# Vertex stabilizers of finite symmetric graphs and a strong version of the Sims conjecture

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# Abstract


Well known Sims conjecture proved by P. J. Cameron, C. E. Praeger, J. Saxl and G. M. Seitz (1983) can be formulated as follows. Let , be an undirected connected finite graph and $G$ be a subgroup of $Aut$ , acting primitively on the vertex set $V$(, ) of , . For $x \in V$(, ) and positive integer $i$, denote by $G_x^{[i]}$ the point-wise stabilizer in $G$ of the closed ball of the radius $i$ with the center $x$ of the graph , . Then there exists a natural number $c$ depending only on the valency of , such that $G_x^{[c]} = 1$. We obtain that $G_x^{[6]} = 1$ for all , and $G$ from the Sims conjecture and, moreover, the constant 6 cannot be decreased. The proof, generalizations and corollaries of this result are discussed.

Sims [20] conjectured that the following is true:

**Sims conjecture.** There exists a function $f$ such that, whenever $G$ is a primitive permutation group on a finite set $\Omega$, if $G_\alpha$ is the stabilizer of a point $\alpha$ in $\Omega$ and if $d$ is the length of any $G_\alpha$-orbit in $\Omega \setminus \{\alpha\}$, then $|G_\alpha| \leq f(d)$.

Fairly many papers have been concerned with this conjecture: Thompson [25], Wielandt [26], Knapp [14], [15], Fomin [10] etc. But only with help of the classification of finite simple groups, Cameron, Praeger, Saxl and Seitz [6] proved this conjecture.

The Sims conjecture may be formulated in a geometrical language as follows.

For an undirected graph , , denote by $V(, )$ and $E(, )$ the vertex set and the edge set of , , respectively.

For an undirected connected graph , , a subgroup $G$ of $Aut$ , , $x \in V(, )$ and positive integer $i$, denote by $G_x^{[i]}$ the point-wise stabilizer in $G$ of the closed ball of the radius $i$ with the center $x$ of the graph , (in the natural metric $d_\Gamma$ of , ).

Let $G$ be a primitive permutation group on a finite set $\Omega, \alpha \in \Omega$ and $M_1 = G_\alpha$. Fix an element $g \in G$ with $M_1^g \neq M_1$ and set $M_2 = M_1^g$. Consider a graph , with the vertex set $V(, ) = \{M_1^x | x \in G\}$ and the edge set $E(, )$ which is defined as follows:

$$\{M_1^x, M_1^y\} \in E(, ) \Longleftrightarrow \exists z \in G : \{M_1^x, M_1^y\}^z = \{M_1, M_2\}.$$

Then , is an undirected connected finite graph, $G$ is a subgroup of $Aut$ , acting primitively on $V(, )$ and the length $d$ of the $M_1$–orbit containing the vertex $M_2$ is the valency of , .

The Sims conjecture is equivalent to the following theorem.

**Theorem 1.** There exists a natural valued function $c$ such that if , is a undirected connected finite graph and $G$ is a subgroup of $Aut$ , acting primitively on $V(, )$ then $G_x^{[c(d)]} = 1$ for $x \in V(, )$, where $d$ is the valency of the graph , .

In [16], we announced the following strengthening of Theorem 1.

**Theorem 2.** There exists a constant $C$ such that, for all pairs $(, , G)$ from Theorem 1, $G_x^{[C]} = 1$.

Next, the following problem naturally arises.

**Problem.** Determine the minimal value of the constant $C$ from Theorem 2.

Very recently we have solved this problem. We proved the following theorem.

**Theorem 3.** If , is an undirected connected graph and $G$ is a primitive on $V(, )$ group of its automorphisms then $G_x^{[6]} = 1$.

In other words, automorphisms of such graphs , are determined by their action on a ball of the radius 6.

In fact, we prove (using classification of finite simple groups) a more strong then Theorem 3 result which is formulated in terms of the subgroup structure of finite groups.

For a finite group $G$ and a pair of its subgroups $M_1$ and $M_2$ , we define, for each $i$, the subgroups $M_1^{(i)}$ and $M_2^{(i)}$ as follows. Set

$$M_1^{(1)} = (M_1 \cap M_2)_{M_1}, M_2^{(1)} = (M_1 \cap M_2)_{M_2}$$

and

$$M_1^{(i+1)} = (M_1^{(i)} \cap M_2^{(i)})_{M_1}, M_2^{(i+1)} = (M_1^{(i)} \cap M_2^{(i)})_{M_2}.$$

For a subgroup $Y$ of a group $X$, we write $Y_X$ for the core of $Y$ in $X$, that is, the maximal normal in $X$ subgroup of $Y$.

Taking in Theorem 3 $M_1 = G_x$ and $M_2 = G_y$, where $x$ and $y$ are adjacent vertices of the graph , , we have $G_x^{[i]} \le M_1^{(i)}$ and $G_y^{[i]} \le M_2^{(i)}$ for all $i$. It is easy to see that the series $M_1 \ge G_x^{[1]} \ge G_x^{[2]} \ge \dots$ consists of normal in $M_1$ subgroups and $G_x^{[i]} = G_x^{[i+1]}$ implies $G_x^{[i]} = 1$. Now Theorem 3 is obtained as a corollary of the following result.

**Theorem 4.** Let $G$ be a finite group and $M_1$, $M_2$ two distinct conjugate maximal subgroup in $G$. Then $M_1^{(6)} = M_2^{(6)}$ is a normal subgroup in $G$.

The following example shows that the constant 6 in Theorems 3 and 4 cannot be decreased.

**Example 1.** Let $G = E_8(q)$, $q = p^m$, $p$ prime, and $M_1$ a maximal parabolic subgroup in $G$ obtained by deleting the root $\alpha_4$ from the Dynkin diagram of $E_8$. Set $Q = O_p(M_1)$, $g = n_{w_4}$ and $M_2 = M_1^g$. Then the series

$$1 = M_1^{(6)} < M_1^{(5)} < M_1^{(4)} < M_1^{(3)} < M_1^{(2)} < O_p(M_1^{(1)}) < Q$$

coincides with the series

$$1 = G_x^{[6]} < G_x^{[5]} < G_x^{[4]} < G_x^{[3]} < G_x^{[2]} < O_p(G_x^{[1]}) < Q$$

and with the upper and lower central series of $Q$.

As another corollary of Theorem 4 we obtain the following result, which was apparently impossible to deduce from the original version of the Sims conjecture.

**Corollary.** Let $G$ be a finite group, $M_1$ a maximal subgroup in $G$ and $M_2$ a subgroup of $G$ which does not belong to $M_1$. Then the subgroup $M_1^{(12)} = M_2^{(12)}$ is normal in $G$.

**Sketch of the proof of Theorem 4.**

Let $G$, $M_1$, $M_2 = M_1^g$, $g \in G$ satisfy the condition of Theorem 4. Without loss of generality we assume that $(M_1)_G = (M_2)_G = 1$ and $1 < |M_1^{(2)}| \leq |M_2^{(2)}|$. Let $\tau$ denote the set of all such triples $(G, M_1, M_2)$. In particular, for a triple in $\tau$, $G_x^{[2]} \neq 1$.

The group $G$ acts faithfully and primitively on the set $\Omega = M_1^G$.

According to Thompson and Wielandt (see [25], [26]), $M_1^{(2)} M_2^{(2)}$ is a nontrivial $p$–group for some prime $p$ and

$$F^*(M_i^{(1)}) = O_p(M_i^{(1)}) \leq F^*(M_i) = O_p(M_i) \text{ for } i = 1, 2.$$

Here $F^*(X)$ is the generalized Fitting subgroup of a group $X$, that is the subgroup generated by all normal nilpotent and subnormal quasisimple subgroups of $X$.

Let $Soc(X)$ denote the socle of a group $X$, that is, the subgroup generated by all minimal normal subgroups of $X$.

Using the classification of finite simple groups, we decompose the set $\tau$ on the following subsets:

$\tau_0$ is the set of the triples $(G, M_1, M_2)$ from $\tau$ with nonsimple $Soc(G)$;

$\tau_1$ is the set of the triples $(G, M_1, M_2)$ from $\tau$ with simple $Soc(G)$ isomorphic to an alternating group;

$\tau_2$ is the set of the triples $(G, M_1, M_2)$ from $\tau - \tau_1$ with simple $Soc(G)$ isomorphic to a group of Lie type over a field of the characteristic $\neq p$;

$\tau_3$ is the set of the triples $(G, M_1, M_2)$ from $\tau - (\tau_1 \cup \tau_2)$ with simple $Soc(G)$ isomorphic to a group of Lie type over a field of the characteristic $p$;

$\tau_4$ is the set of triples $(G, M_1, M_2)$ from $\tau$ with simple $Soc(G)$ isomorphic to one of 26 sporadic groups .

For nonempty subset $\Sigma \subseteq \tau$, we set $c(\Sigma)$ is the maximal integer $c$ such that $M_1^{(c-1)} \neq 1$ or $M_2^{(c-1)} \neq 1$ for some triple $(G, M_1, M_2) \in \Sigma$, or $\infty$, if the maximum cannot be reached.

Theorem 4 is equivalent to the equality $c(\tau) = 6$.

Let $(G, M_1, M_2) \in \tau_0$. Then

$$Soc(G) = T_1 \times \cdots \times T_m, m > 1,$$

all $T_i$'s are isomorphic. According to the O'Nan-Scott theorem (see [3]), $\tau_0 = \tau_0' \cup \tau_0''$, where $(G, M_1, M_2) \in \tau_0'$ means a "diagonal action" for the permutation group $G^\Omega$:

$$M_1 \cap Soc(G) = D_1 \times \ldots \times D_l,$$

where $m = kl$ for some integer $k$ and $D_i$ is the diagonal subgroup of the group

$$T_{(i-1)k+1} \times \ldots \times T_{(i-1)k+k}, |\Omega| = |T_1|^{(k-1)l};$$

$(G, M_1, M_2) \in \tau_0''$ means a "wreath action" for $G^\Omega$:

$G$ is isomorphic to a subgroup of the wreath product $H \wr S_m$ with the product action, where $|\Omega| = s^m$, $H$ is a primitive permutation group on a set $\Delta$ , $|\Delta| = s$ and $Soc(H)$ is isomorphic to $T_1$.

We prove the following result which reduces the problem to the case of groups $G$ with simple socles.

**Reduction theorem.**
(1) $\tau_0' = \emptyset$;
(2) If $(G, M_1, M_2) \in \tau_0''$, then

$$(H, H_\alpha, H_\beta) \in \tau - \tau_0$$

for some $\alpha, \beta \in \Delta$, and

$$c(G, M_1, M_2) \leq c(H, H_\alpha, H_\beta).$$

¿From now on we suppose that $G$ has the simple socle $S$, in particular $S \leq G \leq Aut\ S$. Since $F^*(M_1) = O_p(M_1) \neq 1$, $M_1$ is a $p$–local maximal subgroup in $G$. As $(M_1)_G = 1$, so $S$ does not belong to $M_1$, hence $G = SM_1$. Set $M_0 = M_1 \cap S$. Then it easy to verify that

$$F^*(M_0) = O_p(M_0) \neq 1.$$

Therefore $M_0$ is a $p$-local (not necessary maximal) subgroup in $S$.

Let $(G, M_1, M_2) \in \tau_1$ and $Soc(G) \simeq A_n$, $n \geq 5$. We show that $n \neq 6$ and hence $G$ is isomorphic to $A_n$ or $S_n$ and acts naturally on a set of $n$ points. Using that $M_1$ is $p$-local maximal subgroup of $G$ and considering separately cases of intransitive, transitive imprimitive and primitive action of the subgroup $M_1$ on $n$ points, we prove

**Proposition 1.** $\tau_1 = \emptyset$ and hence $G_x^{[2]} = 1$ for primitive groups $G$ with the alternating socle.

Let $(G, M_1, M_2) \in \tau_2$, where $S = Soc(G)$ is a group of Lie type over a field of the characteristic $r \neq p$. We show that $c(G, M_1, M_2) \leq 3$.

If $S$ is a classic group, then $M_1$ belongs to one of the Aschbacher classes and also $M_0 = S \cap M_1$ is a nonparabolic local subgroup of $S$. We use the desription of maximal in $G$ elements of these classes by Kleidman and Liebeck [13]. Every of the Ascbacher classes is investigated separately.

When $S$ is a exceptional group we apply the classification of local maximal subgroups in $G$ obtained by Cohen, Liebeck, Saxl and Seitz [7].

**Example 2.** Let $S \simeq L_4(3)$ and $G = S < t >$, where $t$ is an involution inducing on $S$ a graph automorphism. Then

$$M_1 = C_G(t) = M_0 \times < t >$$

is a maximal subgroup in $G$ and

$$M_0 \simeq S_4 \times S_4 \simeq PSO_4^+(3).2,$$

i. e. $M_0$ belongs to the Aschbacher class $C_8$. Let $T$ be a Sylow 2-subgroup of $M_1$, $R$ be a Sylow 2-subgroup of G with $t \in T < R$, and $g \in R - T$. Set $M_2 = M_1^g$. Then

$$M_1^{(1)} = O_2(M_1), M_1^{(2)} =< t >\neq 1, M_1^{(3)} = 1.$$

**Example 3.** Let $G = S \simeq E_6^\epsilon(r)$, $\epsilon = \pm 1$, $r \geq 5$, $3|r - \epsilon$. There exists in $G$ an "exotic" maximal subgroup

$$M_1 \simeq 3^{3+3}.SL_3(3).$$

Let $T$ be a Sylow 3-subgroup of $M_1$, $R$ be a Sylow 3-subgroup of $G$ with $T < R$, and $g \in N_R(T) - T$. Set $M_2 = M_1^g$ and $Q = O_3(M_1)$. Then $Q$ is a special group of the order $3^6$ with

$$|Z(Q)| = 3^3,\ M_1^{(1)} = Q,\ M_1^{(2)} = Z(Q),\ M_1^{(3)} = 1.$$

Example 2 shows that there exists a triple $(G, M_1, M_2) \in \tau_2$ such that $S$ is classical group and $c(G, M_1, M_2) = 3$, but $(S, M_1 \cap S, M_2 \cap S) \notin \tau$. Example 3 gives an infinite series of triples $(G, M_1, M_2)$ with exceptional group $S$ and $c(G, M_1, M_2) = 3$.

Thus, the following holds

**Proposition 2.** $c(\tau_2) = 3$.

Let $(G, M_1, M_2) \in \tau_3$, where $S = Soc(G)$ is a group of Lie type over the field $GF(q)$ of characteristic $p$. Then $M_0 = M_1 \cap S$ is a parabolic in $S$. Set $Q = O_p(M_0)$. It is sufficiently easy to prove that the subgroup $Q$ is weakly closed in $M_1$ with respect to $G$, i.e. $O_p(M_1^{(1)}) \cap S < Q$. Using the properties of the group $Aut\ S$ we show that if $M_1^{(i)} \cap S = 1$ for some $i \geq 2$, then $M_1^{(i)} = 1$. Hence $c(G, M_1, M_2)$ is bounded above by the number $\gamma(M_1)$ of the chief factors of $M_1$ in $Q$. It is easy to show that if $G_x^{[i]}/G_x^{[i+1]}$ is isomorphic to $GF(q)H$–module of the dimension 1 ($H$ is a Cartan subgroup in $M_0$) then $G_x^{[i+1]} = 1$. In some cases, this fact allows at once to show that

$$c(G, M_1, M_2) < \gamma(M_1).$$

Further we calculate the function $\gamma(M_1)$.

If $S$ is not isomorphic to

$$B_n(2^m),\ F_4(2^m),\ G_2(2^m),\ G_2(3^m),\ {}^2B_2(2^m),\ {}^2G_2(3^m),\ {}^2F_4(2^m)',$$

then we find the function $\gamma(M_1)$ from the result by Azad, Barry and Seitz [5].

The remaining cases (where $p$ is a very bad prime) are considered case by case with the help of various known results on parabolic subgroups, for example, results by Suzuki [21], [22], Ree [19], Thomas [23], [24], Guterman [12], Parrot [18], Fong and Seitz [11], Curtis, Kantor and Seitz [9], Aschbacher and Seitz [4] etc. Ultimately we prove that $c(G, M_1, M_2) \leq 6$.

Now Example 1 show that the following holds.

**Proposition 3.** $c(\tau_3) = 6$.

At last, let $(G, M_1, M_2) \in \tau_4$. Using the known information about local maximal subgroups of sporadic groups (see for instance citeAtlas and the Aschbacher book citeAsch2), we show that $c(G, M_1, M_2) \leq 5$.

**Example 4.** Let $G \simeq F_2$. There exists in $G$ a maximal subgroup $M_1 \simeq [2^{30}].L_5(2)$. Then there exists an element $g \in G$ such that for $M_2 = M_1^g$ we have

$$O_2(M_1) = M_1^{(1)}, M_1^{(1)}/M_1^{(2)} \simeq M_1^{(2)}/M_1^{(3)} \simeq 2^{10}, M_1^{(3)}/M_1^{(4)} \simeq M_1^{(4)}/M_1^{(5)} \simeq 2^5, M_1^{(5)} = 1.$$

Thus, the following holds.

**Proposition 4.** $c(\tau_4) = 5$

Theorem 4 is proved.

Now, we prove Corollary.

**Proof of Corollary.** Without loss of generality we assume that $(M_1 \cap M_2)_G = 1$. Let , be the graph with $V(,) = \{gM_1, gM_2 | g \in G\}$, $E(,) = \{\{gM_1, gM_2\} | g \in G\}$, and the group $G$ acts on $V(,)$ by left translations. Then , is a connected bipartite graph with the parts

$$V_1 = \{gM_1 | g \in G\} \text{ and } V_2 = \{gM_2 | g \in G\},$$

$G \leq Aut$ , and $G$ acts primitively on $V_1$ and transitively on $V_2$.

Let $x$ and $y$ denote vertices $M_1$ and $M_2$ of the graph , , respectively. Then the groups $M_1$ and $M_2$ act transitively on the neighborhoods of the vertices $x$ and $y$, respectively. By induction on the natural parameter $i$ we prove the equalities

$$M_1^{(i)} = G_x^{[i]} \text{ and } M_2^{[i]} = G_y^{[i]},$$

where $M_1^{(i)}$ and $M_2^{(i)}$ are defined by the triple $(G, M_1, M_2)$ as above.

Consider the graph $,'$ with $V(,') = V_1$ and

$$E(,') = \{\{u, v\} | u, v \in V_1, d_\Gamma(u, v) = 2\}.$$

Then $G$ is a primitive group of the automorphisms of the graph $,'$ and, by Theorem 4, the point-wise stabilizer in $G$ of the ball of the radius 6 of the graph $,'$ with the center $x$ is trivial. But this stabilizer includes the point-wise stabilizer in $G$ of the ball of the radius 12 of the graph , with the center $x$. Corollary is proved.

In connection with the obtained results, the following problems naturally arise.

**Problems.** 1. Describe all triples $(G, M_1, M_2)$ from Theorem 4 with $c(G, M_1, M_2) > 2$.

2. Find the minimal value of $n$ for which, in the condition of Corollary, the subgroup $M_1^{(n)} = M_2^{(n)}$ is normal in the group $G$.

3. Improve, using Theorem 4, known estimates for the order of the point stabilizer in a finite primitive permutation group.


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## References


[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Inv. Math. 76 (1984), 469-514.

[2] M. Aschbacher, Overgroups of Sylow subgroups of sporadic groups, Mem. Amer. Math. Soc. 342 (1986), 1-235.

[3] M. Aschbacher and L. L. Scott, Maximal subgroups of finite groups, J. Algebra 92 (1985), 44-80.

[4] M. Aschbacher and G. M. Seitz, Involutions in Chevalley groups over fields of even order, Nagoya Math. J. 63 (1976), 1-91.

[5] H. Azad, M. Barry and G. M. Seitz, On the structure of parabolic subgroups, Comm. Algebra 18 (1990), 551-562.

[6] P. J. Cameron, C. E. Praeger, J. Saxl and G.M. Seitz, On the Sims conjecture and distance transitive graphs, Bull. London Math. Soc. 15 (1983), 499-506.

[7] A. M. Cohen, M. W. Liebeck, J. Saxl and G. M. Seitz, The local maximal subgroups of exceptional groups of Lie type, finite and algebraic, Proc. London Math. Soc. 64 (1992), 21-48.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of finite groups, Clarendon Press, Oxford, 1985.

[9] C. Curtis, C. Kantor and G. M. Seitz, The two transitive permutation representations of the finite Chevalley groups, Trans. Amer. Math. Soc. 218 (1976), 1-59.

[10] A. N. Fomin, On a conjecture of Sims, In: Sixth All-Union Symp. on Group Theory, Naukova Dumka, Kiev, 1980, 158-165.

[11] P. Fong and G. M. Seitz, Groups with $(B, N)$-pair of rank 2, II, Inv. Math. 24 (1973), 191-239.

[12] M. M. Guterman, A characterization of the groups $F_4(2^n)$, J. Algebra 20 (1972), 1-23.

[13] P. B. Kleidman and M. W. Liebeck, The subgroup structure of the finite classical groups, Cambridge University Press, Cambridge, 1990.

[14] W. Knapp, On the point stabilizer in a primitive permutation group, Math. Z. 133 (1973), 137-168.

[15] W. Knapp, An order bound for a point stabilizer of a primitive permutation group, Arch. Math. 36 (1981), 481-484.

[16] A. S. Kondratiev and V. I. Trofimov, The triviality of the stabilizer of the ball of large radius in primitive groups of authomorphisms of finite graphs, In: Third Intern. Conf. on Algebra, Abstracts, Krasnoyarsk, 1993, p. 159.

[17] M. W. Liebeck, J. Saxl and G. M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, Proc. London Math. Soc. 65 (1992), 297-325.

[18] D. Parrot, A characterization of the Ree groups ${}^2F_4(q)$, J. Algebra 27 (1973), 341-357.

[19] R. Ree, A family of simple groups associated with the simple Lie algebra of Type $(F_4)$, Amer. J. Math. 83 (1961), 401-420.

[20] C. C. Sims, Graphs and finite permutation groups, Math. Z. 95 (1967), 76-86.

[21] M. Suzuki, On a class of doubly transitive groups, Ann. Math. (2) 75 (1962), 105-145.

[22] M. Suzuki, Characterization of linear groups, Bull. Aner. Math. Soc. 75 (1969), 1043-1091.

[23] G. Thomas, A characterization of the groups $G_2(2^n)$, J. Algebra 13 (1969), 87-118.

[24] G. Thomas, A characterization of the Steinberg group $D_4^2(q^3), q = 2^n$, J. Algebra 14 (1970), 373-385.

[25] J. G. Thompson, Bounds for the orders of maximal subgroups, J. Algebra 14 (1970), 135-138.

[26] H. Wielandt, Subnormal subgroups and permutation groups, Ohio State Univ. Lecture Notes, 1971.