

# Fixed-Point Actions in 1-Loop Perturbation Theory\*

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## Abstract

It has been pointed out in recent papers that the example considered earlier in the  $O(N)$   $\sigma$ -model to test whether fixed-point actions are 1-loop perfect actually checked classical perfection only. To clarify the issue we constructed the renormalized trajectory explicitly in 1-loop perturbation theory. We found that the fixed-point action is not exactly 1-loop perfect. The cut-off effects are, however, strongly reduced also on the 1-loop level relative to those of the standard and tree level improved Symanzik actions. Some points on off- and on-shell improvement, Symanzik's program and fixed-point actions are also discussed.

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# 1 Introduction

The fixed-point (FP) action, which lies at the beginning of the renormalized trajectory (RT) in an asymptotically free theory, defines a classically perfect regularization [1]: its classical solutions (instantons) are scale invariant and in quadratic approximation in the fields the spectrum is exact. It has been demonstrated in different models that the cut-off effects are strongly reduced also in the quantum theory [1, 2, 3, 4, 5, 6].

The renormalization group (RG) transformation linearized around the FP (tree level) has a marginal direction. One can raise some formal RG arguments [7, 8] for the statement that on the 1-loop level the FP action begins to move along this marginal direction without changing its form. That would imply that the FP action is automatically 1-loop perfect. The formal RG arguments are not really convincing, however. For this reason an explicit example has been constructed in ref. [9]: it has been shown that the finite volume mass gap  $m(L)$  in the  $d = 2$  non-linear  $\sigma$ -model is free of cut-off effects when calculated with the FP action in 1-loop perturbation theory. The mass gap is, however, a special quantity: any action gives cut-off independent results for  $m(L)L$  on the tree level. It has been conjectured and illustrated through examples recently [10, 11] that in this case tree level improvement will become effective on the 1-loop level cancelling the cut-off effects there. The mass gap therefore is not appropriate to test the issue at hand.

The mass gap  $m(L)$  is determined by the large euclidean time decay of the zero momentum ( $p = 0$ ) two-point function. On the tree level this leads to an effective one-dimensional propagator which, for any action, has no power like cut-off corrections. This is a special situation which does not remain true at  $p \neq 0$ . The smallest energy  $E(L, p)$  in the  $p \neq 0$  channel defined by

$$C(\tau; p) = \frac{1}{L^2} \sum_{x^1, y^1} e^{ip(x^1 - y^1)} \langle \mathbf{S}(x) \mathbf{S}(y) \rangle_{x^0 = -y^0 = \tau} \sim B \exp\{-E(L, p)2\tau\} \quad (1.1)$$

behaves as usual: for a generic action cut-off effects occur already on the tree level and additional cut-off corrections are generated by perturbation theory. We investigate here  $E(L, p)$  with  $p = 2\pi/L$  for different FP actions, and for the tree level improved Symanzik and standard actions. We found small cut-off effects in the FP action predictions. In order to distinguish these effects clearly from the systematic errors entering in the construction of the FP action itself, we studied the RT (quantum perfect action) in 1-loop perturbation theory. We identified explicitly the difference between the FP action and the quantum perfect action and have shown that this difference is responsible for the small cut-off effects seen.

In most of the applications of the FP action until now such block transformations were used where the average of the fine variables in a block was allowed to fluctuate around the block variable. This fluctuation is governed by a parameter  $\kappa$  which was optimized to obtain a short range FP action. As we shall discuss in Section 4, in the formal RG considerations in ref. [8] the  $\kappa = \infty$  case

is special. Unfortunately, it is difficult to test this case because, for most of the block transformations, the FP is not sufficiently short ranged at  $\kappa = \infty$ , which increases the systematic errors of the calculation. We studied the FP action proposed in [12] which has a short ranged quadratic part at  $\kappa = \infty$ . We found, however, that the quartic part and the quantum corrections to the action are much less compact than for the optimised transformation considered in ref. [1]. This might be related to the fact that the block transformation in ref. [12] is close to a decimation, and might explain the poor numerical results obtained there. We found strong indications that this action also produces cut-off effects in 1-loop perturbation theory, but, due to the more extended range of the action, the systematic numerical errors were larger than before. In order to reach a definite conclusion also for the  $\kappa = \infty$  case, we considered another physical quantity: the free-energy density as a function of the chemical potential. This calculation can be done almost completely analytically, no numerical systematic errors influence the results. On the other hand, introducing the chemical potential raises delicate theoretical questions which we were not able to clarify completely. We think, however, that the results, which show small cut-off effects on the 1-loop level in this case as well, are correct.

We recapitulate and extend the arguments on the mass gap [10, 11] in Section 2. We use this occasion to clarify some issues on off-mass shell versus on-mass shell improvement and on the relation between Symanzik improvement and the FP action. The energy  $E(L, p)$  in 1-loop perturbation theory is discussed in Section 3. In Section 4 we consider the relation between the FP action and the marginal operator which makes the  $\kappa = \infty$  case special. In Section 5 we consider the steps leading to the RT in 1-loop perturbation theory. Section 6 treats the dependence of the free energy density on the chemical potential. The numerical results are collected in Section 7.

Our main conclusion was already mentioned before: the FP action is not 1-loop quantum perfect. The authors express regret for having made incorrect statements on this point earlier. On the other hand, as expected intuitively, the FP action, which is classically perfect, generates small cut-off effects on the 1-loop level compared to those of the standard nearest neighbour action. It is interesting to remark in this context that the standardly used next-to-nearest neighbour realization of the tree level improved Symanzik action gives a factor of  $\sim 2$  larger  $O(a^2)$  cutoff effect in  $E(L, p)$  on the 1-loop level than the unimproved nearest neighbour action.

Some questions remained open. We are not able to identify where the formal RG arguments go wrong. In ref. [7] no details are given beyond eq. (20) and the short paragraph following it. In the arguments of ref. [8] there are several questionable points. It is assumed that the FP action and the marginal operator on the classical level are identical. As discussed in Section 4 this is true only at  $\kappa = \infty$ . We find, however, even in this case cut-off effects. The formal arguments in [8] rely further on the assumption that the eigenoperators of the linearized RG transformation around the FP form a complete system. This is not true in general [13]. Finally, it is not quite clear what the condition ‘close to the FP’ used in the discussion means when the coupling constant in front of the action takes the value infinity at the FP.

## 2 The mass gap to 1-loop order

Consider the finite-volume mass gap  $m(L)$  defined on a strip  $0 \leq x \leq L$ ,  $-\infty < t < +\infty$ , with periodic boundary condition in  $x$ . In perturbation theory this has an expansion [14, 15, 10]

$$m(L)L = \frac{N-1}{2} g^2 \sum_{l=0}^{\infty} A_l(L) g^{2l} \quad (2.1)$$

with

$$A_l(L) = \sum_{n=0}^{\infty} \left(\frac{a^2}{L^2}\right)^n \sum_{p=0}^l a_{lp}^{(n)} \left(\ln \frac{L}{a}\right)^{l-p} \quad (2.2)$$

The terms with  $n \geq 1$  in eq. (2.2) yield the  $O(a^{2n})$  lattice artifacts for  $m(L)L$ . As it is known, the tree level result is  $A_0(L) = 1$  for *arbitrary* lattice action — no lattice artifacts appear in this quantity. (It is easy to see why. The propagator for  $p = 0$  states is given by the one-dimensional propagator and  $L$  enters here only as an overall factor. Moreover, the one-dimensional lattice propagator for any regularization differs from the continuum result  $-\frac{1}{2}|t-t'|$  in terms vanishing exponentially fast in  $|t-t'|/a$ . Note also that this is in accordance with the observation that in  $d = 1$  any discretization of the Laplace operator is perfect, which is related to the fact that in this case the equation  $\Delta u = 0$  has only two solutions, 1 and  $t$ , not infinitely many as in higher dimensions.)

The 1-loop contribution  $A_1(L)$  depends already on the form of the action which we write in the general form

$$\begin{aligned} \mathcal{A}(\mathbf{S}) = & -\frac{1}{2} \sum_{n_1, n_2} \rho(n_1 - n_2) (1 - \mathbf{S}_{n_1} \mathbf{S}_{n_2}) \\ & + \sum_{n_1, n_2, n_3, n_4} c(n_1, n_2, n_3, n_4) (1 - \mathbf{S}_{n_1} \mathbf{S}_{n_2}) (1 - \mathbf{S}_{n_3} \mathbf{S}_{n_4}) + \dots \end{aligned} \quad (2.3)$$

In order to discuss the relation between the Symanzik tree level improved [16] and the FP action we shall derive first the tree level on-shell Symanzik conditions for the action in eq. (2.3). The  $O(a^2)$  tree level on-shell Symanzik improvement requires that all  $O(a^2)$  artifacts cancel in all physical quantities calculated on the tree level. These include the spectrum  $E(p)$  related to the 2-point function (influenced only by the coefficients  $\rho$ ), and also the on-shell scattering amplitude related to the 4-point function (to which both the coefficients  $\rho$  and  $c$  contribute).

An alternative way to obtain the tree level on-shell  $O(a^2)$  Symanzik conditions is the following. Consider a lattice action  $\mathcal{A}(\mathbf{S})$  on smooth configurations  $\mathbf{S}$  satisfying the equations of motion, and expand it in powers of  $a^2$ . The coefficients of this expansion are related to higher dimensional operators. The  $O(a^2)$  tree level Symanzik improvement is achieved by choosing the coefficients in  $\mathcal{A}(\mathbf{S})$  so that all the  $O(a^2)$  corrections turn to zero, i.e. the lattice action (on the solutions  $\mathbf{S}$ ) coincides with the continuum action to this accuracy. (For

the case of SU(N) gauge theory Garcia Perez, Snippe and van Baal [17] have derived this expansion for a set of loops. From this expression one recovers the tree level result by Lüscher and Weisz [18] obtained by considering the 2- and 3-point functions of gauge fields.) Note that this procedure is closely related to a remarkable property of the FP actions. Any solution of the FP lattice equations of motion generate a solution of the continuum theory, with exactly the same value of the lattice and continuum actions. On the other hand, all physical quantities calculated at the tree level using FP actions are free of lattice artifacts. Therefore they satisfy the tree level Symanzik conditions to all orders  $O(a^n)$  by construction.

Let us use this alternative procedure to determine the  $O(a^2)$  tree level on-shell Symanzik conditions for the  $O(N)$  sigma model. By expanding the lattice derivatives one obtains

$$\begin{aligned} \mathcal{A}(\mathbf{S}) = & \frac{1}{2} \int d^2x (\partial_\mu \mathbf{S} \cdot \partial_\mu \mathbf{S}) + \\ & a^2 \int d^2x \left\{ R_1 \frac{1}{16} (\partial^2 \mathbf{S} \cdot \partial^2 \mathbf{S}) + R_2 \frac{1}{48} \sum_\mu (\mathbf{S} \cdot \partial_\mu^4 \mathbf{S}) + \right. \\ & \left. C_1 \frac{1}{4} (\mathbf{S} \cdot \partial^2 \mathbf{S})^2 + C_2 \frac{1}{2} \sum_{\mu, \nu} (\partial_\mu \mathbf{S} \cdot \partial_\nu \mathbf{S})^2 + C_3 \frac{1}{4} \sum_\mu (\partial_\mu \mathbf{S} \cdot \partial_\mu \mathbf{S})^2 \right\} + O(a^4). \end{aligned} \quad (2.4)$$

Here we introduced the quartic moments:

$$\sum_n \rho(n) n_\mu n_\nu n_\alpha n_\beta = R_1 (\delta_{\mu\nu} \delta_{\alpha\beta} + \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}) + R_2 \delta_{\mu\nu\alpha\beta}, \quad (2.5)$$

and

$$\begin{aligned} \frac{1}{V} \sum_{n_1 n_2 n_3 n_4} c(n_1, n_2, n_3, n_4) \Delta_\mu \Delta_\nu \Delta'_\alpha \Delta'_\beta = \\ C_1 \delta_{\mu\nu} \delta_{\alpha\beta} + C_2 (\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}) + C_3 \delta_{\mu\nu\alpha\beta}, \end{aligned} \quad (2.6)$$

where  $\Delta = n_1 - n_2$ ,  $\Delta' = n_3 - n_4$ , and  $\delta_{\mu\nu\alpha\beta}$  is 1 when all its indices coincide, otherwise zero. Note that when restricted to solutions of the equations of motion,  $\partial^2 \mathbf{S} = \mathbf{S}(\mathbf{S} \cdot \partial^2 \mathbf{S})$ , the operators multiplying  $R_1$  and  $C_1$  in eq. (2.4) coincide. Accordingly, the corresponding on-shell Symanzik conditions are

$$R_2 = 0, \quad (2.7)$$

$$C_1 + \frac{1}{4} R_1 = 0, \quad C_2 = 0, \quad C_3 = 0. \quad (2.8)$$

These are the most general  $O(a^2)$  conditions since the terms not written out explicitly in eq. (2.3) contribute only to  $O(a^4)$  artifacts. Observe that the  $O(a^2)$  on-shell Symanzik conditions say nothing about  $C_1$  and  $R_1$  separately, only a linear combination of these two moments enters. The tree level spectrum is determined by the quadratic coefficients which has an expansion in momentum space:

$$\bar{\rho}(q) = q^2 + \frac{1}{8} R_1 (q^2)^2 + \frac{1}{24} R_2 \sum_\mu q_\mu^4 + O(q^6). \quad (2.9)$$

The absence of the  $O(a^2)$  artifacts in the tree level spectrum requires only  $R_2 = 0$ , the  $(q^2)^2$  term is allowed. In this case, of course, four-spin interactions should also be present in the action with a quartic coupling  $c$  whose moments satisfy eqs. (2.8). This is the generic realization of Symanzik tree level  $O(a^2)$  on-shell improvement. The FP action which produces no  $O(a^{2n})$  artifacts satisfies the Symanzik conditions this way. On the other hand, if one wants to have an improved action containing only two-spin interactions then  $C_1 = C_2 = C_3 = 0$  and then eq. (2.8) gives  $R_1 = 0$ . In this case, all the terms quartic in  $q$  disappear in eq. (2.9).

We show now that the tree level on-shell Symanzik conditions cancel the  $O(a^2)$  1-loop artifacts in the mass gap  $m(L)$ . The 1-loop contribution  $A_1(L)$  to  $Lm(L)$  can be written as [10]

$$A_1(L) = r_1(L) + (N-2)r_2(L) + s_1(L) + (N-2)s_2(L). \quad (2.10)$$

Here the terms  $r_1, r_2$  come from the quadratic couplings  $\rho$  in eq. (2.3) while  $s_1$  and  $s_2$  from the quartic couplings  $c$ . The corresponding expressions [10] could be written in a common form:

$$X(L) = \frac{1}{L} \sum_{l=0}^{L-1} \int \frac{dk_0}{2\pi} \frac{1}{\tilde{\rho}(k_0, k_1)} F_X(k_0, k_1), \quad (2.11)$$

with  $k_1 = \frac{2\pi}{L}l$ ,  $X = r_1, r_2, s_1, s_2$  and

$$F_{r_1}(k_0, k_1) = 1 - \frac{1}{2} \frac{\partial^2}{\partial k_0^2} \tilde{\rho}(k_0, k_1), \quad (2.12)$$

$$F_{r_2}(k_0, k_1) = 1 - \delta_{k_1 0} \frac{\tilde{\rho}(k_0, 0)}{\hat{k}_0^2}, \quad (2.13)$$

$$F_{s_1}(k_0, k_1) = -8 \left[ \frac{\partial}{\partial q_0} \frac{\partial}{\partial q'_0} \tilde{c}(q_0, 0; q'_0, 0; k_0, k_1) \right]_{q_0=q'_0=k_0/2}, \quad (2.14)$$

$$F_{s_2}(k_0, k_1) = 2 \left[ \frac{\partial^2}{\partial q_0^2} \tilde{c}(q_0, 0; 0; 0) - \frac{\partial^2}{\partial q_0^2} \tilde{c}(q_0, 0; k_0, k_1; 0) \right]_{q_0=0}. \quad (2.15)$$

Here  $\hat{k}_0^2 = 4 \sin^2(k_0/2)$  and the Fourier transforms are defined as

$$\tilde{\rho}(p) = \sum_n \rho(n) e^{-ip \cdot n}, \quad (2.16)$$

$$\tilde{c}(p, q, r) = \frac{1}{V} \sum_{n_1 n_2 n_3 n_4} c(n_1, n_2, n_3, n_4) e^{-ip \cdot \Delta} e^{-iq \cdot \Delta'} e^{-ir \cdot \Delta''}, \quad (2.17)$$

with  $\Delta = n_1 - n_2$ ,  $\Delta' = n_3 - n_4$  and  $\Delta'' = (n_1 + n_2 - n_3 - n_4)/2$ . The omitted terms in eq. (2.3) contain at least three factors of type  $(1 - \mathbf{S}_n \mathbf{S}_{n'})$  and hence do

not contribute to  $A_1(L)$ . One can explicitly separate the leading cut-off effects in eq. (2.11) by integrating over  $k_0$  and observing that the cut-off effects in the sum over  $k_1$  are produced by contribution from the pole at  $k_0 = ik_1 + \dots$ . The final result for the  $O(a^2)$  artifacts is given by

$$\frac{1}{L} \sum_{l=0}^{L-1} \int \frac{dk_0}{2\pi} \frac{1}{\bar{\rho}(k_0, k_1)} F(k_0, k_1) = \text{const} - \frac{\pi}{6} \frac{\alpha}{L^2} + O\left(\frac{1}{L^4}\right), \quad (2.18)$$

where

$$F(ik_1, k_1) = \alpha k_1^2 + O(k_1^4), \quad (2.19)$$

and it is also assumed that the two-point function  $\bar{\rho}(k)$  is improved, i.e.  $R_2 = 0$ . After a straightforward calculation one obtains (restoring  $a$ )

$$r_1(L) + s_1(L) = \text{const} - \frac{\pi}{3} \left( \frac{1}{4} R_1 + C_1 + 2C_2 + C_3 \right) \frac{a^2}{L^2} + O\left(\frac{a^4}{L^4}\right), \quad (2.20)$$

$$r_2(L) + s_2(L) = \text{const} + \frac{1}{2\pi} \ln \frac{L}{a} - \frac{\pi}{6} (2C_2 + C_3) \frac{a^2}{L^2} + O\left(\frac{a^4}{L^4}\right). \quad (2.21)$$

These equations generalize the results obtained by Caracciolo and Pelissetto [11] for two-spin interactions. From the general conditions (2.7,2.8) it follows that there are no  $O(a^2)$  artifacts at the 1-loop level independently, whether they are realized on-shell, or off-shell. This is consistent with the observation in ref. [11] that the  $O(a^2)$  artifacts in  $A_1(L)$  can be cancelled by an on-shell improved  $\rho$  and an appropriately chosen quartic coupling  $c$ . In fact, it is expected on general grounds that no physical distinction could be made between on-shell and off-shell improved actions. Indeed, by changing infinitesimally the field variables in an off-shell improved action, the resulting action will be on-shell improved since new terms proportional to the equations of motion appear.

Let us discuss finally the conjecture in [10] that an  $O(a^{2n})$  Symanzik improved action produces no cut-off effects up to  $O(a^{2n})$  in the 1-loop  $A_1(L)$ . This suggestion has been checked in ref. [10] on a specific example. We provide now a simple argument showing that the FP action does not produce any artifacts in  $A_1(L)$ . Assume that we have calculated the mass gap to 1-loop order with the action  $\beta_0 \mathcal{A}^{\text{FP}}(\mathbf{S})$ . (We use the notation  $\beta = 1/g^2$ ) Alternatively, we can make first a RG step:

$$\beta_0 \mathcal{A}^{\text{FP}}(\mathbf{S}) \xrightarrow{\text{RG}} \beta \mathcal{A}^{\text{FP}}(\mathbf{S}) + \delta \mathcal{A}(\mathbf{S}). \quad (2.22)$$

The quantum corrections shift the overall coupling,  $\beta_0 \rightarrow \beta = \beta_0 - \Delta\beta$ , where  $\Delta\beta = (N-2) \ln 2 / (2\pi)$  and produce, in general, an extra piece  $\delta \mathcal{A}(\mathbf{S})$ . Calculating the mass gap with this new action one should recover the old result including the artifacts. These are, however, associated now with the coarser lattice,  $a' = 2a$ . The part  $\delta \mathcal{A}(\mathbf{S})$  which is not multiplied by  $\beta$ , should be included at tree level. Since at tree level the artifacts to  $m(L)L$  are absent for *any* lattice action, the quantity  $A_1(L)$  in eq. (2.2) evaluated with  $\mathcal{A}^{\text{FP}}(\mathbf{S})$  for lattice spacing

$a$  and  $a' = 2a$  should give the same artifacts. As a consequence, all terms in eq. (2.2) with  $n \geq 1$  should vanish, i.e.  $A_1(L)$  has no artifacts at all. Of course, this argument does not apply to other physical quantities having cut-off effects already on the tree level since in this case the unknown term  $\delta\mathcal{A}(\mathbf{S})$  could also contribute to the cut-off effects in the given order in  $1/\beta$ .

### 3 The energy $E(L, p)$ in 1-loop perturbation theory

The amplitude  $B$  and the energy  $E(L, p)$  in eq. (1.1) can be written as

$$B = g^2 B_0 + g^4 B_1 + \dots, \quad (3.1)$$

$$E(L, p) = E_0 + g^2 E_1 + \dots, \quad (3.2)$$

which leads to the perturbative expansion of the correlator  $C(\tau; p)$ :

$$C(\tau; p) = g^2 C_0(\tau; p) + g^4 C_1(\tau; p) + \dots, \quad (3.3)$$

where, for large  $\tau$  we have:

$$C_0(\tau; p) = B_0 e^{-2\tau E_0}, \quad (3.4)$$

$$C_1(\tau; p) = (B_1 - 2\tau B_0 E_1) e^{-2\tau E_0}. \quad (3.5)$$

We shall consider  $p = 2\pi/L$ . The tree level propagator  $C_0(\tau; p)$  defines  $B_0$  and  $E_0$ , which can be used to determine  $E_1$  from the  $\propto \tau$  part of the 1-loop two-point function  $C_1(\tau; p)$ .

We shall study the cut-off dependence of  $E(L, p)$  using different FP actions, Symanzik tree level improved action with next-to-nearest-neighbour (nnn) coupling and the standard action. The FP actions give the exact continuum value for the tree level result  $LE_0(L, p) = 2\pi$ , while the standard and the Symanzik actions have  $O(a^2/L^2)$  and  $O(a^4/L^4)$  cut-off corrections, respectively. For the 1-loop correction of the energy  $E_1$  we write

$$LE_1 = c_0 + \frac{a^2}{L^2} c_1 + \frac{a^4}{L^4} c_2 + \dots, \quad (3.6)$$

where  $c_0$  is the universal continuum value (turns out to be 0.5 for  $p = 2\pi/L$ ), while  $c_1, c_2, \dots$  are numbers which depend on the form of the action. Since  $E_1$  is the leading  $O(g^2)$  correction to the energy, there are no  $\ln(a^2/L^2)$  type of terms in eq. (3.6). The Feynman graph expansion for the propagator in eq. (1.1) is the same as for  $p = 0$  in the calculation of the mass gap  $m(L)$ . In order to eliminate the quasi-zero modes (they are present even at  $p \neq 0$  on the 1-loop level) we used free boundary conditions in the time direction [15]. The term linear in  $\tau$  in eq. (3.5) can be separated analytically which leads to an easy numerical calculation. We shall summarize the results in Section 7.



## 4 The FP action versus the marginal operator

In most of the applications of the FP action until now such block transformations were used where the average of the fine variables in a block was allowed to fluctuate around the block variable of the coarse lattice. This fluctuation is governed by a parameter  $\kappa$  which is optimized to obtain a short range FP action. In the  $O(N)$   $\sigma$ -model the FP action satisfies the saddle-point equation [1]:

$$\mathcal{A}_\kappa^{\text{FP}}(\mathbf{R}) = \min_{\mathbf{S}} [\mathcal{A}_\kappa^{\text{FP}}(\mathbf{S}) + \kappa T(\mathbf{R}, \mathbf{S})] , \quad (4.1)$$

where  $\mathbf{R}$  and  $\mathbf{S}$  live on the coarse and on the fine lattice, respectively, while  $T$  defines the averaging procedure.

Let us add the operator  $\epsilon \mathcal{O}(\mathbf{S})$  ( $\epsilon$  is small) to the FP action and perform a RG step. Denoting the minimizing configuration in eq. (4.1) by  $\mathbf{S}(\mathbf{R})$  we get in the saddle-point approximation and in linear order in  $\epsilon$ :

$$\begin{aligned} \mathcal{A}_\kappa^{\text{FP}}(\mathbf{S}) + \epsilon \mathcal{O}(\mathbf{S}) &\xrightarrow{\text{RG}} \mathcal{A}_\kappa^{\text{FP}}(\mathbf{S}(\mathbf{R})) + \kappa T(\mathbf{R}, \mathbf{S}(\mathbf{R})) + \epsilon \mathcal{O}(\mathbf{S}(\mathbf{R})) \\ &= \mathcal{A}_\kappa^{\text{FP}}(\mathbf{R}) + \epsilon \mathcal{O}(\mathbf{S}(\mathbf{R})) \end{aligned} \quad (4.2)$$

Using this equation it is easy to see that  $\mathcal{O} = \mathcal{A}^{\text{FP}}$  is the marginal operator of the RG transformation if  $\kappa = \infty$ . Indeed, in this limit the blocking function becomes a  $\delta$ -function constraint and we get

$$\mathcal{A}_\infty^{\text{FP}}(\mathbf{R}) = \min_{\mathbf{S}} \mathcal{A}_\infty^{\text{FP}}(\mathbf{S}) \Big|_{T(\mathbf{R}, \mathbf{S})=0} = \mathcal{A}_\infty^{\text{FP}}(\mathbf{S}(\mathbf{R})) . \quad (4.3)$$

Consequently, for  $\mathcal{O} = \mathcal{A}_\infty^{\text{FP}}$  the r.h.s. of eq. (4.2) reads:

$$\mathcal{A}_\infty^{\text{FP}}(\mathbf{R}) + \epsilon \mathcal{A}_\infty^{\text{FP}}(\mathbf{S}(\mathbf{R})) = \mathcal{A}_\infty^{\text{FP}}(\mathbf{R}) + \epsilon \mathcal{A}_\infty^{\text{FP}}(\mathbf{R}) , \quad (4.4)$$

which has the same form as the l.h.s. of eq. (4.2).

For finite  $\kappa$  the marginal operator is not equal to the FP action, however. In the quadratic approximation (when only the first term in eq. (2.3) is kept) one can construct the marginal operator explicitly for any  $\kappa$ . In Fourier space the marginal operator can be written as

$$\rho_\kappa^{\text{marginal}}(q) = \frac{(\rho_\kappa^{\text{FP}}(q))^2}{\rho_\infty^{\text{FP}}(q)} \quad (4.5)$$

For  $\kappa = \infty$  the marginal operator goes over to the FP action as discussed in the general case above.

The formal manipulations leading to the statement that the FP action is 1-loop perfect assume implicitly that the FP action is identical to the marginal operator [8]. For this reason we studied the cut-off effects in the predictions of the FP actions in the limit  $\kappa = \infty$  also. Using a  $2 \times 2$  block with a flat averaging

defines a FP action in this limit which is rather broad [1]. This feature does not create a problem when investigating the free energy density as the function of the chemical potential since in this calculation the quadratic couplings enter only. In the case of the energy  $E(L, p)$ , where the quartic couplings are needed to a high precision also, we considered the FP action treated in [12]. This action corresponds to a  $\kappa = \infty$  transformation and has a short range  $\rho$  in the notation of eq. (2.3). This case is relevant not only for the issue of 1-loop perfection, but also to understand why the attempt to follow the RT in ref. [12] was not successful. The results discussed in Section 7 shed some light on this problem as well.

## 5 The iterated RG transformation and the RT in 1-loop perturbation theory

Consider the FP action at some very large coupling  $\bar{\beta}$  and perform  $r$  consecutive scale=2 RG steps. (The coupling is defined, as usual, as the coefficient of  $\frac{1}{2}q^2\mathbf{S}(q)\mathbf{S}(-q)$  in Fourier space.) Denote the finest field at the start by  $\mathbf{S}^{(r)}$ , the field after 1 RG step by  $\mathbf{S}^{(r-1)}$ , and the coarsest field at the end by  $\mathbf{S}^{(0)}$ . In the first step we obtain:

$$\begin{aligned} \int D\mathbf{S}^{(r)} \exp \left\{ -\bar{\beta} \left[ \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r)}) + \kappa T(\mathbf{S}^{(r-1)}, \mathbf{S}^{(r)}) \right] \right\} \\ = \exp \left\{ - \left[ \bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r-1)}) + \mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)}) \right] + O(1/\beta) \right\}. \end{aligned} \quad (5.1)$$

The first term in the exponent on the r.h.s. is coming from the leading saddle-point approximation: the contribution of the configuration  $\mathbf{S}^{(r)} = \mathbf{S}^{(r)}(\mathbf{S}^{(r-1)})$  which minimizes the exponent on the l.h.s. (classical result). Expanding  $\mathbf{S}^{(r)}$  around the saddle point solution, the leading quantum corrections are independent of  $\bar{\beta}$  and are denoted by  $\mathcal{A}^{\text{q}}$  in eq. (5.1).

Performing the second step of RG transformation,  $\bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r-1)})$  generates  $\bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r-2)}) + \mathcal{A}^{\text{q}}(\mathbf{S}^{(r-2)})$  as before, while  $\mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)})$  contributes in the leading saddle-point approximation only giving  $\mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)}(\mathbf{S}^{(r-2)}))$  where  $\mathbf{S}^{(r-1)}(\mathbf{S}^{(r-2)})$  is the minimizing solution of the saddle-point equation in the second step. After  $r$  steps we get

$$\bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r)}) \xrightarrow[r \text{ steps}]{} \bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(0)}) + \mathcal{A}_r^{\text{q}}(\mathbf{S}^{(0)}) + O(1/\beta), \quad (5.2)$$

where

$$\mathcal{A}_r^{\text{q}} = \mathcal{A}^{\text{q}}(\mathbf{S}^{(0)}) + \mathcal{A}^{\text{q}}(\mathbf{S}^{(1)}(\mathbf{S}^{(0)})) + \dots + \mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)}(\dots \mathbf{S}^{(1)}(\mathbf{S}^{(0)}) \dots)). \quad (5.3)$$

Define  $\beta = \bar{\beta} - r\Delta\beta$  with  $\Delta\beta = \frac{N-2}{2\pi} \ln 2$  and write

$$\mathcal{A}_r^{\text{q}} = -r\Delta\beta \mathcal{A}^{\text{FP}} + \delta\mathcal{A}_r^{\text{q}}, \quad (5.4)$$

giving

$$\bar{\beta} \mathcal{A}^{\text{FP}}(\mathbf{S}^{(r)}) \xrightarrow[r \text{ steps}]{} \beta \mathcal{A}^{\text{FP}}(\mathbf{S}^{(0)}) + \delta \mathcal{A}_r^{\text{q}}(\mathbf{S}^{(0)}) + O(1/\beta). \quad (5.5)$$

In the limit  $r \rightarrow \infty$ ,  $\beta$  very large, fixed (i.e.  $\bar{\beta} \rightarrow \infty$ ), the r.h.s. of eq. (5.5) defines the RT (the quantum perfect action). If  $\delta \mathcal{A}_\infty^{\text{q}}$  is neither zero, nor redundant (i.e. it gives a non-zero contribution to the cut-off dependence of a physical quantity like  $E(L, p)$  on the 1-loop level) then  $\mathcal{A}^{\text{FP}}$  can not be quantum perfect. In this case the predictions from  $\beta \mathcal{A}^{\text{FP}}$  will have cut-off effects on the 1-loop level which are exactly compensated by the contribution from  $\delta \mathcal{A}_\infty^{\text{q}}$ . In the 1-loop calculation of  $E(L, p)$  only the quadratic part of  $\delta \mathcal{A}_\infty^{\text{q}}$  enters:

$$\delta \mathcal{A}_\infty^{\text{q}}(\mathbf{S}^{(0)}) = -\frac{1}{2} \sum_{n,r} \delta \rho_\infty^{\text{q}}(r) \left(1 - \mathbf{S}_{n+r}^{(0)} \mathbf{S}_n^{(0)}\right) + \dots \quad (5.6)$$

As eqs. (5.3,5.4) show,  $\delta \rho_\infty^{\text{q}}$  can be obtained from the quadratic part of the quantum correction  $\mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)})$  in eq. (5.1) obtained after 1 step of RG transformation. Let us discuss briefly the steps of this calculation.

In order to perform the path integral in eq. (5.1) perturbatively, we write

$$\mathbf{S}_n^{(r)} = \left(\sqrt{1 - \vec{\pi}_n^2}, \vec{\pi}_n\right), \quad \mathbf{S}_{n_B}^{(r-1)} = \left(\sqrt{1 - \vec{\chi}_{n_B}^2}, \vec{\chi}_{n_B}\right), \quad (5.7)$$

and expand  $\mathcal{A}^{\text{FP}}$ ,  $T$  and the measure in the fluctuations  $\vec{\pi}$  and  $\vec{\chi}$ . After shifting the integration variable  $\vec{\pi}_n$  by the classical solution  $\vec{\pi}_n^{\text{c}}(\vec{\chi})$ :

$$\vec{\pi}_n = \vec{\pi}_n^{\text{c}}(\vec{\chi}) + \vec{\xi}_n, \quad (5.8)$$

one obtains terms from  $\mathcal{A}^{\text{FP}} + \kappa T$  which are independent of  $\xi$ , quadratic in  $\xi$ , etc. The terms which are independent of  $\xi$  reproduce the FP action on the coarse lattice (the classical result). This is the first term in the exponent on the r.h.s. of eq. (5.1). The terms quadratic in  $\xi$  of the type ' $\chi\chi\xi\xi$ ' will give ' $\chi\chi$ ' type of corrections after integrating over  $\xi$ . Similar contributions will be produced by the measure and by the term coming from the zero mode fixing, where we can replace  $\vec{\pi}_n$  by  $\vec{\pi}_n^{\text{c}}(\vec{\chi})$ . (These terms are not multiplied by  $\bar{\beta}$ .) Collecting all these contributions one obtains  $\rho^{\text{q}}$  in  $\mathcal{A}^{\text{q}}$ :

$$\mathcal{A}^{\text{q}}(\mathbf{S}^{(r-1)}) = -\frac{1}{2} \sum_{n_B, r_B} \rho^{\text{q}}(r_B) \vec{\chi}_{n_B} \vec{\chi}_{n_B+r_B} + \dots \quad (5.9)$$

The quadratic part of the quantum correction obtained after 1 step of RG determines the quadratic part of  $\mathcal{A}_r^{\text{q}}$  in eq. (5.3) using the relation between  $\vec{\pi}_n$  and  $\vec{\chi}_{n_B}$  (and its iterations) obtained from the saddle-point equation in linear order

$$\vec{\pi}_n^{\text{c}} = \sum_{n_B} Z(n - 2n_B) \vec{\chi}_{n_B}, \quad (5.10)$$

Finally, the relation eq. (5.4) gives  $\delta \rho_r^{\text{q}}$ .

The RT defines a quantum perfect action which will be used as a consistency check on the results in Section 7: the cut off effects generated by the FP action

in  $E(L, p)$  should be exactly cancelled by  $\delta\rho_\infty^q$ . A deviation from this condition reflects the systematic error of the calculation which is mainly due to the error in calculating the quartic couplings  $c$  in eq. (2.3).

Let us add a remark on the finite size effects concerning the action itself. As discussed in the Appendix of ref. [2], the FP action in a small volume can be obtained from the FP action in a large volume by a simple 'wrapping' procedure. This is not true, however, for the quantum corrections, like  $\delta\rho_\infty^q$ . This correction should be calculated separately for each small volume values. The finite size effects in the action go to zero exponentially as the volume is increased.

## 6 The free energy density as the function of the chemical potential in 1-loop perturbation theory with the FP action

The chemical potential is a very convenient tool to probe the system [19, 1]. Unlike the magnetic field, it does not get renormalized and to obtain the free energy on the 1-loop level it is sufficient to expand the action up to quadratic order in the fluctuations.

Technically, the chemical potential  $h$  is a constant, imaginary vector potential:  $A_\mu \rightarrow ih\delta_{\mu,0}Q$ , where  $Q$  is the generator of an  $O(N)$  rotation. Choosing the rotation in the 0-1 plane (the  $O(N)$  indices run as  $i = 0, 1, \dots, N-1$ ) the effect of a non-zero chemical potential is

$$\begin{aligned} S_n^0 &\rightarrow \cosh(n_0 h) S_n^0 + i \sinh(n_0 h) S_n^1, \\ S_n^1 &\rightarrow -i \sinh(n_0 h) S_n^0 + \cosh(n_0 h) S_n^1, \\ S_n^j &\rightarrow S_n^j, \quad \text{for } j = 2, \dots, N-1. \end{aligned} \quad (6.1)$$

Here and in the following we use quantities whose dimension is carried by the lattice unit. Eq. (6.1) gives the dependence on the chemical potential of lattice regularized actions [20, 21].

Consider the generalized action in eq. (2.3). Using eq. (6.1) and writing  $\mathbf{S}_n$  in terms of the fluctuations  $\pi_n$  as in eq. (5.7) we get

$$\beta\mathcal{A}(\mathbf{S}) \rightarrow V f_0(h) + \beta \frac{1}{2} \sum_{n,r} \left[ \rho^{(1)}(r; h) \pi_n^1 \pi_{n+r}^1 + \rho^{(\text{tr})}(r; h) \sum_{j=2}^{N-1} \pi_n^j \pi_{n+r}^j \right] + O(\pi^4), \quad (6.2)$$

where  $V$  is the volume of the system and  $\rho^{(1)}$  and  $\rho^{(\text{tr})}$  are the two different  $h$ -dependent quadratic couplings ( $h$  breaks the symmetry between the 0-1 plane and the transversal directions). Our task is to find  $f_0(h)$ ,  $\rho^{(1)}$  and  $\rho^{(\text{tr})}$  for the FP action. The basic factor  $(1 - \mathbf{S}_n \mathbf{S}_{n'})$  in eq. (2.3) can be written as

$$1 - \mathbf{S}_n \mathbf{S}_{n'} \rightarrow [1 - \cosh((n_0 - n'_0)h)] + O(\pi^2), \quad (6.3)$$

indicating a technical problem: unlike in the case with  $h = 0$ , the terms up to quadratic order in the fluctuations  $\pi$  receive contributions from terms of the action containing arbitrary many factors of the type  $(1 - \mathbf{S}_n \mathbf{S}_{n'})$ .

A way to proceed is to consider first an imaginary  $h$

$$h = -i\mu \quad (6.4)$$

which connects the problem of the free energy in the presence of a chemical potential with that of the free energy of a uniformly twisted solution with fluctuations. Indeed, for real values of  $\mu$  eq. (6.1) at  $\bar{\pi}_n = 0$  describes such a solution where  $\mu$  is the angle (in the 0-1 plane of internal indices) between two neighbouring spins in the time direction, while the configuration is constant in the spatial direction. This is an exact solution of the FP lattice equations of motion and the FP action gives the exact classical value for the action:  $V\mu^2/2$  [1] leading to the well-known continuum result

$$V f_0(h) = -V \frac{h^2}{2g^2}. \quad (6.5)$$

For a uniformly twisted solution  $\mathbf{R}$  with angle  $2\mu$  on the coarse lattice, the minimizing configuration  $\mathbf{S}$  in eq. (4.1) is also a uniformly twisted solution with angle  $\mu$  [1]. Consider now eq. (4.1) with a configuration  $\mathbf{R}$  which contains small fluctuations around the above solution. Under a RG step we get in the saddle point approximation:

$$\begin{aligned} V \frac{\mu^2}{2g^2} + \frac{1}{2g^2} \sum_{n,r} \left[ \rho^{(1)}(r; -i\mu) \pi_n^1 \pi_{n+r}^1 + \rho^{(\text{tr})}(r; -i\mu) \sum_{j=2}^{N-1} \pi_n^j \pi_{n+r}^j \right] + O(\pi^4) \\ \xrightarrow{\text{RG}} V_B \frac{(2\mu)^2}{2g^2} + \frac{1}{2g^2} \sum_{n_B, r_B} \left[ \rho^{(1)}(r_B; -i2\mu) \chi_{n_B}^1 \chi_{n_B+r_B}^1 + \right. \\ \left. \rho^{(\text{tr})}(r_B; -i2\mu) \sum_{j=2}^{N-1} \chi_{n_B}^j \chi_{n_B+r_B}^j \right] + O(\chi^4), \quad (6.6) \end{aligned}$$

where  $V_B = V/4$  is the volume of the coarse lattice.

Consider the RG transformation with the kernel

$$\kappa T(\mathbf{R}, \mathbf{S}) = 2\kappa \sum_{n_B} \left( \mathbf{R}_{n_B} - \frac{\sum_{n_B} \mathbf{S}_{n_B}}{|\sum_{n_B}|} \right)^2, \quad (6.7)$$

where  $\sum_{n_B}$  is the sum over the four  $\mathbf{S}$  spins in the block  $n_B$ . This kernel is a slightly modified version of the transformation used in [1] with the technical advantage of having a trivial norm. For the case of small fluctuations around the twisted solution we have

$$\begin{aligned} \kappa T(\mathbf{R}, \mathbf{S}) \rightarrow 2\kappa \sum_{n_B} \left[ \left( \chi_{n_B}^1 - \frac{1}{4} \sum_{n \in n_B} \pi_n^1 \right)^2 + \right. \\ \left. \sum_{j=2}^{N-1} \left( \chi_{n_B}^j - \frac{1}{\cos(\mu/2)} \frac{1}{4} \sum_{n \in n_B} \pi_n^j \right)^2 \right] + (\text{quartic in the fields}). \quad (6.8) \end{aligned}$$

It is not difficult to find the couplings  $\rho^{(1)}$  and  $\rho^{(\text{tr})}$  satisfying eq. (6.6) under the block transformation in eq. (6.8). After continuing back to real  $h$ , in Fourier space they read

$$\frac{1}{\rho^{(1)}(q; h)} = \sum_{l=-\infty}^{\infty} \frac{1}{(q + 2\pi l)^2} \prod_{i=0}^1 \frac{\sin^2(\frac{1}{2}q_i)}{(\frac{1}{2}q_i + \pi l_i)^2} + \frac{1}{3\kappa}, \quad (6.9)$$

$$\frac{1}{\rho^{(\text{tr})}(q; h)} = d(h) \sum_{l=-\infty}^{\infty} \frac{1}{(q + 2\pi l)^2 + h^2} \prod_{i=0}^1 \frac{\sin^2(\frac{1}{2}q_i)}{(\frac{1}{2}q_i + \pi l_i)^2} + \frac{1}{3\kappa}, \quad (6.10)$$

where

$$d(h) = \prod_{j=1}^{\infty} \frac{1}{\cosh(h/2^{j+1})}. \quad (6.11)$$

(For the transformation with non-trivial norm [1] there enters an extra  $\cos(\mu/2)$  factor in front of the  $n_B$ -sum in eq. (6.8) and in eqs. (6.9,6.10) the term  $1/(3\kappa)$  is replaced by two different  $h$ -dependent functions which we do not quote here explicitly.)

Having the quadratic couplings  $\rho^{(1)}$  and  $\rho^{(\text{tr})}$  one should perform a gaussian integral to get the free energy as the function of the chemical potential. The free energy density on the 1-loop level has the form

$$f(h) = f(0) - h^2 \left[ \frac{1}{2g^2} + (N-2) \frac{1}{8\pi} \ln(ch^2) \right] + O(h^4), \quad (6.12)$$

where  $c$  is a constant and the  $O(h^4)$  terms represent the cut-off effects on the 1-loop level. We shall discuss the results in the next section.

We close this section with remarks on testing 1-loop perfection by calculating the free energy as the function of  $h$  and on the method we applied to obtain eqs. (6.9,6.10). We used the trick of analytic continuation to imaginary  $h$  to connect the problem to a fluctuating uniformly twisted solution. This solution itself is however, unstable against certain transversal fluctuations. (A trace of this instability is the singularity in eq. (6.10) at real  $q$  values when  $h$  is imaginary.) Of course, the original problem with a chemical potential is stable and we do not think that this is a serious problem. A somewhat more delicate question is whether the free energy is a good quantity to consider. We refer here to the special behaviour of the free energy under RG transformations [13]. A useful test would be in this context to check whether at finite  $h$  the contribution from the quantum  $\delta\rho_{\infty}^q$  considered in Section 5 compensates exactly the cut-off effects generated by the FP action. This test has not been done.

## 7 The 1-loop results

We present first the results on the energy  $E(L, p)$  with  $p = 2\pi/L$  for the FP and for the quantum perfect actions which correspond to the block transformation

in eq. (6.7) (trivial norm) with the optimal value for  $\kappa$  ( $\kappa = 2$ ). In this case there are no cut-off effects on the tree level. The 1-loop results are summarized in Table 1.

$L$	$LE_1^{\text{FP}}$	$(LE_1^{\text{FP}} - 0.5)L^2$	$LE_1^{\text{q}}$	$L(E_1^{\text{FP}} + E_1^{\text{q}})$
2	0.676	0.705	-0.217	0.459
3	0.5027	0.0247	0.0135	0.5162
4	0.494654	-0.0855	0.006279	0.500933
5	0.496102	-0.0975	0.004044	0.500146
6	0.497225	-0.0999	0.002794	0.500019
7	0.497946	-0.1006	0.002053	0.499999
8	0.498422	-0.1010	0.001572	0.499994
9	0.498752	-0.1011	0.001242	0.499994
10	0.498988	-0.1012	0.001006	0.499994
12	0.499297	-0.1013	0.000699	0.499996
14	0.499483	-0.1013	0.000513	0.499996

Table 1: Results on the 1-loop contribution  $E_1(L, p)$  to the energy  $E(L, p)$  in a finite spatial volume  $L$  with  $p = 2\pi/L$  using actions related to the block transformation in eq. (6.7) with  $\kappa = 2$ . The continuum value of  $E_1(L, p)$  is 0.5,  $LE_1^{\text{FP}}(L, p)$  is the FP action prediction,  $LE_1^{\text{q}}(L, p)$  is the correction from the operator  $\delta\rho_\infty^{\text{q}}$  in eq. (5.6) which represents the difference between the FP action and the quantum perfect action. The deviation from 0.5 in the last column is a systematic error due to the cuts introduced when calculating the quartic couplings  $c$  of the FP action.

The last column in Table 1 is the prediction of the quantum perfect action. The deviation from the exact value (0.5) is the error of the 1-loop calculation. The source of this error is the approximation introduced in calculating the quartic couplings of the FP action. For  $L > 6$  this error is  $O(10^{-6})$  which is much smaller than the deviation of the prediction of the FP action from 0.5. Since for  $L > 6$  those cut-off effects which are exponentially small (in  $L$ ) are negligible, we can conclude that the FP action of the block transformation in eq. (6.7) at  $\kappa = 2$  has power like cut-off effects in the 1-loop energy. Consequently, it is not 1-loop perfect. The coefficient of the  $O(a^2)$  cut-off correction is  $-0.101$ .

For comparison we give in Table 2 the 1-loop contribution  $E_1(L, p)$  to the energy obtained from the standard and from the nnn realization of the tree level improved Symanzik actions. The cut-off effects seen are significantly larger than those of the FP action. The coefficient of the  $O(a^2)$  correction is 17 and 32 times larger for the standard and Symanzik tree level improved actions, respectively. It is interesting to note that cancelling the tree level  $O(a^2)$  artifacts with an nnn interaction term in the Symanzik program does not have any positive effect on the 1-loop artifacts – actually, the cut-off effects became even larger.

$L$	$LE_1^{\text{st}}$	$(LE_1^{\text{st}} - 0.5)L^2$	$LE_1^{\text{SYM}}$	$(LE_1^{\text{SYM}} - 0.5)L^2$
2	0.707	0.828		
3	0.6248	1.123		
4	0.579786	1.276	0.143802	-5.699
5	0.554845	1.371	0.349222	-3.769
6	0.539873	1.435	0.406457	-3.368
7	0.530223	1.481	0.434246	-3.222
8	0.523654	1.514	0.450680	-3.156
9	0.518991	1.538	0.461444	-3.123
10	0.515566	1.557	0.468957	-3.104
12	0.510983	1.582	0.478568	-3.086
14	0.508152	1.598	0.484293	-3.079

Table 2: Results on the 1-loop contribution to the energy given by the standard and by the tree level improved Symanzik actions.

As discussed in the Introduction and in Section 4, ‘deterministic’ ( $\kappa = \infty$ ) block transformations have the property that the marginal operator is identical to the FP action. Since this is an assumption in the formal considerations on 1-loop perfection [8], we tested this case also. We considered the block transformation investigated in ref. [12]: the block spins sit in the even points of the fine lattice, the fine spin in this point contributes with a weight factor of 0.8 to the block average, the nearest-neighbours have a weight factor of 0.05. Although this transformation is dangerously close to a decimation, it defines at  $\kappa = \infty$  a FP with a quadratic coupling  $\rho$  which is short ranged [12]. Unfortunately, we found that the quartic coupling  $c$  has an extended range already and the quantum propagator related to the fluctuations around the saddle point solution decays less rapidly than for the block transformation in eq. (6.7). This is reflected by the significantly increased systematic errors in the last column of Table 3 which are related to the cuts when calculating the quartic coupling  $c$ . Nevertheless, the numbers in Table 3 suggest strongly that there are cut-off effects in the 1-loop prediction of the FP action in this case also.

For further clarification we studied the free energy density as the function of the chemical potential as discussed in Section 6. From eq. (6.12) follows that the combination

$$R(h) = \frac{d}{dh^2} \left( f(h) + \frac{h^2}{2g^2} \right) + (N-2) \frac{1}{8\pi} \ln(h^2). \quad (7.1)$$

is a constant up to cut-off effects. Table 4 shows the results using the FP action generated by the block transformation in eq. (6.7) with  $\kappa = \infty$ . This calculation has practically no systematic errors. (See, however, the remarks at the end of Section 6.)  $R(h)$  shows small deviations from a constant leading to the conclusion that even those FP actions which correspond to  $\kappa = \infty$  (‘determi-



$L$	$LE_1^{\text{FP}\infty}$	$(LE_1^{\text{FP}\infty} - 0.5)L^2$	$LE_1^{\text{q}}$	$L(E_1^{\text{FP}\infty} + E_1^{\text{q}})$
2	1.709	2.314	-0.407	0.672
3	0.5623	0.5603	-0.0221	0.5402
4	0.52181	0.3490	-0.01289	0.50892
5	0.512134	0.3034	-0.008246	0.503888
6	0.508005	0.2882	-0.005720	0.502285
7	0.505753	0.2819	-0.004203	0.501550
8	0.504358	0.2789	-0.003219	0.501139
9	0.503424	0.2773	-0.002544	0.500880
10	0.502764	0.2764	-0.002060	0.500704
12	0.501914	0.2756	-0.001431	0.500483
14	0.501404	0.2752	-0.001051	0.500353

Table 3: Results on the 1-loop contribution to the energy using the FP action and the quantum perfect action for the  $\kappa = \infty$  block transformation of ref. [12]. The notations are the same as in Table 1.

nistic') block transformations produce non-zero (although small) cut-off effects in 1-loop perturbation theory.

$h$	$R(h)$	$h$	$R(h)$	$h$	$R(h)$
1.4	0.142299	0.8	0.138704	0.4	0.137272
1.2	0.140938	0.7	0.138265	0.3	0.137057
1.0	0.139732	0.6	0.137878	0.2	0.136902
0.9	0.139194	0.5	0.137546	0.1	0.136809

Table 4: The values of  $R(h)$  from eq. (7.1) for different values of the chemical potential  $h$

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