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Multi-Instantons, Three-Dimensional Gauge Theory, and the Gauss-Bonnet-Chern Theorem

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Abstract

We calculate multi-instanton effects in a three-dimensional gauge theory with $N = 8$ supersymmetry and gauge group $SU(2)$. The k -instanton contribution to an eight-fermion correlator is found to be proportional to the Gauss-Bonnet-Chern integral of the Gaussian curvature over the centered moduli space of charge- k BPS monopoles, $\tilde{\mathcal{M}}_k$. For $k = 2$ the integral can be evaluated using the explicit metric on \mathcal{M}_2 found by Atiyah and Hitchin. In this case the integral is equal to the Euler character of the manifold. More generally the integral is the volume contribution to the index of the Euler operator on $\tilde{\mathcal{M}}_k$, which may differ from the Euler character by a boundary term. We conjecture that the boundary terms vanish and evaluate the multi-instanton contributions using recent results for the cohomology of $\tilde{\mathcal{M}}_k$. We comment briefly on the implications of our result for a recently proposed test of M(atrrix) theory.

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The important recent advances in understanding the low-energy dynamics of supersymmetric gauge theory have led to some interesting exact results in three spacetime dimensions (3D). In particular, exact metrics have been proposed for both Coulomb and Higgs branches of the $N = 4$ theories [1]-[7] while exact superpotentials have been determined for the $N = 2$ theories [8][9]. Surprisingly, these results can be seen as consequences of non-perturbative dualities in string theory, where the 3D SUSY field theories appear on the world volume of membranes. Very recently, an interesting application has also been proposed [10] for $N = 8$ SUSY gauge theory in three dimensions as a description of membrane scattering in M(atric) theory.

In all these cases, an important role is played by instanton effects in the three-dimensional gauge theory [2][11][10]. In fact, the exact results yield predictions for all instanton contributions to the low-energy theory. As in four-dimensions [12]-[15], first-principles semiclassical calculations of these effects yield independent quantitative tests of the proposed exact results [16] and therefore of the duality on which they rely. In recent work [11], we calculated a one-instanton contribution to the low-energy effective action in the three-dimensional $N = 4$ theory with gauge group $SU(2)$. This result verifies the conjecture of Seiberg and Witten [2] that the Coulomb branch of this theory is isometric to the Atiyah-Hitchin manifold.

In this paper we will focus on the $N = 8$ theory. Polchinski and Pouliot [10] have calculated a one-instanton contribution in this theory and compared it with a certain scattering amplitude for membranes in eleven-dimensional supergravity. In general, the k -instanton contributions in the $N = 8$ three-dimensional theory can be thought of as an approximation to the M(atric) theory scattering amplitude of two supermembranes with k units of the M-momentum transfer. The k -instanton result can then be compared to an independent calculation of the membrane scattering in eleven-dimensional supergravity. The authors of [10] found agreement between the result of an explicit one-instanton calculation and the corresponding supergravity scattering amplitude for a single unit of M-momentum transfer. This supports the conjectured exactness of the matrix model description of M-theory. In this paper we give the multi-instanton generalization of this result. As shown below, for all values of k , multi-instantons correctly reproduce the M-momentum dependence of the supergravity amplitude calculated in [10], up to perturbative corrections in the multi-instanton background which we have not calculated.

In the three-dimensional gauge theories considered here, the relevant instanton configurations are BPS monopoles. At weak-coupling, path-integration in the sector of topological charge k reduces to a finite-dimensional integral over the moduli-space of k BPS monopoles. In a supersymmetric theory, it is also necessary to integrate over Grassmann parameters corresponding to the fermionic zero-modes of the instanton. For the $N = 8$ theory, we will show that all but eight of these modes are lifted by a four-fermion term in the classical action of the supersymmetric instanton. This means that there are non-zero corrections to the eight-fermion vertex calculated in [10] from all numbers of instantons. Our main result is that the k -instanton contribution is given by the Gauss-Bonnet-Chern (GBC) integral of the Gaussian curvature on the centered moduli-space $\tilde{\mathcal{M}}_k$. This integral is the volume contribution to the index of the Euler operator on $\tilde{\mathcal{M}}_k$. For $k = 2$, the GBC integral has been evaluated by Gauntlett and Harvey [18], using

the explicit metric on the two-monopole moduli space obtained by Atiyah and Hitchin [19]. In this case the volume contribution is exactly equal to the Euler character of $\tilde{\mathcal{M}}_2$ which agrees with the earlier conclusion [17] that the boundary terms for the relevant index theorem for any Asymptotically Locally Flat (ALF) metric should vanish. We propose the extension of this equality for all k . The cohomology of the higher charge monopole moduli-spaces has recently been determined by Segal and Selby [20]. The Euler characters of these spaces then yield the multi-instanton contributions. It is notable that these contributions, which at first sight have a highly non-trivial dependence on the unknown hyper-Kähler metric on $\tilde{\mathcal{M}}_k$, in fact turn out to be topological invariants of these manifolds.

Theories with extended SUSY in three spacetime dimensions are straightforwardly obtained from theories with the same number of component supercharges in four-dimensions (4D) by dimensional reduction. In the following the integer N denotes the number of real two-component Majorana supercharges of the theory in question. Following [6], we will also define $\mathcal{N} = N/2$ which counts the corresponding number of Weyl supercharges for a 4D theory. In Ref. [11] we discussed the $N = 4$ theory in 3D which is the dimensional reduction of the minimal $\mathcal{N} = 2$ theory in 4D. The latter theory consists of an $\mathcal{N} = 2$ gauge multiplet which contains the (4D) gauge field v_m ($m = 0, 1, 2, 3$), a complex scalar \underline{A} and two species of Weyl fermion $\underline{\lambda}_\alpha$ and $\underline{\psi}_\alpha$ all in the adjoint representation of the gauge group.¹ In the following we will be interested in a three-dimensional theory with twice as many supercharges which can be constructed by dimensionally reducing the four-dimensional $\mathcal{N} = 4$ theory [21]. This four-dimensional theory is obtained by augmenting the gauge multiplet introduced above with an adjoint $\mathcal{N} = 2$ hypermultiplet which contains two complex scalars q and \tilde{q} and their Weyl fermion superpartners \underline{f}_α and $\tilde{\underline{f}}_\alpha$. The resulting supersymmetry algebra has an $SU(4)_{\mathcal{R}}$ group of automorphisms and it is convenient to rewrite the four species of Weyl fermions $\{\underline{\lambda}_\alpha, \underline{\psi}_\alpha, \underline{f}_\alpha, \tilde{\underline{f}}_\alpha\}$ as $\underline{\lambda}_\alpha^M$ where $M = 1, 2, 3, 4$.

In the following we will compactify one of the four spatial dimensions, say x_3 , on a circle of radius R . We will be interested in the three-dimensional quantum theory obtained by integrating only over field configurations which are independent of the compactified coordinate. In this case we may integrate over x_3 in the action to obtain,

$$\frac{1}{g^2} \int d^4x \rightarrow \frac{2\pi}{e^2} \int d^3x \quad (1)$$

where $e = g/\sqrt{R}$ defines the dimensionful 3D gauge coupling in terms of the dimensionless 4D counterpart g . The resulting $N = 8$ supersymmetric gauge theory in three-dimensions contains seven real scalar fields and a (3D) gauge field as well as eight real Majorana fermions. The Lagrangian has a $spin(7)$ \mathcal{R} -symmetry group which contains both the $SU(4)_{\mathcal{R}}$ symmetry of the 4D theory as well as a new symmetry group $SU(2)_N$ which appears in 3D [21]. In the following we will not work with the three-dimensional fields directly. In considering instanton effects it will instead be convenient to use the (dimensionally-reduced) fields of the 4D theory introduced above.

¹Our conventions are the same as in [11],[14]: in particular we use undertwiddling for the fields in the $SU(2)$ matrix notation, $\underline{X} \equiv X^a \tau^a/2$.

The theory has a Coulomb branch on which the adjoint scalar fields acquire mutually commuting expectation values. In this phase, two components of the gauge field gain a mass, M_W , by the adjoint Higgs mechanism. The remaining massless photon is eliminated in favour of a periodic scalar $0 \leq \sigma < 2\pi$ via a duality transformation. The resulting classical moduli space is a flat eight-dimensional manifold parametrized by the massless components of the seven adjoint scalars and the dual photon. $N = 8$ supersymmetry forbids quantum corrections to the metric on the Coulomb branch, hence the low-energy effective action with up to two derivatives or four fermions is a free field theory. In fact the first non-trivial quantum corrections appear at the fourth order in the derivative expansion and correspondingly we will calculate corrections to a correlator with eight fermion insertions.

Three-dimensional gauge theories with extended supersymmetry have instanton solutions: the relevant field configurations are BPS multi-monopoles of magnetic charge k . These configurations have finite Euclidean action equal to $|k|S_0 - ik\sigma$ where $S_0 = (8\pi^2 M_W)/e^2$ and the imaginary term involving the dual photon comes from a surface term analogous to the θ -term in four dimensions. To describe the zero modes of these supersymmetric instantons it is convenient to make a particular vacuum choice [11]: the component, v_3 , of the four-dimensional gauge field (which is a scalar in 3D) acquires a non-zero expectation value, as does σ ; all other VEVs are set to zero. As usual we choose the non-zero VEV to lie in the third direction in the $SU(2)$ gauge group: $\langle v_3 \rangle = \sqrt{2}v\tau^3/2$ where v is real and positive and $M_W = \sqrt{2}v$. In this case the static Bogomol'nyi equation satisfied by the gauge and Higgs components of the BPS monopole of charge $k > 0$ can be rewritten as a self-dual Yang-Mills (SDYM) equation for the four-dimensional gauge field, v_m [22]:

$$v_{mn}^{\text{cl}} = *v_{mn}^{\text{cl}} \quad (2)$$

Solutions of negative magnetic charge are obtained from the corresponding anti-self-dual equation.

In order to find the zero modes, $\delta v^m = \tilde{Z}^m$ of the multi-monopole solution v_m^{cl} it is necessary to solve the linearized SDYM equation together with a background gauge condition

$$\mathcal{D}_{\text{cl}}^{[m} \tilde{Z}^{n]} = *\mathcal{D}_{\text{cl}}^{[m} \tilde{Z}^{n]} \quad , \quad \mathcal{D}_{\text{cl}}^m \tilde{Z}^m = 0 \quad . \quad (3)$$

where $\mathcal{D}_{\text{cl}}^m$ is the adjoint gauge-covariant derivative in the self-dual gauge background. The Callias index theorem [23][24] tells us that there are exactly $4k$ normalizable solutions of these equations which we denote $\tilde{Z}_m^{(i)}$ for $i = 1, 2 \dots 4k$. Correspondingly we can introduce collective coordinates X_i for the multi-monopole solutions which (locally) parametrize a charge- k moduli-space, \mathcal{M}_k . The properties of these spaces have been extensively studied by mathematicians and are described in detail in [19]. They are smooth Riemannian manifolds equipped with a natural metric,

$$g_{ij} = \frac{2\pi}{e^2} \int d^3x \, 2\text{Tr} \, \tilde{Z}_m^{(i)} \tilde{Z}_m^{(j)} \quad (4)$$

In addition the monopole moduli-spaces inherit three inequivalent complex structures from the three inequivalent self-dual complex structures on R^4 . These complex structures are covariantly

constant with respect to the metric (4) and generate a representation of the quaternions on the tangent bundle of \mathcal{M}_k . The k -monopole moduli space is therefore a hyper-Kähler manifold of real dimension $4k$.

Spatial translations act freely on \mathcal{M}_k , while global rotations in the unbroken $U(1)$ subgroup of the gauge group act with a \mathbb{Z}_k stabilizer. The moduli space can be isometrically decomposed as,

$$\mathcal{M}_k \simeq R^3 \times \frac{S^1 \times \tilde{\mathcal{M}}_k}{\mathbb{Z}_k} \quad (5)$$

The R^3 and S^1 factors are parametrized by the 3D spacetime position vector X_μ , $\mu = 0, 1, 2$ and overall charge angle $X_3 = \theta$ of the k -monopole solution respectively. The corresponding metric \bar{g}_{mn} , obtained by restriction of g_{ij} to $R^3 \times S^1$, is flat and can be written as,

$$\bar{g}_{mn} = \delta_{mn} g_{XX} + \delta_{m3} \delta_{n3} (g_{\theta\theta} - g_{XX}) \quad , \quad m, n = 0, 1, 2, 3 \quad (6)$$

In order to calculate the instanton contribution to the Euclidean correlators of the theory, it is necessary to integrate over the collective coordinates with the measure obtained from changing variables in the path integral. The contribution of the four global symmetry modes to the measure can be written as [25],

$$\int d\bar{\mu}_B = \int \frac{d^3 X}{(2\pi)^{\frac{3}{2}}} (g_{XX})^{\frac{3}{2}} \int_0^{\frac{2\pi}{k}} \frac{d\theta}{(2\pi)^{\frac{1}{2}}} (g_{\theta\theta})^{\frac{1}{2}} \quad (7)$$

where the limit of integration on the θ -integral reflects the discrete \mathbb{Z}_k symmetry. The overall normalization of the translational and charge rotation zero modes of a single monopole were given in Appendix C of our previous paper [11]. The results generalize trivially to all k giving $g_{XX} = kS_0$ and $g_{\theta\theta} = kS_0/M_W^2$. These constants are related to the mass and moment of inertia of the k -monopole solution respectively.

The remaining factor in (5) is the reduced or centered moduli space $\tilde{\mathcal{M}}_k$. This $4(k-1)$ dimensional hyper-Kähler manifold is the simply-connected k -fold cover of the moduli-space of k monopoles with a fixed centre of mass and global $U(1)$ phase. $\tilde{\mathcal{M}}_k$ is parametrized by coordinates Y_q , with $q = 1, 2, \dots, 4(k-1)$. In an asymptotic regime, these parameters can be identified as the relative separations and charge angles of k distinct BPS monopoles. We will write the restriction of the metric g_{ij} on \mathcal{M}_k to the reduced moduli space, $\tilde{\mathcal{M}}_k$, as \tilde{g}_{pq} . The corresponding contribution to the path integral measure can be expressed as an integral over $\tilde{\mathcal{M}}_k$,

$$\int d\tilde{\mu}_B = \int \frac{\prod_{q=1}^{4(k-1)} dY_q}{(2\pi)^{2(k-1)}} \sqrt{\det(\tilde{g})} \quad (8)$$

The metric \tilde{g}_{pq} is known exactly only in the two-monopole case where it was constructed explicitly by Atiyah and Hitchin [19]. For $k > 2$, only the asymptotic form of the metric in the limit of well-separated monopoles is known [26]. Fortunately our final result will not depend on the metric explicitly but only on the global structure of \mathcal{M}_k .

The problem of finding zero modes of the fermion fields in the monopole background is closely related to the corresponding bosonic problem described above. As in [11], it is convenient to consider the classical equations of motion for the dimensionally-reduced Weyl fermions of the 4D theory.

$$\bar{\mathcal{D}}_{\text{cl}}^{\dot{\alpha}\alpha} \lambda_{\alpha}^{\text{cl}} = 0 \quad (9)$$

$$\mathcal{D}_{\text{cl}}^{\alpha\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}^{\text{cl}} = 0 \quad (10)$$

In a monopole background of positive magnetic charge, Eq. (9) has normalizable solutions while Eq. (10) has none. In fact there is a simple relation between the zero modes of $\bar{\mathcal{D}}_{\text{cl}}$ and the zero modes of the corresponding linearized equation (3) for the gauge field. For each bosonic zero mode, $Z_m^{(i)}$, the spinor $\lambda_{\alpha}^{(i,\dot{\alpha})} = \sigma_{\alpha\dot{\alpha}}^m Z_m^{(i)}$ with $\dot{\alpha} = 1$ and 2 , yields two linearly-independent solutions for the adjoint Dirac equation (9).

Naively, the above argument indicates that each species of Weyl fermion has $8k$ zero modes. However, as explained in [27][28], this overcounts the number of linearly independent solutions by a factor of four. The essential observation is that the three inequivalent complex structures on the moduli space, $(J^a)_i{}^j$ for $a = 1, 2, 3$, act on the vector zero modes as,

$$(J^a)_i{}^j Z_m^{(i)} = \eta_{mn}^a Z_n^{(j)} \quad (11)$$

where η_{mn}^a are the self-dual generators of $SO(4)$ rotations of the 4D vector index m . It then follows easily that (for fixed indices j and α) the six zero modes $(J^a)_i{}^j \lambda_{\alpha}^{(i,\dot{\alpha})}$, for $a = 1, 2, 3$ and $\dot{\alpha} = 1, 2$, are linear combinations of the two independent modes $\lambda_{\alpha}^{(j,\dot{\alpha})}$. Eliminating this overcounting, the total number of normalizable zero modes of the Dirac operator in the charge- k monopole background is $2k$ for each species of Weyl fermion.

Just as for the bosonic zero modes, it is convenient to consider separately the fermionic modes which correspond to the action of symmetry generators on the monopole configuration. The bosonic fields of the BPS monopole are invariant under half the SUSY generators. The action of the remaining generators yields a total of eight zero modes of the left-handed Weyl fermion fields. These can be parametrized by four Grassmann spinor collective coordinates, ξ_{β}^M with $M = 1, 2, 3, 4$, as

$$\lambda_{\alpha}^{\text{cl}M} = \frac{1}{2} \xi_{\beta}^M (\sigma^m \bar{\sigma}^n)_{\alpha}{}^{\beta} v_{mn}^{\text{cl}} \quad (12)$$

The corresponding contribution to the multi-instanton measure is,

$$\int d\bar{\mu}_F = \int \prod_{M=1}^4 d^2 \xi_M (k \mathcal{J}_{\xi})^{-4} \quad (13)$$

where the normalization constant \mathcal{J}_{ξ} is independent of k . In Appendix C of [11], we obtained $\mathcal{J}_{\xi} = 2S_0$. The modes (12) are protected by $N = 8$ SUSY and cannot be lifted. Hence to obtain a non-zero contribution to the path integral, it will be necessary to saturate the Grassmann integrations appearing in (13) by inserting at least eight fermion fields (or alternatively four fermion-bilinear parts of boson fields).

The $8(k-1)$ remaining fermion zero modes do not correspond to the action of any symmetries and their explicit forms are not known. In the following it will suffice to relate these modes to their bosonic counterparts, $Z_m^{(q)}$, where as before $q = 1, 2, \dots, 4(k-1)$. We will use the fact that the tangent space T_k at any point on $\tilde{\mathcal{M}}_k$ is a vector field over the quaternions [19]. The three inequivalent complex structures can then be represented as the unit quaternions which act on the tangent vectors by quaternionic multiplication. Equivalently we can find a real basis for T_k in which the zero modes $Z_m^{(q)}$ (where $q = 1, 2, \dots, 4(k-1)$) can be partitioned into $k-1$ blocks of four on which the $su(2)$ algebra generated by the complex structures acts irreducibly. Specifically we will choose a basis so that, for each value $M = 1, 2, 3, 4$, the index q picks out one zero mode from each quaternionic block of four as it runs from 1 to $4(k-1)$ over all values satisfying $q = M \bmod 4$. A simple consequence is that, as q runs over such a set of $k-1$ values, $\lambda_\alpha^{(q, \dot{\alpha})}$ (with $\dot{\alpha} = 1$ and 2) form a set of $2(k-1)$ linearly independent fermion zero modes. It is convenient to parametrize the resulting set of $8(k-1)$ zero modes of the four species of 4D Weyl fermions, λ_α^M , in terms of $4(k-1)$ Grassmann 2-component real spinor parameters α_β^q as

$$\lambda_\alpha^{\text{cl}M} = \sum_{q=M \bmod 4} \epsilon^{\beta\dot{\alpha}} \alpha_\beta^q \lambda_\alpha^{(q, \dot{\alpha})} \quad (14)$$

The corresponding contribution to the multi-instanton measure then has the simple form,

$$\int d\tilde{\mu}_F = \int \frac{\prod_{q=1}^{4(k-1)} d\alpha_1^q d\alpha_2^q}{\det(\tilde{g})} \quad (15)$$

As the modes appearing in (15) are not protected by any (super)symmetry it is natural to expect that they will be lifted. In fact, in the corresponding instanton calculation in the four-dimensional theory one finds [29], that this lifting occurs precisely due to Grassmann quadrilinear terms in the classical action of the instanton. It is straightforward to show that similar terms arise in the present case. By an explicit calculation we find,

$$S_{\text{quad}}^{(k)} = \frac{1}{4} \tilde{R}_{pqrs} \alpha_1^p \alpha_1^q \alpha_2^r \alpha_2^s \quad (16)$$

where \tilde{R}_{pqrs} is the Riemann tensor formed from the metric \tilde{g}_{pq} on $\tilde{\mathcal{M}}_k$. In particular one can show that the vertex (16) is invariant under the transformations of the collective coordinates which are induced by $N = 8$ supersymmetry transformations on the fields. Similarly (16) is invariant under the abelian \mathcal{R} -symmetry, denoted $U(1)_N$ in [2], which prevents a similar lifting occurring in the $N = 4$ theory.

A simple way to derive $S_{\text{quad}}^{(k)}$ is to view the BPS monopoles which are instantons in R^3 as solitons in $R^3 \times S^1$, by introducing a compactified Euclidean ‘time’ dimension, τ , $0 < \tau < \beta$. The $N = 8$ SUSY Yang-Mills theory on R^3 is the $\beta \rightarrow 0$ limit of the corresponding theory on $R^3 \times S^1$ with periodic boundary conditions in τ for fermions as well as bosons. To correctly include leading order semiclassical effects, the monopole solutions must now be made ‘time’-dependent by allowing their bosonic and fermionic collective coordinates to depend of τ . After

substituting such τ -dependent k -monopole configurations into the action and integrating over R^3 one obtains an effective action describing τ -dependent geodesic motion of k monopoles on the reduced moduli space $\tilde{\mathcal{M}}_k$ with the metric \tilde{g}_{pq} , [27][28]

$$S_{\text{eff}}^{(k)} = kS_0 + \int_0^\beta d\tau \frac{1}{2} \left[\tilde{g}_{pq} d_\tau Y^p d_\tau Y^q + \tilde{g}_{pq} i\bar{\alpha}^p \gamma^0 D_\tau \alpha^q + \frac{1}{12} \tilde{R}_{pqrs}(\bar{\alpha}^p \alpha^r)(\bar{\alpha}^q \alpha^s) \right] \quad (17)$$

where $\bar{\alpha} = \alpha \gamma^0$ with $\gamma^0 = \sigma^2$, and $D_\tau \alpha^q = d_\tau \alpha^q + d_\tau Y^r \tilde{\Gamma}_{rq}^p \alpha^q$ is the covariant derivative on $\tilde{\mathcal{M}}_k$ formed from the metric \tilde{g}_{pq} . Equation (17) is the action of an $N = 8 \times (1/2)$ supersymmetric Euclidean quantum mechanics² on $\tilde{\mathcal{M}}_k$. The Riemann tensor term arises in (17) naturally as the supersymmetric completion³ of the kinetic terms in $S_{\text{eff}}^{(k)}$. We can now dimensionally reduce and consider the τ -independent contributions to $S_{\text{eff}}^{(k)}$. In this case only the four-fermion term survives and, rearranging the spinor indices, we obtain Eq. (16). An equivalent procedure is to let $\beta \rightarrow 0$ as in Ref. [30], after rescaling constant fermionic coordinates by a factor of $\beta^{-1/4}$. In fact, as we will see below, the instanton contribution is formally equal to the regularized Witten index, $\text{Tr}[(-1)^F \exp(-\beta H)]$, of the quantum mechanics defined by the action (17) and is therefore independent of β .

As in any semiclassical instanton calculation, to complete the specification of the measure we must also consider the contribution of the non-zero modes. Usually in supersymmetric gauge theories these contributions cancel between boson and fermi degrees of freedom. However, in our recent work on $N = 4$ SUSY gauge theory in three dimensions [11], we found that this cancellation is not complete due to the spectral asymmetry of the Dirac operator in the monopole background. Specifically we found that the contribution of the gauge multiplet (including ghosts) was equal to a non-trivial ratio of functional determinants,

$$R = \left[\frac{\det(\Delta_+)}{\det'(\Delta_-)} \right]^{\frac{1}{2}} \quad (18)$$

where $\Delta_+ = \bar{\mathcal{P}}_{\text{cl}} \mathcal{P}_{\text{cl}}$, $\Delta_- = \mathcal{P}_{\text{cl}} \bar{\mathcal{P}}_{\text{cl}}$ and \det' denotes the removal of zero eigenvalues. In the present case of $N = 8$ SUSY, we must also include the contribution of the adjoint hypermultiplet. A straightforward calculation shows that this contribution is equal to R^{-1} . Hence, the extra supersymmetries of the present case lead to a complete cancellation of the non-zero mode contributions. Collecting all the factors together, the complete multi-instanton measure can be written as,

$$\begin{aligned} \int d\mu^{(k)} &= \int d\bar{\mu}_B d\tilde{\mu}_B \int d\bar{\mu}_F d\tilde{\mu}_F \exp(-kS_0 + ik\sigma - S_{\text{quad}}^{(k)}) \\ &= \int d\bar{\mu} \int d\tilde{\mu}^{(k)} k^{-3} \exp(-kS_0 + ik\sigma) \end{aligned} \quad (19)$$

²As explained above, N counts the number of two-component Majorana spinor supercharges. The notation $N = 8 \times (1/2)$ indicates that the SUSY QM described by (17) has eight one-component supercharges

³In the case of $\mathcal{N} = 2$ SUSY in 4D considered in [27], the corresponding $S_{\text{eff}}^{(k)}$ is an $N = 4 \times (1/2)$ quantum mechanics on the moduli space and the Riemann tensor term is not generated.

where

$$\int d\bar{\mu} = \frac{1}{2^5 \pi M_W S_0^2} \int d^3 X \int \prod_{M=1}^4 d^2 \xi_M \quad (20)$$

and $d\tilde{\mu}^{(1)} = 1$. For $k > 1$,

$$\int d\tilde{\mu}^{(k)} = \int \frac{\prod_{q=1}^{4(k-1)} dY^q d\alpha_1^q d\alpha_2^q}{(2\pi)^{2(k-1)} \sqrt{\det(\tilde{g})}} \exp(-S_{\text{quad}}^{(k)}) \quad (21)$$

As discussed above, in order to obtain a non-zero contribution, it is necessary to saturate the Grassmann integrals in (20) which correspond to the eight unlifted zero modes protected by supersymmetry. The simplest correlator which is non-zero in the instanton background must therefore have eight fermion insertions. This part of the calculation is a straightforward extension of the calculation given in [11] of a four-fermion correlator in the $N = 4$ theory. Specifically we consider the instanton contribution to a correlator with two insertions of each species of adjoint fermion,

$$G^{(8)}(x_1, x_2, \dots, x_8) = \langle \lambda_\alpha(x_1) \lambda_\beta(x_2) \psi_\gamma(x_3) \psi_\delta(x_4) f_\epsilon(x_5) f_\kappa(x_6) \tilde{f}_\rho(x_7) \tilde{f}_\sigma(x_8) \rangle \quad (22)$$

where, as in [11], we have defined low-energy massless fermion fields by⁴ $\lambda_\alpha^M = \text{Tr}(\tilde{\lambda}_\alpha^M \tau^3)$. At leading semiclassical order each of these fields is replaced by its large-distance (LD) behaviour of the corresponding zero modes in the monopole background:

$$(\lambda_\alpha^M)^{\text{LD}} = 8\pi k \xi_\beta^M \mathcal{S}_F(x - X)_\alpha^\beta \quad (23)$$

This asymptotic form is valid for $|x - X| \gg M_W^{-1}$ where X is the centre of mass of the k instanton configuration and $\mathcal{S}_F(x) = \gamma_\mu x_\mu / (4\pi|x|^2)$ is the three-dimensional Weyl fermion propagator. The factor k in (23) reflects the coefficient of the long-range magnetic Coulomb fields of the charge- k monopole.

The final result can be expressed as a contribution to an eight anti-fermion vertex which is a correction to the classical low-energy effective action for the massless fermions. The latter is just a free massless action for the Weyl degrees of freedom.

$$S_F = \frac{2\pi}{e^2} \int d^3 x i \bar{\lambda}^M \bar{\sigma}_\mu \partial_\mu \lambda^M \quad (24)$$

where a sum over the $SU(4)_{\mathcal{R}}$ index is implied. The resulting eight anti-fermion vertex can be written as a sum over contributions from sectors of different (positive) topological charge k ,

$$S_I = \sum_{k=1}^{\infty} \mathcal{V}_k \exp(-k S_0 + ik\sigma) \int d^3 x \prod_{M=1}^4 \bar{\lambda}_M^2 \quad (25)$$

⁴Recall that the $SU(4)_{\mathcal{R}}$ index M labels the four species of Weyl fermions.

Evaluating the integrals over ξ_α^M we obtain,

$$\begin{aligned}\mathcal{V}_k &= \frac{\mathcal{V}k^5}{S_0^3} \int d\tilde{\mu}^{(k)} \\ \mathcal{V} &= \frac{2^{14}\pi^9}{e^2} \left(\frac{2\pi}{e^2}\right)^8\end{aligned}\quad (26)$$

where the eight powers of $2\pi/e^2$ in (26) reflect our choice of normalization for the kinetic term in (24). For $k = 1$, after a comparison of normalizations and definitions, we find agreement with the one-instanton result of Polchinski and Pouliot [10] (their equation (3.23)).

For $k > 1$, the remaining problem is to evaluate the integral (21) over the parameters which correspond to the relative moduli of the k -monopole solution and their superpartners. The fermionic integrals in this expression are saturated by bringing down $2(k-1)$ powers of quadrilinear term $S_{\text{quad}}^{(k)}$. After performing the Grassmann integrations we are left with a bosonic integral over the reduced charge- k moduli space. Setting $d = 4(k-1)$ we have,

$$\int d\tilde{\mu}^{(k)} = \frac{1}{(8\pi)^{d/2}(d/2)!} \int \frac{\prod_{q=1}^d dY^q}{\sqrt{\det(\tilde{g})}} \varepsilon^{p_1 p_2 \dots p_d} \varepsilon^{q_1 q_2 \dots q_d} \tilde{R}_{p_1 p_2 q_1 q_2} \dots \tilde{R}_{p_{d-1} p_d q_{d-1} q_d} \quad (27)$$

This integral has a familiar form: it is precisely the volume contribution to the index of the Euler operator on the non-compact manifold $\tilde{\mathcal{M}}_k$. If we were integrating over a compact manifold it would simply be equal to the Euler characteristic of that manifold by the Gauss-Bonnet-Chern theorem. As usual, an extension to the case of a non-compact manifold can be obtained from a limiting case of the index theorem on a manifold with boundary⁵. In the present case the Euler character of the manifold can be written as the sum of the volume term (27) and a surface term which involves the integral of a certain Chern-Simons form over the boundary. As we discuss below, these surface terms are known to vanish for the case $k = 2$ and we will conjecture that they also vanish for all $k > 2$.

The centered moduli-space of two BPS monopoles has real dimension four. In this case, the GBC integral has been evaluated explicitly by Gauntlett and Harvey [18] using the known metric on $\tilde{\mathcal{M}}_2$. They obtained,

$$\int d\tilde{\mu}^{(2)} = -\frac{1}{32\pi^2} \int_{\tilde{\mathcal{M}}_2} \varepsilon^{abcd} \tilde{R}_{ab} \wedge * \tilde{R}_{cd} = 2 \quad (28)$$

where the integrand is written in a vierbein basis in terms of the curvature 2-forms \tilde{R}_{ab} . In fact, the Euler character $\chi(\tilde{\mathcal{M}}_2)$ is also equal to two, reflecting the fact that the manifold contracts onto a two-sphere which is the double-cover of the bolt. Hence, in this case, the boundary terms in the index theorem must vanish in the infinite volume limit where the boundary is removed to infinity. This is consistent with a previous analysis of Gibbons, Pope and Römer [17], who showed that the boundary contribution vanishes for a class of metrics, known as Asymptotically Locally Flat (ALF) metrics, of which the Atiyah-Hitchin metric on $\tilde{\mathcal{M}}_2$ is a particular example.

⁵Index theory on manifolds with boundary is reviewed in [31].

Unfortunately, it is much harder to make a precise statement about the boundary contributions to the index theorem for the case $k > 2$. Here, it is known that the metric approaches a multi-centre Taub-NUT metric in the limit in which the separation between each pair of monopoles becomes large [26]. This metric has the same asymptotic flatness properties as the two-monopole metric and it is very plausible that the surface contributions from these parts of the boundary vanish as they do for $k = 2$. However, the ‘boundary at infinity’ of the $\tilde{\mathcal{M}}_k$ for $k > 2$ also contain clustering regions in which at least one pair of monopoles remain at finite separation. Heuristically one may argue that these regions represent a fraction of the boundary which tends to zero in the infinite volume limit. As the metric and therefore the surface-integrand is known to be non-singular, it should be possible show that the contribution of these regions to the surface integral is bounded above by some inverse power of the volume. In the absence of a more precise analysis of this issue we will simply assume that the surface contribution does vanish in this limit and therefore that the GBC integral (27) is equal to the Euler character $\chi(\tilde{\mathcal{M}}_k)$ for all k .

The cohomology of the centered multi-monopole moduli spaces, $\tilde{\mathcal{M}}_k$, has recently been determined from description of these spaces as spaces of rational maps [20]. The cohomology with complex coefficients, $H^*(\tilde{\mathcal{M}}_k)$, is divided into different sectors according to the action of the discrete symmetry group, \mathbb{Z}_k . Let $H^*(\tilde{\mathcal{M}}_k)_p$ denote the part of the cohomology where $\xi \in \mathbb{Z}_k$ acts as $\xi \rightarrow \xi^p$. Then the result of Ref. [20] is that $H^i(\tilde{\mathcal{M}}_k)_p$ has complex dimension one whenever $i = 2k - 2(p, k)$ where (p, k) is the greatest common divisor of p and k , and dimension zero otherwise. As the non-vanishing cohomology is even, the Euler character is obtained by counting the total number of solutions of the condition $i = 2k - 2(p, k)$;

$$\chi(\tilde{\mathcal{M}}_k) = \sum_{i=0}^{4(k-1)} \sum_{p=0}^{k-1} \delta_{i, 2k - (p, k)} = k \quad (29)$$

Hence our final result for the multi-instanton contribution to the vertex (25) is $\mathcal{V}_k = k^6 \mathcal{V} / S_0^3$. Now the instanton series may be summed straightforwardly to give;

$$S_I = \mathcal{V} \left[\frac{1}{2S_0^3} \left(\frac{\partial}{\partial S_0} \right)^6 \coth \left(\frac{S_0}{2} + \frac{i\sigma}{2} \right) \right] \int d^3x \prod_{M=1}^4 \bar{\lambda}_M^2 \quad (30)$$

Including the effects of multi-instantons of negative topological charge yields a vertex for the left-handed Weyl fermions. The result is obtained by making the replacements, $\bar{\lambda}_M^2 \rightarrow \lambda_M^2$ and $\sigma \rightarrow -\sigma$ in (30).

We finish by comparing the resummed multi-instanton vertex obtained above with the amplitude for membrane scattering in eleven-dimensional supergravity calculated in [10]. The amplitude given in equation (3.39) of this reference yields a prediction for a term, $S_{4-\partial}$, in the low-energy effective action of the 3D SUSY gauge theory which has four time derivatives. In our notation this term involves four powers of $\partial_0 v_3$, where $v_3 = \text{Tr}(v_3 \tau^3)$ is the only low-energy scalar field (other than σ) with a non-zero VEV,

$$S_{4-\partial} = K \int d^3x (\partial_0 v_3)^4 \sum_{n=-\infty}^{+\infty} \frac{1}{(S_0^2 + (2\pi n - \sigma)^2)^3} \quad (31)$$

Here K is an overall constant which is independent of the VEV. We will evaluate the sum on the RHS of (31) using the identity,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{S_0^2 + (2\pi n - \sigma)^2} = \frac{1}{4S_0} \left[\coth\left(\frac{S_0}{2} + \frac{i\sigma}{2}\right) + \coth\left(\frac{S_0}{2} - \frac{i\sigma}{2}\right) \right] \quad (32)$$

The required sum is obtained by differentiating the above relation twice with respect to S_0^2 . The presence of four-derivative term (31) can be checked directly in the instanton approach by saturating the eight Grassmann integrals in (20) with four insertions of the fermion bilinear terms in the scalar field v_3 . Equivalently, we may compare the instanton-induced vertex (30) with the eight-fermion vertex which arises as the supersymmetric completion of (31) in the low-energy effective action. Up to a constant, the coefficient of the eight-fermion vertex is obtained from that of the four-derivative term by differentiating the latter with respect to the VEV four times⁶. The vertex for the right-handed fermions obtained in this way can be written as,

$$\tilde{S}_I = 2\mathcal{V} \left[\left(\frac{\partial}{\partial S_0} \right)^4 \left(\frac{1}{2S_0} \frac{\partial}{\partial S_0} \right)^2 \frac{1}{S_0} \coth\left(\frac{S_0}{2} + \frac{i\sigma}{2}\right) \right] \int d^3x \prod_{M=1}^4 \bar{\lambda}_M^2 \quad (33)$$

The predicted vertex has a double expansion in powers of $1/S_0 \sim e^2/v$ and in powers of the instanton action $\exp(-S_0 + i\sigma)$. The latter expansion is an expansion in instanton number and, to compare with multi-instanton result (30) we will retain terms of all orders. The former expansion corresponds to a series of perturbative corrections and, as we wish to compare with a leading order semiclassical calculation, we must retain only the leading order term in each instanton sector. To this order the partial derivatives in (33) can all be moved to the right to act on the hyperbolic cotangent and we find agreement with (30).

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⁶This is analogous to the relation between the instanton contributions to the kinetic terms and to the four-fermion term in the four-dimensional theories studied in [14].

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