Instantons, Three-Dimensional Gauge Theory, and the Atiyah-Hitchin Manifold

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Abstract

We investigate quantum effects on the Coulomb branch of three-dimensional $N = 4$ supersymmetric gauge theory with gauge group $SU(2)$. We calculate perturbative and one-instanton contributions to the Wilsonian effective action using standard weak-coupling methods. Unlike the four-dimensional case, and despite supersymmetry, the contribution of non-zero modes to the instanton measure does not cancel. Our results allow us to fix the weak-coupling boundary conditions for the differential equations which determine the hyper-Kähler metric on the quantum moduli space. We confirm the proposal of Seiberg and Witten that the Coulomb branch is equivalent, as a hyper-Kähler manifold, to the centered moduli space of two BPS monopoles constructed by Atiyah and Hitchin.

1 Introduction

Recent work by several authors [1]-[7] has provided exact information about the low-energy dynamics of $N = 4$ supersymmetric gauge theory in three dimensions. In particular, an interesting connection has emerged between the quantum moduli spaces of these theories and the classical moduli spaces of BPS monopoles in $SU(2)$ gauge theory [2]. Chalmers and Hanany [4] have proposed that the Coulomb branch of the $SU(n)$ gauge theory is equivalent as a hyper-Kähler manifold to the centered moduli space of n BPS monopoles. For $n > 2$ this correspondence is intriguing because the hyper-Kähler metric on the manifold in question is essentially unknown. Subsequently, Hanany and Witten [6] have shown that the equivalence, for all n, is a consequence of S-duality applied to a certain configuration of D-branes in type IIB superstring theory.

The case $n = 2$, which is the main topic of this paper, provides an important test for these ideas because the two-monopole moduli space and its hyper-Kähler metric have been found explicitly by Atiyah and Hitchin [8]. We will refer to the four-dimensional manifold which describes the relative separation and charge angle of two BPS monopoles as the Atiyah-Hitchin (AH) manifold. In fact these authors effectively classified all four-dimensional hyper-Kähler manifolds with an $SO(3)$ action which rotates the three inequivalent complex structures. Correspondingly, the moduli space of the $N = 4$ theory with gauge group $SU(2)$ was analysed by Seiberg and Witten [2]. The effective low-energy theory in this case is a non-linear σ-model with the four-dimensional Coulomb branch of the $SU(2)$ theory as the target manifold. The $N = 4$ supersymmetry of the low-energy theory requires that the metric induced on the target space by the σ-model kinetic terms be hyper-Kähler [9]. By virtue of its global symmetry structure, the Coulomb branch of this theory necessarily fits into Atiyah and Hitchin's classification scheme. Seiberg and Witten compared the weak coupling behaviour of the SUSY gauge theory with the asymptotic form of the metric on the AH manifold in the limit of large-spatial separation between the monopoles, $r \gg 1$ [1]. They found exact agreement between perturbative effects in the SUSY gauge theory, and the expansion of the AH metric in inverse powers of r. In fact, as we will review below, there are an infinite number of inequivalent hyper-Kähler four-manifolds with the required isometries which share this asymptotic behaviour. However, Atiyah and Hitchin showed that only one of these, the AH manifold itself, is singularity-free. Motivated by expectations from string theory [1], Seiberg and Witten proposed that the Coulomb branch of the three-dimensional $N = 4$ theory should also have no singularities. The correspondence between the two manifolds then follows automatically.

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[1]r is the separation between the monopoles in units of the inverse gauge-boson mass in the $3 + 1$ dimensional gauge theory in which the two BPS monopoles live.

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Clearly the arguments reviewed above come close to a first-principles demonstration that the Coulomb branch of the $SU(2)$ theory is the AH manifold. The only assumption made about the strong coupling behaviour of the SUSY gauge theory which is not automatically guaranteed by symmetries alone is the absence of singularities. In this paper we will proceed without this assumption[2]. One must then choose between an infinite number of possible metrics with the same asymptotic form. By virtue of the hyper-Kähler condition and the global symmetries, the components of these metrics each satisfy the same set of coupled non-linear ODE's as the AH metric. In fact, as we review below, there is precisely a one-parameter family of solutions of these differential equations which have the same asymptotic behaviour as the AH metric to all finite orders in $1/r$ but differ by terms of order $\exp(-r)$. Seiberg and Witten showed that the exponentially suppressed terms correspond to instanton effects in the weakly-coupled SUSY gauge theory. In this paper we will calculate the one-loop perturbative and one-instanton contributions to the low-energy theory using standard background field and semiclassical methods respectively. The results of these calculations suffice to fix the boundary conditions for the differential equations which determine the metric. We find that the resulting metric is equal to the AH metric, thereby confirming the prediction of Seiberg and Witten.

The paper is organised as follows. In Section 2, we introduce the model and review its relation to $N = 2$ SUSY Yang-Mills theory in four dimensions. We also review the classical form of the Coulomb branch and perform an explicit one-loop evaluation of the metric. In Section 3 we discuss the properties of supersymmetric instantons in three-dimensional (3D) gauge theory. The field configurations in question themselves correspond to the BPS monopoles of the four-dimensional (4D) gauge theory[3]. Much of Section 3 is devoted to obtaining the measure for integration over the instanton collective coordinates. The number of bosonic and fermionic zero modes of the BPS monopole is determined by the Callias index theorem which we briefly review. Like their four-dimensional counterparts, the three-dimensional instantons have a self-duality property which, together with supersymmetry, ensures a large degree of cancellation between non-zero modes of the bose and fermi fields. However, unlike the four-dimensional case and despite supersymmetry, this cancellation is not complete because of the spectral asymmetry of the Dirac operator in a monopole background. We calculate, for the first time, the residual term which arises from the non-cancelling ratio of determinants of the quadratic fluctuation operators of the scalar, fermion, gauge and ghost degrees of freedom. We apply the result-

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[2]In the following, we will, however, retain the weaker assumption that the moduli space has at most isolated singularities.

[3]To avoid confusion with the proposed connection to monopole moduli spaces, from now on, the term 'monopole' will always refer to the instantons of the three-dimensional SUSY gauge theory unless otherwise stated.

ing one-instanton measure to calculate the leading non-perturbative correction to a four-fermion vertex in the low-energy effective action.

In Section 4 we show that the one-loop and one-instanton data calculated in Sections 2 and 3, together with the (super-)symmetries of the model, are sufficient to to determine the exact metric on the Coulomb branch. We begin by reviewing the arguments leading to the exact solution of the low-energy theory proposed by Seiberg and Witten. We analyze the solutions of the non-linear ODE's which determine the metric and exhibit a one-parameter family of solutions which agree with the metric on the Coulomb branch determined up to one-loop in perturbation theory. We show that each of these solutions leads to a different prediction for the one-instanton effect calculated in Section 3 and precise agreement is obtained only for the solution which corresponds to the AH-manifold: the singularity-free case. For the most part, calculational details are relegated to a series of Appendices.

2 $N = 4$ Supersymmetry in 3D

2.1 Fields, symmetries and dimensional reduction

In this Section we will briefly review some basic facts about the $N = 4$ supersymmetric $SU(2)$ gauge theory in three dimensions considered in [2]. It is particularly convenient to obtain this theory from the four-dimensional Euclidean $N = 2$ SUSY Yang-Mills theory by dimensional reduction. In discussing the 4D theory we will adopt the notation and conventions of [10]: the $N = 2$ gauge multiplet contains the gauge field $\underset{\sim}{v}_m$ $(m = 0, 1, 2, 3)$, a complex scalar $\underset{\sim}{A}$ and two species of Weyl fermion $\underset{\sim}{\lambda}_\alpha$ and $\underset{\sim}{\psi}_\alpha$ all in the adjoint representation of the gauge group. As in [10] we use undertwiddling for the fields in the $SU(2)$ matrix notation, $\underset{\sim}{X} \equiv X^a \tau^a / 2$. The resulting $N = 2$ SUSY algebra admits an $SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ group of automorphisms. In the 4D quantum theory, the Abelian factor of the $\mathcal{R}$-symmetry group is anomalous due to the effect of 4D instantons.

Following Seiberg and Witten, the three-dimensional theory is obtained by compactifying one spatial dimension[4], say x_3, on a circle of radius R. In the following we will restrict our attention to field configurations which are independent of the compactified dimension. This yields a classical field theory in three spacetime dimensions which we will then quantize. Seiberg and Witten also consider the distinct problem

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[4]For simplicity of presentation we choose x_3 to be the compactified dimension. In practice, however, in order to flow from the standard chiral basis of gamma matrices in 4D to gamma matrices in 3D one has to dimensionally reduce in the x_2 direction. To remedy this one can always reshuffle gamma matrices in 4D. See Appendix A for more details.

of quantizing the $N = 2$ theory on $R^3 \times S^1$. In this approach quantum fluctuations of the fields which depend on x_3 are included in the path integral and one can interpolate between the 3D and 4D quantum theories by varying R. We will not consider this more challenging problem here. Integrating over x_3 in the action gives,

$$\frac{1}{g^2} \int d^4x \;\; \rightarrow \;\; \frac{2\pi}{e^2} \int d^3x \tag{1}$$

where $e = g/\sqrt{R}$ defines the dimensionful 3D gauge coupling in terms of the dimensionless 4D counterpart g.

Compactifying one dimension breaks the $SO(4)_E \simeq SU(2)_l \times SU(2)_r$ group of rotations of four-dimensional Euclidean spacetime down to $SO(3)_E$. Following the notation of [2], the double-cover of the latter group is denoted $SU(2)_E$. The 4D gauge field $\underset{\sim}{v}_m$ splits into a 3D gauge field $\underset{\sim}{v}_\mu$, and a real scalar $\underset{\sim}{\phi}_3$ such that $\underset{\sim}{v}_\mu = \underset{\sim}{v}_m$, for $m = \mu = 0, 1, 2$ and $\underset{\sim}{\phi}_3 = \underset{\sim}{v}_3$. It is also convenient to decompose the 4D complex scalar $\underset{\sim}{A}$ into two real scalars: $\underset{\sim}{\phi}_1 = \sqrt{2}\mathrm{Re}\,\underset{\sim}{A}$ and $\underset{\sim}{\phi}_2 = \sqrt{2}\mathrm{Im}\,\underset{\sim}{A}$. The 4D Weyl spinors $\underset{\sim}{\lambda}_\alpha$, $\underset{\sim}{\psi}_\alpha$ ($\underset{\sim}{\bar\lambda}_{\dot\alpha}$, $\underset{\sim}{\bar\psi}_{\dot\alpha}$) of $SU(2)_l$ ($SU(2)_r$) can be rearranged to form four 3D Majorana spinors of $SU(2)_E$: $\underset{\sim}{\chi}_\alpha^A$ for $A = 1, 2, 3, 4$. Correspondingly, the two Weyl supercharges of the $N = 2$ theory are reassembled as four Majorana supercharges which generate the $N = 4$ supersymmetry of the three-dimensional theory. Details of dimensional reduction, field definitions and our conventions for spinors in three and four dimensions are given in Appendix A.

While the number of spacetime symmetries decreases reducing from 4D down to 3D, the number of $\mathcal{R}$-symmetries increases. First, the $SU(2)_\mathcal{R}$ symmetry which rotates the two species of Weyl fermions and supercharges in the four-dimensional theory remains unbroken in three dimensions. In addition, the $U(1)_\mathcal{R}$ symmetry is enlarged to a simple group which, following [2], we will call $SU(2)_N$. Unlike $U(1)_\mathcal{R}$ in the 4D case, this symmetry remains unbroken at the quantum level. The real scalars, $\underset{\sim}{\phi}_i$ with $i = 1, 2, 3$, transform as a **3** of $SU(2)_N$. The index $A = 1, 2, 3, 4$ on the Majorana spinors $\underset{\sim}{\chi}_\alpha^A$ introduced above reflects the fact that they transform as a **4** of the combined $\mathcal{R}$-symmetry group, $SO(4)_\mathcal{R} \simeq SU(2)_\mathcal{R} \times SU(2)_N$.

2.2 The low-energy theory

The three-dimensional $N = 4$ theory has flat directions along which the three adjoint scalars $\underset{\sim}{\phi}_i$ acquire mutually commuting expectation values. By a gauge rotation, each of the scalars can be chosen to lie in the third isospin component:

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$\langle \underset{\sim}{\phi_i} \rangle = \sqrt{2} v_i \tau^3 / 2$. After modding out the action of the Weyl group, $v_i \to -v_i$, the three real parameters v_i describe a manifold of gauge-inequivalent vacua. As we will see below, this manifold is only part of the classical Coulomb branch. For any non-zero value of $\mathbf{v} = (v_1, v_2, v_3)$, the gauge group is broken down to $U(1)$ and two of the gauge bosons acquire masses $M_W = \sqrt{2}|\mathbf{v}|$ by the adjoint Higgs mechanism. At the classical level, the global $SU(2)_N$ symmetry is also spontaneously broken to an Abelian subgroup $U(1)_N$ on the Coulomb branch. The remaining component of the gauge field is the massless photon $v_\mu = \text{Tr}(\underset{\sim}{v_\mu} \tau_3)$ with Abelian field-strength $v_{\mu\nu}$. For each matrix-valued field $\underset{\sim}{X}$ in the microscopic theory, we define a corresponding massless field in the Abelian low-energy theory: $X = \text{Tr}(\underset{\sim}{X} \tau^3)$. Hence, at the classical level, the bosonic part of the low-energy Euclidean action is simply given by the free massless expression,

$$S_{\text{B}} = \frac{2\pi}{e^2} \int d^3x \left[\tfrac{1}{4} v_{\mu\nu} v_{\mu\nu} + \tfrac{1}{2} \partial_\mu \phi_i \partial_\mu \phi_i \right] \qquad (2)$$

The presence of 3D instantons in the theory means we must also include a surface term in the action, which is analogous to the θ-term in four dimensions. In the low-energy theory this term can be written as,

$$S_{\text{S}} = \frac{i\sigma}{8\pi} \int d^3x \, \varepsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho} \qquad (3)$$

A dual description of the low-energy theory can be obtained by promoting the parameter σ to be a dynamical field [11]. This field serves as a Lagrange multiplier for the Bianchi identity constraint. In the presence of this constraint one may integrate out the Abelian field strength to obtain the bosonic effective action,

$$S_{\text{B}} = \frac{2\pi}{e^2} \int d^3x \, \tfrac{1}{2} \partial_\mu \phi_i \partial_\mu \phi_i + \frac{2e^2}{\pi(8\pi)^2} \int d^3x \, \tfrac{1}{2} \partial_\mu \sigma \partial_\mu \sigma \qquad (4)$$

The Dirac quantization of magnetic charge, or, equivalently, the 3D instanton topological charge,

$$k = \frac{1}{8\pi} \int d^3x \, \varepsilon^{\mu\nu\rho} \partial_\mu v_{\nu\rho} \in Z , \qquad (5)$$

means that, with the normalization given in (3), σ is a periodic variable with period 2π. In the absence of magnetic charge σ only enters through its derivatives and the action has a trivial symmetry, $\sigma \to \sigma + c$ where c is a constant. The VEV of the σ-field spontanteously breaks this symmetry and provides an extra compact dimension for the Coulomb branch. Modding out the action of the Weyl group, the classical Coulomb branch can then be thought of as $(R^3 \times S^1)/Z_2$.

It will be convenient to write the fermionic terms in the low-energy action in terms of the (dimensionally reduced) Weyl fermions. At the classical level, the action contains free kinetic terms for these massless degrees of freedom,

$$S_F = \frac{2\pi}{e^2} \int d^3x \left(i\bar{\lambda}\bar{\sigma}_\mu \partial_\mu \lambda + i\bar{\psi}\bar{\sigma}_\mu \partial_\mu \psi \right) \tag{6}$$

The Weyl fermions in 4D can be related to the 3D Majorana fermions χ_α^A by going to a complex basis for the $SO(4)_\mathcal{R}$ index. In Appendix A we define a basis such that the holomorphic components χ_α^a are equal to $\epsilon_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}$ and $\epsilon_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}$ for $a = 1$ and $a = 2$ respectively. Similarly the anti-holomorphic components $\chi_\alpha^{\bar{a}}$ are equal to the left-handed Weyl fermions λ_α and ψ_α for $\bar{a} = \bar{1}$ and $\bar{a} = \bar{2}$. In this basis, the effective action (6) can be rewritten as,

$$S_F = -\frac{2\pi}{e^2} \int d^3x \, \delta_{a\bar{b}} \chi^a \gamma_\mu \partial_\mu \chi^{\bar{b}} \tag{7}$$

where γ_μ are gamma-matrices in 3D.

Perturbative corrections lead to finite corrections to the classical low-energy theory in powers of e^2/M_W. At the one-loop level several effects occur. First, there is a finite renormalization of the gauge coupling appearing in (4) and (6):

$$\frac{2\pi}{e^2} \quad \rightarrow \quad \frac{2\pi}{e^2} - \frac{1}{2\pi M_W} \tag{8}$$

This result is demonstrated explicitly in Appendix B. Second, there is a more subtle one-loop effect discussed in [2]. By considering the realization of the $U(1)_N$ symmetry in an instanton background, Seiberg and Witten showed that a particular coupling between the dual photon σ and the other scalars ϕ_i appears in the one-loop effective action. More generally the one-loop effective action will contain vertices with arbitrary numbers of boson and fermion legs. However, as we will see in Section 4, the exact form of the low-energy effective action is essentially determined to all orders in perturbation theory by the finite shift in the coupling (8) together with the constraints imposed by $N = 4$ supersymmetry. In the next Section we will turn our attention to the non-perturbative effects which modify this description.

3   Supersymmetric Instantons in Three Dimensions

3.1   Instantons in the microscopic theory

The four-dimensional $N = 2$ theory has static BPS monopole solutions of finite energy [12], [13]. In the three-dimensional theory obtained by dimensional reduction,

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these field configurations have finite Euclidean action, $S_{\rm cl}^{(k)} = |k| S_{\rm cl}$ for magnetic charge k, with $S_{\rm cl} = (8\pi^2 M_W)/e^2$. For each value of k, the monopole solutions are exact minima of the action which yield contributions of order $\exp(-|k| S_{\rm cl})$ to the partition function and Greens functions of the theory at weak coupling. Hence BPS monopoles appear as instantons in the $N = 4$ supersymmetric gauge theory in 3D. In this Section (together with Appendix C), as well as presenting several general results, we will provide a quantitative analysis of these effects for the case $k = 1$. We begin by determining the bosonic and fermionic zero modes of the instanton and the corresponding collective coordinates. We then consider the contribution of non-zero frequency modes to the instanton measure which requires the evaluation of a ratio of functional determinants. In subsection 3.2, we use these results to calculate the leading non-perturbative correction to a four-fermion vertex in the low-energy effective action.

To exhibit the properties of the 3D instantons, it is particularly convenient to work with the (dimensionally-reduced) fields of the four-dimensional theory and also with the specific vacuum choice $v_i = v\delta_{i3}$. In this case, the static Bogomol'nyi equation satisfied by the gauge and Higgs components of the monopole can be concisely rewritten as a self-dual Yang-Mills equation for the four-dimensional gauge field, $\underset{\sim}{v}_m$ of Section 2.1 [14][5]

$$\underset{\sim}{v}_{mn}^{\rm cl} = {}^* \underset{\sim}{v}_{mn}^{\rm cl} \tag{9}$$

Because of this self-duality, instantons in three dimensions have many features in common with their four-dimensional counterparts. In the following, we will focus primarily on the new features which are special to instanton effects in three-dimensional gauge theories. One important difference is that instantons in 3D are exact solutions of the equations of motion in the spontaneously broken phase. In four dimensions, instantons are only quasi-solutions in the presence of a VEV and this leads to several complications which do not occur in the 3D case.

Bosonic zero modes, $\delta \underset{\sim}{v}^m = \underset{\sim}{Z}^m$, of the 3D k-instanton configuration are obtained by solving the linearized self-duality equation subject to the background field gauge constraint (see Appendix C),

$$\mathcal{D}_{\rm cl}^{[m} \underset{\sim}{Z}^{n]} = {}^* \mathcal{D}_{\rm cl}^{[m} \underset{\sim}{Z}^{n]} \quad , \quad \mathcal{D}_{\rm cl}^m \underset{\sim}{Z}^m = 0 \ . \tag{10}$$

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[5]Of course these is no loss of generality here, for any choice of three-dimensional vacuum v_i it is possible to construct an analogous four-dimensional Euclidean gauge field which is self-dual in the monopole background: one simply needs to include the component $v_i \phi_i$ of the scalar as the 'fourth' component of the gauge field. However, only if we choose $v_i = v\delta_{i3}$ does this correspond to the four-dimensional gauge field of Section 2.1.

where, $\mathcal{D}^m_{\mathrm{cl}}$ is the adjoint gauge-covariant derivative in the self-dual gauge background, $v^{\mathrm{cl}}_{\sim}$. As a consequence of self-duality, there is an exact correspondence between these bosonic zero modes and the fermionic zero modes, which are solutions of the adjoint Dirac equation in the self-dual background. The latter is precisely the equation of motion for the 4D Weyl fermions in the monopole background,

$$\bar{\not{D}}^{\dot{\alpha}\alpha}_{\mathrm{cl}} \, \lambda^{\mathrm{cl}}_{\sim\alpha} \;=\; 0 \tag{11}$$

$$\not{D}^{\alpha\dot{\alpha}}_{\mathrm{cl}} \, \bar{\lambda}^{\mathrm{cl}}_{\sim\dot{\alpha}} \;=\; 0 \tag{12}$$

Following Weinberg [15], we form the bispinor operators, $\Delta_+ = \bar{\not{D}}_{\mathrm{cl}}\not{D}_{\mathrm{cl}}$ and $\Delta_- = \not{D}_{\mathrm{cl}}\bar{\not{D}}_{\mathrm{cl}}$. In general, for a self-dual background field[6], $v^{\mathrm{cl}}_{\sim m}$,

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\begin{aligned}
(\Delta_+)_{\dot{\alpha}}{}^{\dot{\beta}} &= \mathcal{D}^2_{\mathrm{cl}}\delta_{\dot{\alpha}}{}^{\dot{\beta}} + (\bar{\sigma}^{mn})_{\dot{\alpha}}{}^{\dot{\beta}} \, v^{\mathrm{cl}}_{\sim mn} = \mathcal{D}^2_{\mathrm{cl}}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \;, \\
(\Delta_-)_{\alpha}{}^{\beta} &= \mathcal{D}^2_{\mathrm{cl}}\delta_{\alpha}{}^{\beta} + (\sigma^{mn})_{\alpha}{}^{\beta} \, v^{\mathrm{cl}}_{\sim mn}
\end{aligned}
\tag{13}
$$

Δ_- can have normalizable zero modes, while Δ_+ is positive and has none. Let the number of normalizable zero modes of Δ_- be q. Then correctly accounting for spinor indices, the number of normalizable zero modes, $Z_{\sim m}$, of the gauge field $v^{\mathrm{cl}}_{\sim m}$ is $2q$ [15].

Adapting the Callias index theorem [16] to the context of BPS monopoles, Weinberg [15] showed that q could be obtained as the $\mu \to 0$ limit of the regularized trace,

$$\mathcal{I}(\mu) = \mathrm{Tr}\left[\frac{\mu}{\Delta_- + \mu} - \frac{\mu}{\Delta_+ + \mu}\right] \tag{14}$$

defined for $\mu > 0$. It turns out that the only non-zero contribution to $\mathcal{I}(\mu)$ comes from a surface term that can be evaluated explicitly for arbitrary k. Weinberg's result is,

$$\mathcal{I}(\mu) = \frac{2k\,M_W}{(M_W^2 + \mu)^{\frac{1}{2}}} \tag{15}$$

Setting $\mu = 0$ yields $q = 2k$. Another consequence of this analysis is that the adjoint Dirac equation (11) has $2k$ independent solutions while (12) has none. Although the index information is contained in the $\mu \to 0$ limit of equation (15) we will need this result for general μ in the following.

The upshot of Weinberg's index calculation is the conventional wisdom that the BPS monopole of charge k, or, equivalently, the 3D k-instanton, has $4k$ bosonic collective coordinates. For $k = 1$ these simply correspond to the three components,

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[6]We define standard self-dual and anti-self-dual projectors, $\sigma^{mn} = \frac{1}{4}\sigma^{[m}\bar{\sigma}^{n]}$ and $\bar{\sigma}^{mn} = \frac{1}{4}\bar{\sigma}^{[m}\sigma^{n]}$.

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X_m, of the instanton position in three-dimensional spacetime and an additional angle, $\theta \in [0, 2\pi]$, which describes the orientation of the instanton in the unbroken $U(1)$ gauge subgroup. In an instanton calculation we must integrate over these coordinates with the measure obtained by changing variables in the path integral. Explicitly,

$$\int d\mu_B = \int \frac{d^3 X}{(2\pi)^{\frac{3}{2}}} (\mathcal{J}_X)^{\frac{3}{2}} \int_0^{2\pi} \frac{d\theta}{(2\pi)^{\frac{1}{2}}} (\mathcal{J}_\theta)^{\frac{1}{2}} \tag{16}$$

In Appendix C, we calculate the Jacobian factors, $\mathcal{J}_X = S_{\rm cl}$ and $\mathcal{J}_\theta = S_{\rm cl}/M_W^2$.

Similarly the two species of Weyl fermions, $\underset{\sim}{\lambda}$ and $\underset{\sim}{\psi}$ each have $2k$ independent zero-mode solutions in the instanton-number k background. For $k = 1$ these four modes correspond to the action of the four supersymmetry generators under which the 3D instanton transforms non-trivially. As in the four-dimensional case, the modes in question can be parametrized in terms of two-component Grassmann collective coordinates ξ_α and ξ'_α as

$$\begin{aligned}
\underset{\sim}{\lambda}_\alpha^{\rm cl} &= \tfrac{1}{2}\xi_\beta(\sigma^m \bar{\sigma}^n)_\alpha{}^\beta \underset{\sim}{v}_{mn}^{\rm cl} \\
\underset{\sim}{\psi}_\alpha^{\rm cl} &= \tfrac{1}{2}\xi'_\beta(\sigma^m \bar{\sigma}^n)_\alpha{}^\beta \underset{\sim}{v}_{mn}^{\rm cl}
\end{aligned} \tag{17}$$

The corresponding contribution to the instanton measure is,

$$\int d\mu_F = \int d^2\xi \, d^2\xi' (\mathcal{J}_\xi)^{-2} \tag{18}$$

In Appendix C we find that $\mathcal{J}_\xi = 2S_{\rm cl}$.

As usual in any saddle-point calculation, to obtain the leading-order semiclassical result it is necessary to perform Gaussian integrals over the small fluctuations of the fields around the classical background. In general these integrals yield functional determinants of the operators which appear at quadratic order in the expansion around the instanton. A simplifying feature that holds for all self-dual configurations is that each of the fluctuation operators for scalars, spinors gauge fields and ghosts are related in a simple way to one of the operators Δ_+ or Δ_-. This standard connection is as follows. In the chiral basis for 4D Majorana fermions (see Appendix A), the quadratic fluctuation operator is

$$\Delta_F = \begin{pmatrix} 0 & \not{P}_{\rm cl} \\ \not{P}_{\rm cl} & 0 \end{pmatrix} \tag{19}$$

Performing the Grassmann Gaussian integration over $\underset{\sim}{\lambda}$ yields

$$\frac{{\rm Pf}(\Delta_F)}{{\rm Pf}(\Delta_F^{(0)})} = \left(\frac{\det'(\Delta_F^2)}{\det(\Delta_F^{(0)2})} \right)^{\frac{1}{4}} = \left(\frac{\det'(\Delta_-)\det(\Delta_+)}{\det(\Delta^{(0)})^2} \right)^{\frac{1}{4}} \tag{20}$$

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where $\det'$ denotes the removal of zero eigenvalues and the superscript (0) denotes the fluctuation operator for the corresponding field in the vacuum sector. In particular we define the operator $\Delta^{(0)} = \bar{\mathcal{D}}^{(0)}\mathcal{D}^{(0)} = \mathcal{D}^{(0)}\bar{\mathcal{D}}^{(0)}$, formed from the vacuum Dirac operators $\mathcal{D}^{(0)}$ and $\bar{\mathcal{D}}^{(0)}$. Integrating out the $\underset{\sim}{\psi}$ field yields a second factor equal to (20). For the bose fields we introduce the gauge fixing and ghost terms using the 4D background gauge, $\mathcal{D}^{cl}_m \delta \underset{\sim}{v}^a_m = 0$, and expand the action around the classical configuration. Quadratic fluctuation operators for the complex scalar, $\underset{\sim}{A}$, the gauge field, $\underset{\sim}{v}$, and the ghosts, $\underset{\sim}{c}$ and $\underset{\sim}{\bar{c}}$ are

$$\Delta_A = \Delta_c = \mathcal{D}^2_{cl} = \tfrac{1}{2}\text{Tr}(\Delta_+)$$
$$\Delta_v = -\mathcal{D}^2_{cl}\delta_{mn} + 2\underset{\sim}{v}^{cl}_{mn} = -\tfrac{1}{2}\text{Tr}(\bar{\sigma}^n \Delta_- \sigma^m) \tag{21}$$

where Tr is over the spinor indices. The bosonic Gaussian integrations now give,

$$\left(\frac{\det'(\Delta_v)}{\det(\Delta_v^{(0)})}\right)^{-\frac{1}{2}} \left(\frac{\det(\Delta_A)}{\det(\Delta_A^{(0)})}\right)^{-1} \left(\frac{\det(\Delta_c)}{\det(\Delta_c^{(0)})}\right)^{+1} = \left(\frac{\det'(\Delta_-)}{\det(\Delta^{(0)})}\right)^{-1} \tag{22}$$

Combining the fermion and boson contributions, we find that the total contribution of non-zero modes to the instanton measure is given by,

$$R = \left[\frac{\det(\Delta_+)}{\det'(\Delta_-)}\right]^{\frac{1}{2}} \tag{23}$$

In any supersymmetric theory the total number of non-zero eigenvalues of bose and fermi fields is precisely equal. This corresponds to the fact that, for any self-dual background, the non-zero eigenvalues of Δ_+ and Δ_- are equal [17]. Naively, this suggests that the ratio R is unity. As the spectra of these two operators contain a continuum of scattering states in addition to normalizable bound states, this assertion must be considered carefully. For the continuum contributions to the determinants of Δ_+ and Δ_- to be equal, it is necessary not only that the continuous eigenvalues have the same range, but that the density of these eigenvalues should also be the same. Following the original approach of 't Hooft [18] in four dimensions, one can regulate the problem by putting the system in a spherical box with fixed boundary conditions. In this case the spectrum of scattering modes becomes discrete and the resulting eigenvalues depend on the phase shifts of the scattering eigenstates. In the limit where the box size goes to infinity, these phase shifts determine the density of continuum eigenvalues. For a four-dimensional instanton, 't Hooft famously discovered that the phase shifts in question were equal for the small fluctuation operators of each of the fields. As a direct result of this, the ratio R is equal to unity in the background of any number of instantons in a 4D supersymmetric gauge theory. However, the phase-shifts associated with the operators Δ_+ and Δ_- are not equal in a monopole

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background. In the four-dimensional theory, Kaul [19] noticed that this effect leads to a non-cancellation of quantum corrections to the monopole mass. In our case, as we will show below, it will yield a non-trivial value of R.

In fact the mismatch in the continuous spectra of Δ_+ and Δ_- can be seen already from the index function $\mathcal{I}(\mu)$. Comparing with the definition (14), we see that the fact that Weinberg's formula (15) has a non-trivial dependence on μ and is not simply equal to $2k$ precisely indicates a difference between the non-zero spectra of the two operators. This observation can be made precise by the following steps. First, dividing (14) by μ and then performing a parametric integration we obtain.

$$\int_\mu^\infty \frac{d\mu'}{\mu'} \mathcal{I}(\mu') = \text{Tr}\left[\log\left(\Delta_+ + \mu\right) - \log\left(\Delta_- + \mu\right)\right] \tag{24}$$

The amputated determinant appearing in the ratio R is properly defined as,

$$\text{det}'(\Delta_-) = \lim_{\mu \to 0} \left[\frac{\det(\Delta_- + \mu)}{\mu^{2k}}\right] \tag{25}$$

The power of μ appearing in the denominator reflects the number of zero eigenvalues of Δ_- calculated above. Using the relation $\text{Tr}\log(\hat{O}) = \log\det(\hat{O})$ in (24), we obtain a closed formula for R:

$$R = \lim_{\mu \to 0} \left[\mu^{2k} \exp\left(\int_\mu^\infty \frac{d\mu'}{\mu'} \mathcal{I}(\mu')\right)\right]^{\frac{1}{2}} \tag{26}$$

Evaluating this formula on (15) we obtain the result, $R = (2M_W)^{2k}$. For $k = 1$, we can combine the various factors (16), (18) and (23) to obtain the final result for the one-instanton measure,

$$\begin{aligned}
\int d\mu^{(k=1)} &= \int d\mu_B \int d\mu_F \, R \, \exp(-S_{\text{cl}} + i\sigma) \\
&= \frac{M_W}{2\pi} \int d^3X \, d^2\xi d^2\xi' \exp(-S_{\text{cl}} + i\sigma) \tag{27}
\end{aligned}$$

Here we have performed the integration over θ, anticipating the fact that the integrands we are ultimately interested in will not depend on this variable. The term $i\sigma$ is the contribution of the surface term (3) which we will discuss further below.

3.2 Instanton effects in the low energy theory

Because of their long-range fields, instantons and anti-instantons have a dramatic effect on the low-energy dynamics of three-dimensional gauge theories. As they can

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be thought of as magnetic charges in $(3 + 1)$-dimensions, 3D instantons and anti-instantons experience a long-range Coulomb interaction. In the presence of a massless adjoint scalar, the repulsive force between like magnetic charges is cancelled. This cancellation is reflected in the existence of static multi-monopole solutions in the BPS limit. However, in the case of an instanton and anti-instanton, there is always an attractive force. In a purely bosonic gauge theory, Polyakov [11] showed that a dilute gas of these objects leads to the confinement of electric charge. The long-range effects of instantons and anti-instantons can be captured by including terms in the low-energy effective Lagrangian of the characteristic form,

$$\mathcal{L}_I \quad \sim \quad \exp\left(-\frac{8\pi^2|\phi|}{e^2} \pm i\sigma\right) \tag{28}$$

where $|\phi|^2 = \phi_1^2 + \phi_2^2 + \phi_3^2$. This is simply the instanton action of the previous section, with the VEVs v_i and σ being promoted to dynamical fields.

In the presence of massless fermions, the instanton contribution to the low-energy action necessarily couples to n fermion fields, where n is the number of zero modes of the Dirac operator in the monopole background. In the three-dimensional $N = 2$ theory considered in [20], instantons induce a mass term for the fermions in the effective action. In fact this corresponds to an instanton-induced superpotential which lifts the Coulomb branch. In the present case, the effective action will contain an induced four-fermion vertex due to the contribution of a single instanton [2]. This vertex has a simple form when written in terms of the (dimensionally-reduced) Weyl fermions of (6),

$$S_I \quad = \quad \kappa \int d^3x \, \bar{\lambda}^2 \bar{\psi}^2 \, \exp\left(-\frac{8\pi^2|\phi|}{e^2} + i\sigma\right) \tag{29}$$

where the coefficient κ will be calculated explicitly below. In this case, $N = 4$ SUSY does not allow the generation of a superpotential and, as we will review below, the four-fermion vertex is instead the supersymmetric completion of an instanton correction to the metric on the moduli space.

Just like the vertex induced by (four-dimensional) instantons in the four-dimensional $N = 2$ theory, self-duality means that the above vertex contains only Weyl fermions of a single chirality. Equivalently, the vertex contains only the holomorphic components of the 3D Majorana fermions in the complex basis of (7). In the 4D case, the chiral form of the vertex signals the presence of an anomaly in the $U(1)_\mathcal{R}$ symmetry. By virtue of our dimensional reduction and choice of vacuum, the $U(1)_\mathcal{R}$ symmetry of the 4D theory corresponds to $U(1)_N$ defined in Section 1, the unbroken subgroup of $SU(2)_N$ global symmetry in the 3D theory. More precisely the corresponding charges

are related as $Q_N = Q_{\mathcal{R}}/2$. Hence each of the right-handed 4D Weyl fermions carries $U(1)_N$ charge $-1/2$, which suggests that the induced vertex (29) violates the conservation of $U(1)_N$ by -2 units. However, as explained in [2], the symmetry is non-anomalous in 3D due to the presence of the surface term $i\sigma$ in the instanton exponent. Assigning σ the $U(1)_N$ transformation,

$$\sigma \rightarrow \sigma + 2\alpha \tag{30}$$

under the action of $\exp(i\alpha Q_N)$, the symmetry of the effective action is restored. However, because the VEV of σ transforms non-trivially, the symmetry is now spontaneously broken and hence $SU(2)_N$ has no unbroken subgroup. An equivalent statement is that the orbit of $SU(2)_N$ on the classical moduli space is three-dimensional, a fact that will play an important role in the considerations of the next Section.

In the following we will be interested in the instanton corrections to the leading terms in the derivative expansion of the low-energy effective action. As in the four-dimensional theory, this restricts our attention to terms with at most two derivatives or four fermions. However, an important difference with that case is that the 3D instantons are exact solutions of the equations of motion and there is no mechanism for the VEV to lift fermion zero modes in a way that preserves the $U(1)_N$ symmetry [7]. It follows that the only sectors of the theory which can contribute to the leading terms in the low-energy effective action are those with instanton number (magnetic charge) -1, 0 or $+1$. Another important difference with the 4D case is that, because there is no holomorphic prepotential in 3D, there can also be contributions to the effective action from configurations containing arbitrary numbers of instanton/anti-instanton pairs.

Finally we will compute the exact coefficient of the four-anti-fermion vertex (29) by examining the large distance behaviour of the correlator

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle \lambda_\alpha(x_1)\lambda_\beta(x_2)\psi_\gamma(x_3)\psi_\delta(x_4)\rangle \tag{31}$$

where the Weyl fermion fields are replaced by their zero-mode values in the one-instanton background. The explicit formulae for the zero-modes of the low-energy fermions are straightforward to extract, via (17) from the long-range behaviour of the magnetic monopole fields. Using the formulae of Appendix C for the large-distance (LD) limit of the fermion zero modes we find,

$$\begin{aligned}
\lambda_\alpha^{\mathrm{LD}} &= 8\pi \left(\mathcal{S}_{\mathrm{F}}(x-X)\right)_\alpha^{\ \beta} \xi_\beta \\
\psi_\alpha^{\mathrm{LD}} &= 8\pi \left(\mathcal{S}_{\mathrm{F}}(x-X)\right)_\alpha^{\ \beta} \xi'_\beta
\end{aligned} \tag{32}$$

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[7]A more general analysis of this issue will be presented elsewhere.

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This asymptotic form is valid for $|x - X| \gg M_W^{-1}$ where X is the instanton position and $\mathcal{S}_F(x) = \gamma_\mu x_\mu / (4\pi |x|^2)$ is the three-dimensional Weyl fermion propagator. The leading semiclassical contribution to the correlator (31) is given by,

$$G^{(4)}(x_1, x_2, x_3, x_4) = \int d\mu^{(k=1)} \lambda_\alpha^{\mathrm{LD}}(x_1) \lambda_\beta^{\mathrm{LD}}(x_2) \psi_\gamma^{\mathrm{LD}}(x_3) \psi_\delta^{\mathrm{LD}}(x_4) \tag{33}$$

where $d\mu^{(k=1)}$ is the one-instanton measure (27). Performing the ξ and ξ' integrations we obtain,

$$\begin{aligned}
G^{(4)}(x_1, x_2, x_3, x_4) &= 2^9 \pi^3 M_W \exp(-S_{\mathrm{cl}} + i\sigma) \int d^3 X \; \epsilon^{\alpha'\beta'} \mathcal{S}_F(x_1 - X)_{\alpha\alpha'} \\
&\times \; \mathcal{S}_F(x_2 - X)_{\beta\beta'} \epsilon^{\gamma'\delta'} \mathcal{S}_F(x_3 - X)_{\gamma\gamma'} \mathcal{S}_F(x_1 - X)_{\delta\delta'}
\end{aligned} \tag{34}$$

This result is equivalent to the contribution of the vertex (29) added to the classical low-energy action (6). Our calculation shows that the coefficient κ takes the value,

$$\kappa = 2^7 \pi^3 M_W \left(\frac{2\pi}{e^2} \right)^4 \tag{35}$$

where the four powers of $(2\pi/e^2)$ reflect our choice of normalization for the fermion kinetic terms in (6).

4   The Exact Low-Energy Effective Action

In this Section, following the arguments of [2], we will determine the exact low energy effective action of the $N = 4$ SUSY gauge theory in three dimensions. Below, we will write down the most general possible ansatz for the terms in the low-energy effective action with at most two derivatives or four fermions which is consistent with the symmetries of the model. As we will review, the combined restrictions of $N = 4$ supersymmetry and the global $SU(2)_N$ symmetry lead to a set of non-linear ordinary differential equations for the components of the hyper-Kähler metric which in turn determines the relevant terms in the effective action. We will find a one-parameter family of solutions of these equations which agree with the one-loop perturbative calculation of Section 2. Our main result is that the one-instanton contribution to the four-fermion vertex, calculated from first principles in Section 3, uniquely selects the metric of the Atiyah-Hitchin manifold from this family of solutions.

We begin by discussing the general case of a low-energy theory with scalar fields $\{X_i\}$ and Majorana[8] fermion superpartners $\{\Omega_i^\alpha\}$ where $i = 1, \ldots, d$. As usual the scalars define coordinates on the quantum moduli space, $\mathcal{M}$, which is a manifold of

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[8]The 3D Majorana condition is $\bar{\Omega} = i\Omega$, see Appendix A.

real dimension d. The low-energy effective action has the form of a three-dimensional supersymmetric non-linear σ-model with $\mathcal{M}$ as the target manifold,

$$S_{\text{eff}} = K \int d^3x \ \left\{ \tfrac{1}{2} g_{ij}(X) \ \left[\partial_m X^i \partial_m X^j - \Omega^i \not{D} \Omega^j \right] - \tfrac{1}{12} R_{ijkl}(\Omega^i \cdot \Omega^k)(\Omega^j \cdot \Omega^l) \right\} \quad (36)$$

where K is an overall constant included for later convenience. The kinetic terms in the action define a metric g_{ij} on the moduli space $\mathcal{M}$, and $\not{D}$ and R_{ijkl} denote the corresponding covariant Dirac operator and Riemann tensor respectively.

Following Alvarez-Gaume and Freedman [9], the supersymmetries admitted by the above action are written in the form,

$$\delta X_i = \epsilon^{[1]} \cdot \Omega_i + \sum_{q=2}^{N} J_i^{[q]j}(X)\epsilon^{[q]} \cdot \Omega_j \quad (37)$$

The action (36) is automatically invariant under the $N = 1$ supersymmetry parametrized by $\epsilon^{[1]}$. However, $N > 1$ supersymmetry requires the existence of $N - 1$ linearly independent tensors $J_i^{[q]j}$ which commute with the infinitesimal generators of the holonomy group H of the target. It is also necessary that these tensors form an $su(2)$ algebra. In general the holonomy group of a d-dimensional Riemannian manifold is a subgroup of $SO(d)$. For $N = 4$, the existence of three such tensors which commute with the holonomy generators imply that d must divisible by 4 and that H is restricted to be a subgroup of $Sp(d/4)$. Such manifold of symplectic holonomy is by definition hyper-Kähler. The tensors $J_i^{[q]j}$ ($q = 2, 3, 4$) define three inequivalent complex structures on $\mathcal{M}$. An equivalent statement of the hyper-Kähler condition is to require that these complex structures be covariantly constant with respect to the metric g_{ij}.

In the case $d = 4$, the holonomy can be chosen to lie in the $Sp(1) \simeq SU(2)$ subgroup of $SO(4)$ generated by the self-dual tensors η_{ij}^a $a = 1, 2, 3$ (defined in Appendix A). In fact the holonomy generators are components of the Riemann tensor R_{ijkl} and a sufficient condition for a hyper-Kähler four-manifold is for this tensor to be self-dual[9]. Correspondingly the complex structures $J_i^{[q]j}(X)$ ($q = 2, 3, 4$) can be taken as linear combinations of the anti-self-dual $SO(4)$ generators $\bar{\eta}_{ij}^a$. As indicated, the relevant linear combinations appearing in (37) will vary as one moves from one point on the manifold to another. Another restriction on the moduli space $\mathcal{M}$ comes from the action of the global $SU(2)_N$ symmetry. As we have seen above, there is no anomaly in this symmetry either in perturbation theory or from non-perturbative effects, hence we expect that the exact quantum moduli space has an $SU(2)_N$ isometry. Further, because of the non-trivial transformation of the dual photon described above, we

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[9]The more conventional choice of an anti-self-dual Riemann tensor and self-dual complex structures is related to this one by a simple redefinition of the fields and the parameters $\epsilon^{[q]}$.

know that $SU(2)_N$ will generically have three-dimensional orbits on $\mathcal{M}$. It can also be checked explicitly that the three complex structures on $\mathcal{M}$ introduced above transform as a **3** of $SU(2)_N$.

Remarkably, the problem of classifying all the hyper-Kähler manifolds of dimension four with the required isometry has been solved in an entirely different physical context. As discussed in Section 1, these are exactly the properties of the reduced or centered moduli space of two BPS monopoles of gauge group $SU(2)$. Atiyah and Hitchin [8] considered all manifolds with these properties and showed that there is only one manifold which has no singularities: the AH manifold. In the original context, the absence of singularities was required because of known properties of multi-monopole solutions. In particular, the metric on the moduli space of an arbitrary number of BPS monopoles was known to be complete. This means that every curve of finite length on the manifold has a limit point (see Chapter 3 of [8] and references therein). In the present context, the absence of singularities on the quantum moduli space $\mathcal{M}$ can be taken as an assumption about the strong-coupling behaviour of the 3D SUSY gauge theory. This assumption leads directly to Seiberg and Witten's proposal that the quantum moduli space has the same hyper-Kähler metric as the the AH manifold. In the following we will show that the one-instanton contribution, Eqs (35) and (29), calculated in the previous section, together with the results of one-loop perturbation theory (8), provides a direct proof of the SW proposal without assuming the absence of strong-coupling singularities.

Following Gibbons and Manton [21], we parametrize the orbits of $SO(3)_N$ (whose double-cover is $SU(2)_N$) by Euler angles θ, ϕ and ψ with ranges, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 2\pi$ and introduce the standard left-invariant one-forms,

$$
\begin{aligned}
\sigma_1 &= -\sin\psi \ d\theta + \cos\psi \sin\theta \ d\phi \\
\sigma_2 &= \cos\psi \ d\theta + \sin\psi \sin\theta \ d\phi \\
\sigma_3 &= d\psi + \cos\theta \ d\phi
\end{aligned}
\tag{38}
$$

The remaining dimension of the moduli space $\mathcal{M}$, transverse to the orbits of $SO(3)_N$, is labelled by a parameter r and the most general possible metric with the required isometry takes the form [22],

$$
g_{ij}dX_i dX_j = f^2(r)dr^2 + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2
\tag{39}
$$

The function $f(r)$ depends on the definition of the radial parameter r. We will also define the corresponding Cartesian coordinates,

$$
\begin{aligned}
X &= r\sin\theta\cos\phi \\
Y &= r\sin\theta\sin\phi \\
Z &= r\cos\theta
\end{aligned}
\tag{40}
$$

We start by identifying the parameters introduced above in terms of the low-energy fields of Section 2.2 in a way which is consistent with their transformation properties under $SO(3)_N$. The triplet (X, Y, Z) transforms as a vector of $SO(3)_N$ and will be identified, up to a rescaling, with the triplet of scalar fields appearing in the Abelian classical action (4): $(X, Y, Z) = (S_{cl}/M_W)(\phi_1, \phi_2, \phi_3)$. This means that we are defining the parameter r to be equal to S_{cl}. The remaining Euler angle, ψ, by definition transforms as $\psi \to \psi + \alpha$ under a rotation through an angle α about the axis defined by the vector (X, Y, Z). After comparing this transformation property with the $U(1)_N$ transformation (30) of the dual photon σ, we set $\psi = \sigma/2$. In the following we will refer to (X, Y, Z, σ) as standard coordinates.

The weak-coupling behaviour of the metric g_{ij} can be deduced by comparing the general low-energy effective action (36) with its classical counterpart (4), together with the one-loop correction (8). Choosing the constant K in (36) to be equal to $2e^2/(\pi(8\pi)^2)$, the classical metric is just the flat one, δ_{ij}, in the standard coordinates introduced above. Including the one-loop renormalization of the coupling (8), the metric functions, a^2, b^2 and c^2 are given by,

$$
\begin{aligned}
a^2 = b^2 &\simeq S_{cl}(S_{cl} - 2) \\
c^2 &\simeq 4 + 8/S_{cl}
\end{aligned}
\tag{41}
$$

Corrections to the RHS of the above formulae come from two-loops and higher and are down by powers of $1/S_{cl}$. In fact, the equality $a^2 = b^2$ persists to all orders in perturbation theory, reflecting the fact that $U(1)_N$ is only broken (spontaneously) by non-perturbative effects. To this order, the function f^2 is also determined to be equal to $1 - 2/S_{cl} + O(1/S_{cl}^2)$.

The Majorana fermions Ω_α^i appearing in (36), will be real linear combinations of the fermions χ_α^A of the low-energy action (6). In general, this linear relation will have a non-trivial dependence on the bosonic coordinates. For our purpose it will be sufficient to determine this relation at leading order in the weak-coupling limit, $r = S_{cl} \to \infty$. In the standard coordinates we set,

$$
\Omega_\alpha^i \simeq M^{iA}(\theta, \phi, \sigma)\chi_{A\alpha}
\tag{42}
$$

up to corrections of order $1/S_{cl}$. It is convenient to complexify the $SO(4)_{\mathcal{R}}$ index as in Section 2.2 and consider instead coefficients M^{ia} and $M^{i\bar{a}} = (M^{ia})^*$. In order to reproduce the fermion kinetic term in (7) these coefficients must obey the relations,

$$
\begin{aligned}
\delta_{ij} M^{ia} M^{j\bar{b}} &= \delta^{a\bar{b}} \left(\frac{S_{cl}}{M_W}\right)^2 \\
\delta_{ij} M^{ia} M^{jb} &= \delta_{ij} M^{i\bar{a}} M^{j\bar{b}} = 0
\end{aligned}
\tag{43}
$$

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The first condition fixes the overall normalization of the fermions in (36) while the second is required for their kinetic term to be $U(1)_N$ invariant. The remaining freedom present in this identification is related to the $\mathcal{R}$-symmetry of the $N = 4$ SUSY algebra.

The hyper-Kähler condition can be formulated as a set of non-linear ordinary differential equations for the functions a, b, c and f:

$$\frac{2bc}{f}\frac{da}{dr} = (b - c)^2 - a^2 \tag{44}$$

together with the two equations obtained by cyclic permutation of a, b and c. The solutions of these equations are analysed in detail in Chapter 9 of [8] and we will adapt the analysis given there to our current purposes. The equations completely determine the behaviour of a, b and c as functions of r only once one makes a specific choice for the function f, for example the choice $f = -b/r$ made by Gibbons and Manton [21]. If one makes such a choice in the present context, then the corresponding relation between r and the weak-coupling parameter $S_{\rm cl}$ will receive quantum corrections which cannot be determined. In fact we have chosen instead to define r to be equal to $S_{\rm cl}$ and, correspondingly, this implies some choice for f which can only be determined order by order in perturbation theory. This reflects an important feature of the exact results for the low-energy structure of SUSY gauge theories which is familiar from four-dimensions. In these theories the constraints of supersymmetry and (in the 4D case) duality allow one to specify the exact quantum moduli space as a Riemannian manifold. However the Lagrangian fields and couplings of the conventional weak-coupling description define a particular coordinate system on the manifold and sometimes the relation of these parameters to the parameters of the exact low-energy effective Lagrangian can not be determined explicitly. As discussed in [24], just such an ambiguity arises in the relation between the tree-level coupling constant and the exact low-energy coupling in the finite four-dimensional $N = 2$ theory with four hypermultiplets.

In the light of the above discussion we will eliminate both f and r from the equations, and focus on the information about the metric which is independent of the choice of parametrization. Following [8] we obtain a single differential equation for $x = b/a$ and $y = c/a$,

$$\frac{dy}{dx} = \frac{y(1 - y)(1 + y - x)}{x(1 - x)(1 + x - y)} \tag{45}$$

The solutions of this equation are described by curves or trajectories in the (x, y)-plane. In the weak coupling limit $S_{\rm cl} \to \infty$ we have $a^2 = b^2 \to \infty$ and $c^2 \to 4$. Hence we must find all the solutions of (45) which pass through the point Q with coordinates $x = 1$, $y = 0$ (see Diagram 7 on p74 of [8]). The relevant features of these solutions are as follows:

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1: A particular solution which passes through Q is the trajectory $x = 1$. Returning to the full equations (44), we find that this corresponds to the solution:

$$a = b = \frac{c}{(1 - c^2/4)} \tag{46}$$

Eliminating S_{cl} in (41), we find this relation is obeyed up to one-loop in perturbation theory as long as we choose square roots so that $ac = bc < 0$. This solution describes the singular Taub-NUT geometry and, as discussed in [2], this is the exact solution of the low-energy theory up to non-perturbative corrections. In other words the resulting effective action (36) includes the sum of all corrections from all orders in perturbation theory. However, as we have commented above, the function f is not determined by the solution, so the all-orders effective action cannot be written explicitly in terms of the weak coupling parameter S_{cl}.

2: There is precisely a one-parameter family of solutions passing through Q, each of which is exponentially close to the line $x = 1$ $(y < 0)$ near Q. Linearizing around $x = 1$, $y = 0$ we may integrate (45) to obtain the leading asymptotic behaviour,

$$a - b \;\simeq\; B\frac{a^2}{c}\exp\left(\frac{2a}{c}\right) \tag{47}$$

where B is a constant of integration. Corrections to the RHS are down by powers of $y = c/a$ or by powers of $\exp(2/y)$.

3: Using numerical methods to examine the behaviour of these solutions away from the point Q, one finds that there is a unique critical trajectory which originates at the point P' with coordinates $(0, -1)$. All other trajectories originate either at the origin $(0, 0)$ or at negative infinity $(0, -\infty)$. Atiyah and Hitchin show that only the critical trajectory corresponds to a complete manifold. The critical solution can be constructed explicitly in terms of elliptic functions, its asymptotic form near Q is given in Gibbons and Manton [21] (see equation (3.14) of this reference) and agrees with (47) with a specific value for the integration constant $B = B_{cr} = 16\exp(-2)$. All trajectories with $B \neq B_{cr}$ correspond to singular geometries.

Using the identifications (41), the asymptotic behaviour (47) becomes,

$$a - b \;\simeq\; -8qS_{cl}^2\exp\left(-S_{cl}\right) \tag{48}$$

where $q = B/B_{cr}$. Hence the leading deviation from the perturbative relation $a = b$ comes with exactly the exponential suppression characteristic of a one-instanton effect. When substituted in the metric (39) and the effective action (36), this term yields a contribution to the boson kinetic terms which also comes with the phase

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factor $\exp(\pm i\sigma)$ expected for the (anti-)instanton term. Further, each member of the family of solutions parametrized by the constant q yields a different prediction for this one-instanton effect. In principle, it is straightforward to check the coefficient of this term against the results of a semiclassical calculation of the scalar propagator. This would involve calculating the Grassmann bilinear contributions to the scalar field which come from the fermion zero modes in the monopole background. In the following, we will choose instead to extract a prediction for the four-fermion vertex in (36) and compare this directly with the result (35) of Section 3.

The effective action (36) contains a four-fermion vertex proportional to the Riemann tensor. To make contact with the results of Section 3 where the vacuum is chosen to lie in the ϕ_3 direction in orbit of $SU(2)_N$, we evaluate the Riemann tensor corresponding to the metric (39) at the point on the manifold with standard coordinates $(0, 0, r = S_{\text{cl}}, \sigma)$. This calculation is presented in Appendix D (Many of the necessary results have been given previously by Gauntlett and Harvey in Appendix B of [23]). In particular, we evaluate the leading-order contribution to the σ-dependent terms in the Riemann tensor in the weak-coupling limit: $S_{\text{cl}} \to \infty$. The result is best expressed in the complex basis with coordinates,

$$z_1 = \frac{1}{\sqrt{2}}(X - iY) \qquad\qquad z_2 = \frac{1}{\sqrt{2}}(Z - i\sigma) \qquad (49)$$

In this basis, the relevant contribution to the Riemann tensor is pure holomorphic and is given by,

$$R_{1212} = 8qS_{\text{cl}} \exp\left(-S_{\text{cl}} + i\sigma\right) \qquad (50)$$

where the other pure holomorphic components are related to this one by the usual symmetries of the Riemann tensor: $R_{abcd} = -R_{bacd} = R_{cdab} = -R_{abdc}$. The anti-instanton contribution is pure anti-holomorphic and all components of mixed holomorphy are independent of σ.

The above result for the Riemann tensor means that the corresponding four-fermion vertex in the non-linear σ-model (36), has exactly the chiral form expected from the discussion of Section 3.2. In particular, invariance of the vertex under $U(1)_N$ transformations implies that the holomorphic (anti-holomorphic) components Ω^a ($\Omega^{\bar{a}}$) of the target space fermions have $U(1)_N$ charge $-1/2$ $(+1/2)$. Hence we complete our identification of the fermions by demanding that the transformation (42) maps fermions of positive (negative) $U(1)_N$ charge to fermions of positive (negative) $U(1)_N$ charge. At the chosen point $(0, 0, r = S_{\text{cl}}, \sigma)$, we write the relation between fermions, $(\Omega^a, \Omega^{\bar{a}})$ in the complex basis (49) and $(\chi^b, \chi^{\bar{b}})$ in the complex basis of (7) as,

$$\begin{aligned} \Omega_a^\alpha &\simeq M_{ab}\chi_b^\alpha \\ \Omega_{\bar{a}}^\alpha &\simeq M_{\bar{a}\bar{b}}\chi_{\bar{b}}^\alpha \end{aligned} \qquad (51)$$

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where $M_{\bar{a}\bar{b}} = (M_{ab})^*$ and, as in (42), corrections to the RHS are down by inverse powers of S_{cl}. The second condition in (43) is automatically satisfied. Regarding M_{ab} as a 2×2 matrix, the first condition in (43) is satisfied by choosing $M = (S_{\mathrm{cl}}/M_W)\hat{M}$, where $\hat{M} \in SU(2)$ is a residual degree of freedom associated with the unbroken $SU(2)_{\mathcal{R}}$ symmetry[10].

Finally we calculate the four-fermion vertex which follows from the instanton contribution to the Riemann tensor (50). After taking into account the symmetries of the Riemann tensor described above and performing a Fierz rearrangement the resulting vertex becomes,

$$\mathcal{L}_{4\mathrm{F}} = \frac{1}{4}K\, R_{1212}\left(\Omega^1 \cdot \Omega^1\right)\left(\Omega^2 \cdot \Omega^2\right) \tag{52}$$

We rewrite the vertex in terms of Weyl fermions using,

$$\begin{aligned}
\left(\Omega^1 \cdot \Omega^1\right)\left(\Omega^2 \cdot \Omega^2\right) &= (\det(M))^2\left(\chi^1 \cdot \chi^1\right)\left(\chi^2 \cdot \chi^2\right) \\
&= \left(\frac{S_{\mathrm{cl}}}{M_W}\right)^4 \bar{\lambda}^2\bar{\psi}^2
\end{aligned} \tag{53}$$

Collecting together the various factors, the final result for the induced four-fermi vertex is,

$$\mathcal{L}_{4\mathrm{F}} = 2^7\pi^3 q M_W \left(\frac{2\pi}{e^2}\right)^4 \bar{\lambda}^2\bar{\psi}^2 \, \exp\left(-S_{\mathrm{cl}} + i\sigma\right) \tag{54}$$

Comparing this with the calculated value (35) for the coefficient κ in the instanton-induced vertex (29), we deduce that $q = 1$. This implies that the quantum moduli space of the theory is in fact the Atiyah-Hitchin manifold.

<center>******************</center>

Appendix A: Dimensional Reduction

In this Appendix we present our conventions and give the details of dimensional reduction. We work in Minkowski[11] space in 4D and 3D with the metric signature

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[10]Strictly speaking M is only restricted to lie in $U(2)$. However the additional phase can be reabsorbed by changing the identification $\psi = \sigma/2$ made above by an additive constant.

[11]With the exception of this Appendix and the next one, the calculations in the rest of the paper are performed in Euclidean space

$(+,-,-,...)$ and $m, n = 0, 1, 2, 3, \ \mu, \nu = 0, 1, 2$.

We start with the $N = 2$ supersymmetric Yang-Mills in 4D

$$S_{\text{4D}} \ = \ \frac{1}{g^2} \int d^4 x \ \text{Tr} \ \Big\{ \ -\tfrac{1}{2} v_{mn} v^{mn} + i \, \bar{\lambda} \, \slashed{\mathcal{D}} \, \lambda + i \, \lambda \, \slashed{\mathcal{D}} \, \bar{\lambda} + \text{D}^2 \tag{55}$$
$$+ 2 \mathcal{D}_m \, A^\dagger \mathcal{D}^m \, A + i \, \bar{\psi} \, \slashed{\mathcal{D}} \, \psi + i \, \psi \, \slashed{\mathcal{D}} \, \bar{\psi} + 2 \, F^\dagger \, F$$
$$- 2 \, \text{D} \, [\, A, \, A^\dagger] + 2 \sqrt{2} \, i \Big([\, A^\dagger, \, \psi] \, \lambda + \bar{\lambda} [\, A, \, \bar{\psi}] \Big) \Big\} \ .$$

Here v_m is the gauge field, A is the complex scalar field, Weyl fermions λ and ψ are their superpartners, while D and F are auxiliary fields. Also $\slashed{\mathcal{D}}_{\alpha \dot{\alpha}} = \mathcal{D}_m \sigma^m_{\alpha \dot{\alpha}}$, and $\bar{\slashed{\mathcal{D}}}^{\dot{\alpha} \alpha} = \mathcal{D}^m \bar{\sigma}^{\dot{\alpha} \alpha}_m$, where $\mathcal{D}_m \, X = \partial_m \, X - i \, [\, v_m, \, X]$. Wess and Bagger [25] spinor summation conventions are used throughout and sigma-matrices in Minkowski space are, $\sigma^m_{\alpha \dot{\alpha}} = (-1, \tau^a)$, $\bar{\sigma}^{m \, \dot{\alpha} \alpha} = (-1, -\tau^a)$.

The three-dimensional theory is obtained by making all the fields independent of one spatial dimension and decoupling this dimension from the theory, Eq. (1). The only subtle point in this program is the dimensional reduction of the fermions. In 4D the 2-component Weyl spinors can be combined into the 4-component Majorana spinors,

$$\lambda_{\text{ch}} = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} , \qquad \bar{\lambda}_{\text{ch}} = (\lambda^\alpha , \ \bar{\lambda}_{\dot{\alpha}}) ,$$

$$\bar{\psi}_{\text{ch}} = (\psi^\alpha , \ \bar{\psi}_{\dot{\alpha}}) , \qquad \psi_{\text{ch}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} , \tag{56}$$

in such a way that,
$$i \, \bar{\lambda} \, \slashed{\mathcal{D}} \, \lambda + i \, \lambda \, \slashed{\mathcal{D}} \, \bar{\lambda} \ = \ i \, \bar{\lambda}_{\text{ch}}, \ ^m_{\text{ch}} \mathcal{D}_m \, \lambda_{\text{ch}} \ , \tag{57}$$
where , $^m_{\text{ch}}$ is the gamma matrix of the 4D theory in the standard chiral basis,

$$, \ ^m_{\text{ch}} \ = \ \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \ . \tag{58}$$

Thus, λ_{ch} and ψ_{ch} are the Majorana fermions in the chiral basis.

For the purposes of dimensional reduction it is more convenient to choose a different – real – basis for Majorana spinors in 4D related by a unitary transformation to the

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chiral basis above,

$$\underset{\sim}{\lambda}_{\text{re}} = \begin{pmatrix} \underset{\sim}{\chi}_\alpha \\ \underset{\sim}{\tilde{\chi}}_\alpha \end{pmatrix} , \qquad \underset{\sim}{\bar{\lambda}}_{\text{re}} = i\left(\underset{\sim}{\tilde{\chi}}^\alpha , \underset{\sim}{\chi}^\alpha \right) ,$$

$$\underset{\sim}{\bar{\psi}}_{\text{re}} = i\left(\underset{\sim}{\tilde{\eta}}^\alpha , \underset{\sim}{\eta}^\alpha \right) , \qquad \underset{\sim}{\psi}_{\text{re}} = \begin{pmatrix} \underset{\sim}{\eta}_\alpha \\ \underset{\sim}{\tilde{\eta}}_\alpha \end{pmatrix} , \tag{59}$$

where new (real) 2-spinors are simply the 'real' and 'imaginary' parts of the Weyl 2-spinors,

$$\underset{\sim}{\chi}_\alpha = \frac{1}{\sqrt{2}}\left(\underset{\sim}{\lambda}_\alpha + \underset{\sim}{\bar{\lambda}}_{\dot\alpha} \right) \qquad \underset{\sim}{\tilde{\chi}}_\alpha = -\frac{i}{\sqrt{2}}\left(\underset{\sim}{\lambda}_\alpha - \underset{\sim}{\bar{\lambda}}_{\dot\alpha} \right)$$

$$\underset{\sim}{\eta}_\alpha = \frac{1}{\sqrt{2}}\left(\underset{\sim}{\psi}_\alpha + \underset{\sim}{\bar{\psi}}_{\dot\alpha} \right) \qquad \underset{\sim}{\tilde{\eta}}_\alpha = -\frac{i}{\sqrt{2}}\left(\underset{\sim}{\psi}_\alpha - \underset{\sim}{\bar{\psi}}_{\dot\alpha} \right) . \tag{60}$$

The 4D gamma matrices in this basis are

$$\gamma^0_{\text{re}} = \begin{pmatrix} 0 & -\tau^2 \\ -\tau^2 & 0 \end{pmatrix} , \qquad \gamma^1_{\text{re}} = \begin{pmatrix} 0 & i\tau^3 \\ i\tau^3 & 0 \end{pmatrix} ,$$

$$\gamma^2_{\text{re}} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix} , \qquad \gamma^3_{\text{re}} = \begin{pmatrix} 0 & -i\tau^1 \\ -i\tau^1 & 0 \end{pmatrix} . \tag{61}$$

Now the dimensional reduction from 4D to 3D is straightforward, one has to decouple the second dimension, $(x_0, x_1, x_2, x_3) \to (x_0, x_1, x_3) \equiv (y_0, y_1, y_2)$. The four real 2-spinors $\underset{\sim}{\chi}_\alpha$, $\underset{\sim}{\tilde{\chi}}_\alpha$ and $\underset{\sim}{\eta}_\alpha$, $\underset{\sim}{\tilde{\eta}}_\alpha$ become Majorana spinors in 3D and the three gamma matrices satisfying the Clifford algebra in 3D can be read off from (61): $\gamma^0 = \tau^2, \gamma^1 = -i\tau^3, \gamma^2 = i\tau^1$. Note that the 3D Majorana condition $\psi^T C = \psi^\dagger \gamma_0$ is now the condition that the spinor is real since the charge conjugation matrix is $C = \tau^2$. We have defined the Dirac conjugate in 3D as $\bar\psi = \psi^\dagger \gamma_0$. For a Majorana spinor this means $\underset{\sim}{\bar{\chi}}^\alpha = i\underset{\sim}{\chi}^\alpha$.

By renaming the gamma matrices in 4D we can always choose the third and not the second dimension to decouple. This will always be assumed, see the footnote on page 4.

Finally we define bosonic fields $\underset{\sim}{\phi}_{1,2,3}$ and $\underset{\sim}{v}_\mu$ in 3D as follows,

$$\underset{\sim}{v}_m = \begin{cases} \underset{\sim}{v}_\mu , & m = 0, 1, 2 \\ \underset{\sim}{\phi}_3 , & m = 3 \end{cases} , \qquad \underset{\sim}{A} = \frac{\underset{\sim}{\phi}_1 + i\underset{\sim}{\phi}_2}{\sqrt{2}} , \tag{62}$$

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and the 4D action (56) becomes

$$
S_{3D} = \frac{2\pi}{e^2} \int d^3x\ \text{Tr}\{ -\tfrac{1}{2}\underset{\sim}{v}_{\mu\nu}\underset{\sim}{v}^{\mu\nu} + \mathcal{D}_\mu\underset{\sim}{\phi_i}\mathcal{D}^\mu\underset{\sim}{\phi_i} - \underset{\sim}{\chi}\hat{\mathcal{D}}\underset{\sim}{\chi} - \underset{\sim}{\tilde\chi}\hat{\mathcal{D}}\underset{\sim}{\tilde\chi} - \underset{\sim}{\eta}\hat{\mathcal{D}}\underset{\sim}{\eta} - \underset{\sim}{\tilde\eta}\hat{\mathcal{D}}\underset{\sim}{\tilde\eta}
$$
$$
+2\underset{\sim}{\phi_3}([\underset{\sim}{\chi},\underset{\sim}{\tilde\chi}] + [\underset{\sim}{\eta},\underset{\sim}{\tilde\eta}]) + 2\underset{\sim}{\phi_2}([\underset{\sim}{\eta},\underset{\sim}{\chi}] + [\underset{\sim}{\tilde\chi},\underset{\sim}{\tilde\eta}]) + 2\underset{\sim}{\phi_1}([\underset{\sim}{\chi},\underset{\sim}{\tilde\eta}] + [\underset{\sim}{\tilde\chi},\underset{\sim}{\eta}])
$$
$$
+([\underset{\sim}{\phi_1},\underset{\sim}{\phi_2}]^2 + [\underset{\sim}{\phi_2},\underset{\sim}{\phi_3}]^2 + [\underset{\sim}{\phi_3},\underset{\sim}{\phi_1}]^2)\} \ . \tag{63}
$$

Here $\hat{\mathcal{D}}_\alpha{}^\beta = \mathcal{D}_\mu (\gamma^\mu)_\alpha{}^\beta$ with $\mathcal{D}^\mu = \partial^\mu - i[\underset{\sim}{v}^\mu,\ ]$ and $\gamma^{0,1,2}$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

This action can be written in a manifestly $SO(4)_{\mathcal{R}}$ invariant form. First, define $\underset{\sim}{\chi}^A = (\underset{\sim}{\chi}, \underset{\sim}{\tilde\chi}, \underset{\sim}{\eta}, -\underset{\sim}{\tilde\eta})$, where $A = 1, ..., 4$ is the $SO(4)_{\mathcal{R}}$ index. Second, introduce the self-dual and anti-self-dual 't Hooft η^i-matrices [18]

$$
\eta^i_{AB} = \begin{cases} \epsilon_{iAB} & A, B = 1, 2, 3 \\ -\delta_{Bi} & A = 4 \\ \delta_{Ai} & B = 4 \end{cases} \qquad \bar\eta^i_{AB} = \begin{cases} \epsilon_{iAB} & A, B = 1, 2, 3 \\ \delta_{Bi} & A = 4 \\ -\delta_{Ai} & B = 4 \end{cases} . \tag{64}
$$

With this definition, η is self-dual and $\bar\eta$ anti-self-dual with respect to $\epsilon^{1234} = +1$. Moreover, they form two sets of commuting $su(2)$ algebras, as $[\eta^i, \bar\eta^j] = 0$. In this notation the action (63) takes the form,

$$
S_{3D} = \frac{2\pi}{e^2} \int d^3x\ \text{Tr}\{ -\tfrac{1}{2}\underset{\sim}{v}_{\mu\nu}\underset{\sim}{v}^{\mu\nu} + \mathcal{D}_\mu\underset{\sim}{\phi_i}\mathcal{D}^\mu\underset{\sim}{\phi_i} - \underset{\sim}{\chi}^A\hat{\mathcal{D}}\underset{\sim}{\chi}^A
$$
$$
+ \sum_{i<j}[\underset{\sim}{\phi_i},\underset{\sim}{\phi_j}]^2 + 2\underset{\sim}{\phi_i}\bar\eta^i_{AB}\underset{\sim}{\chi}^A\underset{\sim}{\chi}^B\} \ . \tag{65}
$$

The Lagrangian has a global $SO(4)_{\mathcal{R}} \simeq SU(2)_N \times SU(2)_{\mathcal{R}}$ symmetry. The $SU(2)_R$ leaves the three scalar fields invariant, and acts on the fermions. In terms of the four-dimensional Weyl fermions $(\underset{\sim}{\lambda}\ \underset{\sim}{\psi})$ forms a doublet under $SU(2)_R$. Rewriting this in terms of the Majorana's in 3D, one finds

$$
\underset{\sim}{\chi}^A \mapsto \exp\left(\tfrac{1}{2}\alpha^k \eta^k_{AB}\right) \underset{\sim}{\chi}^B \ . \tag{66}
$$

The $SU(2)_N$ group is the remnant of a 3-dimensional rotation group in the $N = 1$, $D = 6$ theory. The scalar fields transform as

$$
\underset{\sim}{\phi}^i \mapsto \exp\left(\beta^k R^k\right)^i{}_j \underset{\sim}{\phi}^j \ , \tag{67}
$$

where $(R^k)^i{}_j = \epsilon^{kij}, \epsilon^{123} = 1$ are the standard rotation group generators. On the fermions, $SU(2)_N$ acts as

$$
\underset{\sim}{\chi}^A \mapsto \exp\left(\tfrac{1}{2}\beta^k \bar\eta^k_{AB}\right) \underset{\sim}{\chi}^B \tag{68}
$$

These transformations leave the microscopic theory invariant, as one can check explicitly.

The supersymmetry transformation rules for the scalar fields are given by

$$\delta\underset{\sim}{\phi}^i = \bar{\epsilon}^{A\alpha}\bar{\eta}^i_{AB}\underset{\sim}{\chi}^B_\alpha \tag{69}$$

They can be obtained from a dimensional reduction of the SUSY rules in 4D. One can check that $SU(2)_N$ and $SU(2)_{\mathcal{R}}$ are invariances of this transformation. Finally we can relate the Majorana fermions $\underset{\sim}{\chi}^A$ to the (dimensionally-reduced) Weyl fermions $\underset{\sim}{\lambda}$ and $\underset{\sim}{\psi}$ by complexifying in the $SO(4)_{\mathcal{R}}$ index. We define a complex basis;

$$\underset{\sim}{\chi}^1_\alpha = \frac{1}{\sqrt{2}}(\underset{\sim}{\chi}_\alpha - i\underset{\sim}{\tilde{\chi}}_\alpha) = \epsilon_{\alpha\dot\beta}\underset{\sim}{\bar\lambda}^{\dot\beta} \quad ; \quad \underset{\sim}{\chi}^{\bar1}_\alpha = \frac{1}{\sqrt{2}}(\underset{\sim}{\chi}_\alpha + i\underset{\sim}{\tilde{\chi}}_\alpha) = \underset{\sim}{\lambda}_\alpha$$

$$\underset{\sim}{\chi}^2_\alpha = \frac{1}{\sqrt{2}}(\underset{\sim}{\eta}_\alpha - i\underset{\sim}{\tilde\eta}_\alpha) = \epsilon_{\alpha\dot\beta}\underset{\sim}{\bar\psi}^{\dot\beta} \quad ; \quad \underset{\sim}{\chi}^{\bar2}_\alpha = \frac{1}{\sqrt{2}}(\underset{\sim}{\eta}_\alpha + i\underset{\sim}{\tilde\eta}_\alpha) = \underset{\sim}{\psi}_\alpha \tag{70}$$

Appendix B: Wilsonian Effective Action at 1-loop

In this Appendix we extract the 1-loop effective $U(1)$ Wilsonian action from the microscopic $SU(2)$ Lagrangian.

In order to calculate the Wilsonian effective action, it is customary to split up all fields (except the ghosts) into a background part and a fluctuating part, e.g.,

$$\underset{\sim}{\phi}_i = \underset{\sim}{\phi}_{i\,\text{bkgd}} + \delta\underset{\sim}{\phi}_i \,, \quad \text{etc.} \tag{71}$$

The background fields should be thought of as comprising large-wavelength modes which will justify a gradient expansion; in particular the background scalar field includes the VEV **v** which we can choose to point in the third direction in both colour and $SU(2)_N$ space:

$$\underset{\sim}{\phi}_{i\,\text{bkgd}} = \sqrt{2}\text{v}\delta_{i3}\tau^3/2 + \delta\underset{\sim}{\phi}_{i\,\text{bkgd}} \tag{72}$$

By convention, under an $SU(2)$ gauge transformation, the variation of the total field is entirely assigned to the fluctuating parts while the background parts are held fixed, thus:

$$
\begin{aligned}
\delta_{\theta(x)}\,\delta v^a_\mu &= \epsilon^{abc}\theta^b(x)(v^c_{\mu\,\text{bkgd}} + \delta v^c_\mu) -, \partial_\mu\theta^a(x) \\
&= \epsilon^{abc}\theta^b(x)\delta v^c_\mu - (D_\mu\underset{\sim}{\theta})^a \\
\delta_{\theta(x)}\,\delta\phi^a_k &= \epsilon^{abc}\theta^b(x)\left(\phi^c_{k\,\text{bkgd}} + \delta\phi^c_k\right) \\
\delta_{\theta(x)}\,v^a_{\mu\,\text{bkgd}} &= \delta_{\theta(x)}\,\phi^a_{k\,\text{bkgd}} = 0
\end{aligned}
\tag{73}
$$

and likewise for the fermions. Here $D^\mu = \partial^\mu - i[v^\mu_{\text{bkgd}}, \ ]$ is the background covariant derivative. It is convenient to specialize to the one-parameter family of "background R_ξ gauges," linear in the fluctuating fields, defined by the gauge-fixing term

$$\mathcal{L}_{\text{g.f.t.}} = -\frac{2\pi}{e^2}\frac{1}{2\xi}\sum_{a=1,2,3}(f^a_\xi)^2 \tag{74}$$

where

$$f^a_\xi \tau^a/2 = D_\mu\,\delta\underset{\sim}{v}^\mu + i\xi\sum_{k=1,2,3}[\underset{\sim}{\phi}_{k\,\text{bkgd}},\,\delta\underset{\sim}{\phi}_k] \tag{75}$$

As in the usual R_ξ gauges, for any ξ, $\mathcal{L}_{\text{g.f.t.}}$ is constructed to cancel out the troublesome quadratic cross term $-\sqrt{2}\text{v}(\delta\phi^1_3\partial^\mu\delta v^2_\mu - \delta\phi^2_3\partial^\mu\delta v^1_\mu)$ induced in the $SU(2)$ Lagrangian by the VEV (72) (with an integration by parts). The corresponding action for the triplet of complex ghosts c^i follows straightforwardly from Eq. (73):

$$\begin{aligned}
\mathcal{L}_{\text{ghost}} &= \frac{2\pi}{e^2}\,c^{i\dagger}\frac{\delta f^i_\xi}{\delta\theta^j}\,c^j \\
&= \frac{2\pi}{e^2}\,\vec{c}^{\,\dagger}\cdot[\,-D^2\,-\,(D^\mu\circ\delta\vec{v}_\mu\times\ )\,+\,\xi\vec{\phi}_{k\,\text{bkgd}}\times((\vec{\phi}_{k\,\text{bkgd}}+\delta\vec{\phi}_k)\times\ )]\vec{c}
\end{aligned} \tag{76}$$

using an obvious 3-vector notation for the adjoint fields, e.g., $\vec{v}_\mu = (v^1_\mu, v^2_\mu, v^3_\mu)$.

We will calculate (part of) the one-loop effective action for the massless quanta $\{\phi^3_{1\,\text{bkgd}}, \phi^3_{2\,\text{bkgd}}, \delta\phi^3_{3\,\text{bkgd}}, v^3_{\mu\,\text{bkgd}}, \chi^3_{\text{bkgd}}, \tilde{\chi}^3_{\text{bkgd}}, \eta^3_{\text{bkgd}}, \tilde{\eta}^3_{\text{bkgd}}\}$. Here $\phi^3_{1\,\text{bkgd}}$ and $\phi^3_{2\,\text{bkgd}}$ are the Goldstone bosons associated with the spontaneous breaking of the $SU(2)_N$ symmetry down to $U(1)$ under which they transform as a doublet, whereas $\delta\phi^3_{3\,\text{bkgd}}$ is the singlet dilaton (cf Eq (72)). Since these scalars have different charges under the unbroken $U(1)$, one generically expects their effective couplings to renormalize differently at the loop level, hence:

$$\begin{aligned}
\mathcal{L}_{\text{eff}} = \frac{2\pi}{e^2}\frac{1}{2}\left(1 - \frac{C_1 e^2}{M_W} + \mathcal{O}(e^4)\right)\sum_{i=1,2}\left(\partial_\mu\phi^3_{i\,\text{bkgd}}\right)^2 \\
+\frac{2\pi}{e^2}\frac{1}{2}\left(1 - \frac{C_3 e^2}{M_W} + \mathcal{O}(e^4)\right)(\partial_\mu\delta\phi^3_{3\,\text{bkgd}})^2\ +\ \cdots
\end{aligned} \tag{77}$$

where the one-loop numerical constants C_1 and C_3 are not necessarily equal, and we omit spinor and gauge fields as well as higher derivative terms. However, since our choice of gauge fixing, (75), respects both scale and $SU(2)_N$ invariance, Eq. (77) must come from an $O(3)$-invariant expression containing no explicit factors of M_W nor of

$\delta\phi^3_{3\,\text{bkgd}}$; only the total background fields $\phi^3_{i\,\text{bkgd}}$ can appear. In particular, explicit factors of M_W are replaced by

$$M_W \;\to\; \rho\,, \qquad \rho = [\;\sum_{i=1,2,3} (\phi^3_{i\,\text{bkgd}})^2]^{1/2} \tag{78}$$

Thus (77) must come from

$$\begin{aligned}
\mathcal{L}^{\text{1-loop}}_{\text{eff}} \;=\;& \frac{2\pi}{e^2}\frac{1}{2}\left(1 - \frac{C_1 e^2}{\rho}\right)\sum_{i=1,2,3}(\partial_\mu\phi^3_{i\,\text{bkgd}})^2\; +\\
& \frac{2\pi}{e^2}\frac{(C_1 - C_3)e^2}{2\rho^3}\left(\sum_{i=1,2,3}\phi_{i\,\text{bkgd}}\partial_\mu\phi_{i\,\text{bkgd}}\right)^2 \;+\;\cdots
\end{aligned} \tag{79}$$

in terms of the two possible $SU(2)$ invariants at the 2-derivative level. The difference between (77) and (79) lies in the 3-point and higher-point functions when one expands about the VEV.

Below we shall explicitly calculate

$$C_1 \;=\; C_3 \;=\; \frac{1}{4\pi^2} \tag{80}$$

which constitutes our 1-loop prediction.

Calculation of the effective action. Extracting the 1-loop Wilsonian effective action is an intricate calculation in a generic "background R_ξ" gauge, but for the specific value $\xi = 1$ we can exploit the following observation. Let us extend the gauge field $\underset{\sim}{v}_\mu$ to a 6-dimensional vector field incorporating the three scalars $\underset{\sim}{\phi}_i$:

$$\underset{\sim}{v}^{\text{6D}}_\mu \;=\; (\,\underset{\sim}{v}_0\,,\,\underset{\sim}{v}_1\,,\,\underset{\sim}{v}_2\,,\,\underset{\sim}{v}_3 \equiv \underset{\sim}{\phi}_1\,,\,\underset{\sim}{v}_4 \equiv \underset{\sim}{\phi}_2\,,\,\underset{\sim}{v}_5 \equiv \underset{\sim}{\phi}_3\,) \tag{81}$$

With the convention that all relevant field configurations are constant in the final three spatial directions,

$$\frac{\partial}{\partial x^3} \;=\; \frac{\partial}{\partial x^4} \;=\; \frac{\partial}{\partial x^5} \;\equiv\; 0 \tag{82}$$

then the 3-dimensional $N = 4$ $SU(2)$ Lagrangian is known to be simply that of 6-dimensional $N = 1$ SYM theory. Furthermore, for the specific choice $\xi = 1$, the gauge-fixing conditions (75) may be rewritten compactly as

$$f^a_{\xi=1}\tau^a/2 \;=\; D^{\text{6D}}_\mu\,\delta\underset{\sim}{v}^{\text{6D}\,\mu} \tag{83}$$

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where D_μ^{6D} is the 6-dimensional background covariant derivative defined subject to (82), and the metric signature is $(+,-,-,-,-,-)$. Likewise the ghost action (76) simplifies to

$$\mathcal{L}_{\mathrm{ghost}} = \frac{2\pi}{e^2}\, \vec{c}\,^\dagger \cdot \left[\, - (D^{\mathrm{6D}})^2 \; - \; D^{\mathrm{6D}\,\mu} \circ \delta\vec{v}_\mu^{\,\mathrm{6D}} \times \; \right]\vec{c} \tag{84}$$

With these simplifications the problem is now reduced, quite literally, to a textbook exercise (see Sec. 16.6 of [28], which we follow closely). At the one-loop level we focus solely on terms quadratic in the fluctuating fields, dropping for example the second term in (84). The resulting Gaussian functional determinants may be exponentiated in the usual way, and lead to a contribution

$$\sum_{s=0,\frac{1}{2},1} \eta_s\, \mathrm{Tr}\log\Delta_s \tag{85}$$

to the effective action. Here s indexes the spin, and the weight factor η_s takes the values $\eta_1 = -\frac{1}{2}$ for the 6D vector, $\eta_{1/2} = +\frac{1}{2}$ for the single 6D Dirac spinor formed from the four 3D Majorana fermions[12], and $\eta_0 = +1$ for the complex ghosts. Δ_s is the Gaussian quadratic form sandwiched between the spin-s fluctuating fields in the adjoint representation of $SU(2)_{\mathrm{color}}$. As shown in [28] it has the universal form:

$$\Delta_s = -\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta_s^{(\mathcal{J})} \tag{86}$$

where in the present model

$$\Delta^{(1)} = i\{\partial^\mu,\, v_{\mathrm{bkgd}\,\mu}^{\mathrm{6D}\,a} t^a\}\,, \qquad \Delta^{(2)} = v_{\mathrm{bkgd}\,\mu}^{\mathrm{6D}\,a} t^a\, v_{\mathrm{bkgd}}^{\mathrm{6D}\,\mu\,b} t^b\,, \qquad \Delta_s^{(\mathcal{J})} = v_{\mathrm{bkgd}\,\mu\nu}^{\mathrm{6D}\,a} t^a\, \mathcal{J}_s^{\mu\nu} \tag{87}$$

Here t^a is the color generator in the adjoint representation, $(t^a)_{bc} = -i\epsilon_{abc}$, and $\mathcal{J}_s^{\mu\nu}$ is the generator of 6-dimensional Lorentz transformations in the spin-s representation:

$$(\mathcal{J}_1^{\mu\nu})_{\lambda\sigma} = i(\delta_\lambda^\mu \delta_\sigma^\nu - \delta_\sigma^\mu \delta_\lambda^\nu)\,, \quad \mathcal{J}_{1/2}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,, \quad \mathcal{J}_0^{\mu\nu} = 0\,. \tag{88}$$

We now depart from [28] in two obvious ways, in order to incorporate spontaneous symmetry breaking. First, we require the background fields to live purely in the third direction in color space,

$$\underset{\sim}{v}_{\mathrm{bkgd}\,\mu}^{\mathrm{6D}} = v_{\mathrm{bkgd}\,\mu}^{\mathrm{6D}\,3}\,\tau^3/2 \tag{89}$$

Second, rather than truncating the expansion of the logarithm at second order in the total background field, we instead expand about the VEV, rewriting Eq. (72) as

$$\underset{\sim}{v}_{\mathrm{bkgd}\,\mu} = \sqrt{2}\mathrm{v}\,\delta_{\mu 5}\,\tau^3/2 + \delta\underset{\sim}{v}_{\mathrm{bkgd}\,\mu} \tag{90}$$

---

[12]NB: We take $\eta_{1/2} = +1/2$ rather than $\eta_{1/2} = +1$ for the spinor because in the definition of $\Delta_{1/2}$ we will square the $\displaystyle{\not{D}}$ operator following [28].

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and keep terms to second order in the deviation field $\delta \underset{\sim}{v}_{\text{bkgd}\,\mu}$ but to all orders in the VEV itself. In terms of the deviation field, Eq. (86) becomes

$$
\begin{aligned}
\Delta^{(1)} &= i\{\partial^\mu,\, \delta v^{6\text{D}\,3}_{\text{bkgd}\,\mu} t^3\}\,, \\
\Delta^{(2)} &= \tilde{\Delta}^{(2)} - 2M_W \delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3 - M_W^2 t^3 t^3\,, \\
\Delta^{(\mathcal{J})}_s &= \delta v^{6\text{D}\,3}_{\text{bkgd}\,\mu\nu} t^3 \mathcal{J}^{\mu\nu}_s
\end{aligned}
\tag{91}
$$

Here

$$
\tilde{\Delta}^{(2)} = \delta v^{6\text{D}\,3}_{\text{bkgd}\,\mu} t^3\, \delta v^{6\text{D}\,\mu\,3}_{\text{bkgd}} t^3
\tag{92}
$$

Also, thanks to (89), $\delta v^{6\text{D}\,3}_{\text{bkgd}\,\mu\nu}$ is now the abelian field strength $\partial_{[\mu} \delta v^{6\text{D}\,3}_{\text{bkgd}\,\nu]}$ subject as always to the constancy conditions (82). Taylor expanding the logarithm then gives

$$
\sum_{s=0,\frac{1}{2},1} \eta_s \,\text{Tr} \log \Big(-(\partial^2 + M_W^2 t^3 t^3) + \Delta^{(1)} + \tilde{\Delta}^{(2)} + \Delta^{(\mathcal{J})}_s - 2M_W \delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3 \Big)
$$

$$
= \text{const.} + \sum_{s=0,\frac{1}{2},1} \eta_s \,\text{Tr}\Big(\mathcal{G}\cdot(\Delta^{(1)} + \tilde{\Delta}^{(2)} + \Delta^{(\mathcal{J})}_s - 2M_W \delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3)
$$

$$
-\mathcal{G}\cdot(\Delta^{(1)} + \Delta^{(\mathcal{J})}_s - 2M_W \delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3)\cdot\mathcal{G}\cdot(\Delta^{(1)} + \Delta^{(\mathcal{J})}_s - 2M_W \delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3)\Big)
$$

$$
\tag{93}
$$

where corrections to the RHS are $\mathcal{O}(\delta v^3_{\text{bkgd}})$ Here $\mathcal{G}$ is the diagonal colour-space matrix,

$$
\mathcal{G} = \text{diag}\Big(\frac{1}{p^2 - M_W^2}\,,\, \frac{1}{p^2 - M_W^2}\,,\, \frac{1}{p^2} \Big)
\tag{94}
$$

as follows from $(t^3 t^3)_{ab} = \delta_{ab} - \delta_{a3}\delta_{b3}$. Apart from the masses in the propagators (94), the chief effect of the spontaneous symmetry breaking in (93) are the terms linear and quadratic in M_W. Let us dispose of these first. Terms linear in M_W are

$$
\sum_{s=0,\frac{1}{2},1} \eta_s M_W \,\text{Tr}\Big(-2\mathcal{G}\cdot\delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3 + 2\mathcal{G}\cdot\delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3\cdot\mathcal{G}\cdot\Delta^{(1)}
$$

$$
+ 2\mathcal{G}\cdot\delta v^{6\text{D}\,3}_{\text{bkgd}\,5} t^3 t^3\cdot\mathcal{G}\cdot\Delta^{(\mathcal{J})}_s \Big)
\tag{95}
$$

These are would-be tadpole contributions to the effective action. Respectively, the second and third terms here vanish because $\text{tr}_{\text{color}}\, t^3 t^3 t^3 = 0$ and $\text{tr}_{\text{rep}}\, \mathcal{J}^{\mu\nu}_s = 0$. The first term is a nonvanishing Feynman diagram whose only spin dependence comes from the trace over the Lorentz representation, giving a relative weight of $6, 4, 1$, respectively, for the vector, spinor and ghost loop. From the above values of η_s one sees that

$$
6\eta_1 + 4\eta_{1/2} + \eta_0 = 0
\tag{96}
$$

so that the tadpoles do cancel among the three types of loops (a manifestation of supersymmetry). And the same arithmetic kills the cross-term in (93) proportional

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to M_W^2. (Since these mass-dependent terms were the only potential source of difference between $v_{\text{bkgd}\,5}^{\text{6D}\,3} \equiv \phi_{\text{bkgd}3}^3$ and the other two massless scalars, their vanishing implies that $C_1 = C_3$ in the notation of Eq. (77).) In fact these three arguments kill almost all of the M_W^0 terms in (93) as well, leaving only

$$\sum_{s=0,\frac{1}{2},1} \eta_s \operatorname{Tr}\big(-\tfrac{1}{2}\mathcal{G} \cdot \Delta_s^{(\mathcal{J})} \cdot \mathcal{G} \cdot \Delta_s^{(\mathcal{J})} \big) = -\tfrac{1}{2} \int \frac{d^3 k}{(2\pi)^3} v_{\mu\nu}^3(k) v_{\rho\sigma}^3(-k)$$

$$\times \Big(\int \frac{d^3 p}{(2\pi)^3} 2 \cdot \frac{1}{p^2 - M_W^2} \cdot \frac{1}{(p+k)^2 - M_W^2} \Big) \sum_{s=0,\frac{1}{2},1} \eta_s \operatorname{tr} \mathcal{J}_s^{\mu\nu} \mathcal{J}_s^{\rho\sigma} \tag{97}$$

The p integration yields $i/(4\pi M_W) + \mathcal{O}(k^2)$ (the factor of 2 inside the integrand counts the two massive colors $a = 1, 2$). The spin sum simplifies using [28]

$$\operatorname{tr} \mathcal{J}_s^{\mu\nu} \mathcal{J}_s^{\rho\sigma} = (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) C(s) \tag{98}$$

where an explicit calculation using (88) gives $C(1) = 2$, $C(\frac{1}{2}) = 1$, and $C(0) = 0$. So the net contribution is

$$\frac{i}{8\pi M_W} \int \frac{d^3 k}{(2\pi)^3} v_{\text{bkgd}\,\mu\nu}^3(k) v_{\text{bkgd}}^{3\,\mu\nu}(-k) \tag{99}$$

where we drop terms of $\mathcal{O}(k^4)$. A comparison with the tree-level action $-\frac{2\pi}{e^2} \frac{i}{4} \int (v_{\mu\nu}^3)^2$ then implies

$$\frac{2\pi}{e^2} \rightarrow \frac{2\pi}{e^2} - \frac{1}{2\pi M_W} + \mathcal{O}(e^2) \tag{100}$$

which is our one-loop prediction $C_1 = C_3 = 1/4\pi^2$.

Appendix C: Three-dimensional Instantons

In this Appendix we set up and give the details of the 1-instanton calculus in three spacetime dimensions.

First we consider the bosonic part of the three-dimensional Euclidean action (63),

$$S_{\text{B}} = \frac{2\pi}{e^2} \int d^3 x \operatorname{Tr}\{ \tfrac{1}{2} v_{\mu\nu} v^{\mu\nu} + \mathcal{D}_\mu \phi_i \mathcal{D}^\mu \phi_i + ([\phi_1, \phi_2]^2 + [\phi_2, \phi_3]^2 + [\phi_3, \phi_1]^2) \} \ . \tag{101}$$

To find the instanton configuration we employ Bogomol'nyi's lower bound approach [12],

$$S_{\text{B}} \geq \frac{2\pi}{e^2} \int d^3 x \tfrac{1}{2} \Big(B_\mu^a B_\mu^a + \mathcal{D}_\mu \phi_3^a \mathcal{D}_\mu \phi_3^a \Big) \tag{102}$$

$$= \frac{2\pi}{e^2} \int d^3 x \ \tfrac{1}{2} \Big((B_\mu^a + \mathcal{D}_\mu \phi_3^a)^2 + (-2 B_\mu^a \mathcal{D}_\mu \phi_3^a) \Big) \geq \frac{2\pi}{e^2} \int d^3 x (-B_\mu^a \mathcal{D}_\mu \phi_3^a) \ ,$$

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where $B_\mu^a = \frac{1}{2}\epsilon_{\mu\nu\rho}v_{\nu\rho}^a$. The 3D bosonic instanton, being a minimum of S_B, saturates Bogomol'nyi bound (103),

$$\underset{\sim}{\phi}{}^{\text{cl}}_1 = 0 \ , \qquad \underset{\sim}{\phi}{}^{\text{cl}}_2 = 0 \ , \tag{103}$$

$$B_\mu^{\text{cl}\ a} = \mathcal{D}^{\text{cl}}{}_\mu \phi_3^{\text{cl}\ a} \ , \tag{104}$$

where (103) ensures the vanishing of the commutator terms in (101) and the remaining components of the bosonic instanton satisfy Bogomol'nyi equation (104) and are given by [13],

$$
\begin{aligned}
\phi_3^{\text{cl}\ a} &= \left(M_W |x| \coth M_W |x| - 1 \right) \frac{x^a}{x^2} \ , \\
v_\mu^{\text{cl}\ a} &= \left(1 - \frac{M_W |x|}{\sinh M_W |x|} \right) \epsilon_{a\mu\nu} \frac{x^\nu}{x^2} \ ,
\end{aligned}
\tag{105}
$$

with the boundary conditions as $|x| \to \infty$,

$$\phi_3^{\text{cl}\ a} \to \frac{x^a}{x} M_W \ , \qquad B_\mu^{\text{cl}\ a} \to -\frac{x^a x^\mu}{x^4} \ . \tag{106}$$

The instanton action follows from (103), (106),

$$S_{\text{cl}} = \frac{2\pi}{e^2} \int d^3x\, \partial_\mu (-B_\mu^{\text{cl}\ a} \phi_3^{\text{cl}\ a}) = \frac{2\pi}{e^2}\, 4\pi M_W \tag{107}$$

From the above formulae it is obvious that the bosonic components of the 3D instanton are those of the BPS (anti-)monopole in the corresponding four-dimensional theory in the $\underset{\sim}{v}_0 = 0$ gauge.

The fermi-field components of the instanton can be determined by infinitesimal supersymmetry transformations of the bosonic components (105) and will be discussed later.

Isolated one-instanton contribution to the functional integral is of the generic form [18],

$$\mathcal{Z}_1 = \int d\mu_B \int d\mu_F\ R\ \exp[-S_{\text{cl}}] \ , \tag{108}$$

where $d\mu_B$ and $d\mu_F$ are the measures of integrations over collective coordinates of bosonic and fermionic zero modes, R is the ratio of functional determinants over non-zero eigenvalues of the operators of quadratic fluctuations in the instanton background, Eq. (23). In this Appendix we determine $d\mu_B$ and $d\mu_F$.

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It will prove to be particularly convenient to arrange the 3D instanton analysis in a four-dimensional way. The four-vector field $\underset{\sim}{v}_m$ of the dimensionally reduced 4D theory (translationally invariant along x_3) is given by (62). The Bogomol'nyi equations (104) in this language are equivalent to the 4D self-duality equations [14],

$$v_{mn}^{\text{cl } a} = {}^* v_{mn}^{\text{cl } a} \tag{109}$$

and the instanton action (107) is

$$S_{\text{cl}} = \frac{2\pi}{e^2} \int d^3x \ \tfrac{1}{4} v_{mn}^{\text{cl } a} v_{mn}^{\text{cl } a} = \frac{8\pi^2}{e^2} M_W \ . \tag{110}$$

The 3D instanton or, equivalently, the BPS monopole, is in this way similar to the 4D Yang-Mills instanton [26]. The functional integration over the (x_3-independent) fluctuations δv_m^a around the instanton will be performed in the (four-dimensional) covariant background gauge,

$$\mathcal{D}_m^{\text{cl}} \delta v_m^a = 0 \ , \tag{111}$$

and the bosonic instanton zero modes will be of the general form [27],

$$Z_m^{a \ [k]} = \frac{\partial v_m^{\text{cl } a}}{\partial \gamma^{[k]}} + \mathcal{D}_m^{\text{cl}} \Lambda^a \ , \tag{112}$$

where $\gamma^{[k]}$ are the zero modes collective coordinates – the three-translations, X_μ, and the $U(1)$ rotation, θ. The second term on the right hand side of (112) is necessary to keep Z_m in the background gauge (111). In general, these bosonic zero modes, Z_m, can be written in a more compact notation,

$$\mathcal{D}_{[m}^{\text{cl}} Z_{n]} = {}^* \mathcal{D}_{[m}^{\text{cl}} Z_{n]} \quad , \quad \mathcal{D}^{\text{cl } m} Z_m = 0 \ . \tag{113}$$

The measure $d\mu_B$ is now simply [27],

$$d\mu_B = \prod_{k=1}^{4} \frac{d\gamma^{[k]}}{\sqrt{2\pi}} \left(\det_{rs} \frac{2\pi}{e^2} \int d^3x \ Z_m^{a \ [r]} \ Z_m^{a \ [s]} \right)^{1/2} \ . \tag{114}$$

First, consider translational zero modes, cf. (112),

$$Z_m^{a \ [\nu]} = -\partial_\nu v_m^{\text{cl } a} + \mathcal{D}_m^{\text{cl}} v_\nu^{\text{cl } a} = v_{m\nu}^{\text{cl } a} \ . \tag{115}$$

Their overlap is

$$\begin{aligned}
\mathcal{O}_{\mu\nu} &= \frac{2\pi}{e^2} \int d^3x \ Z_m^{a \ [\mu]} \ Z_m^{a \ [\nu]} = \frac{2\pi}{e^2} \int d^3x \ v_{m\mu}^{\text{cl } a} \ v_{m\nu}^{\text{cl } a} \\
&= \frac{\delta_{\mu\nu}}{4} \frac{2\pi}{e^2} \int d^3x \ v_{mn}^{\text{cl } a} \ v_{mn}^{\text{cl } a} = \delta_{\mu\nu} \ S_{\text{cl}} \ .
\end{aligned} \tag{116}$$

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Now we consider the $U(1)$ orientation zero mode. To do this we need to, first, gauge rotate the instanton (105) into the unitary (singular) gauge, where

$$v_{0 \text{ sing}}^{\text{cl } a} = \phi_{3 \text{ sing}}^{\text{cl } a} \sim \delta^{a3} , \tag{117}$$

and, second, allow the further global gauge transformations consistent with (117), they obviously form a $U(1)$ subgroup of the $SU(2)$,

$$\tilde{V}_{m \text{ sing}}^{\text{cl}} = \exp[i\theta\tau^3/2] \, \tilde{v}_{m \text{ sing}}^{\text{cl}} \, \exp[-i\theta\tau^3/2] . \tag{118}$$

The $U(1)$ zero mode in the form (112) is

$$\tilde{Z}_m^{[3]} = \partial_\theta \tilde{V}_{m \text{ sing}}^{\text{cl}} + \frac{1}{M_W} \mathcal{D}_m^{\text{cl}} \left(\tilde{\phi}_{\text{sing}}^{\text{cl}} - M_W \tau^3/2 \right) = \frac{1}{M_W} \tilde{v}_{m3 \text{ sing}}^{\text{cl}} , \tag{119}$$

with the overlaps,

$$\mathcal{O}_{33} = \frac{2\pi}{e^2} \int d^3x \, Z_m^{a\,[3]} \, Z_m^{a\,[3]} = \frac{1}{M_W} \frac{2\pi}{e^2} \int d^3x \, v_{m3}^{\text{cl } a} \, v_{m3}^{\text{cl } a} = \frac{1}{M_W} S_{\text{cl}}$$

$$\mathcal{O}_{\mu 3} = \frac{2\pi}{e^2} \int d^3x \, Z_m^{a\,[\mu]} \, Z_m^{a\,[3]} = 0 . \tag{120}$$

Finally combining (114), (116) and (120) we obtain the desired expression (16) for $d\mu_B$,

$$\int d\mu_B = \int \frac{d^3X}{(2\pi)^{3/2}} \, S_{\text{cl}}^{3/2} \int_0^{2\pi} \frac{d\theta}{(2\pi)^{1/2}} \frac{S_{\text{cl}}^{1/2}}{M_W} . \tag{121}$$

Our next goal is $d\mu_F$. The 1-instanton solution of the $N = 4$ supersymmetric Yang-Mills in 3D has four fermion zero modes which can be determined by infinitesimal $N = 4$ supersymmetry transformations of the bosonic components (105). Equivalently they can be understood in terms of the Weyl spinors of the four-dimensional theory, (17),

$$\lambda_\alpha^{\text{cl}} = \tfrac{1}{2}\xi_\beta(\sigma^m\bar\sigma^n)_\alpha^{\ \beta} v_{mn}^{\text{cl}}$$
$$\psi_\alpha^{\text{cl}} = \tfrac{1}{2}\xi'_\beta(\sigma^m\bar\sigma^n)_\alpha^{\ \beta} v_{mn}^{\text{cl}} , \tag{122}$$

where the two-component Grassmann collective coordinates ξ_α and ξ'_α are the parameters of infinitesimal $N = 2$ supersymmetry transformations in (dimensionally reduced) 4D theory. Since $\alpha = 1, 2$ there are two λ's and two ψ's which is equivalent to four zero modes in terms of Majorana spinors in 3D. There are no anti-fermion zero modes due to the self-duality, (109). The fermion collective coordinates integration measure is in general,

$$\int d\mu_F = \int d^2\xi \, d^2\xi' \, (\mathcal{J}_\xi)^{-2} , \tag{123}$$

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where the fermion Jacobian is,

$$\mathcal{J}_\xi = \frac{2\pi}{e^2} \int d^3x \; d^2\xi \, \mathrm{Tr} \; \underset{\sim}{\lambda}^{\mathrm{cl}\; \alpha} \underset{\sim}{\lambda}^{\mathrm{cl}}_\alpha \; , \tag{124}$$

and it is understood[13] that $\int d^2\xi \, \xi^2 \equiv 1$. The right hand side of (124) is easily evaluated with the use of (122) and the sigma-matrix algebra, and gives $\mathcal{J}_\xi = 2S_{\mathrm{cl}}$. Comparing with (18) we obtain,

$$\int d\mu_F = \int d^2\xi \, d^2\xi' \, (2S_{\mathrm{cl}})^{-2} \; . \tag{125}$$

Finally we need the long distance asymptotics of the fermion zero modes (122). These can be readily obtained by, first, switching from σ^m matrices in (122) to $\underset{\mathrm{ch}}{\sigma}^m$ of Eq. (58), second, rotating into the real basis, (61), decoupling the x_2 direction and, finally, switching to γ^μ matrices in 3D. The result is then rotated into the singular gauge, (117), and the large distance limit $|x| \to \infty$ is considered

$$\lambda_\alpha^{\mathrm{LD}}(x) = 2\gamma_\alpha^{\mu\;\beta} \, \xi_\beta \, \frac{x_\mu}{x^3} \; , \quad \psi_\alpha^{\mathrm{LD}}(x) = 2\gamma_\alpha^{\mu\;\beta} \, \xi'_\beta \, \frac{x_\mu}{x^3} \; , \tag{126}$$

which confirms Eq. (32).

Appendix D: The Riemann Tensor

In this appendix we calculate the leading exponentially suppressed terms in the Riemann tensor for the metric

$$ds^2 = f^2(r)dr^2 + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2$$

with σ_i defined in (38) and the functions a, b and c satisfying (44). Following Appendix B of [23], we define the vierbein one-forms $\hat{\theta}^\alpha = \theta_i^\alpha dX_i$ with

$$\hat{\theta}^1 = a\sigma_1 \; , \quad \hat{\theta}^2 = b\sigma_2 \; , \quad \hat{\theta}^3 = c\sigma_3 \quad \hat{\theta}^4 = f dr \tag{127}$$

which, using (44) gives us the spin connection

$$\omega_r^1 = (a'/f)\sigma_1 \; , \; \omega_r^2 = (b'/f)\sigma_2 \; , \; \omega_r^3 = (c'/f)\sigma_3$$

$$\omega_2^1 = (1 + c'/f)\sigma_3 \; , \; \omega_3^2 = (1 + a'/f)\sigma_1 \; , \; \omega_1^3 = (1 + b'/f)\sigma_2 \tag{128}$$

---

[13]Note that in (124) we could have summed over the isospin $a = 1, 2, 3$ rather than trace over the $SU(2)$ matrices. This would have produced an extra factor of $1/2$ which then would require an alternative prescription for Grassmanian integration, $\int d^2\xi \, \xi^2 \equiv 2$ and the final answer for $\mathcal{J}_\xi$ would be the same.

The curvature 2-forms are now found to be

$$R^{12} = \hat{c}'dr \wedge \sigma_3 + [-\hat{c} + \hat{a} + \hat{b} + 2\hat{a}\hat{b}]\sigma_1 \wedge \sigma_2$$
$$R^{23} = \hat{a}'dr \wedge \sigma_1 + [-\hat{a} + \hat{b} + \hat{c} + 2\hat{b}\hat{c}]\sigma_2 \wedge \sigma_3$$
$$R^{31} = \hat{b}'dr \wedge \sigma_2 + [-\hat{b} + \hat{c} + \hat{a} + 2\hat{c}\hat{a}]\sigma_3 \wedge \sigma_1 \tag{129}$$

Where $\hat{a} = a'/f$, $\hat{b} = b'/f$ and $\hat{c} = c'/f$. The other components are determined by $R^{34} = R^{12}$ and cyclic, giving us a Riemann tensor self-dual with respect to the epsilon tensor $\epsilon^{1234} = +1$. Explicitly we have,

$$R_{\alpha\beta\gamma\delta} = \eta^a_{\alpha\beta} T_{ab} \eta^b_{\gamma\delta} \tag{130}$$

with $T = \text{diag}(-\hat{a}'/fa, -\hat{b}'/fb, -\hat{c}'/fc)$ where,

$$R_{ijkl} = \theta^\alpha_i \theta^\beta_j \theta^\gamma_k \theta^\delta_l R_{\alpha\beta\gamma\delta} \tag{131}$$

Calculating the Riemann tensor at the point $(0, 0, r, \sigma)$ in the standard coordinates, we then change to the complex basis (49) with coordinates

$$z_1 = \frac{1}{\sqrt{2}}(X - iY) \qquad\qquad z_2 = \frac{1}{\sqrt{2}}(Z - i\sigma) \tag{132}$$

Labelling the coordinates $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ by an index $P = 1, 2, 3, 4$, the Riemann tensor is conveniently given in terms of the following (4×4) matrices,

$$A = \begin{pmatrix} 0 & A_+ e^{+i\psi} & 0 & A_- e^{+i\psi} \\ -A_+ e^{+i\psi} & 0 & -A_- e^{-i\psi} & 0 \\ 0 & A_- e^{-i\psi} & 0 & A_+ e^{-i\psi} \\ -A_- e^{+i\psi} & 0 & -A_+ e^{-i\psi} & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -B_+ e^{+i\psi} & 0 & -B_- e^{+i\psi} \\ B_+ e^{+i\psi} & 0 & -B_- e^{-i\psi} & 0 \\ 0 & B_- e^{-i\psi} & 0 & B_+ e^{-i\psi} \\ B_- e^{+i\psi} & 0 & -B_+ e^{-i\psi} & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & \frac{ab}{r^2} & 0 \\ 0 & 0 & 0 & \frac{cf}{2} \\ -\frac{ab}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{cf}{2} & 0 & 0 \end{pmatrix} \tag{133}$$

where $A_\pm = af \pm \frac{1}{2}bc$ and $B_\pm = bf \pm \frac{1}{2}ac$. The final result is

$$R_{PQRS} = -\frac{\hat{a}'}{4far^2}A_{PQ}A_{RS} + \frac{\hat{b}'}{4fbr^2}B_{PQ}B_{RS} + \frac{\hat{c}'}{fc}C_{PQ}C_{RS} \tag{134}$$

The leading exponentially suppressed terms are proportional to $(a - b)$ whose asymptotic form is given in (48) as,

$$a - b \quad \simeq \quad -8qr^2 e^{-r} \tag{135}$$

Hence, to compare with the instanton calculation, it suffices to approximate all other factors by their leading weak coupling behaviour. From (41) we have,

$$
\begin{aligned}
a + b &\simeq 2r + O(1/r) \\
c &\simeq -2 + O(1/r) \\
f &\simeq -1 + O(1/r)
\end{aligned}
\tag{136}
$$

Expanding the exact expression (134) we find that, to the required order,

$$
\begin{aligned}
R_{1212} &= 8qre^{-r+i\sigma} \\
R_{\bar{1}\bar{2}\bar{1}\bar{2}} &= 8qre^{-r-i\sigma}
\end{aligned}
\tag{137}
$$

As discussed in Section 4, the remaining pure (anti)-holomorphic components are related to these by symmetries of the Riemann tensor. All components of mixed holomorphy are independent of σ.

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