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Orientifold Limit of F-theory Vacua

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Abstract

We show how an F-theory compactified on a Calabi-Yau $(n+1)$-fold in appropriate weak coupling limit reduces formally to an orientifold of type IIB theory compactified on an auxiliary complex n-fold. In some cases (but not always) if the original $(n+1)$-fold is singular, then the auxiliary n-fold is also singular. We illustrate this by analysing F-theory on elliptically fibered Calabi-Yau 3-folds on base F_n.

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F-theory[1, 2] and orientifolds[3, 4, 5, 6] are two different classes of compactifications of type IIB string theory. In recent years we have seen several examples of dual pairs of theories, of which one is an F-theory compactification, and the other is an orientifold[7, 8, 9, 10, 11, 12, 13]. Typically in these examples the F-theory background describes the non-perturbative correction to the orientifold background[7, 14, 15, 13].

In this note we shall show that this feature is quite general, – namely given any F-theory compactification on a Calabi-Yau $(n+1)$-fold, we can go to appropriate region in the moduli space of the theory where the background (at least formally) looks like that of an orientifold of type IIB theory on an n-dimensional complex manifold. We start from the Weierstrass form of an elliptically fibered manifold:

$$y^2 = x^3 + f(\vec{u})x + g(\vec{u}), \tag{1}$$

where f and g are appropriate polynomials on some base $\mathcal{B}$ labelled by complex coordinates $\vec{u}$. F-theory compactified on such a manifold is by definition type IIB theory compactified on the base $\mathcal{B}$ with background axion-dilaton field $\lambda(\vec{u})$ given by:

$$j(\lambda) = \frac{4 \cdot (24f)^3}{4f^3 + 27g^2}, \tag{2}$$

where $j(\lambda)$ is the modular invariant function of λ with a pole at $i\infty$, zero at $e^{i\pi/3}$, and normalized such that $j(i) = (24)^3$. The locations of the seven branes on the base are at the zeroes of

$$\Delta = 4f^3 + 27g^2 \tag{3}$$

where $\lambda \to i\infty$ up to an SL(2,Z) transformation. We shall first consider a special family of points in the moduli space of this F-theory where f and g take special form[13]:

$$f(\vec{u}) = C\eta(\vec{u}) - 3h(\vec{u})^2, \tag{4}$$

and

$$g(\vec{u}) = h(\vec{u})(C\eta(\vec{u}) - 2h(\vec{u})^2). \tag{5}$$

Here h and η are appropriate polynomials on $\mathcal{B}$, and C is a constant denoting the overall normalization of η. This gives

$$j(\lambda) = \frac{4 \cdot (24)^3 (C\eta - 3h^2)^3}{C^2\eta^2(4C\eta - 9h^2)}, \tag{6}$$

and

$$\Delta = C^2\eta^2(4C\eta - 9h^2)\,. \tag{7}$$

From eq.(6) we see that as $C \to 0$ with η and h fixed, $j(\lambda)$ goes to infinity almost everywhere on $\mathcal{B}$ except at the zeroes of the numerator. This corresponds to $\lambda \to i\infty$ (up to an SL(2,Z) transformation) almost everywhere on $\mathcal{B}$, and choosing the convention that this limit is $i\infty$ and not one of its images under SL(2,Z), we can identify this as the weak coupling limit. In particular the average λ is related to C as

$$e^{2\pi i\langle\lambda\rangle} \sim C^2\,, \tag{8}$$

for small C. We shall now show that in this limit the background λ given in (6) can be identified to that of an orientifold, with orientifold seven plane situated at $h = 0$, and a pair of Dirichlet seven branes situated at $\eta = 0$.

To see this first of all note from (7) that at $\eta = 0$, Δ has a double zero. Since neither f nor g vanish there, this is an A_1 singularity,[3] and $\lambda \to i\infty$ on this hypersurface up to an SL(2,Z) transformation. However, our previous convention that for small C, $Im(\lambda)$ is large at a generic point in $\mathcal{B}$, and the fact that we can pass from a generic point in $\mathcal{B}$ to the $\eta = 0$ surface keeping $j(\lambda)$ given in (6) always large, shows that on the surface $\eta = 0$ λ actually goes to $i\infty$ and not to any of its SL(2,Z) transform. The SL(2,Z) monodromy around the hypersurface $\eta = 0$ is then given by T^2 where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{9}$$

Thus the singularity at $\eta = 0$ can be interpreted as due to the presence of a pair of coincident D-7 branes.

The other zeroes of Δ are situated at

$$9h(\vec{u})^2 - 4C\eta(\vec{u}) = 0\,, \tag{10}$$

which can be rewritten as

$$h(\vec{u}) = \pm\frac{2}{3}\sqrt{C\eta(\vec{u})}\,. \tag{11}$$

[3]We are implicitly assuming that η and h do not have any common factor, and that each has only simple zeroes. Otherwise we might get more complicated singularities. We shall encounter such examples later.

For small C, this represents a pair of hypersurfaces close to the hypersurface $h = 0$. The monodromy around each of these hypersurfaces is conjugate to T. Let us take them to be[4]

$$MTM^{-1} \quad \text{and} \quad NTN^{-1}, \tag{12}$$

respectively, where M and N are some SL(2,Z) matrices. Thus the effect of going around both these branches is given by $MTM^{-1}NTN^{-1}$. We shall now explicitly compute this monodromy from (6). As seen from this equation, for small C, $j(\lambda)$ is large everywhere along a contour enclosing the hypersurface $h = 0$ *as long as the contour remains at a finite distance from this hypersurface, thereby enclosing both branches of (11).* Thus $Im(\lambda)$ remains large along this contour and comes back to its original value as we travel once around the contour. Since $j(\lambda) \sim exp(-2\pi i\lambda)$ for large $Im(\lambda)$, the change in $Re(\lambda)$ is given by

$$-\frac{1}{2\pi i}\oint \frac{dj}{j}, \tag{13}$$

where the integral is performed along the contour around $h = 0$. For small C, (6) gives

$$j(\lambda) \sim h^4/C^2\eta^2 . \tag{14}$$

Substituting this in (13) and picking up the contribution from the pole at $h = 0$, we see that $Re(\lambda)$ changes by -4 along this contour.[5] Thus we get

$$MTM^{-1}NTN^{-1} = \pm T^{-4} . \tag{15}$$

Starting with the most general ansatz for the SL(2,Z) matrices M and N, and substituting in eq.(15), one can verify that the most general solutions for MTM^{-1} and NTN^{-1} are of the form:

$$MTM^{-1} = \begin{pmatrix} 1-p & p^2 \\ -1 & 1+p \end{pmatrix}, \qquad NTN^{-1} = \begin{pmatrix} -1-p & (p+2)^2 \\ -1 & 3+p \end{pmatrix}, \tag{16}$$

where p is an arbitrary integer. This gives:

$$MTM^{-1}NTN^{-1} = -T^{-4} . \tag{17}$$

[4]Instead of characterizing these hypersurfaces by the SL(2,Z) monodromy around them, we could also characterize them by the value of λ on the hypersurface. A monodromy matrix MTM^{-1} with $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ will correspond to λ being equal to p/r on that hypersurface.

[5]Since we are interested in calculating the monodromy around the two branches of the hypersurface given in (11) we choose the contour in such a way that it does not enclose any branch of the hypersurface $\eta = 0$.

Now since in the weak coupling limit $C \to 0$ the two branches collapse onto the surface $h = 0$, only this monodromy will be visible in this limit. But this is precisely the monodromy around an orientifold seven plane. In order to describe this in more concrete terms, let us consider an auxiliary manifold $\mathcal{M}$ which is a double cover of the base $\mathcal{B}$, and is defined by the equation:

$$\xi^2 - h(\vec{u}) = 0\,, \tag{18}$$

where ξ is a complex variable. We now consider type IIB theory on this auxiliary manifold $\mathcal{M}$, and mod out this theory by the transformation

$$(-1)^{F_L} \cdot \Omega \cdot \sigma\,, \tag{19}$$

where $(-1)^{F_L}$ is the discrete Z_2 symmetry of type IIB theory that changes the sign of all the Ramond sector states in the left-moving sector of the world-sheet, Ω is the world-sheet parity transformation, and σ denotes the transformation

$$\xi \to -\xi\,. \tag{20}$$

This transformation leaves fixed the seven plane $\xi = 0$, which using (18) can also be written as $h(\vec{u}) = 0$. This represents the orientifold plane. Then a closed curve around this plane in the quotient manifold $\mathcal{B}$ will have precisely the monodromy given in (17). The -1 factor in the monodromy (17) is the effect of the transformation $(-1)^{F_L} \cdot \Omega$, whereas the T^{-4} factor reflects the fact that such an orientifold seven plane carries -4 units of magnetic charge of the Ramond-Ramond scalar field that forms the real part of λ[6, 7]. The splitting of the orientifold plane into two seven-branes for non-zero C reflects the phenomenon already observed in ref.[7].

This establishes that the F-theory background, in the particular limit we have described, does represent an orientifold of IIB on $\mathcal{M}$ with a pair of D-branes situated at $\eta(\vec{u}) = 0$ on the quotient $\mathcal{B}$. We shall now consider deformation of (4), (5) to most general f and g. Since η is an arbitrary polynomial (subject to the same restrictions as f, and its overall normalization specified separately in the constant C), we see from (4) that by choosing the most general η and C we can get the most general f. Thus we only need to deform g. To this end consider the following form of g:

$$g = h(C\eta - 2h^2) + C^2\chi\,, \tag{21}$$

where χ is a general polynomial subject to the same restrictions as g. This will certainly give us the most general g. Of course some of the deformations of χ will simply correspond to deformations of h and η keeping f fixed, but this will not concern us here as we are only interested in showing that we have the most general g. The factor of C in (21) has been chosen such that Δ still has an overall factor of C^2 and the relation (8) remains valid. Indeed with this choice,

$$\Delta = C^2\{\eta^2(4C\eta - 9h^2) + 54h(C\eta - 2h^2)\chi + 27C^2\chi^2\}\,. \tag{22}$$

This no longer has a factorized form. In order to identify the orientifold planes and D-branes, we need to take the weak coupling limit $C \to 0$ keeping h, η and χ fixed. In this limit:

$$\Delta = -9C^2h^2(\eta^2 + 12h\chi)\,. \tag{23}$$

From this we see that the orientifold plane is still located at $h = 0$, but the D-brane positions have shifted to

$$\eta = \pm\sqrt{-12h\chi}\,. \tag{24}$$

Thus the D-branes are split, but they coincide at the point of intersection with the orientifold plane. Also as we travel once around the orientifold plane $h = 0$ remaining on the surface of the D-brane, we see from (24) that the two D-branes get exchanged. This is entirely in agreement with the results found in [15]. The splitting of the D-brane pair can be described, in the orientifold language, as a result of switching on vacuum expectation values of fields that are charged under the SU(2) gauge group associated with the original D-brane pair.

We would also like to know if the complex n-fold defined by eq.(18) satisfies the Calabi-Yau condition. Due to physics reasons we know that this must be the case, since otherwise type IIB compactified on (18) will not have any supersymmetry, and hence its orientifold will also not have any unbroken space-time supersymmetry. But we can also try to verify this directly. First let us consider the case where the base is a toric variety. In this case we can introduce homogeneous coordinates $(u_1, \cdots u_m)$ on the base with identifications of the form:

$$(u_1, \cdots u_m) \equiv ((\lambda^{(i)})^{w_1^{(i)}} u_1, \cdots (\lambda^{(i)})^{w_m^{(i)}} u_m) \quad \text{for} \quad 1 \le i \le p\,, \tag{25}$$

where $\lambda^{(i)}$ for $1 \le i \le p$ are non-zero complex numbers. This describes a toric variety of complex dimension $n = (m - p)$. Let us use the notation that the vector

$\vec{w}_k \equiv (w_k^{(1)}, \ldots w_k^{(p)})$ denotes the weight of the kth coordinate u_k. f and g appearing in eq.(1) must be homogeneous polynomials of the coordinates u_k with weight $2\vec{w}$ and $3\vec{w}$ respectively for some p dimensional vector $\vec{w}$, so that we can assign weights $\vec{w}$ and $(3\vec{w}/2)$ to the coordinates x and y respectively. In order that (1) describes a Calabi-Yau $(n+1)$-fold, the total weight of the coordinates x, y and the u_k's must be equal to the weight of the polynomial on the left hand side of (1). This gives

$$3\vec{w} = \vec{w} + \frac{3}{2}\vec{w} + \sum_{k=1}^{m} \vec{w}_k \,. \tag{26}$$

Let us now turn to eq.(18). From eqs.(4) and (21) we see that χ, η and h must be homogeneous polynomials in $\{u_k\}$ with weights $3\vec{w}$, $2\vec{w}$ and $\vec{w}$ respectively, and hence ξ in eq.(18) must have weight $\vec{w}/2$. In order that (18) describes a Calabi-Yau manifold, the sum of weights of ξ and the u_k's must be equal to that of the polynomial on the right hand side of (18). This requires

$$\vec{w} = \frac{1}{2}\vec{w} + \sum_{k=1}^{m} \vec{w}_k \,. \tag{27}$$

But this is simply a consequence of (26). Thus we see that if the original $(n+1)$-fold that we started with describes a Calabi-Yau $(n+1)$-fold, then the auxiliary n-fold (18) also satisfies the Calabi-Yau condition. Of course we shall also need to analyse the possible singularities of this n-fold in each case separately.

We can also give a more general argument that does not depend on the base being a toric variety. For this let us take f and g appearing in (1) to be sections of line bundles $L^{\otimes 4}$ and $L^{\otimes 6}$ respectively for some line bundle L on $\mathcal{B}$, so that x and y appearing in (1) can be regarded as taking values in $L^{\otimes 2}$ and $L^{\otimes 3}$ respectively. In order that (1) describes a Calabi-Yau manifold, its first Chern class must vanish. This requires,

$$c_1(\mathcal{B}) + c_1(L)(3+2-6) = 0 \,. \tag{28}$$

The coefficients 3, 2 and 6 of $c_1(L)$ on the left hand side of this equation reflect the degree of y, x and the constraint (1) respectively. (Here by degree we simply refer to the power of L that appears in the corresponding line bundle). This gives

$$c_1(\mathcal{B}) = c_1(L) \,. \tag{29}$$

Let us now turn to eq.(18). From eqs.(4) and (21) we see that χ, η and h must describe sections of the line bundles $L^{\otimes 6}$, $L^{\otimes 4}$ and $L^{\otimes 2}$ respectively, and hence ξ in eq.(18) must be valued in the line bundle L. The condition for the vanishing of the first Chern class of the manifold described by (18) is then given by,

$$c_1(\mathcal{B}) + c_1(L)(1-2) = 0\,. \tag{30}$$

Again the coefficients 1 and 2 of $c_1(L)$ on the left hand side of (27) reflect the degree of ξ and the constraint (18) respectively. But (30) is simply a consequence of (29). Thus the auxiliary n-fold (18) satisfies the Calabi-Yau condition.

Let us now illustrate this in the context of a class of F-theory compactifications discussed in ref.[2], namely on elliptically fibered Calabi-Yau 3-folds on base F_n. Let (u, v) denote the affine coordinates on the base. Then the polynomials f and g appearing in eq.(1) are of the form[2]:

$$f(u,v) = \sum_{k=0}^{8} \sum_{l=0}^{8-4n+nk} f_{kl} u^k v^l\,, \tag{31}$$

and,

$$g(u,v) = \sum_{k=0}^{12} \sum_{l=0}^{12-6n+nk} g_{kl} u^k v^l\,. \tag{32}$$

Thus we can take

$$h(u,v) = \sum_{k=0}^{4} \sum_{l=0}^{4-2n+nk} h_{kl} u^k v^l\,, \tag{33}$$

$$\eta(u,v) = \sum_{k=0}^{8} \sum_{l=0}^{8-4n+nk} \eta_{kl} u^k v^l\,, \tag{34}$$

$$\chi(u,v) = \sum_{k=0}^{12} \sum_{l=0}^{12-6n+nk} \chi_{kl} u^k v^l\,, \tag{35}$$

so that f and g given in eqs.(4) and (21) are of the form given in (31), (32). Let us now try to determine under what condition the surface (18) with h as given in (33) is non-singular. For this we note from (33) that for a given n, h_{kl} is non-zero only for

$$k \geq 2 - \frac{4}{n}\,, \qquad l \leq 4 - 2n + nk\,. \tag{36}$$

Thus for $n \leq 2$, h_{00} is non-vanishing, and hence the surface defined in eq.(18) has no singularity at $u = v = 0$. On the other hand for $n \geq 5$, h_{0l} and h_{1l} vanish for all l, and (18) takes the form:

$$\xi^2 - u^2 \sum_l h_{2l} v^l + O(u^3) = 0 \tag{37}$$

This surface is clearly singular at $u = \xi = 0$ for all v since the the left hand side of eq.(37), as well as all its derivatives vanish at $u = \xi = 0$. This is related to the fact that our original assumption that h has only simple zeroes breaks down in this case.

Thus it remains to analyse the two cases $n = 3$ and $n = 4$. For $n = 4$ the surface takes the form:

$$\xi^2 - h_{10}u + O(u^2) = 0\,. \tag{38}$$

Thus the derivative of the left hand side of this equation with respect to u does not vanish at $u = \xi = 0$ and the surface is non-singular there. For $n = 3$, on the other hand, the equation of the surface is

$$\xi^2 - u(h_{10} + h_{11}v) + O(u^2) = 0\,. \tag{39}$$

In this case the left hand side of this equation, as well as all its derivatives vanish at $\xi = u = 0$, $v = -h_{10}/h_{11}$, and the surface is singular at that point.

Thus the manifold $\mathcal{M}$ is non-singular for $n = 0, 1, 2$ and 4. The $n = 2$ case is already known to be equivalent to the $n = 0$ case[2], so we have three independent cases. These are precisely the cases for which the Calabi-Yau manifold elliptically fibered over F_n were found to be equivalent to Voicin-Borcea orbifolds[16, 17] in ref.[2]. Our construction makes the orientifold limit of these models explicit, and at the same time provides us with the general configuration of D-branes allowed for this orientifold, namely on the hypersurface (24) with h, η and χ as given in eqs.(33)-(35). Note also that for $n = 4$, $h \sim u$, $\eta \sim u^2$ and $\chi \sim u^3$ for small u. Thus there is a D_4 singularity at $u = 0$. From the orientifold viewpoint this corresponds to four D-branes on top of an orientifold plane at $u = 0$. From the general form of h, η and χ it can easily be seen that the $u = 0$ plane does not intersect any other component of D-branes or orientifold planes; hence the unbroken gauge group at a generic point in the moduli space is $SO(8)$[2].

In special cases one may be able to take the limit where the auxiliary manifold $\mathcal{M}$ itself can be regarded as an orbifold of a torus. In this case the model is mapped to an orientifold of type IIB compactified on a torus, as was the case in refs.[7, 8, 9, 13]. However we should add a cautionary remark here. Typically conformal field theory orbifolds do not correspond to geometric orbifolds, but represent geometric orbifolds accompanied by half unit of background $B_{\mu\nu}$ flux through the collapsed two cycles[18]. On the other hand in F-theory we do not normally have this flux. Thus these two theories share the same background axion-dilaton field, but different background tensor fields. If the tensor field flux is even under the orientifold projection (as is the case in ref.[13]), then the two are in the same moduli space, and we can continuously go from the orientifold to the F-theory by switching off the tensor field flux. On the other hand if the tensor field flux is odd under the orientifold projection (as is the case in the model of refs.[8, 9, 19]) then we cannot switch it off continuously and the two theories are in different moduli spaces in general. (Note that half unit of tensor field flux is still even, and hence will survive the orientifold projection.)

The results of this paper can also be used in reverse, namely to construct orientifolds of type IIB theory compactified on a general Calabi-Yau n-fold with a Z_2 isometry that leaves fixed a surface of codimension one. We can mod out the theory by a combination of this Z_2 transformation and $(-1)^{F_L} \cdot \Omega$ to get an orientifold. Let $\vec{u}$ denote the complex coordinates on the quotient, and $h(\vec{u}) = 0$ denote the location of the orientifold plane in this quotient. In order to cancel the RR charge carried by the orientifold plane, we need to place appropriate D-branes on this quotient manifold. This can be done by placing the Dirichlet 7-branes along the surface $\eta^2 + 12h\chi = 0$, where η, h and χ satisfy the condition that the surface described in (1) with f and g given in eqs.(4), (21), describes a Calabi-Yau $(n+1)$-fold. λ given in (2) provides a non-perturbative description of the background around such a configuration of D-brane pairs and orientifold planes. For $n = 3$, we also need to place appropriate number of three branes filling non-compact part of space-time to cancel all the tadpoles[20].

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