi\, \psi^N) \tag{4.1}$$

where D is the superspace supercovariant derivative The conditions for conformal invariance are derived from the action

$$\begin{aligned} S = \int d^D x e^{-2\Phi} \sqrt{|g|} \Big(& R - \frac{1}{3} H^2 + 4(\nabla\Phi)^2 \\ & - \frac{\alpha'}{2} [\mathrm{tr}(F_{ij} F^{ij}) - R^{(-)}_{abij} R^{(-)baij}] + O(\alpha'^2) \Big) \end{aligned} \tag{4.2}$$

where H is now given by

$$H = \frac{1}{2} db + \alpha' [\Omega(A) - \Omega(\omega^-)] \tag{4.3}$$

and Ω is the Chern-Simons 3-form

$$\mathrm{tr}(F^2) = d\Omega(A), \qquad \Omega(A) = \mathrm{tr}(A dA + \tfrac{2}{3} A^3) \tag{4.4}$$

The curvatures $R^{(\pm)}$ and connections $\Gamma^{(\pm)}$ are given by (2.3),(2.5) with the torsion (4.3). A vielbein e_i^a has been introduced, with the corresponding spin connections

$\omega_i^{(\pm)ab}$, curvatures $R^{(\pm)a}{}_{bij}$ and curvature 2-forms $R^{(\pm)a}{}_b$. The gravitational Chern-Simons term is given by

$$\mathrm{tr}(R^{(\pm)2}) = R^{(\pm)a}{}_b R^{(\pm)b}{}_a = d\Omega(\omega^{(\pm)}), \qquad \Omega(\omega^{(\pm)}) = \mathrm{tr}(\omega^{(\pm)} d\omega^{(\pm)} + \tfrac{2}{3}\omega^{(\pm)3}) \tag{4.5}$$

The new torsion satisfies

$$dH = \alpha'[\mathrm{tr}(F^2) - \mathrm{tr}(R^{(-)2})] \tag{4.6}$$

and the condition

$$\int_\sigma \mathrm{tr}(F^2) = \int_\sigma \mathrm{tr}(R^{(-)2}) \tag{4.7}$$

is required over any 4-cycle σ for H to be well-defined. A key identity is

$$R^{(+)}_{ijkl} - R^{(-)}_{klij} = -2H_{[ijk,l]} \tag{4.8}$$

which can be rewritten using (4.6). As H appears in the gravitational Chern-Simons term on the right hand side of (4.3), the equations (4.3),(4.5) only implicitly define H, but H can be constructed perturbatively in α'.

The model has (2,0) supersymmetry classically if (i) (M, g_{ij}, b_{ij}) is a (2,1) geometry, i.e. a hermitian space with torsion whose complex structure is covariantly constant (2.7) with respect to the connection $\Gamma^{(+)}$ defined by (2.3),(4.3), and (ii) V is a holomorphic vector bundle, i.e. the field strength $F = dA + A^2$ is a (1,1) form [9]. This implies that the (1,0) part of the connection $\mathcal{A} = A_\alpha dz^\alpha$ satisfies

$$\mathcal{A} = V^{-1}\partial V \tag{4.9}$$

for some complex G-valued function V, i.e. V takes values in the complexification of G. (A group element in a neighbourhood of the identity is of the form $g = \exp \alpha_m t^m$ where α_m are real coordinates and t^m are elements of the Lie algebra

}. The prepotential is of the form $V = \exp v_m t^m$ where v_m are complex, and $\bar{V} = \exp \bar{v}_m t^m$.) Under a gauge transformation with parameter $g(x) \in G$

$$A \to g^{-1} dg + g^{-1} A g \tag{4.10}$$

As $A = \mathcal{A} + \mathcal{A}^*$, the connection will be pure gauge if V is real. The prepotential V transforms as

$$V \to \bar{\lambda} V g \tag{4.11}$$

under a gauge transformation and under a pre-gauge transformation with holomorphic G-valued parameter $\lambda(z) \in G$; the pre-gauge transformations leave $\mathcal{A}$ invariant. It is also useful to define

$$U = V \bar{V}^{-1} \tag{4.12}$$

which is invariant under the gauge transformations since g is real, but transforms under the pre-gauge transformations as

$$U \to \bar{\lambda} U \lambda^{-1} \tag{4.13}$$

The gauge transformations (4.10),(4.11) have parameter g taking values in G. Consider the (1,0) form

$$a = U^{-1} \partial U \tag{4.14}$$

which can be rewritten as

$$a = \bar{V} A \bar{V}^{-1} + \bar{V} d \bar{V}^{-1} \tag{4.15}$$

Thus a is related to A by a complex gauge transformation (4.10),(4.11) with parameter $g = \bar{V}^{-1}$ taking values in the *complexification* of G. Thus the complexified

vector bundle V_c is a holomorphic bundle with holomorphic connection a (see e.g. [26]). Similarly, the complex gauge transformation

$$A \to \bar{a} \equiv VAV^{-1} + VdV^{-1} = U\bar{\partial}U^{-1} \tag{4.16}$$

defines an anti-holomorphic connection $\bar{a}$ which is a (0,1) form. Under the pre-gauge transformations (4.13),

$$a \to \lambda^{-1}a\lambda + \lambda^{-1}\partial\lambda \tag{4.17}$$

and the field strength is

$$f = da - a^2 = \bar{\partial}a \tag{4.18}$$

since the (0,1) part of a vanishes. This is related to F by

$$F = \bar{V}f\bar{V}^{-1} = V\bar{f}V^{-1} \tag{4.19}$$

so that

$$\mathrm{tr}(F^n) = \mathrm{tr}(f^n) = \mathrm{tr}(\bar{f}^n) \tag{4.20}$$

As the (2,2) form $\mathrm{tr}(F^2)$ satisfies $\partial\mathrm{tr}(F^2) = \bar{\partial}\mathrm{tr}(F^2) = 0$, then by lemma (i) there is a (1,1) form $\Upsilon(V,\bar{V})$ such that

$$\mathrm{tr}(F^2) = i\partial\bar{\partial}\Upsilon \tag{4.21}$$

$\Upsilon(V,\bar{V})$ is the Bott-Chern 2-form [17], constructed in [18,16,26]. The Chern-Simons form $\Omega(A)$ given by (4.4) then satisfies

$$\Omega(A) = \hat{d}\Upsilon + d\chi \tag{4.22}$$

for some 2-form $\chi(V,\bar{V})$. Note that the Bott-Chern form can be written entirely in terms of U, $\Upsilon(V,\bar{V}) = \Upsilon(U)$ but χ cannot.

An instructive example is that in which G is abelian. Then $F^m = dA^m$ and there are real scalars ϕ^m, θ^m $(m = 1, \ldots, rank(G))$ such that

$$\begin{aligned} A^m &= \mathcal{A}^m + \bar{\mathcal{A}}^m = d\theta^m + \hat{d}\phi^m, \qquad \mathcal{A}^m = \partial(\theta^m + i\phi^m), \\ a^m &= 2i\partial\phi^m \qquad \bar{a}^m = -2i\bar{\partial}\phi^m \end{aligned} \tag{4.23}$$

and

$$V = \exp(\theta + i\phi), \qquad U = \exp(2i\phi) \tag{4.24}$$

Then

$$F^m = dA^m = \partial a = \bar{\partial}\bar{a} = -2i\partial\bar{\partial}\phi^m \tag{4.25}$$

and

$$\mathrm{tr}(F^2) = -4\partial\bar{\partial}\phi^m\partial\bar{\partial}\phi^m$$

and the Bott-Chern form can be chosen to be

$$\Upsilon = -4i\partial\phi^m\bar{\partial}\phi^m$$

The Chern-Simons form AdA then satisfies (4.22) with

$$\chi = -2i\theta^m F^m \tag{4.26}$$

Under a gauge transformation (4.10),(4.11) with $g = e^{\alpha}$ and $\lambda = e^{2l}$ (with $\bar{\partial}l^m = 0$)

$$\delta A_i^m = \partial_i\alpha^m, \qquad \delta\phi^m = i(l^m - \bar{l}^m), \qquad \delta\theta^m = \alpha^m + (l^m + \bar{l}^m) \tag{4.27}$$

In the non-abelian case, introducing coordinates ϕ^m on the group manifold M, one has

$$\Upsilon = (G_{mn} + B_{mn})\partial\phi^m\bar{\partial}\phi^n \tag{4.28}$$

defining a metric $G_{mn}(\phi)$ and anti-symmetric tensor $B_{mn}(\phi)$. This can be constructed explicitly as follows [16]. Let $A(t, x^\mu)$ be a 1-parameter family of connections labelled by $0 \leq t \leq 1$, constructed from pre-potentials $V(t, x^\mu)$ with

corresponding t-dependent U, a, f, F defined as above. Then

$$\frac{\partial}{\partial t} f = \bar{\partial} \dot{a} = \bar{\partial} \partial_a (U^{-1} \dot{U}) \tag{4.29}$$

where

$$\partial_a (U^{-1} \dot{U}) \equiv \partial (U^{-1} \dot{U}) + [a, U^{-1} \dot{U}] \tag{4.30}$$

so that

$$\begin{aligned} \frac{\partial}{\partial t} \mathrm{tr} F^n = \frac{\partial}{\partial t} \mathrm{tr} f^n &= n \, \mathrm{tr} \left(\bar{\partial}_a (U^{-1} \dot{U}) f^{n-1} \right) \\ &= n \bar{\partial} \partial_a \mathrm{tr} \left((U^{-1} \dot{U}) f^{n-1} \right) \\ &= n \bar{\partial} \partial \mathrm{tr} \left((U^{-1} \dot{U}) f^{n-1} \right) \end{aligned} \tag{4.31}$$

Thus if $F(1, x^\mu) = F(x^\mu)$ and $F(0, x^\mu) = \hat{F}(x^\mu)$,

$$\mathrm{tr}(F^n) = \mathrm{tr}(\hat{F}^n) + i \bar{\partial} \partial \Upsilon_n \tag{4.32}$$

where

$$\Upsilon_n = -i \frac{1}{n} \int dt \, \mathrm{tr}(U^{-1} \dot{U} f^{n-1}) = -i \frac{1}{n} \int dt \, \mathrm{tr}(U^{-1} \dot{U} [\bar{\partial} (U^{-1} \partial U)]^{n-1}) \tag{4.33}$$

The case $n = 2$ defines the form needed here, $\Upsilon(U) = \Upsilon_2$, which will exist locally. Note that it is only defined by (4.21) up to the addition of a Δ-closed term, $\Upsilon \to \Upsilon + \partial X + \bar{\partial} Y$.

In four dimensions, the Donaldson action

$$\int J \wedge \Upsilon \tag{4.34}$$

gives an action on any hermitian space with complex structure 2-form J whose variation with respect to U or ϕ implies that F is self-dual. In $2n + 2$ dimensions,

the action

$$\int J^n \wedge \Upsilon \tag{4.35}$$

implies that the (2,0) part of F vanishes, and F satisfies the Uhlenbeck-Yau equation

$$J^{ij}F_{ij} = 0 \tag{4.36}$$

The two-dimensional case $S = \int \Upsilon$ gives a Wess-Zumino-Witten model for the complexification of G.

For geometries in which $dH = 0$ (e.g. for the (2,1) sigma-model or for the (2,0) model in the classical limit $\alpha' \to 0$), the fact that $\omega^{(+)}$ has $U(n_1, n_2)$ holonomy together with (4.8) implies that $R^{(-)}$ is a (1,1) form, so that the tangent bundle $T(M)$ with connection $\omega^{(-)}$ is holomorphic. Then there are complex $U(n_1, n_2)$-valued scalars W such that

$$\omega^{(-)} = W^{-1}\partial W + (W^{-1}\partial W)^*$$

and there is a Bott-Chern form $\Upsilon(Y)$ and a 2-form $\chi(W, \bar{W})$ such that

$$\Omega(\omega^{(-)}) = \hat{d}\Upsilon + d\chi \tag{4.37}$$

where

$$Y = W\bar{W}^{-1} \tag{4.38}$$

In the quantum case, the Chern-Simons corrections to H and hence to $\omega^{(-)}$ give α' corrections to these equations, but again there are forms $\Upsilon(Y)$ and $\chi(W, \bar{W})$ satisfying (4.37) which can be constructed order by order in α'.

There will be (2,0) supersymmetry in the quantum theory if the complex structure is covariantly constant with respect to the connection given by (2.3),(4.3), whose torsion now includes the Chern-Simons terms (4.3) [9]; thus this connection

has $U(n_1, n_2)$ holonomy. This again implies that H is given by (2.9), but now (4.6) implies [11,12]

$$i\partial\bar{\partial}J = \alpha'[\mathrm{tr}(F^2) - \mathrm{tr}(R^{(-)2})] \tag{4.39}$$

This implies the local existence of a (1,0) form k such that

$$J = \alpha'\hat{\Upsilon} + i(\partial\bar{k} + \bar{\partial}k) \tag{4.40}$$

where

$$\hat{\Upsilon} = \Upsilon(U) - \Upsilon(Y) \tag{4.41}$$

which will be well-defined if (4.7) holds. Then the metric is given by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha + i\alpha' \hat{\Upsilon}_{\alpha\bar{\beta}} \tag{4.42}$$

while the torsion potential can be chosen to be

$$b_{\alpha\bar{\beta}} = \partial_\alpha \bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha + \alpha' \hat{\chi}_{\alpha\bar{\beta}} \tag{4.43}$$

where

$$\hat{\chi}_{\alpha\bar{\beta}} = \chi_{\alpha\bar{\beta}}(V, \bar{V}) - \chi_{\alpha\bar{\beta}}(W, \bar{W}) \tag{4.44}$$

This is in agreement with the results of Howe and Papadopoulos [26], in which it was shown that all anomalies in the (2,0) sigma-model can be cancelled by adding finite local counterterms to the g_{ij}, b_{ij}, so that

$$g_{ij} \to g_{ij} + \alpha'\hat{\Upsilon}_{ij}, \qquad b_{ij} \to b_{ij} + \alpha'\hat{\chi}_{ij} \tag{4.45}$$

together with attributing to b_{ij} the standard anomalous transformations

$$\delta b_{ij} = \alpha' \mathrm{tr}(Ad\alpha - \omega^{(-)} d\Lambda) \tag{4.46}$$

under Lorentz and gauge symmetries with parameters Λ, α respectively [9,26]. Note that whereas shifting g_{ij} by a counterterm proportional to $\mathrm{tr}A_iA_j$, which was used

in the arguments of [3], is sufficient to remove the sigma-model anomalies in the (1,0) model, this is not consistent with (2,0) supersymmetry and it is necessary to use the counterterms (4.45), as shown in [26].

The Yang-Mills field equation is, to lowest order in α',

$$D^{(+)i}F_{ij} - 2\nabla^i\Phi F_{ij} = 0 \tag{4.47}$$

where $D^{(+)}$ is the gauge and gravitational covariant derivative involving the connections $\Gamma^{(+)}$ and A. This can be integrated to give the Uhlenbeck-Yau equation (4.36), which can be rewritten as

$$g^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = 0 \tag{4.48}$$

Indeed, differentiating (4.36) and using (2.7),(2.14),(2.20) gives (4.47). The Uhlenbeck-Yau equation (4.36) will receive higher order corrections in α' in general. Note that the complex structure J^{ij} in (4.36) is the modified one containing the Bott-Chern form $\hat{\Upsilon}$.

The conditions given above are sufficient for the sigma-model to be conformally invariant to lowest order in α'. These are not the most general solutions, but they are precisely the ones that will admit Killing spinors and so be invariant under spacetime supersymmetries when considered as superstring backgrounds [11,12]. The more general backgrounds, which necessarily have an isometry, will be discussed in [23]. These field equations are obtained by varying the action (2.23) with respect to k_α and V, where $g_{\alpha\bar{\beta}}$ is given by (4.42). Then (2.23) is the effective action generating the conformal invariance conditions for (2,0) sigma models to lowest order in α', and so is the leading part of the effective action for (2,0) strings.

Consider now the conditions for the (1,0) action (4.1) to have a twisted (2,0) supersymmetry. As in the last section, this requires the existence of a real structure S on M satisfying (3.5),(3.6). Invariance of the terms in (4.1) involving the

fermionic superfields ψ requires that the Yang-Mills field strength satisfies

$$S_{[i}{}^{k}F_{j]k} = 0 \tag{4.49}$$

so that the field strength F is a (1,1) form ($F_{\alpha\beta} = 0$, $F_{\tilde{\alpha}\tilde{\beta}} = 0$) and this implies that

$$A = \mathcal{A} + \tilde{\mathcal{A}} \tag{4.50}$$

where

$$\mathcal{A} = V^{-1}\partial_u V, \qquad \tilde{\mathcal{A}} = \tilde{V}^{-1}\partial_v \tilde{V} \tag{4.51}$$

for two independent real potentials $V, \tilde{V}$, each taking values in G (not its complexification). The potential A will be pure gauge if $V = \tilde{V}$. The Uhlenbeck-Yau equation is replaced by

$$S^{ij}F_{ij} = 0 \tag{4.52}$$

which is equivalent to

$$g^{\alpha\tilde{\beta}}F_{\alpha\tilde{\beta}} = 0 \tag{4.53}$$

The results described above for the usual (2,0) model generalise straightforwardly to this twisted case. In particular, there is a Bott-Chern-type form $\tilde{\Upsilon}$ and a form $\tilde{\chi}$ such that

$$\mathrm{tr}(F^2) = \alpha' \partial_u \partial_v \tilde{\Upsilon} \tag{4.54}$$

and the Chern-Simons form $\Omega(A)$ (4.3) is given by

$$\Omega(A) = (\partial_u - \partial_v)\tilde{\Upsilon}(U) + d\tilde{\chi}(V, \tilde{V}) \tag{4.55}$$

where $U = V\tilde{V}^{-1}$. Similarly, the spin-connection has prepotentials $W, \tilde{W}$ and the gravitational Chern-Simons term gives a form $\tilde{\Upsilon}(Y)$ with $Y = W\tilde{W}^{-1}$. and the

quantum metric is

$$g_{\alpha\tilde{\beta}} = \alpha' \hat{\Upsilon}_{\alpha\tilde{\beta}} + \partial_\alpha \tilde{k}_{\tilde{\beta}} + \tilde{\partial}_{\tilde{\beta}} k_\alpha \tag{4.56}$$

where

$$\hat{\Upsilon} = \tilde{\Upsilon}(U) - \tilde{\Upsilon}(Y) \tag{4.57}$$

The field equations are again obtained by varying (2.23), where the metric is given by (4.56).

5. The (2,1) String

For the (2,0) sigma-model to have (2,1) supersymmetry, it is necessary that the fermions ψ^M split into a set $\psi^i = e^i_a \psi^a$ which can combine with the (2,0) superfields ϕ^i to form (2,1) supermultiplets, and a set $\psi^{M'}$ on which the extra supersymmetry is non-linearly realised. Thus the vector bundle V should be of the form $TM \times V'$ where TM is the tangent bundle and V' is some other bundle with structure group G' [1]. The structure group G of V is then in $G' \times O(n)$. The fermions ψ^M then split into $(\psi^a, \psi^{M'})$, with $M' = 1, \ldots dim(V')$. The ψ^a are sections of $TM \times S_+$ and are the superpartners of the (2,0) scalar multiplets. The supercurrent generating the extra supersymmetry on the fermions $\psi^{m'}$ is of the form

$$G = \frac{1}{6} f_{M'N'P'} \psi^{M'} \psi^{N'} \psi^{P'} \tag{5.1}$$

where $f_{M'N'P'}$ are the structure constants of some Lie group, so that this supersymmetry is realised non-linearly on the fermions.

The connection A decomposes into a connection on TM and a connection A' on V'. The connection on TM given by restricting the V connection A to TM must be gauge-equivalent to $\omega^{(-)}$ [10-13], so that in a suitable gauge $A = A' + \omega^{(-)}$ and there is a prepotential V' for A'. Substituting this in the conditions obtained

above for (2,0) supersymmetry, we obtain

$$H = \frac{1}{2}db + \alpha'\Omega(A') \tag{5.2}$$

As there are no gravitational Chern-Simons terms, H does not appear on the right hand side, so that (5.2) gives H explicitly. The global condition (4.7) now becomes $\int_\sigma \mathrm{tr}(F'F') = 0$ for all 4-cycles σ. Then

$$i\partial\bar{\partial}J = \alpha'\mathrm{tr}(F')^2 \tag{5.3}$$

and there is (1,0) form k such that

$$J = \alpha'\Upsilon(U') + i(\partial\bar{k} + \bar{\partial}k) \tag{5.4}$$

and the metric and torsion potential are given by

$$\begin{aligned} g_{\alpha\bar{\beta}} &= i\alpha'\Upsilon_{\alpha\bar{\beta}}(U') + \partial_\alpha\bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}}k_\alpha \\ b_{\alpha\bar{\beta}} &= i\alpha'\chi_{\alpha\bar{\beta}}(V',\bar{V}') + \partial_\alpha\bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}}k_\alpha \end{aligned} \tag{5.5}$$

The Yang-Mills equation becomes

$$J^{ij}F'_{ij} = 0 \tag{5.6}$$

These equations can be obtained by varying the action (2.23).

It will be useful to write the metric in terms of a fixed background metric $\hat{g}_{\alpha\bar{\beta}}$ (e.g. a flat metric) which is given in terms of a potential $\hat{k}$ by $\hat{g}_{\alpha\bar{\beta}} = \partial_\alpha\hat{k}_{\bar{\beta}} + \partial_{\bar{\beta}}\hat{k}_\alpha$, and a fluctuation given in terms of a vector field B_i defined by

$$B_\alpha = -i(k_\alpha - \hat{k}_\alpha), \qquad B_{\bar{\alpha}} = i(\bar{k}_{\bar{\alpha}} - \hat{k}^*_{\bar{\alpha}}) \tag{5.7}$$

with field strength $\mathcal{F} = dB$. Then

$$g_{\alpha\bar{\beta}} = \hat{g}_{\alpha\bar{\beta}} + i\mathcal{F}_{\alpha\bar{\beta}} + \alpha'\Upsilon_{\alpha\bar{\beta}} \tag{5.8}$$

The gauge symmetry (2.13) has become the usual gauge transformation of an

abelian gauge field

$$\delta B_i = \partial_i \chi \tag{5.9}$$

and the action (2.22) becomes

$$S = \int d^D x \sqrt{|det(\hat{g}_{\alpha\bar{\beta}} + i\mathcal{F}_{\alpha\bar{\beta}} + \alpha' \Upsilon_{\alpha\bar{\beta}})|} \tag{5.10}$$

which is similar to a Born-Infeld action. Note that the (2,0) part of $\mathcal{F}$ is non-zero. Note that

$$*det g_{\alpha\bar{\beta}} \propto J_\wedge J = J^0_\wedge J^0 + 2\alpha' J_\wedge \Upsilon + (\alpha')^2 \Upsilon_\wedge \Upsilon \tag{5.11}$$

where $J^0 = J - \alpha' \Upsilon = \partial \bar{k} + \bar{\partial} k$ is the classical complex structure, so that the expansion of the action (2.22) includes a Donaldson term (4.34), plus other terms such as Υ^2. These are needed to ensure that the connection is holomorphic and satisfies the Uhlenbeck-Yau equation with respect to the quantum complex structure J which has gauge-field dependence, instead of with respect to the classical complex structure J^0.

Instead of the usual (2,1) string or sigma-model, one can construct a string or sigma-model based on the twisted form of the (2,1) algebra. Much of the analysis is similar to that for the usual (2,1) string, but different factors of i and -1. There is a real structure S and the classical metric is (4.43) and the gauge potential A' is given by (4.50),(4.51) in terms of pre-potentials $V', \tilde{V}'$ for A'. In the quantum case, the 2-form S is

$$S = \alpha' \tilde{\Upsilon}(V', \tilde{V}') + \partial_u \tilde{k} + \partial_v k \tag{5.12}$$

and the metric and torsion potential are given by

$$\begin{aligned} g_{\alpha\tilde{\beta}} &= \alpha' \tilde{\Upsilon}_{\alpha\tilde{\beta}}(V', \tilde{V}') + \partial_\alpha \tilde{k}_{\tilde{\beta}} + \tilde{\partial}_{\tilde{\beta}} k_\alpha \\ b_{\alpha\tilde{\beta}} &= \alpha' \chi_{\alpha\tilde{\beta}}(V', \tilde{V}') + \partial_\alpha \tilde{k}_{\tilde{\beta}} - \tilde{\partial}_{\tilde{\beta}} k_\alpha \end{aligned} \tag{5.13}$$

The Yang-Mills equation becomes

$$J^{ij}F'_{ij} = 0 \tag{5.14}$$

and these equations can also be obtained by varying the action (2.23).

For target spaces of signature (2,2) with $SL(2,\mathbb{R}) = SU(1,1)$ holomnomy, the sigma-model has twisted (4,1) supersymmetry which contains both the usual (2,1) algebra and the twisted one, and both approaches give the same result, but in terms of different variables. The action in either approach is (2.23), but can be viewed as depending on the variables $k_\alpha, k_{\bar{\alpha}}, U'$ through (5.5) or on $k_\alpha, k_{\tilde{\alpha}}, \tilde{U}'$ through (5.13).

For the action based on the twisted (2,1) formalism, it is useful to define B now by

$$B_\alpha = (k_\alpha - \hat{k}_\alpha), \qquad B_{\tilde{\alpha}} = -(\tilde{k}_{\tilde{\alpha}} - \hat{k}_{\tilde{\alpha}}) \tag{5.15}$$

in terms of the potential $k, \tilde{k}$ of the twisted (2,1) sigma-model. The gauge symmetry is again (5.9) and the metric is now given by

$$g_{\alpha\bar{\beta}} = \hat{g}_{\alpha\bar{\beta}} - \mathcal{F}_{\alpha\bar{\beta}} + \alpha'\Upsilon_{\alpha\bar{\beta}} \tag{5.16}$$

This formalism based on the real structure may be better suited to performing a null reduction with respect to a a null Killing vector, as coordinates could be chosen so that the Killing vector represents translation in one of the null coordinate directions e.g. $\partial/\partial u^1$.

6. The Schild Action

The action (2.23) is of the form

$$S_{NG} = \int d^D x \, g^\nu \tag{6.1}$$

where $g = |\det g_{ij}|$ and $\nu = 1/4$, whereas the covariant Nambu-Goto action is given by (6.1) with $\nu = 1/2$. Instead of introducing an intrinsic metric on the world-volume to obtain a Polyakov-type action, one can introduce a scalar world-volume gauge field V to obtain a Schild-type action

$$S_S = \frac{1}{2\nu} \int d^D x \;\; V^{1-2\nu} g^\nu - (1-2\nu)\beta V \tag{6.2}$$

where ν, β are constants. This action is invariant under world-sheet diffeomorphisms:

$$\delta\phi = \xi^a \partial_a \phi, \qquad \delta V = \partial_a (V \xi^a) \tag{6.3}$$

so that V is a scalar density.

The V field equation is

$$V = \beta^{\frac{1}{2\nu}} \sqrt{g} \tag{6.4}$$

Substituting for V in the Schild action gives the Nambu-Goto action

$$S_S \rightarrow \;\; \beta^{\frac{2\nu-1}{2\nu}} \int d^D x \; \sqrt{g} \tag{6.5}$$

so that these two models are classically equivalent.

The diffeomorphism symmetry (6.3) can be partially fixed by imposing the gauge condition $V = a$ to the Schild action, where a is some constant. The gauge-

fixed lagrangian is the Eguchi-type Lagrangian

$$L_E = g^\nu + C \tag{6.6}$$

where C is a constant. The field equation

$$\partial L_E/\partial\phi = 0 \;\Rightarrow\; g = \text{const} \tag{6.7}$$

The Eguchi action is invariant under the volume preserving diffeomorphisms

$$\delta\phi^\mu = \xi^a\partial_a\phi^\mu, \quad \partial_a\xi^a = 0 \tag{6.8}$$

Thus the action (2.23) can be obtained from gauge-fixing the Schild action (6.1) with $\nu = 1/4$. Note that the condition $detg_{\alpha\bar{\beta}} = e^{-\Phi}$ from (2.19) together with the field equation $V = \sqrt{g}$ implies that V can be identified with $e^{-\Phi}$, at least in a special coordinate system.

Acknowledgements

I would like to thank M. Abou Zeid, D. Kutasov, E. Martinec and G. Papadopoulos for useful discussions and G. Gibbons for drawing my attention to the references [14,15].

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