

# Duality and Supersymmetric Monopoles

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Exact duality in supersymmetric gauge theories leads to highly non-trivial predictions about the moduli spaces of BPS monopole solutions. These notes attempt to be a pedagogical review of the current status of these investigations and are based on lectures given at the 33rd Karpacz Winter School: Duality - Strings and Fields, February 1997.

## 1. INTRODUCTION

Electromagnetic duality has emerged as a powerful tool to study strongly coupled quantum fields. In  $N=4$  super-Yang-Mills theory and some special theories with  $N=2$  supersymmetry, the duality is conjectured to be exact in the sense that it is valid at all energy scales. These theories provide the most natural setting for Montonen and Olive's original idea [1] since they have vanishing  $\beta$ -functions and hence the quantum corrections are under precise control. In many theories with  $N=2$  and  $N=1$  supersymmetry duality plays an important role in elucidating the infrared dynamics. In these models one can study strong coupling phenomenon such as confinement and chiral symmetry breaking in an exact context<sup>2</sup>.

The purpose of these lectures is to review some aspects of exact duality focusing on theories with  $N=4$  supersymmetry. In these theories the duality group is  $SL(2, \mathbf{Z})$  which includes a  $\mathbf{Z}_2$  corresponding to the interchange of electric and magnetic charges along with the interchange of strong and weak coupling. Sen was the first to realise [2] that  $SL(2, \mathbf{Z})$  or “ $S$ -duality” leads to highly non-trivial predictions about the BPS spectrum of magnetic monopoles and dyons in the theory. BPS states are important for testing duality because they form short representations of the supersymmetry algebra and hence we have good control over their behaviour as we vary the cou-

pling. At weak coupling the predicted BPS spectrum can be translated into statements about certain geometric structures on the moduli space of BPS monopole solutions.

We begin with bosonic  $SU(2)$  BPS monopoles, reviewing some aspects of the moduli space approximation and discussing how quantised dyons appear in the semiclassical spectrum. Next we describe some features of  $N=4$  super-Yang-Mills theory before studying the  $S$ -duality predictions. We analyse the  $SU(2)$  case followed by the higher rank gauge groups. We conclude by outlining some open problems in the study of exact duality.

## 2. $SU(2)$ BPS MONOPOLES AND THE MODULI SPACE APPROXIMATION

Consider the Yang-Mills-Higgs Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2}D_\mu \Phi^a D^\mu \Phi^a, \quad (1)$$

where  $A_\mu = A_\mu^a T^a$  is an  $SU(2)$  connection with field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \epsilon[A_\mu, A_\nu]$ , and  $\Phi = \Phi^a T^a$  transforms in the adjoint representation with the covariant derivative given by  $D_\mu \Phi = \partial_\mu \Phi + \epsilon[A_\mu, \Phi]$ . We choose the Lie algebra generators  $T^a$  to be anti-hermitian. There is no potential term for the Higgs field and we are thus considering the “BPS limit” [3,4] which is relevant for the supersymmetric extension. The moduli space of Higgs vacua is obtained by imposing  $\langle \Phi^a \Phi^a \rangle = v^2$  and is thus a two-sphere. If  $v^2 \neq 0$  then  $SU(2)$  is spontaneously broken to  $U(1)$ . The electric and magnetic charge with respect to the  $U(1)$  specified by the Higgs field are

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<sup>2</sup>See the contributions of other speakers at the school.

given by

$$\begin{aligned} Q_e &= \frac{1}{v} \int dS^i (E_i^a \Phi^a), \\ Q_m &= \frac{1}{v} \int dS^i (B_i^a \Phi^a), \end{aligned} \quad (2)$$

where  $E_i^a = F_{0i}$  and  $B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a$  are the non-abelian electric and magnetic field strengths, respectively, and the integration is over a surface at spatial infinity.

The perturbative states consist of a massless photon, a massless neutral scalar and  $W^\pm$  bosons with electric charge  $Q_e = \pm e$  and mass  $\epsilon v$ . To analyse the monopole and dyon spectrum we need to construct classical static monopole solutions and then perform a semi-classical analysis. Let us begin by noting that for all finite energy configurations the Higgs field must lie in the vacuum at spatial infinity. The Higgs field of these configurations thus provides a map from the two sphere at spatial infinity to the two sphere of Higgs vacua. These maps are characterised by a topological winding number  $k$  and one can show that this implies that the magnetic charge is quantised

$$Q_m = \frac{4\pi}{e} k. \quad (3)$$

The minimal magnetic monopole charge is twice the Dirac unit because we could add electrically charged fields in the fundamental representation of  $SU(2)$  that would carry 1/2 integer electric charges in contrast to the integer charged  $W$ -bosons.

To proceed with the construction of static monopole solutions it will be convenient to work in the  $A_0 = 0$  gauge. We must then impose Gauss' Law, the  $A_0$  equation of motion, as a constraint on the physical fields:

$$D_i \dot{A}_i + e[\Phi, \dot{\Phi}] = 0. \quad (4)$$

In this gauge the Hamiltonian is  $H = T + V$  where the kinetic and potential energies are given by

$$T = \frac{1}{2} \int d^3x (\dot{A}_i^a \dot{A}_i^a + \dot{\Phi}^a \dot{\Phi}^a), \quad (5)$$

$$V = \frac{1}{2} \int d^3x (B_i^a B_i^a + D_i \Phi^a D_i \Phi^a), \quad (6)$$

respectively. Noting that  $V$  can be rewritten [4] as

$$V = \frac{1}{2} \int d^3x [(B_i^a \mp D_i \Phi^a)(B_i^a \mp D_i \Phi^a)] \pm v Q_m, \quad (7)$$

we deduce that in each topological class  $k$  corresponding to magnetic charge given by (3) there is a Bogomol'nyi bound on the mass of any static classical monopole solution:

$$M \geq v |Q_m| = \frac{4\pi v}{e} |k|. \quad (8)$$

The static energy is minimised when the bound is saturated which is equivalent to the Bogomol'nyi (or BPS) equations

$$B_i = \pm D_i \Phi. \quad (9)$$

The upper sign corresponds to positive  $k$  or "monopoles" and the lower sign corresponds to negative  $k$  or "anti-monopoles". From now on we will restrict our considerations to monopoles, the extension to anti-monopoles being trivial. In the  $A_0 = 0$  gauge there are no static dyon solutions; the dyons emerge as time dependent solutions as we will see.

The moduli space of gauge inequivalent solutions to the Bogomol'nyi equations will be denoted  $\mathcal{M}_k$ . Let us discuss some of the geometry of this manifold. We begin by recalling that in the  $A_0 = 0$  gauge the configuration space of fields is given by  $\mathcal{C} = \mathcal{A}/\mathcal{G}$  where  $\mathcal{A} = \{A_i(x), \Phi(x)\}$  is the space of finite energy field configurations and we have divided out by  $\mathcal{G}$ , the group of gauge transformations that go to the identity at spatial infinity. Tangent vectors  $\{\dot{A}, \dot{\Phi}\}$  to  $\mathcal{C}$  must satisfy Gauss Law (4). From this point of view, the kinetic energy in (5) is simply the metric on  $\mathcal{C}$ . The moduli  $Z^\alpha$  that appear in the general solution to the Bogomol'nyi equations  $\{A(x, Z), \Phi(x, Z)\}$ , are natural coordinates on  $\mathcal{M}_k \subset \mathcal{C}$ . Tangent vectors to  $\mathcal{M}_k$  must also satisfy the linearised Bogomol'nyi equations

$$\epsilon_{ijk} D_j \dot{A}_k = D_i \dot{\Phi} + e[\dot{A}_i, \Phi]. \quad (10)$$

Using the coordinates  $Z^\alpha$  we have

$$\{\dot{A}, \dot{\Phi}\} = \dot{Z}^\alpha \{\delta_\alpha A_i, \delta_\alpha \Phi\}, \quad (11)$$

where  $\{\delta_\alpha A_i, \delta_\alpha \Phi\}$  satisfy

$$D_i \delta_\alpha A_i + e[\Phi, \delta_\alpha \Phi] = 0, \quad (12)$$

$$\epsilon_{ijk} D_j \delta_\alpha A_k = D_i \delta_\alpha \Phi + e[\delta_\alpha A_i, \Phi], \quad (13)$$

which are simply the equations for a physical zero mode. The zero modes can be obtained by differentiating the general solution with respect to the moduli but in general one has to include a gauge transformation to ensure that it satisfies (13) i.e.,

$$\begin{aligned} \delta_\alpha A_i &= \partial_\alpha A_i - D_i \epsilon_\alpha, \\ \delta_\alpha \Phi &= \partial_\alpha \Phi - e[\Phi, \epsilon_\alpha]. \end{aligned} \quad (14)$$

The metric on  $\mathcal{C}$  gives rise to a metric on  $\mathcal{M}_k$  which can be written in terms of the zero modes:

$$\mathcal{G}_{\alpha\beta}(Z) = \int d^3x [\delta_\alpha A_i^a \delta_\beta A_i^a + \delta_\alpha \Phi^a \delta_\beta \Phi^a]. \quad (15)$$

$\mathcal{M}_k$  is  $4k$ -dimensional which can be established, for example, by counting zero modes using an index theorem. The space of field configurations  $\mathcal{A}$  inherits three almost complex structures, from those on  $\mathbf{R}^4$  and they descend to give a hyper-Kähler structure on  $\mathcal{M}_k$ . Explicit formulae for the complex structures on  $\mathcal{M}_k$  in terms of the zero modes can be found in [5]. More details on the geometry of  $\mathcal{M}_k$  can be found in [6].

The moduli space for a single BPS monopole can be determined by explicitly constructing the most general solution and we find  $\mathcal{M}_1 = \mathbf{R}^3 \times S^1$ . The  $\mathbf{R}^3$  piece simply corresponds to the position of the monopole in space. The  $S^1$  arises from the gauge transformation  $g = e^{\chi\Phi/v}$  on any solution. Since this does not go to the identity at infinity, it is a “large” gauge transformation, it corresponds to a physical motion. Since all fields are in the adjoint a  $2\pi$  rotation in  $SU(2)$  is the identity and we conclude that  $0 \leq \chi < 2\pi$ . We will see that this coordinate is a dyon degree of freedom.

We have noted that the dimension of  $\mathcal{M}_k$  is  $4k$ . The physical reason for the existence of these multimonopole configurations is that in the BPS limit there is a cancellation between the vector repulsion and scalar attraction between two monopoles. Heuristically one can think of the  $4k$  dimensions as corresponding to a position in  $\mathbf{R}^3$  and a phase for each monopole but the structure of  $\mathcal{M}_k$  turns out to be much more subtle and interesting. For general  $k$  we can separate out a

piece corresponding to the motion of the centre of mass of the multi-monopole configuration and we have  $\mathcal{M}_k = \mathbf{R}^3 \times (S^1 \times \tilde{\mathcal{M}}_k^0)/\mathbf{Z}_k$ . The  $S^1$  factor is related to the total electric charge.  $\tilde{\mathcal{M}}_k^0$  is  $4(k-1)$  dimensional and hyper-Kähler. It admits an  $SO(3)$  group of isometries which corresponds to a rotation of the multi monopole configuration in space. Although the topology of these spaces is well understood, the metric is explicitly known only for  $k=2$  [6].

To determine the semi-classical spectrum of states with magnetic charge  $k$  we start with a classical solution  $(A^{cl}(x, Z), \Phi^{cl}(x, Z))$ . To have a well defined perturbation scheme with  $e \ll 1$ , we need to introduce a collective co-ordinate for each zero mode; these are the moduli  $Z^\alpha$ . We then expand an arbitrary time dependent field as a sum of the massive modes with time dependent coefficients and allow the collective coordinates to become time-dependent (see, e.g., [7]). A low-energy ansatz for the fields is obtained by ignoring the massive modes and demanding that the only time dependence is via the collective co-ordinates. Thus we are led to the ansatz<sup>3</sup>

$$\begin{aligned} A_i(x, t) &= A_i^{cl}(x, Z(t)), \\ \Phi(x, t) &= \Phi^{cl}(x, Z(t)), \\ A_0 &= \dot{Z}^\alpha \epsilon_\alpha. \end{aligned} \quad (16)$$

After substituting this into the action (1) we obtain an effective action

$$S = \frac{1}{2} \int dt \mathcal{G}_{\alpha\beta} \dot{Z}^\alpha \dot{Z}^\beta - \frac{4\pi v}{e} k, \quad (17)$$

which is precisely that of a free particle propagating on the moduli space  $\mathcal{M}_k$  with metric (15). This is the moduli space approximation [9]. The classical equations of motion are simply the geodesics on  $\mathcal{M}_k$ .

To proceed with the semiclassical analysis we need to study the quantum mechanics of (17). Let us show how a quantised spectrum of dyons emerges in the quantum theory. For  $k=1$  we

<sup>3</sup>Note that the  $A_0$  term is included to ensure that the motion is orthogonal to gauge transformations. One could do a gauge transformation if one wants to remain in the  $A_0=0$  gauge (see also the discussion in [8]).

have  $\mathcal{M}_1 = \mathbf{R}^3 \times S^1$  and including various constants we have:

$$S = \frac{1}{2} \int dt \left[ \left( \frac{4\pi v}{e} \right) \dot{\mathbf{Z}}^2 + \frac{4\pi}{ve^3} \dot{\chi}^2 \right] - \frac{4\pi v}{e} . \quad (18)$$

The wavefunctions are plane waves of the form  $e^{i\mathbf{P} \cdot \mathbf{Z}} e^{in_e \chi}$  where  $n_e$  is an integer. In the moduli space approximation  $Q_e = -ie\partial_\chi$  and we see that we have a tower of dyons with  $Q_e = n_e e$ . The mass of these states can be calculated and we get

$$\begin{aligned} M &= n_e^2 v e^3 / 8\pi + 4\pi v / e \\ &\approx v [Q_e^2 + Q_m^2]^{1/2} , \end{aligned} \quad (19)$$

where we have used the fact that we are assuming  $e \ll 1$  in our approximations. By generalising the argument that led to (8) it can be shown [10] that  $M \geq v [Q_e^2 + Q_m^2]^{1/2}$  for all classical solutions to the equations of motion (static dyons can be obtained if we do not work in the  $A_0 = 0$  gauge). We thus see that in the moduli space approximation the bound is saturated. Of course in the purely bosonic theory we are considering here this could get higher order quantum corrections.

For  $k > 1$  we can perform a similar analysis on  $\mathcal{M}_k$ , looking for scattering states and bound states of the Hamiltonian in the usual fashion. This has been pursued in the bosonic theory in [11,12]. The momentum conjugate to the coordinate on the  $S^1$  gives the total electric charge  $Q_e$  of the configuration<sup>4</sup> and the bound states have masses  $M = v [Q_e^2 + Q_m^2]^{1/2} + \Delta E$  where  $\Delta E$  is the relative kinetic energy.

We conclude this section by considering a renormalisable term that we can add to the Lagrangian (1) that plays an important role in duality:

$$\delta \mathcal{L} = -\frac{\theta e^2}{32\pi^2} F_{\mu\nu}^a * F^{a\mu\nu} . \quad (20)$$

As it is a total derivative it doesn't affect the equations of motion. It is related to instanton effects and it also affects the electric charge of dyons. Recall that the dyon collective coordinate arose from doing a gauge transformation

<sup>4</sup>The individual electric charge of each monopole is not a good quantum number due to the possibility of  $W$ -boson exchange which the moduli space approximation incorporates.

about the  $\Phi$  axis. The Noether charge picks up a  $\theta$  dependent contribution and one finds that  $Q_e = n_e e + \epsilon\theta/2\pi n_m$  [13]. In the moduli space approximation this manifests itself via  $Q_e = -ie\partial_\chi + \epsilon\theta/2\pi n_m$ . At this point it is convenient to rescale the fields  $\{A, \Phi\} \rightarrow \{A, \Phi\}/e$ . Our combined Lagrangian then takes the simple form

$$\mathcal{L} = -\frac{1}{16\pi} \text{Im}\tau [F^2 + iF * F] - \frac{1}{2e^2} D\Phi^2 , \quad (21)$$

where we have introduced the complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} . \quad (22)$$

The BPS mass formula for dyons (19) is then given by

$$M = v |n_e + n_m \tau| . \quad (23)$$

Due to the rescaling, here and in the following  $v$  contains a hidden factor of the coupling constant  $e$ .

### 3. $N=4$ SUPER-YANG-MILLS

$N=4$  super-Yang-Mills theory has the maximal amount of supersymmetry with spins less than or equal to one. It has vanishing beta-function and is thought to describe a conformally invariant theory. In addition it is supposed to exhibit  $S$ -duality, which we shall define below. We consider  $N=4$  super-Yang-Mills with arbitrary simple gauge group  $G$ . It can be obtained as the dimensional reduction on a six-torus of  $N=1$  super-Yang-Mills theory in ten dimensions (see, e.g., [14]). The ten-dimensional Lorentz group reduces to  $SO(3,1) \times SO(6)$  and  $SO(6)$  becomes a global symmetry of the theory. The bosonic fields in the supermultiplet come from the ten dimensional gauge field and consist of a gauge field and 6 Higgs fields  $\phi^I$ , transforming as a  $\mathbf{6}$  of  $SO(6)$ , all taking values in the adjoint representation of  $G$ . There are four Weyl fermions in the adjoint transforming as a  $\mathbf{4}$  of  $Spin(6)$  that come from the reduction of the Majorana-Weyl spinor in ten dimensions. Including a  $\theta$  parameter, the bosonic part of the action is

$$S = -\frac{1}{16\pi} \text{Im} \int \tau \text{Tr} (F \wedge F + i * F \wedge F)$$

$$- \frac{1}{2e^2} \int [\text{Tr} D_\mu \phi^I D^\mu \phi^I + V(\phi^I)] , \quad (24)$$

where the potential is given by

$$V(\phi^I) = \sum_{1 \leq I < J \leq 6} \text{Tr}[\phi^I, \phi^J]^2 , \quad (25)$$

and here we have taken  $\text{Tr} T^a T^b = \delta^{ab}$ .

The classical vacua of the theory are given by  $V(\phi^I) = 0$  or equivalently  $[\phi^I, \phi^J] = 0$  for all  $I, J$ . In this theory, there are no quantum corrections to the moduli space of vacua. For generic vacua, i.e., generic expectation values  $\langle \phi^I \rangle$ , the gauge symmetry is broken down to  $U(1)^r$  where  $r$  is the rank of the gauge group. A given  $N=4$  theory is specified by  $G$ ,  $\langle \phi^I \rangle$  and  $\tau$ .

The six Higgs fields define a set of conserved electric and magnetic charges which appear as central charges in the  $N=4$  supersymmetry algebra:

$$\begin{aligned} Q_e^I &= \frac{1}{ev} \int d\mathbf{S} \cdot \text{Tr}(\mathbf{E}\phi^I) , \\ Q_m^I &= \frac{1}{ev} \int d\mathbf{S} \cdot \text{Tr}(\mathbf{B}\phi^I) , \end{aligned} \quad (26)$$

For BPS saturated states, i.e., states in the short 16 dimensional representation of the supersymmetry algebra, the mass is exactly given by the formula

$$M^2 = \frac{v^2}{e^2} ((Q_e^I)^2 + (Q_m^I)^2) . \quad (27)$$

The spin content of the short BPS multiplet is the same as the massless multiplet and has spins  $\leq 1$ . There are also medium sized representations consisting of 64 states with spins  $\leq 3/2$ , but these only arise when  $Q_e^I$  is not proportional to  $Q_m^I$ . These can only appear when the rank of the gauge group is greater than one and we will see that  $S$ -duality makes no predictions about the existence of these states as they don't appear in the perturbative spectrum. The generic representation of the  $N=4$  algebra has 256 states with spins  $\leq 2$  and the masses can be renormalised.

It is important to emphasise that the mass formula for BPS states (27) is derived from the supersymmetry algebra and hence it is valid in the quantum theory [15,14] in contrast to the bosonic

case. Thus the mass of BPS states is exactly given by their electric and magnetic quantum numbers. This is an important property of BPS states which enables us to use them to test  $S$ -duality. It will also be useful to note that half of the supersymmetry generators are realised as zero on a BPS multiplet. This is sometimes rephrased as saying that BPS states preserve (or break) half of the supersymmetry.

In a generic vacuum  $\langle \phi^I \rangle$  at weak coupling we deduce that there are massive  $W$ -boson BPS multiplets. To determine the dyon spectrum we need to quantise the BPS monopole solutions in a semi-classical context. For simplicity we will restrict our attention in the following to a single direction in the moduli space of vacua:

$$\begin{aligned} \langle \phi^2 \rangle &= \dots = \langle \phi^6 \rangle = 0 , \\ \phi^1 &\equiv \Phi , \quad \langle \text{Tr} \Phi^2 \rangle = v^2 , \end{aligned} \quad (28)$$

which clearly satisfies  $V(\phi^I) = 0$ . Classical BPS monopole solutions with zero electric charge are then obtained by solving the Bogomol'nyi equations we considered before

$$B_i = D_i \Phi . \quad (29)$$

Note that for the vacua (28) only the first component of the electric and magnetic charges are non-zero and we will write  $Q_e^1 = Q_e$ ,  $Q_m^1 = Q_m$ . More general vacua have been considered in [16] but the region we will analyse seems to lead to the richest monopole physics.

#### 4. $N=4$ $G=SU(2)$ AND $S$ -DUALITY

We now restrict our attention to gauge group  $G = SU(2)$ . Since we are focusing on a single Higgs field (28) we will be able to directly use many of the results in section two. We will assume  $v^2 \neq 0$  so that  $SU(2) \rightarrow U(1)$ . BPS states with charges  $(n_m, n_e)$  satisfy the mass formula (23). It is an important fact that BPS states with  $(n_m, n_e)$  relatively prime integers are absolutely stable for all values of  $\tau$ . This is deduced by charge conservation and the triangle inequality.

We now state the  $S$ -duality conjecture: the  $SL(2, \mathbf{Z})$  transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} ,$$

$$(n_m, n_e) \rightarrow (n_m, n_e) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \quad (30)$$

where  $a, b, c, d \in \mathbf{Z}$ ,  $ad - bc = 1$ , give the same theory [2]. The  $SL(2, \mathbf{Z})$  group is generated by  $T : \tau \rightarrow \tau + 1$  which is equivalent to the transformation  $\theta \rightarrow \theta + 2\pi$  that can be deduced in perturbation theory (after a relabeling of states) and  $S : \tau \rightarrow -1/\tau$ , which for  $\theta = 0$ , is equivalent to strong-weak coupling and electric-magnetic duality originally considered in [1,14].

A simple check of S-duality is that the BPS mass formula (23) is invariant. This is a necessary condition because the BPS mass formula can be derived from the supersymmetry algebra and hence it holds in the quantum theory. To see the invariance one should note that  $v \rightarrow v' = v|c\tau + d|$  under a  $SL(2, \mathbf{Z})$  transformation because we rescaled the higgs field by a factor of the coupling constant  $e$ .

We now argue that there are more sophisticated tests of S-duality. We begin by noting that the perturbative spectrum can be determined at weak coupling and consists of a neutral massless photon multiplet  $(0, 0)$  and massive  $W^\pm$ -boson BPS multiplets with charge  $(0, \pm 1)$ . S-duality maps the  $W$ -boson multiplets to BPS states  $(k, l)$ , with  $k$  and  $l$  relatively prime integers, typically at strong coupling. But since these are precisely the absolutely stable BPS states they cannot decay as we vary  $\tau$  and we deduce that they must also exist at weak coupling where we can search for them using semi-classical techniques. We will argue that they can be translated into the existence of certain geometric structures on the moduli space  $\mathcal{M}_k$ .

If we assume that the entire spectrum of BPS states does not vary as we change the coupling then we can deduce that the above BPS states are the only BPS states in the theory. Any extra states would necessarily be BPS states at threshold, i.e., at threshold to decay into other BPS states. For example the mass of a potential BPS state  $(2, 2)$  is only marginally stable into the decay of two  $(1, 1)$  states. If there were such states at threshold then we could use S-duality to map them to purely electrically charged states  $(0, n)$  with  $n \neq \pm 1$ . Using our assumption that the spectrum of BPS states doesn't change as we

vary the coupling we conclude that these states should exist at weak coupling but this contradicts what we see in perturbation theory. We believe that the additional assumption is weak due to the very strong constraints that  $N=4$  supersymmetry imposes on the quantum theory. What is now known about the BPS spectrum, and will be reviewed below, supports this assumption<sup>5</sup>.

Let us translate the prediction of the spectrum of BPS states with relatively prime charges  $(k, l)$  into statements about the moduli space of monopoles. The semiclassical analysis begins with the moduli space  $\mathcal{M}_k$  of BPS monopole solutions. We have noted that the  $4k$  coordinates on  $\mathcal{M}_k$  can be interpreted as collective coordinates that must be introduced for  $4k$  bosonic zero modes. In the  $N=4$  context we also have fermionic zero modes. These arise from solving the Dirac equation for the fermion fields in the presence of a given monopole solution. There are four Weyl or two Dirac spinors in the adjoint of  $SU(2)$  and an index theorem [18] tells us that there are  $4k$  fermionic  $c$ -number zero modes that require the introduction of  $4k$  complex Grassmann odd fermionic "collective coordinates"  $\psi^\alpha$ . This means the low-energy ansatz for the fermions will include terms of the schematic form

$$\lambda(x, t) \sim \psi(t) \lambda^{cl}(x, Z(t)) \quad (31)$$

where  $\lambda^{cl}(x, Z)$  is a  $c$ -number fermion zero mode for the monopole solution specified by the moduli  $Z$ . We noted above that BPS states preserve half of the supersymmetry. This manifests itself in the fact that half of the supersymmetry generators leave the classical BPS monopole solution invariant. It can be shown that the bosonic and fermionic zero modes form a multiplet of the unbroken supersymmetries. This is essential in obtaining a supersymmetric low-energy ansatz for the fields. The ansatz for the low-energy fields is technically quite involved and has been carried out in [5,19]. The result of substituting the ansatz into the spacetime Lagrangian leads to the

<sup>5</sup>See [17] for an alternative way of determining the BPS spectrum that also bears on this issue.

following supersymmetric quantum mechanics

$$S = \frac{1}{2} \int dt (\mathcal{G}_{\alpha\beta} [\dot{Z}^\alpha \dot{Z}^\beta + i\bar{\psi}^\alpha \gamma^0 D_t \psi^\beta] + \frac{1}{6} R_{\alpha\beta\gamma\delta} \bar{\psi}^\alpha \psi^\gamma \bar{\psi}^\beta \psi^\delta) - \frac{4\pi v}{e^2} k, \quad (32)$$

where we have traded the complex  $\psi^\alpha$  for a real two component Majorana spinor  $\psi_i^\alpha$  and the covariant derivative of these fermions is obtained using the pullback of the Christoffel connection:  $D_t \psi^\alpha = \dot{\psi}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{Z}^\beta \psi^\gamma$ . For a general metric the supersymmetric quantum mechanics has  $N=1$  supersymmetry specified by a real two component spinor  $\epsilon$ . In the case that the target is hyper-Kähler there are an additional three supersymmetries with parameters  $\epsilon^{(m)}$  [20]. Since the monopole moduli spaces are hyper-Kähler there are eight real supersymmetry parameters which precisely correspond to the half of the spacetime supersymmetry that is preserved by BPS states.

The quantisation of this model is discussed in [21]. The states are in one to one correspondence with differential forms on  $\mathcal{M}_k$ . There are four real two component supercharges. Replacing one of these with a complex one component charge  $Q$  we can write the Hamiltonian as  $H = \{Q, Q^\dagger\} + 4\pi n_m v / e^2$  where we have included the topological term. The supersymmetry charge  $Q$  is realised as the exterior derivative acting on forms,  $Q = d$ , and  $Q^\dagger$  as its adjoint  $Q^\dagger = d^\dagger = *d*$  with  $*$  being the Hodge star acting on forms. As a consequence, the Hamiltonian is the Laplacian acting on differential forms

$$H = dd^\dagger + d^\dagger d + \frac{4\pi v}{e^2} n_m. \quad (33)$$

For  $n_m = 1$  we have  $\mathcal{M}_1 = \mathbf{R}^3 \times S^1$ . A basis of forms is given by  $\{1, dZ^\alpha, \dots, dZ^1 \wedge dZ^2 \wedge dZ^3 \wedge dZ^4\}$  which gives 16 states corresponding to a BPS multiplet. To be more precise we need to check that the spins of these states are the same as those of the BPS multiplet. For  $n_m = 1$  all of the fermionic zero modes can be constructed explicitly as Goldstinos by acting with the broken supersymmetry generators. One can check the angular momentum content and one finds that the spin content is that of a BPS multiplet [14]. The wave functions multiplying these forms are just

as in the bosonic case,  $e^{i\mathbf{P} \cdot \mathbf{Z}_e^{in_e \chi}}$ , corresponding to dyons with  $Q_e = n_e \epsilon + \epsilon \theta / 2\pi$ . The Laplacian on  $\mathbf{R}^3 \times S^1$  is trivial and by following the same arguments as in the bosonic case, we deduce that the mass of these states is given by (23). Putting this together we deduce that for  $n_m = 1$  there is a tower of BPS dyon states  $(n_m, n_e) = (1, n_e)$  exactly as predicted by duality.

Now we turn to  $n_m = k > 1$ . In this case  $\mathcal{M}_k = \mathbf{R}^3 \times (S^1 \times \tilde{\mathcal{M}}_k^0) / \mathbf{Z}_k$ . If we first ignore the  $\mathbf{Z}_k$  identification then the states are tensor products of forms on  $\mathbf{R}^3 \times S^1$  with forms on  $\tilde{\mathcal{M}}_k^0$ , respectively,  $|s\rangle = |\omega\rangle_{n_e} \otimes |\alpha\rangle$ . The analysis for the states  $|\omega\rangle_{n_e}$  is similar to the  $n_m = 1$  case: there are 16 differential forms that are again associated with Goldstinos and these make up the spin content of a BPS multiplet. The wave functions give rise to quantised electric charge with  $Q_e = n_e \epsilon + \epsilon \theta n_m / 2\pi$ . The energy of the states  $|s\rangle$  can be determined and we find

$$H|s\rangle = \left(\frac{P^2}{2M} + M\right) |\omega_{n_e}\rangle \otimes |\alpha\rangle + |\omega\rangle_{n_e} \otimes |\Delta\alpha\rangle \quad (34)$$

with  $M$  given by the BPS mass formula (23). Thus to get a BPS state with charges  $(n_m, n_e)$  we need a normalisable (i.e.,  $L^2$ ) harmonic<sup>6</sup> form  $\alpha$  on  $\tilde{\mathcal{M}}_k^0$ . The action of  $\mathbf{Z}_k$  on the  $S^1$  is a cyclic shift which leads to the action  $|\omega\rangle_{n_e} \rightarrow e^{2\pi i n_e / k} |\omega\rangle_{n_e}$ . Hence for the state  $|s\rangle$  to be well defined on  $\mathcal{M}_k$  we need the form  $|\alpha\rangle$  to transform as  $|\alpha\rangle \rightarrow e^{-2\pi i n_e / k} |\alpha\rangle$ .

Recalling that duality predicts that for each relatively prime integers  $(k, l)$  there is a unique BPS state, we conclude that  $\tilde{\mathcal{M}}_k^0$  must have a unique normalisable harmonic form which picks up a phase  $e^{-2\pi i l / k}$  under the  $\mathbf{Z}_k$  action, for every relatively prime pair of integers  $(k, l)$ . The uniqueness implies that the form must either be self-dual or anti-self dual, since otherwise acting with the Hodge star  $*$  would generate another harmonic form with the above properties. This conjecture was formulated by Sen who also found

<sup>6</sup>Note that if  $\Delta\alpha = \epsilon\alpha$  with  $\epsilon \neq 0$  then by acting with the supersymmetry charges one can show that it always comes in multiplets of 16. Combining this with the 16 states  $|\omega\rangle$  gives rise to a 256 multiplet of  $N=4$  supersymmetry. These states are relevant for studying the scattering of BPS states.

the harmonic form for  $k = 2$ . [2]. For  $k > 2$  substantial evidence was provided in [22] (see also [23]).

## 5. HIGHER RANK GAUGE GROUPS

We now turn to  $N=4$  theories with simple gauge groups  $G$  with rank  $r > 1$  with maximal symmetry breaking to  $U(1)^r$ . For simplicity of notation we will often discuss the case  $G = SU(3) \rightarrow U(1)^2$ . The Lie algebra of  $G$  has a maximal abelian subalgebra  $H$  with  $r$  generators  $H_i$ . We can define raising and lowering operators  $E_{\pm\alpha}$  that satisfy

$$\begin{aligned} [H_i, E_{\alpha}] &= \alpha_i E_{\alpha}, \\ [E_{\alpha}, E_{-\alpha}] &= \sum_{i=1}^r \alpha^i H_i. \end{aligned} \quad (35)$$

(a linear combination of these generators give the  $T^a$  satisfying  $\text{Tr} T^a T^b = \delta_{ab}$  that we used before).  $\alpha$  is an  $r$ -component root vector. A basis of simple roots,  $\beta^{(a)}$  ( $a = 1, \dots, r$ ), may be chosen such that any root is a linear combination of  $\beta^{(a)}$  with integral coefficients all of the same sign. Positive roots are those with positive coefficients.

We continue to work with a single Higgs field  $\Phi$  by restricting our attention to the special part of moduli space (28). We may choose the Cartan subalgebra such that our vacuum is specified by  $\langle \Phi \rangle = v \mathbf{h} \cdot \mathbf{H}$  with  $v^2 = \langle \text{Tr} \Phi^2 \rangle$ . If  $\alpha \cdot \mathbf{h} = 0$  for some root  $\alpha$  then the unbroken gauge group is nonabelian. Otherwise, maximal symmetry breaking occurs, and  $\langle \Phi \rangle$  picks out a unique set of simple roots which satisfy the condition  $\mathbf{h} \cdot \beta^{(a)} > 0$  [24].

Since the fields are in the adjoint representation, the electric quantum numbers of states live on the  $r$ -dimensional root lattice spanned by the simple roots  $\beta^{(a)}$ ,

$$\mathbf{q} = \sum n_a^e \beta^{(a)}, \quad (36)$$

where the  $n_a^e$  are integer. The electric charge (for  $\theta = 0$ ) is then given by

$$Q_e = e \mathbf{h} \cdot \mathbf{q}. \quad (37)$$

At weak coupling we deduce that for each root  $\alpha$  there is a BPS  $W$ -boson with  $\mathbf{q} = \alpha$ . For  $SU(3)$

we have  $W$ -bosons with  $\mathbf{n}^e = \pm(1, 0), \pm(0, 1)$  and  $\pm(1, 1)$  corresponding to the two simple roots  $\beta^{(1)}, \beta^{(2)}$  and the non-simple positive root  $\gamma = \beta^{(1)} + \beta^{(2)}$ , respectively. From (27) we deduce that the  $W$ -bosons corresponding to simple roots are stable, while those corresponding to the non-simple roots are only neutrally stable. In  $SU(3)$  we have that  $M\gamma = M\beta^{(1)} + M\beta^{(2)}$ .

Magnetic quantum numbers arise from topologically nontrivial field configurations. For any finite energy solution the Higgs field must approach the vacuum: let the asymptotic value along the positive  $z$ -axis be  $\Phi_0 = v \mathbf{h} \cdot \mathbf{H}$  (the value in any other direction can only differ from this by a gauge transformation). Asymptotically we also have

$$B_i = \frac{r_i}{4\pi r^3} G(\Omega), \quad (38)$$

where  $G$  is covariantly constant, and takes the value  $G_0$  along the positive  $z$ -axis. The Cartan subalgebra may be chosen so that  $G_0 = \mathbf{g} \cdot \mathbf{H}$ . For a smooth solution this quantity must satisfy a topological quantization condition [25,26]

$$e^{iG_0} = I. \quad (39)$$

The solution to this equation is

$$\mathbf{g} = 4\pi \sum n_a^m \beta^{(a)*}, \quad (40)$$

where the  $n_a^m$  are integers and the  $\beta^{(a)*}$  are the the simple coroots, defined as

$$\beta^{(a)*} = \frac{\beta^{(a)}}{\beta^{(a)2}}. \quad (41)$$

The magnetic quantum numbers thus live on the coroot lattice spanned by the  $\beta^{(a)*}$ . For maximal symmetry breaking, all of the  $n_a^m$  are conserved topological charges, labeling the homotopy class of the Higgs field configuration. For solutions of the Bogomol'nyi equations (29) all of the integers in  $n_a^m$  have the same sign. The topological charge  $\mathbf{g}$  determines the magnetic charge by the formula

$$Q_m = \frac{1}{e} \mathbf{g} \cdot \mathbf{h}. \quad (42)$$

A general dyon state may be labeled either by the electric and magnetic charge  $r$ -vectors  $\mathbf{q}, \mathbf{g}$  or



by the integer valued  $r$ -vectors  $\mathbf{n}^e$  and  $\mathbf{n}^m$ . For a BPS state the mass is given by the BPS mass formula (27) which, using (37) and (42), can be recast in the form

$$M = v |(\mathbf{h} \cdot \boldsymbol{\beta}^{(a)})n_a^e + \tau(\mathbf{h} \cdot \boldsymbol{\beta}^{(a)*})n_a^m|, \quad (43)$$

where we have reinstated  $\theta$ .

We now have enough definitions to define the action of  $S$ -duality. It is the natural generalisation of the  $SU(2)$  case (30): the  $SL(2, \mathbf{Z})$  duality on a general dyon state is given by

$$\begin{aligned} \tau &\rightarrow \frac{a\tau + b}{c\tau + d}, \\ (\mathbf{n}^m, \mathbf{n}^e) &\rightarrow (\mathbf{n}^m, \mathbf{n}^e) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \end{aligned} \quad (44)$$

and when we act with the  $S$ -generator  $S : \tau \rightarrow -1/\tau$  we must replace the group  $G$  with its dual group  $G^*$  [25]. For simply laced groups i.e., all the roots have the same length (the  $ADE$  groups), the  $N=4$  supersymmetric Lagrangian with gauge group  $G$  is the same as that of  $G^*$  since all fields are in the adjoint representation. For non-simply-laced groups this is not true since for example  $SO(2N+1)^* = Sp(N)$ . In this case one does not expect the theory to be invariant under the full  $SL(2, \mathbf{Z})$  duality group, but rather a  $\Gamma_0(2)$  subgroup [27] (see also [28]). We restrict our considerations to simply-laced gauge groups in the following.

Just as in the  $SU(2)$  case  $S$ -duality maps the perturbative  $W$ -boson states into an infinite number of dyon BPS states. For the  $SU(3)$  case we generate the following  $SL(2, \mathbf{Z})$  orbits:

$$\begin{aligned} (\mathbf{n}^m, \mathbf{n}^e) = & (k(1, 0), l(1, 0)), \\ & (k(0, 1), l(0, 1)), \\ & (k(1, 1), l(1, 1)), \end{aligned} \quad (45)$$

for relatively prime integers  $k$  and  $l$ . Like the  $SU(2)$  case we have again typically been mapped to strong coupling. For the first two classes of states we note from the BPS mass formula (43) that they are absolutely stable and hence we conclude that they also exist at weak coupling. The whole orbit of states coming from the  $(1, 1)$   $W$ -boson are only marginally stable. Consequently

we have to again employ the additional assumption that in the  $N=4$  theory the spectrum of marginal states does not change as we vary the coupling. In this case we should see these states at weak coupling also.

It is perhaps worth noting here that by starting with the perturbative spectrum of  $W$ -bosons  $S$ -duality only makes predictions about the short BPS representations of the  $N=4$  supersymmetry algebra. This is because the purely electrically charged  $W$ -bosons have parallel electric and magnetic charge vectors  $Q_e^I$  and  $Q_m^I$ . If any medium sized representations of the  $N=4$  algebra existed they would necessarily have non-parallel charge vectors and lie on separate  $SL(2, \mathbf{Z})$  orbits. It would be interesting to know if they existed.

To test the  $S$ -duality predictions (45) we begin by reviewing some aspect of BPS monopole solutions. Using an index theorem Weinberg has argued that the moduli space of monopoles of charge  $\mathbf{n}^m$  has dimension

$$d = 4 \sum_a n_a^m. \quad (46)$$

A number of explicit monopole solutions can be constructed by embedding  $SU(2)$  monopoles as follows [29]. Let  $\phi^s, A_i^s$  be an  $SU(2)$  monopole solution with charge  $k$  and Higgs expectation value  $\lambda$ . If we let  $\boldsymbol{\alpha}$  be any root satisfying  $\boldsymbol{\alpha} \cdot \mathbf{h} > 0$  then we can define an  $SU(2)$  subgroup with generators

$$\begin{aligned} t^1 &= (2\boldsymbol{\alpha}^2)^{-1/2}(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}) \\ t^2 &= -i(2\boldsymbol{\alpha}^2)^{-1/2}(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}) \\ t^3 &= (\boldsymbol{\alpha}^2)^{-1}\boldsymbol{\alpha} \cdot \mathbf{H}. \end{aligned} \quad (47)$$

A monopole with magnetic charge

$$\mathbf{g} = 4\pi k \boldsymbol{\alpha}^* \quad (48)$$

is then given by

$$\begin{aligned} \Phi &= \sum_s \phi^s t^s + v(\mathbf{h} - \frac{\mathbf{h} \cdot \boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} \boldsymbol{\alpha}) \cdot \mathbf{H} \\ A_i &= \sum_s A_i^s t^s \\ \lambda &= v \mathbf{h} \cdot \boldsymbol{\alpha}. \end{aligned} \quad (49)$$

Since the moduli space of  $SU(2)$  monopoles with charge  $k$  has dimension  $4k$  these solutions provide

a  $4k$  dimensional submanifold of monopole solutions with charge (48). Note that by embedding an  $SU(2)$  monopole with charge one we obtain spherically symmetric monopole solutions.

Weinberg has shown that there is a distinguished set of  $r$  “fundamental monopoles” with  $\mathbf{g} = 4\pi\beta^{(a)*}$  i.e., they have magnetic charge vectors  $\mathbf{n}^m$  consisting of a one in the  $a$ th position and zeroes elsewhere. The reason for calling them fundamental is twofold. First, they have no “internal” degrees of freedom: all of these solutions can be constructed by embedding an  $SU(2)$  monopole of unit charge using the corresponding simple root and consequently they have only four zero modes: three translation zero modes and a  $U(1)$  phase zero mode corresponding to dyonic excitations of the same  $U(1)$  as where the magnetic charge lies<sup>7</sup>. Secondly, the index theorem (46) is consistent with thinking of a general monopole with charge  $\mathbf{n}^m$  as a multimonopole configuration consisting of  $n_a^m$  fundamental monopoles of type  $a$ .

Note that for magnetic monopoles with charge vector  $\mathbf{g} = 4\pi k\beta^{(a)*}$  i.e., consisting of  $k$  fundamental monopoles of the same type, the dimension of moduli space is  $4k$ . Thus we deduce that these solutions can all be obtained by embedding  $SU(2)$  monopoles of charge  $k$ , using the embedding based on the same simple root.

Let us now return to the BPS states predicted by  $S$ -duality. We need to study the semiclassical quantisation for a given magnetic charge  $\mathbf{n}^m$ . Just as in the  $SU(2)$  case the bosonic zero and fermionic zero modes are paired by the unbroken supersymmetry and a low-energy ansatz again leads to the  $N=4$  supersymmetric quantum mechanics (32) on the moduli space of solutions  $\mathcal{M}_{\mathbf{n}^m}$ . First consider monopoles with  $\mathbf{n}^m = (k, 0)$  or  $\mathbf{n}^m = (0, k)$  i.e.,  $k$  fundamental monopoles of the same type. The moduli space of these monopoles is the  $SU(2)$  moduli space  $\mathcal{M}_k$ . The dyonic states with charges  $(k(1, 0), l(1, 0))$  and  $(k(0, 1), l(0, 1))$  predicted by duality are equivalent to the harmonic forms on  $\mathcal{M}_k$  required by  $S$ -duality in the  $SU(2)$  theory. The results of [2,22] thus constitute tests of duality for higher

rank gauge groups.

The new predictions for  $SU(3)$  monopoles arise in the sectors with both magnetic quantum numbers non-zero. In particular, the  $(k(1, 1), l(1, 1))$  dyon states should arise as bound states of  $k(1, 0)$  and  $k(0, 1)$  monopoles. Note from the BPS mass formula that these states are only neutrally stable and consequently they should emerge as bound states at threshold. At present only for  $k = 1$  have these states been shown to exist. Let us make some comments on this case.

It was shown in [30–32] that the moduli space for  $\mathbf{n}^m = (1, 1)$  is given by

$$\mathcal{M}_{(1,1)} = \mathbf{R}^3 \times \frac{\mathbf{R} \times \mathcal{M}_{\text{TN}}}{\mathbf{Z}} \quad (50)$$

where  $\mathcal{M}_{\text{TN}}$  is four-dimensional Taub-NUT space. The  $\mathbf{R}^3$  factor corresponds to the centre of mass of the  $(1, 0)$  and  $(0, 1)$  monopole configuration. Taub-NUT space is a hyper-Kähler manifold as required for the quantum mechanics (32) to have  $N=4$  supersymmetry. Taub-NUT space has  $U(2)$  isometry, of which a  $SU(2)_L$  subgroup corresponds to the action of rotating the multimonopole configuration in space. That this is  $SU(2)$  and not  $SO(3)$  can be demonstrated by studying the zero modes about the spherically symmetric  $(1, 1)$  solution that can be obtained via the  $SU(2)$  embedding using the root  $\gamma$  [30]. Note that the fixed point set of the  $SU(2)_L$  action is a single point in Taub-NUT space (the “nut”) and this corresponds to the spherically symmetric solutions. The extra  $U(1)_R$  isometry in  $U(2)$  combined with the factor  $\mathbf{R}$  in (50) and the identification under the integers  $\mathbf{Z}$  lead to dyon states with electric charge  $\mathbf{n}^e$ . One might have expected an  $S^1$  factor rather than  $\mathbf{R}$  for the total electric charge in the  $(1, 1)$  direction (i.e., parallel to the magnetic charge), but this is not quite correct due to the fact that in general the masses of the two fundamental monopoles  $(1, 0)$  and  $(0, 1)$  are not equal.

The basis of 16 forms on  $\mathbf{R}^3 \times \mathbf{R}$  leads to a BPS supermultiplet of 16 states in the  $N=4$  supersymmetric quantum mechanics. In order to get the dyon BPS states predicted by duality (45) with magnetic charge  $(1, 1)$  there must exist a unique normalizable harmonic (anti)-self-dual two-form

<sup>7</sup>One can check that the embedded  $SU(2)$  solutions are invariant under gauge transformations of the other  $U(1)$ ’s.

on Taub-NUT space that is invariant under the  $U(1)_R$  isometry to ensure that the electric charge is  $l(1, 1)$ . Such a harmonic form exists [30,31].

## 6. SOME OPEN PROBLEMS

As we have discussed a number of highly non-trivial checks of exact  $S$ -duality can be carried out by studying the geometry of monopole moduli spaces. Let us conclude by discussing some open issues.

In the  $N=4$  theory with maximal symmetry breaking the BPS states predicted by duality correspond to normalisable harmonic forms on BPS monopole moduli spaces. For gauge group  $SU(2)$  the work of [22] provided substantial evidence for the appropriate harmonic forms. It would be desirable to have a similar analysis for higher rank gauge groups. For  $SU(3) \rightarrow U(1)^2$  we have seen that the duality predictions for monopoles with  $k$  fundamental monopoles of the same type, i.e.,  $\mathbf{n}^m = k(1, 0)$  or  $k(0, 1)$  reduce to those of the  $SU(2)$  case. The new  $SU(3)$  predictions with  $\mathbf{n}^m = (1, 1)$  correspond to a harmonic form on Taub-NUT space. It remains to be shown that the BPS states with  $\mathbf{n}^m = k(1, 1)$  exist for  $k \neq 1$  and that the BPS states in (45) are the only ones in the spectrum. The  $SU(2)$  and  $SU(3)$  results can be embedded in higher rank gauge groups. This can be illustrated by considering  $G = SU(4) \rightarrow U(1)^3$ . In this case  $S$ -duality predicts BPS states with  $\mathbf{n}^m = k(1, 0, 0)$ ,  $k(0, 1, 0)$ ,  $k(0, 0, 1)$ , which are equivalent to the  $SU(2)$  predictions,  $k(1, 1, 0)$ ,  $k(0, 1, 1)$  which are equivalent to the  $SU(3)$  predictions, and  $k(1, 1, 1)$  which are the new  $SU(4)$  predictions. Apart from the cases we have discussed there is only one more class of moduli spaces that are explicitly known: when there are no more than a single fundamental monopole of each type (e.g.  $(1, 1, 1)$  for the  $SU(4)$  example) [33–35]. It is a natural generalisation of Taub-NUT space and the harmonic form predicted by duality has been shown to exist [36]. The major obstacle in verifying more of the  $S$ -duality predictions is our lack of knowledge about the monopole moduli spaces.

New issues arise in the  $N=4$  case when a non-abelian gauge group remains unbroken. In this

case the existence of massless  $W$ -bosons might seem to require dual massless monopoles which cannot be studied as conventional semi-classical solitons. There are also massive monopoles that can be studied. If they carry non-abelian magnetic charge there are subtleties to do with the moduli space approximation due to the non-normalisability of zero modes corresponding to global gauge rotations (see e.g., [37,38]). The moduli spaces of monopoles that have net abelian magnetic charge can in some cases be determined as limits of moduli spaces in which the symmetry is maximally broken. Curiously, it is claimed that the harmonic forms found by [36] become non-normalisable in this limit [38].

In this paper we have only discussed  $N=4$  theories. Special theories with  $N=2$  supersymmetry and vanishing  $\beta$ -function are also candidates for exhibiting exact duality. For gauge group  $SU(2)$  with  $N_f=4$  hypermultiplets in the fundamental representation, it is conjectured that the duality group is the semi-direct product of  $SL(2, \mathbf{Z})$  with the global flavour symmetry group  $Spin(8)$  [39].  $SL(2, \mathbf{Z}) \bmod 2$  is isomorphic to  $S_3$  the permutation group of three objects, which is also the group of outer automorphisms of  $Spin(8)$  which acts on the  $\mathbf{v}, \mathbf{s}, \mathbf{c}$ , conjugacy classes. This duality predicts an orbit of vector multiplets at threshold with charges  $(n_m, n_e) = 2(k, l)$  and an orbit of hypermultiplets with charges  $\pm(k, l)$ . In this theory there are half as many fermionic zero modes coming from the vector multiplet as in the  $N=4$  theory and the net result is that one should study an  $N=2$  supersymmetric quantum mechanics on the moduli space of  $SU(2)$  monopoles  $\mathcal{M}_k$ . As a consequence, the states are spinors on  $\mathcal{M}_k$  not forms. In addition there are fermionic zero modes coming from the hypermultiplets that give rise to a natural  $O(k)$  bundle on  $\mathcal{M}_k$  [40]. The BPS states predicted by duality correspond to certain harmonic spinors coupled to this bundle. For monopole charge  $k = 2$  these were found using index theory [41,42]. Perhaps an analysis similar to [22] is possible for higher monopole charge. See also [17] for a different approach.

For higher rank theories with  $N=2$  supersymmetry and vanishing  $\beta$ -function less is known about exact duality. A straightforward attempt

to find the duality group acting on the lattice of electric and magnetic charges was attempted in [43] for  $G = SU(3)$  (see also [44]) but the results were inconclusive.

All of these tests we have been discussing concern the spectrum of BPS states. If the exact duality conjectures are true then they should also apply to non-BPS states, for example the scattering of BPS states. Since such process are not protected by supersymmetry it remains a challenging problem to find evidence for  $S$ -duality in this sector.

More generally we would like to know the underlying reasons for duality in field theory. String theory duality would seem to provide one answer since we can embed these gauge theories in various string theory settings. Of course this still leaves the more involved issue of elucidating the deeper principles that underly string theory duality.

## REFERENCES

1. C. Montonen and D. Olive, *Phys. Lett.* **72B** (1977) 117.
2. A. Sen, *Phys. Lett.* **B329** (1994) 217.
3. M.K. Prasad and C.M. Sommerfeld, *Phys. Rev. Lett.* **35** (1975) 760.
4. E.B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24** (1976) 449.
5. J.P. Gauntlett, *Nucl. Phys.* **B411** (1994) 443.
6. M. F. Atiyah and N. J. Hitchin, *The Geometry and Dynamic of Magnetic Monopoles*, Princeton Univ. Press, Princeton, NJ, 1988.
7. R. Rajaraman, *Solitons and Instantons*, North-Holland 1982.
8. E.J. Weinberg, hep-th/9610065.
9. N.S. Manton, *Phys. Lett.* **110B** (1982) 54.
10. S. Coleman, S. Parke, A. Neveu and C.M. Sommerfeld, *Phys. Rev.* **D15** (1977) 544.
11. G.W. Gibbons and N. S. Manton, *Nucl. Phys.* **274** (1986) 183.
12. B. Schroers, *Nucl. Phys.* **367** (1991) 177.
13. E. Witten, *Phys. Lett.* **86B** (1979) 283.
14. H. Osborn, *Phys. Lett.* **83B** (1979) 321.
15. E. Witten and D. Olive, *Phys. Lett.* **78B** (1978) 97.
16. C. Fraser and T. Hollowood, hep-th/9704011.
17. F. Ferrari, hep-th/9702166.
18. C. Callias, *Comm. Math. Phys.* **62** (1978) 213.
19. J. D. Blum, *Phys. Lett.* **B333** (1994) 92.
20. L. Alvarez-Gaumé and D. Freedman, *Comm. Math. Phys.* **80** (1981) 443.
21. E. Witten, *Nucl. Phys.* **B202** (1982) 253.
22. G. Segal and A. Selby, *Comm. Math. Phys.* **177** (1996) 775.
23. M. Porrati, *Phys. Lett.* **B377** (1996) 67.
24. E. J. Weinberg, *Nucl. Phys.* **B167** (1980) 500.
25. P. Goddard, J. Nuyts and D. Olive, *Nucl. Phys.* **B125** (1977) 1.
26. F. Englert and P. Windey, *Phys. Rev.* **D14** (1976) 2728.
27. L. Girardello, A. Giveon, M. Porrati and A. Zaffaroni, *Phys. Lett.* **B334** (1994) 331; *Nucl. Phys.* **B448** (1995) 127.
28. N. Dorey, C. Fraser, T. Hollowood and M.A. Kneipp, *Phys. Lett.* **383B** (1996) 422.
29. F. A. Bais, *Phys. Rev.* **D18** (1978) 1206.
30. J.P. Gauntlett and D.A. Lowe, *Nucl. Phys.* **B472** (1996) 194.
31. K. Lee, E. Weinberg and P. Yi, *Phys. Lett.* **B376** (1996) 97.
32. S.A. Connell, *The dynamics of the  $SU(3)$  charge (1, 1) magnetic monopoles*, University of South Australia preprint.
33. K. Lee, E. Weinberg and P. Yi, *Phys. Rev.* **D54** (1996) 1633.
34. M.K. Murray, hep-th/9605054.
35. G. Chalmers, hep-th/9605182
36. G.W. Gibbons, *Phys. Lett.* **B382** (1996) 53.
37. N. Dorey, C. Fraser, T. Hollowood and M.A. Kneipp, hep-th/9512116.
38. K. Lee, E. Weinberg and P. Yi, *Phys. Rev.* **D54** (1996) 6354.
39. N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484.
40. N.S. Manton and B.J. Schroers, *Ann. Phys.* **225** (1993) 290.
41. J.P. Gauntlett and J. Harvey *Nucl. Phys.* **B463** (1996) 287.
42. S. Sethi, M. Stern and E. Zaslow, *Nucl. Phys.* **B457** (1995) 484.
43. M. Cederwall and M. Holm, hep-th/9603134.
44. O. Aharony and S. Yankielowicz, *Nucl. Phys.* **B473** (1996) 93.