

TRIANGULATED CATEGORIES IN THE MODULAR REPRESENTATION THEORY OF FINITE GROUPS

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1. INTRODUCTION AND NOTATION

1.1. **Introduction.** There are two major examples of triangulated categories that have assumed importance in modular representation theory in recent years: derived categories and stable module categories. These are not unrelated: the stable module category of a group algebra is equivalent to a natural quotient (in the sense of triangulated categories) of its derived category: this is proved in [17]. However, in this survey we shall describe some applications of the two kinds of category that are rather different in flavour.

In the first half, we shall discuss some important conjectures on equivalences between derived categories of block algebras. Much recent work on the representation theory of general finite groups in characteristic p has been concerned with investigating the relationship between the representation theory of a group G and of its ' p -local subgroups', i.e., normalizers of non-trivial p -subgroups. A theme running through many recent conjectures has been the belief that many aspects of the representation theory of G should be 'determined locally'. For example, a famous conjecture of Alperin claims that the number of isomorphism classes of non-projective irreducible representations of G over a field of characteristic p should be determined locally in a precise manner. But among this circle of conjectures, perhaps the most satisfying is a conjecture of Broué's claiming that, in certain circumstances, blocks of G and of its local subgroups should have equivalent derived categories. The reason that this seems so satisfying is that it makes such a precise connection between the 'global' and the 'local' representation theory; it is to be hoped that, at least conjecturally, a similarly precise statement will be found for general blocks.

In the second half of this survey, we shall discuss some recent work on stable module categories whose flavour is more related to group cohomology and the theory of varieties for modules. The theme of this work is to exploit an analogy between stable module categories and the stable homotopy category of algebraic topology, borrowing techniques from homotopy theory. These techniques require the use of limiting procedures which necessarily take us outside the world of finite dimensional modules: this is probably unsurprising to homotopy theorists, who have habitually used huge infinite dimensional spaces for many years, but might seem more shocking to a representation theorist. We shall see that this use of infinite dimensional modules sheds new light on the usual finite dimensional representation theory.

1.2. Notation. By a ‘module’ for a ring A , we shall mean a right module unless we specify otherwise. We shall denote the category of A -modules by $\text{Mod}(A)$ and the category of finitely presented A -modules by $\text{mod}(A)$.

Usually the rings we consider will be algebras over some commutative ring R (most often, R is a field or a discrete valuation ring). When A and B are two such R -algebras, then by an ‘ A - B -bimodule’ we shall mean a bimodule on which the left and right actions of R coincide: in other words an $A^{\text{op}} \otimes_R B$ -module.

When we deal with complexes of modules, we shall try to be consistent in using cohomological notation: i.e., our complexes will be cochain complexes, with the differential increasing degree. When we refer to a complex X , we shall use X^i for the degree i cochains, and $X[n]$ will be the complex X ‘shifted n places to the left’: i.e., $X[n]^i = X^{i+n}$, with the differentials of $X[n]$ obtained from those of X by multiplying by $(-1)^n$. We shall use the standard notation for the various categories of complexes: for example, the category of cochain complexes of A -modules will be denoted by $C(\text{Mod}(A))$, and the category of bounded complexes by $C^b(\text{Mod}(A))$, the homotopy category of complexes will be denoted by $K(\text{Mod}(A))$ and the derived category by $D(\text{Mod}(A))$.

2. EQUIVALENCES OF DERIVED CATEGORIES

Although equivalences of derived categories of module categories occur for all kinds of rings, it is probably for blocks of finite group algebras that they are being studied most intensively at present. This is because of conjectures of Broué [10] that predict that such equivalences are widespread in modular representation theory and that they give a structural explanation of why some long-standing character-theoretic conjectures should be true.

There are some special properties of group algebras and their blocks that simplify and enhance the general ‘Morita theory’ for derived categories when applied to such algebras. For one thing, blocks are always symmetric algebras, and we shall see that this allows us to slightly simplify the general theory. Another feature of group algebras is that they have many special classes of representations, such as permutation representations, that seem particularly natural to study; we shall see that such special representations also seem to shed light on the derived equivalences that are expected to exist between blocks.

2.1. Some remarks on symmetric algebras. The usual definition of a symmetric algebra over a field k (e.g., [2, Definition 1.6.1]) is as follows.

Definition 2.1. A finite dimensional k -algebra A is **symmetric** if there is a linear map

$$\theta : A \longrightarrow k$$

with the following two properties:

- (i) θ is symmetric, meaning that for all pairs of elements $a, b \in A$, $\theta(ab) = \theta(ba)$.
- (ii) The kernel of θ contains no non-zero left or right ideals of A .

The important motivating example is the group algebra kG of a finite group G over k , where θ can be taken to be the map

$$\theta\left(\sum_{g \in G} \lambda_g g\right) = \lambda_1,$$

taking a typical element of the group algebra to the coefficient of the identity element of G . More generally, if kB is a block algebra of kG , then kB is also symmetric, via the restriction of θ to kB .

There is a well-known equivalent definition that is more convenient to use when it comes to investigating module categories, and which generalizes more easily to coefficient rings other than fields. For the readers' convenience we shall include a proof that this alternative definition is equivalent.

Theorem 2.2. *A finite dimensional algebra A over a field k is symmetric if and only if A and its k -linear dual $A^\vee = \text{Hom}_k(A, k)$ are isomorphic as A -bimodules.*

Proof. Recall first that if M is any A -bimodule (e.g., $M = A$), then the natural A -bimodule structure on $M^\vee = \text{Hom}_k(M, k)$, is given by the formula

$$(a\psi b)(m) = \psi(bma),$$

for any $m \in M$, $\psi \in M^\vee$ and $a, b \in A$.

Suppose that A is symmetric and that $\theta : A \rightarrow k$ is a map with the properties required by Definition 2.1. Then we can define a map

$$\phi : A \rightarrow A^\vee$$

by

$$\phi(b) = b\theta$$

for $b \in A$. Clearly ϕ is a left A -module homomorphism, since if $a, b \in A$, then

$$\phi(ab) = ab\theta = a\phi(b).$$

Using property (ii) of θ , it is also easy to see that ϕ is a right A -module homomorphism, as

$$\phi(ba)(x) = (ba\theta)(x) = \theta(xba) = \theta(axb) = (\phi(b)a)(x)$$

for $a, b, x \in A$. Suppose that b is in $\ker(\phi)$, so $b\theta = 0$, and hence

$$0 = (b\theta)(a) = \theta(ab)$$

for all $a \in A$. Then the left ideal Ab is contained in $\ker(\theta)$, and so $b = 0$ by property (ii) of θ . So we have proved that ϕ is an injective A -bimodule homomorphism, and so ϕ must be an isomorphism, since A and A^\vee have the same (finite) k -dimension.

Conversely, suppose that

$$\phi : A \rightarrow A^\vee$$

is an A -bimodule isomorphism, and let

$$\theta = \phi(1) \in A^\vee.$$

Then, if $a, b \in A$,

$$\theta(ab) = (b\theta)(a) = (b\phi(1))(a) = \phi(b)(a) = (\phi(1)b)(a) = (\theta b)(a) = \theta(ba),$$

and so θ is symmetric. If $b \in A$ is an element of a right ideal of A contained in $\ker(\theta)$, then

$$0 = \theta(ba) = (\theta b)(a)$$

for all $a \in A$, and so

$$0 = \theta b = \phi(b),$$

and hence $b = 0$. Therefore $\ker(\theta)$ contains no non-zero right ideals and similarly contains no non-zero left ideals. Thus θ satisfies both conditions of Definition 2.1, and so A is symmetric. \square

In the case of a finite group algebra kG , the kG -bimodule isomorphism between kG and kG^\vee takes an element $h \in G$ to the element $\theta_h \in kG^\vee$ with

$$\theta_h\left(\sum_{g \in G} \lambda_g g\right) = \lambda_{h^{-1}}.$$

In fact, for any commutative coefficient ring R , this formula gives an RG -bimodule isomorphism between RG and $RG^\vee = \text{Hom}_R(RG, R)$, so it makes sense to define a symmetric R -algebra to be an R -algebra A which is finitely generated and projective as an R -module and which is isomorphic to its R -linear dual A^\vee as an A -bimodule.

Let us give an easy but important application of this way of thinking of symmetric algebras. The following theorem tells us that, when dealing with symmetric algebras, all reasonable notions of duality are equivalent.

Theorem 2.3. *Let A and B be symmetric R -algebras, for some commutative ring R .*

(i) *The functors $\text{Hom}_R(?, R)$ and $\text{Hom}_A(?, A)$ are isomorphic as functors from right A -modules to left A -modules (or vice versa).*

(ii) *The functors $\text{Hom}_R(?, R)$, $\text{Hom}_A(?, A)$ and $\text{Hom}_B(?, B)$ are all isomorphic as functors from A - B -bimodules to B - A -bimodules.*

Proof. The isomorphism of (i) holds since

$$\begin{aligned} \text{Hom}_A(?, A) &\cong \text{Hom}_A(?, \text{Hom}_R(A, R)) \\ &\cong \text{Hom}_R(? \otimes_A A, R) \\ &\cong \text{Hom}_R(?, R), \end{aligned}$$

and the isomorphisms of (ii) follow from this by naturality. \square

The following corollary will be important when we look at functors between module categories and derived categories of blocks.

Corollary 2.4. *Let A and B be symmetric R -algebras, and let M be an A - B -bimodule that is finitely generated and projective as a left A -module and as a right B -module. Then the functor*

$$? \otimes_B M^\vee : \text{Mod}(B) \longrightarrow \text{Mod}(A),$$

where $M^\vee = \text{Hom}_R(M, R)$, is both left and right adjoint to

$$? \otimes_A M : \text{Mod}(A) \longrightarrow \text{Mod}(B).$$

Proof. The right adjoint to the functor $? \otimes_A M$ is $\text{Hom}_B(M, ?)$. For right B -modules X and Y , consider the natural map of R -modules

$$X \otimes_B \text{Hom}_B(Y, B) \longrightarrow \text{Hom}_B(Y, X)$$

defined by

$$x \otimes \alpha \longmapsto [y \mapsto x\alpha(y)]$$

for $x \in X$, $y \in Y$ and $\alpha \in \text{Hom}_B(Y, B)$. If $Y = B$, it is straightforward to check that this map is an isomorphism between the modules $X \otimes_B \text{Hom}_B(B, B)$ and $\text{Hom}_B(B, X)$, which are both naturally isomorphic to X . Hence, by additivity, the map is an isomorphism whenever Y is a finitely generated projective B -module. Taking $Y = M$, we get, for any B -module X , a natural isomorphism

$$X \otimes_B \text{Hom}_B(M, B) \longrightarrow \text{Hom}_B(M, X)$$

which, by naturality in M_B , is an A -module homomorphism. Since $\text{Hom}_B(M, B) \cong M^\vee$ by Theorem 2.3, it follows that

$$? \otimes_B M^\vee \cong \text{Hom}_B(M, ?)$$

is right adjoint to $? \otimes_A M$.

Using Theorem 2.3 again, the R -linear dual $\text{Hom}_R(P, R)$ of a finitely generated projective left A -module P is isomorphic to $\text{Hom}_A(P, A)$, which is a finitely generated projective right A -module (as can be seen by considering the case $P = A$), and similarly the dual of a finitely generated projective right B -module is finitely generated and projective as a left B -module. Hence the B - A -bimodule M^\vee is finitely generated and projective as a left B -module and as a right A -module. We can therefore apply what we have already proved to M^\vee to deduce that $? \otimes_B M^\vee$ is left adjoint to $? \otimes_A (M^\vee)^\vee$. However, M is finitely generated and projective as an R -module, and so the natural map

$$M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$$

sending $m \in M$ to $\beta \mapsto \beta(m)$ is an isomorphism: even, by naturality, an isomorphism of A - B -bimodules. Hence $(M^\vee)^\vee \cong M$, and we are done. \square

It is fairly straightforward to generalize Corollary 2.4 to deal with functors between chain homotopy categories or derived categories.

Corollary 2.5. *Let A and B be symmetric R -algebras, and let X be a bounded complex of A - B -bimodules which are finitely generated and projective as left A -modules and as right B -modules. Then the functor $? \otimes_A X$ is both left and right adjoint to the functor $? \otimes_B X^\vee$, where $X^\vee = \text{Hom}_R(X, R)$. This is true whether we regard them as functors between categories of complexes, chain homotopy categories of complexes or (since all the functors involved are exact) between derived categories.*

Proof. The only slight refinement of Corollary 2.4 that we need is that the adjunctions described there are natural in M , in the sense that if M and N are two bimodules satisfying

the conditions of that corollary, and if $\gamma : M \rightarrow N$ is a bimodule homomorphism, then, for example, the diagram of bifunctors

$$\begin{array}{ccc} \mathrm{Hom}_B(? \otimes_A N, ?) & \longrightarrow & \mathrm{Hom}_A(? , ? \otimes_B N^\vee) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_B(? \otimes_A M, ?) & \longrightarrow & \mathrm{Hom}_A(? , ? \otimes_B M^\vee) \end{array}$$

commutes, where the horizontal maps are the adjunction isomorphisms and the vertical maps are induced by γ . This follows easily from the proof of Corollary 2.4.

Now if C is a complex of A -modules and D is a complex of B -modules, then we have, by Corollary 2.4 and the remark we have just made, a natural isomorphism

$$\mathrm{Hom}_B(C \otimes_A X, D) \cong \mathrm{Hom}_A(C, D \otimes_B X^\vee)$$

of triple complexes of R -modules. Taking the ‘completed’ total complexes (i.e., the analogue of the total complex where direct products rather than direct sums are used to form the terms in each degree), we have a natural isomorphism of degree zero cocycles

$$\mathrm{Hom}_{C(\mathrm{Mod}(B))}(C \otimes_A X, D) \cong \mathrm{Hom}_{C(\mathrm{Mod}(A))}(C, D \otimes_B X^\vee)$$

and a natural isomorphism of degree zero cohomology

$$\mathrm{Hom}_{K(\mathrm{Mod}(B))}(C \otimes_A X, D) \cong \mathrm{Hom}_{K(\mathrm{Mod}(A))}(C, D \otimes_B X^\vee),$$

giving adjunctions at the level of categories of complexes and of the chain homotopy categories. Since the functors involved are all exact, we also get an adjunction at the level of derived categories.

Similarly $? \otimes_B X^\vee$ is also left adjoint to $? \otimes_A X$.

□

Some more detailed results along these lines can be found in [9].

2.2. Derived equivalences between symmetric algebras. Throughout this section R will be a commutative noetherian coefficient ring. The examples we will have in mind are a field or a complete discrete valuation ring.

In [18] it was proved that if A and B are two R -algebras that are projective as R -modules (which is certainly the case if they are symmetric R -algebras according to the definition proposed above), and if they are derived equivalent, then there is an object X of the derived category $D^b(\mathrm{Mod}(A^{op} \otimes_R B))$ of A - B -bimodules, called a **two-sided tilting complex**, so that

$$? \otimes_A^L X : D^b(\mathrm{Mod}(A)) \rightarrow D^b(\mathrm{Mod}(B))$$

is an equivalence.

Lemma 2.6. *If A and B are symmetric R -algebras that are derived equivalent via a two-sided tilting complex, then X is isomorphic (in $D^b(\mathrm{Mod}(A^{op} \otimes_R B))$) to a bounded complex of finitely generated A - B -bimodules, projective as left A -modules and as right B -modules.*

Proof. By Proposition 3.1 of [18], X is isomorphic in $D^b(\text{Mod}(B))$ to a bounded complex of finitely generated projective B -modules. Therefore X has cohomology in only finitely many degrees, and this cohomology is finitely generated over R . We can therefore choose a projective A - B -bimodule resolution P of X whose terms are finitely generated. Since P is isomorphic in $D^b(\text{Mod}(B))$ to a bounded complex of projective modules, it follows that the degree n cocycles of P form a projective B -module for all but finitely many n . Similarly they form a projective left A -module for all but finitely many n , and so for n small enough, the complex

$$\dots \rightarrow 0 \rightarrow Z^n \rightarrow P^n \rightarrow P^{n+1} \rightarrow \dots,$$

where Z^n is the bimodule of degree n cocycles of P , will be isomorphic to X in the derived category (since all the homology of P will be in degrees greater than n) and will be a complex of finitely generated bimodules projective as left A -modules and right B -modules. □

Because of this lemma, we can always take our two-sided tilting complexes to satisfy the conditions described: i.e., to be bounded complexes of finitely generated bimodules, projective both as left and as right modules. This has the benefit that the functor $? \otimes_A X$ is exact, and so we do not need to take the derived functor to get an equivalence between the derived categories of A and B . Also, if X is such a complex, then Corollary 2.5 tells us that the functor $? \otimes_B X^\vee$ is left and right adjoint to $? \otimes_A X$, and so (at the level of derived categories, at least, where $? \otimes_A X$ is an equivalence) $? \otimes_B X^\vee$ is a ‘quasi-inverse’ to $? \otimes_A X$: in other words, the two functors compose in either order to give functors isomorphic to the identity functor. In terms of complexes, this is equivalent to the isomorphisms

$$X \otimes_B X^\vee \cong A$$

in $D^b(\text{Mod}(A^{op} \otimes_R A))$, and

$$X^\vee \otimes_A X \cong B$$

in $D^b(\text{Mod}(B^{op} \otimes_R B))$. In other words, $X \otimes_B X^\vee$ and $X^\vee \otimes_A X$ both have cohomology concentrated in degree zero and isomorphic to ${}_A A_A$ and ${}_B B_B$ respectively.

In fact, the construction described in Lemma 2.6 gives more. The complex constructed there has the property that all but one term is projective as a *bimodule*. The dual of this complex has the same property, and so if we take X to be this complex, then every term of $X \otimes_B X^\vee$ except for the degree zero term is projective as an A -bimodule. This is because the tensor product of an A - B -bimodule M that is projective as a left A -module and a projective B - A -bimodule is projective as an A -bimodule, since it is a direct summand of a direct sum of terms isomorphic to

$$M \otimes_B (B^{op} \otimes_R A) \cong M \otimes_R A.$$

Since we also know that $X \otimes_B X^\vee$ is self-dual and has homology concentrated in degree zero, it follows that it must be split. In other words,

$$X \otimes_B X^\vee \cong A$$

in the chain homotopy category $K^b(\text{Mod}(A^{op} \otimes_R A))$, not just in the derived category. Similarly,

$$X^\vee \otimes_A X \cong B$$

in $K^b(\text{Mod}(B^{op} \otimes_R B))$. In [20], we call a two-sided tilting complex with this extra property a **split endomorphism tilting complex**, since $X \otimes_B X^\vee$ is naturally isomorphic to the complex $\text{End}_B(X)$. We have proved the following.

Proposition 2.7. *If A and B are symmetric R -algebras that are derived equivalent, then there is a split endomorphism tilting complex that induces an equivalence of derived categories.*

From the definition of a split endomorphism tilting complex, the following is clear.

Theorem 2.8. *If A and B are symmetric R -algebras and ${}_A X_B$ is a split endomorphism tilting complex, then $? \otimes_A X$ and $? \otimes_B X^\vee$ are quasi-inverse equivalences between the chain homotopy categories $K^b(\text{Mod}(A))$ and $K^b(\text{Mod}(B))$.*

Actually, although we have proved that a two-sided tilting complex for symmetric algebras can always be chosen to be a split endomorphism tilting complex, the complex that our proof provides, with only one non-projective term, is not always the most suitable one: in general there are many choices of such complexes that are isomorphic in the derived category but not in the homotopy category.

2.3. Derived equivalences between blocks: generalities. Of course, all that we have said in the last two sections applies to blocks of finite group algebras, and it is in this situation that some of the most intriguing examples of derived equivalences are conjectured to occur.

Let us specialize our general coefficient ring R . From now on, we shall be considering blocks with coefficients in a field or a complete discrete valuation ring. Let us choose a prime p , and let \mathcal{O} be a complete discrete valuation ring with residue field k of characteristic p and with field of fractions K of characteristic 0. Since we shall not be concerned with rationality questions, we shall also assume that these rings are all ‘large enough’, meaning that k and K contain all the $|G|$ th roots of unity for every group G under consideration.

First, let us recall the basic ideas of block theory. If G is a finite group, then there is a unique decomposition

$$(1) \quad 1 = e_1 + e_2 + \cdots + e_n$$

of the multiplicative identity of kG into primitive orthogonal central idempotents: in other words, the e_i are non-zero elements of the centre of kG , $e_i^2 = e_i$ (i.e., e_i is idempotent) and $e_i e_j = 0$ if $i \neq j$ (i.e., e_i is orthogonal to e_j), and there is no way of writing any e_i as the sum of two non-zero orthogonal central idempotents. This implies that the group algebra kG is a direct product

$$kG = A_1 \times A_2 \times \cdots \times A_n$$

of k -algebras, where $A_i = kG.e_i$ is an algebra with multiplicative identity e_i , called a **block algebra**, and is indecomposable as a two-sided ideal of kG .

One of the important elementary results of block theory is that the decomposition (1) lifts uniquely to a decomposition

$$(2) \quad 1 = \tilde{e}_1 + \tilde{e}_2 + \cdots + \tilde{e}_n$$

of $1 \in \mathcal{O}G$ into primitive orthogonal central idempotents. Hence

$$\mathcal{O}G = \tilde{A}_1 \times \tilde{A}_2 \times \cdots \times \tilde{A}_n,$$

where $\tilde{A}_i = \mathcal{O}G.\tilde{e}_i$. Also $A_i \cong \tilde{A}_i \otimes_{\mathcal{O}} k$ is the reduction modulo the maximal ideal of \mathcal{O} of \tilde{A}_i . Hence there is a natural 1-1 correspondence between block algebras over \mathcal{O} and over k .

Of course, the decomposition (2) can also be regarded as a decomposition of $1 \in KG$ into orthogonal central idempotents, but \tilde{e}_i is usually not primitive in KG . Since KG is a semisimple K -algebra, and so is a direct product of matrix algebras over K (remember that we are assuming that K is big enough), each factor $KG.\tilde{e}_i$ is just the product of some subset of this set of matrix algebras. Each irreducible character χ of G corresponds to one of these matrix algebras, and hence is associated to one of the idempotents \tilde{e}_i : we say then that the character χ ‘belongs’ to the block \tilde{e}_i . Perhaps we should point out that there are differing opinions as to what a block ‘really is’. To some people it is the collection of characters that belongs to the block, to some it is the block algebra, and to some it is the corresponding central idempotent: these three sets of people are probably not disjoint.

There is one block that is of special interest; this is the block to which the trivial character belongs, and is called the **principal block**. The corresponding principal block algebra over k is the only one which does not annihilate the trivial module k (i.e., the one-dimensional module on which each element of G acts as the identity). The structure of the principal block is in many senses at least as complicated as that of any other block. In what follows we shall tend to concentrate on the principal block, as this will allow us to cover the more important ideas while avoiding many of the technicalities of block theory.

Let us now start to look at general consequences of equivalences between derived categories of block algebras; everything we will say here has been noted by Broué [10], but we will try to give a slightly different perspective.

Beginning with the most trivial case, what can we say about derived equivalences over K , where the algebras involved are just products of matrix algebras? Well, if A is any semisimple K -algebra, then the indecomposable objects of the derived category $D^b(\text{mod}(A))$ are, up to isomorphism, just the objects $S[n]$, where S is a simple A -module and $n \in \mathbb{Z}$. So if A and B are both products of matrix algebras over K , with isomorphism classes of simple modules represented by $\{S_1, \dots, S_l\}$ and $\{T_1, \dots, T_m\}$ respectively, then if there is an equivalence

$$F : D^b(\text{mod}(A)) \longrightarrow D^b(\text{mod}(B))$$

we must have a bijection

$$\sigma : \{1, \dots, l\} \longrightarrow \{1, \dots, m\}$$

and an l -tuple (n_1, \dots, n_l) of integers, such that, for each $i \in \{1, \dots, l\}$,

$$F(S_i) \cong T_{\sigma(i)}[n_i].$$

Conversely, for any such choice of data, there is a corresponding equivalence.

If we have two finite groups G and H , central idempotents $e \in KG$ and $f \in KH$, and a derived equivalence between $KG.e$ and $KH.f$, we therefore have a 1-1 correspondence between the corresponding sets of characters. In fact, the equivalence of derived categories induces an isomorphism of groups of virtual characters, where the virtual character of a bounded complex C of finitely generated modules is

$$\sum_{i \in \mathbb{Z}} (-1)^i \text{ch}(C^i).$$

This isomorphism takes $\text{ch}(S_i)$ to $(-1)^{n_i} \text{ch}(T_i)$, so we have a ‘correspondence with signs’ between the sets of irreducible characters, where the signs measure the parity of the corresponding integers n_i . Equivalently, this isomorphism is an isometry with respect to the usual inner product of characters.

Any such isometry arises from some derived equivalence over K , but the isometries that arise from derived equivalences over \mathcal{O} have extra properties, as we shall now see.

Suppose that e and f are central idempotents in $\mathcal{O}G$ and $\mathcal{O}H$, and that X is a two-sided tilting complex of $\mathcal{O}G.e$ - $\mathcal{O}H.f$ -bimodules, which, as we saw in Section 2.2, we may as well choose to be a bounded complex of finitely generated bimodules that are projective as left and as right modules. Thus

$$? \otimes_{\mathcal{O}G} X : D^b(\text{mod}(\mathcal{O}G.e)) \longrightarrow D^b(\text{mod}(\mathcal{O}H.f))$$

is an equivalence. It is trivial to check that $X \otimes_{\mathcal{O}} K$ and $X \otimes_{\mathcal{O}} k$ are also two-sided tilting complexes, inducing equivalences

$$? \otimes_{KG} (X \otimes_{\mathcal{O}} K) : D^b(\text{mod}(KG.e)) \longrightarrow D^b(\text{mod}(KH.f))$$

and

$$? \otimes_{kG} (X \otimes_{\mathcal{O}} k) : D^b(\text{mod}(kG.e)) \longrightarrow D^b(\text{mod}(kH.f))$$

respectively (it should cause no confusion if we use the same letters e and f to denote the images of these idempotents under the quotient maps $\mathcal{O}G \longrightarrow kG$ and $\mathcal{O}H \longrightarrow kH$).

As we have seen, the derived equivalence over K induces an isometry of characters, but we have the extra information that if P is a projective $\mathcal{O}G.e$ -module, then $P \otimes_{\mathcal{O}G} X$ is a bounded complex of projective $\mathcal{O}H.f$ -modules. It follows that the isometry takes the character of any projective $\mathcal{O}G.e$ -module to a \mathbb{Z} -linear combination of characters of projective $\mathcal{O}H.f$ -modules. A similar remark applies to the inverse isometry, so the following theorem holds.

Theorem 2.9 (Broué). *If G and H are finite groups, and e and f are central idempotents in $\mathcal{O}G$ and $\mathcal{O}H$ respectively, then a derived equivalence between $\mathcal{O}G.e$ and $\mathcal{O}H.f$ induces an isometry between the groups of virtual characters of $KG.e$ and $KH.f$ that preserves the subgroups spanned by characters of projective modules for $\mathcal{O}G.e$ and $\mathcal{O}H.f$.*

An isometry with this property is called a **perfect isometry** by Broué [10], who also gives a characterization of such isometries in terms of arithmetic properties of character values, which allows the existence of perfect isometries to be tested with character tables. Compared to proving that two derived categories are equivalent, therefore, it is a relatively easy task to prove that there is a perfect isometry, and one often hears perfect isometries

referred to as the character-theoretic ‘shadow’ of the deeper structural phenomenon of a derived equivalence.

We have also seen that a derived equivalence between $\mathcal{O}G.e$ and $\mathcal{O}H.f$ induces an equivalence

$$F : D^b(\text{mod}(kG.e)) \longrightarrow D^b(\text{mod}(kH.f))$$

over k . In the same way that the equivalence over K induces an isomorphism of character groups, this equivalence induces an isomorphism between the groups of virtual Brauer characters of $kG.e$ and $kH.f$, which are free abelian groups with bases consisting of the Brauer characters of the simple $kG.e$ -modules and $kH.f$ -modules. Of course, this implies that the number of simple modules is the same in each case, but in contrast to the situation over K , there is no natural bijection, since if V is a simple $kG.e$ -module, there is no reason why $F(V)$ should be isomorphic to a shift $U[n]$ of any simple $kH.f$ -module U , and so the Brauer character of $F(V)$ need not be an irreducible Brauer character, even up to a sign.

In fact, as Broué proves in [10], this equality between the numbers of simple modules over k follows from the existence of a perfect isometry, which is *a priori* a weaker hypothesis.

Let us summarize the results of the preceding discussion.

Theorem 2.10. *Let G and H be finite groups, and let e and f be central idempotents in $\mathcal{O}G$ and $\mathcal{O}H$ respectively. If there is a derived equivalence between $\mathcal{O}G.e$ and $\mathcal{O}H.f$ then*

(a) *$KG.e$ and $KH.f$ have the same number of irreducible characters.*

(b) *$kG.e$ and $kH.f$ have the same number of isomorphism classes of simple modules, although there need be no natural bijection.*

2.4. Derived equivalences between blocks: Broué’s conjectures. In [10], Broué conjectured that there are many examples of pairs of blocks of related groups that have equivalent derived categories. As we saw in the previous section, this would have the consequence that these blocks have the same number of characters and the same number of simple modules over k ; in many situations where the conjectures apply, even these simple numerical equalities are not yet known to be true.

The most well-known of his conjectures is the ‘abelian defect group’ conjecture; to avoid going into such aspects of block theory as Brauer correspondence, we shall state it here only for principal blocks.

Conjecture 1 (Broué). Let G be a finite group with an abelian Sylow p -subgroup P and let $N_G(P)$ be the normalizer of P in G . There is a derived equivalence between the principal block algebras of $\mathcal{O}G$ and $\mathcal{O}N_G(P)$.

In fact, it is reasonable to hope that this equivalence should preserve the trivial module, in the sense that the trivial module \mathcal{O}_G for $\mathcal{O}G$ should be sent to the trivial module $\mathcal{O}_{N_G(P)}$ for $\mathcal{O}N_G(P)$.

This conjecture is known to be true in rather few cases. If P is cyclic, the equivalence over k is known from [17], and it was then proved by Linckelmann [15] that this equivalence also works over \mathcal{O} . It is also known when $p = 2$ and G is $SL_2(4)$ or $SL_2(8)$ [22]. Since, by Theorem 2.1 of [18], derived equivalences behave well with respect to taking tensor products of algebras, it follows that if G and G' are two groups for which Broué’s conjecture is true then it is also true for $G \times G'$.

By Theorem 2.9, the following weaker conjecture would be a consequence of Broué's conjecture.

Conjecture 2. Let G be a finite group with an abelian Sylow p -subgroup P . There is a perfect isometry between the principal blocks of G and $N_G(P)$ in characteristic p .

This conjecture is much easier to check for a particular group than the previous one, and has been verified in far more cases, although it is not known to be true in general. In particular, Fong and Harris [13] have proved that it is a consequence of the classification of finite simple groups that it is always true in characteristic two, and Fong has proved a corresponding result in characteristic three. Of course, this provides evidence for the more difficult conjecture on derived equivalences.

By Theorem 2.10, an even weaker consequence of Broué's conjecture is the following.

Conjecture 3. Let G be a finite group with an abelian Sylow p -subgroup P . The principal blocks of G and $N_G(P)$ in characteristic p have the same number of irreducible characters and the principal block algebras of kG and $kN_G(P)$ have the same number of isomorphism classes of simple modules.

Even this conjecture is not known to be true in general. It is a special case of Alperin's weight conjecture [1], which gives a more complicated conjectural description of the number of simple modules of a general block algebra in terms of normalizers of p -subgroups. An interesting problem, on which little progress has been made, is to find a structural explanation of Alperin's conjecture along the lines of Broué's conjecture.

Finally, the existence of derived equivalences between blocks is certainly not restricted to the case of abelian defect groups, although it is in this case that the neatest general conjecture has been formulated. To indicate a few examples, there are several cases of tame blocks in characteristic two that are derived equivalent although not Morita equivalent. Also, Enguehard [12] has proved that given *any* two blocks of two symmetric groups that have the same defect group, there is a perfect isometry between the two blocks; it should be true that there is also a derived equivalence, although this has only been proved in a few cases.

2.5. Splendid equivalences. In the paper [10] of Broué studying perfect isometries, he actually introduced a stronger version, called an **isotypy**. Again we shall restrict our attention to principal blocks, for the sake of simplicity. He noticed that in all cases of a group G for which he could find a perfect isometry between the principal blocks of G and the normalizer $H = N_G(P)$ of a Sylow p -subgroup P , there were actually families of perfect isometries: for each subgroup $Q \leq P$ there was a perfect isometry between the principal blocks of the centralizers $C_G(Q)$ and $C_H(Q)$. What is more, these families were compatible in a sense that he made precise. This raised the problem of formulating the correct definition of a 'compatible family' of derived equivalences. For principal blocks, at least, this problem was addressed in [20].

The solution depends on looking at permutation modules and their direct summands. Recall that if G is a group and R is any commutative ring, then a **permutation RG -module** is one which has an R -basis which is fixed setwise by the action of G : in other

words, there is a G -set Ω (i.e., a set on which G acts) so that the module is isomorphic to $R[\Omega]$, the free R -module with basis Ω on which the action of G is determined by extending linearly its action on Ω . There are various different names in currency for direct summands of permutation modules over \mathcal{O} or k : they are sometimes called **trivial source modules**, because in Green's theory of vertices and sources the source of each of their indecomposable summands is the trivial module for some subgroup, and they are sometimes called **p -permutation modules**, because they become permutation modules on restriction to any p -subgroup.

Since, when we look at two-sided tilting complexes, we must deal with bimodules, let us recall that an RG - RH -bimodule M for two group algebras can be regarded as an $R[G \times H]$ -module via the action

$$m.(g, h) = g^{-1}.m.h,$$

for $m \in M$, $g \in G$ and $h \in H$. It therefore makes sense to refer to a bimodule as being a 'permutation bimodule', meaning it is a permutation module when regarded as a module for the direct product of the two groups involved.

Of course, most modules for group algebras are not permutation modules or even p -permutation modules. However, many of those that arise naturally are, and so it is not unreasonable to hope that if a two-sided tilting complex arises naturally, then its terms might be p -permutation bimodules. Indeed, many familiar exact functors are induced by taking the tensor product with a p -permutation bimodule: for example, induction, restriction, inflation, projection onto a block, and (for a subgroup $U \leq G$ of order prime to p) the functor from $\text{mod}(\mathcal{O}G)$ to $\text{mod}(\mathcal{O}[N_G(U)/U])$ given by taking U -fixed points. Also, it was proved in [19] that if a finite group G acts on a variety V , then the l -adic cohomology with compact support $H_c^*(V, \mathbb{Z}_l)$ can be realized as the cohomology of a complex of l -permutation $\mathbb{Z}_l G$ -modules (note that, for reasons of tradition, the characteristic has changed briefly from p to l): this is relevant to the present discussion, since Broué [10] makes precise conjectures relating derived equivalences for finite reductive groups in non-defining characteristic l to the l -adic cohomology of Deligne-Lusztig varieties.

There is a construction, sometimes referred to as the **Brauer construction**, that is of key importance when using p -permutation modules. To motivate this, consider first a G -set Ω . We can form the fixed point set Ω^G , and this gives a functor

$$?^G : G\text{-sets} \longrightarrow \text{sets}.$$

More generally, if H is a subgroup of G , then $N_G(H)$ acts naturally on Ω^H , and so we have a functor

$$?^H : G\text{-sets} \longrightarrow N_G(H)\text{-sets}.$$

Of course, H acts trivially on Ω^H , so we could regard this functor as taking values in $N_G(H)/H$ -sets.

If we try to linearize this, we run into a problem. If Ω is a G -set and H is a subgroup of G as before, and if R is a commutative ring, then of course we can form the permutation module $R[\Omega^H]$ for $N_G(H)$. However, this is *not* in general a functor from permutation RG -modules to permutation $RN_G(H)$ -modules, the problem being that $R[\Omega^H]$ depends on the choice of permutation basis for $R[\Omega]$. In fact, it is not too hard to produce examples

of G -sets Ω_0 and Ω_1 so that $R[\Omega_0]$ and $R[\Omega_1]$ are isomorphic as RG -modules, but Ω_0^G and Ω_1^G have different numbers of elements.

There is one special case where we can recover functoriality. This is when $R = k$ is a field of characteristic p and $H = Q$ is a p -group. In this case, we can give an alternative description of $k[\Omega^Q]$ that makes it clear that it is functorial in $k[\Omega]$. Recall first that if K is a subgroup of H and M is a $\mathbb{Z}H$ -module, then the relative trace map

$$\mathrm{Tr}_K^H : M^K \rightarrow M^H$$

is defined by

$$\mathrm{Tr}_K^H(m) = \sum_{K \setminus H} mh$$

for $m \in M^K$, where h runs over a set of representatives for the cosets $K \setminus H$. Now it is easy to check that $k[\Omega^Q]$ is naturally isomorphic, as a $kN_G(Q)$ -module, to

$$k[\Omega]^Q / \sum_{Q' < Q} \mathrm{Tr}_{Q'}^Q(k[\Omega]^{Q'}),$$

which is clearly functorial in $k[\Omega]$: the point is that $k[\Omega]^Q$ has a basis given by the sums of the Q -orbits in Ω , and a basis element corresponding to an orbit of length greater than one is in the image of the relative trace from the stabilizer of an element of that orbit. This functor, the Brauer construction with respect to Q , extends additively to p -permutation kG -modules, and the p -permutation $kN_G(Q)$ -module (or $kN_G(Q)/Q$ -module) obtained by applying the functor to a p -permutation kG -module M is denoted by $M(Q)$. More details about the Brauer construction can be found in [8], where it is used systematically to study p -permutation modules. One final comment that we should make is that it is very important that the coefficient ring is a field of characteristic p : the Brauer construction does *not* work over a discrete valuation ring \mathcal{O} .

At first sight it seems that the Brauer construction does not mix very well with derived categories, since it is far from being exact, and so does not induce a well-defined functor on the derived category of kG -modules, or even the subcategory of complexes whose terms are p -permutation modules. However, it is additive, and so does induce a well-defined functor on the subcategory of $K(\mathrm{mod}(kG))$ consisting of complexes of p -permutation modules. It seems a good idea, then, to look at split endomorphism tilting complexes, as we did in Section 2.3, since we are then essentially working with the homotopy category rather than the derived category.

After this rather lengthy motivation, let us give the definition of a ‘**splendid tilting complex**’: this is an abbreviation of ‘**SPLit ENDomorphism tilting complex of p -permutation bimodules Induced from Diagonal subgroups**’. Here we give the definition in a special case that will suffice for studying Broué’s conjecture for principal blocks; a slightly more general definition is given in [20].

Definition 2.11. Let G be a finite group with a Sylow p -subgroup P , and let $H \leq G$ be a subgroup with $P \leq H$. Let R be either k or \mathcal{O} , and let e and f be the principal block idempotents in RG and RH respectively. A **splendid tilting complex** for $RG.e$ and $RH.f$ is a split endomorphism tilting complex of $RG.e$ - $RH.f$ -bimodules that are,

considered as $R[G \times H]$ -modules, direct sums of direct summands of permutation modules of the form

$$\text{Ind}_{\Delta Q}^{G \times H}(R)$$

for subgroups $Q \leq P$, where ΔQ is the ‘diagonal’ embedding of Q in $G \times H$; i.e.,

$$\Delta Q = \{(q, q) \in G \times H : q \in Q\}.$$

A derived equivalence induced by a splendid tilting complex is called a **splendid equivalence**.

Notice the two main features of the definition: firstly, the split endomorphism property, and secondly, the use of p -permutation modules. Both of these turn out to be important for the main theorems about splendid tilting complexes.

The main motivation for introducing this idea was to provide a good concept of a ‘compatible family of derived equivalences’, at least for principal blocks, in order to give a structural explanation of Broué’s isotypies. We do this over k by using the Brauer construction. Note that, with the notation of the definition, $N_{G \times H}(\Delta Q)$ contains $C_G(Q) \times C_H(Q)$ as a subgroup, and so the Brauer construction with respect to ΔQ can be regarded as a functor from p -permutation $k[G \times H]$ -modules to $k[C_G(Q) \times C_H(Q)]$ -modules.

The following theorem is a special case of [20, Theorem 4.1].

Theorem 2.12. *With the notation of Definition 2.11, suppose that P is abelian, and that $H = N_G(P)$. If X is a splendid tilting complex for the principal block algebras $kG.e$ and $kH.f$, then for each subgroup $Q \leq P$, $X(\Delta Q)$ is a splendid tilting complex for the principal block algebras of $kC_G(Q)$ and $kC_H(Q)$.*

Our proof of this theorem, which we shall not give here, depended on the fact that H ‘controls fusion of p -subgroups’ in G : i.e., if Q_0 and Q_1 are subgroups of P that are conjugate in G , so $Q_1 = Q_0^g$ for some $g \in G$, then they are conjugate in H and the isomorphism from Q_0 to Q_1 given by conjugating by g is also given by conjugation by some element of H . This is not in general true if P is non-abelian, but Harris [14] has shown that this condition is not essential for the proof. Also, Puig [16] has given a remarkable generalization of this theorem to arbitrary blocks, and he shows that the analogue of the ‘control of fusion’ condition is actually a consequence of the existence of a splendid equivalence.

It should be fairly clear why the two main features of the definition are important for this theorem: in both cases it is because we are using the Brauer construction.

As it stands, this theorem is not very satisfactory as an explanation of isotypies, since the use of the Brauer construction requires us to work over k , but to deduce perfect isometries we need derived equivalences over \mathcal{O} . Of course, it is easy to prove that a derived equivalence over \mathcal{O} gives one over k , as we saw in Section 2.3, but the converse is not true in general. Luckily, it turns out that it is true when we have the ‘splendid’ condition. The next theorem is Theorem 5.2 of [20].

Theorem 2.13. *With the notation of Definition 2.11, let X be a splendid tilting complex for $kG.e$ and $kH.f$. There is a splendid tilting complex \tilde{X} for $\mathcal{O}G.e$ and $\mathcal{O}H.f$, unique up to isomorphism, so that $X \cong \tilde{X} \otimes_{\mathcal{O}} k$.*

The way that the two main features of the definition enter the proof of this theorem is as follows. An important property of p -permutation modules over k is that they, *and maps between them*, can be lifted to \mathcal{O} . This means that the complex X can at least be lifted to a sequence of p -permutation bimodules over \mathcal{O} with maps between them: however, when we do this we do not necessarily get a complex, since the composition of two ‘differentials’ is not necessarily zero. It turns out that there is a sequence of obstructions to being able to lift the differentials so that they do compose to zero, and all of these obstructions lie in the space

$$\mathrm{Hom}_{K^b(\mathrm{mod}(k[G \times H]))}(X, X[2]).$$

Now the split endomorphism property enters: this implies that the complex $\mathrm{Hom}_{kG}(X, X)$ is split (as a complex of kH -bimodules) with cohomology concentrated in degree zero. Applying the H -fixed point functor (for the diagonal action of H), we deduce that the complex $\mathrm{Hom}_{k[G \times H]}(X, X)$ also has cohomology concentrated in degree zero, and the space in which the obstructions lie is the degree two cohomology. We could not do this without splitness, since the H -fixed point functor is not exact.

Putting the two theorems together, we get a very satisfactory structural counterpart of isotypies: given a splendid equivalence over \mathcal{O} between the principal block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, we can reduce modulo the maximal ideal of \mathcal{O} to get a splendid equivalence over k , then we can apply Theorem 2.12 to get splendid equivalences between the principal block algebras of $kC_G(Q)$ and $kC_H(Q)$ for each $Q \leq P$, and finally we can apply Theorem 2.13 to get splendid equivalences between the principal block algebras of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$.

Of course, all of this would be of little interest if splendid equivalences did not occur, but to date the evidence suggests that the obvious strengthening of Broué’s conjecture is true:

Conjecture 4. Let G be a finite group with an abelian Sylow p -subgroup P and let $N_G(P)$ be the normalizer of P in G . There is a splendid equivalence between the principal block algebras of $\mathcal{O}G$ and $\mathcal{O}N_G(P)$.

One interesting phenomenon that we explore in [20] arises when we look at Morita equivalences, which are of course a simple special case of derived equivalences, in the light of the idea of splendid equivalences. The obvious way that a Morita equivalence can be ‘splendid’ is that it can be induced by a p -permutation bimodule. However, there are natural examples that do not arise in this way. In many cases these are splendid nevertheless: the bimodule inducing the Morita equivalence is the cohomology of a splendid tilting complex (with more than one term) which happens to have its cohomology concentrated in one degree. So even for Morita equivalences we are led to look at tilting complexes!

3. BOUSFIELD LOCALIZATION IN THE STABLE MODULE CATEGORY

In this section we shall give a brief taste of some new techniques that have recently been introduced into the study of the stable module category of a finite group in work of Benson, Carlson and the author [5, 21, 6, 7]. When we say these techniques are ‘new’, we mean that they are new to representation theory: they are well-known in algebraic topology, where they are applied to the stable homotopy category, and the stable module

category (at least if we include infinite dimensional modules) is similar enough that the ideas can be borrowed wholesale. The key construction is called ‘Bousfield localization’ and was introduced into stable homotopy theory in work of Brown, Adams and Bousfield.

We shall be working over a field k of characteristic p , which from now on we shall choose to be algebraically closed.

3.1. The stable module category and varieties for modules. Let us start by recalling the basic idea of the stable module category; we shall deal with *all* modules, not just the finitely generated ones.

Definition 3.1. Let G be a finite group. The **stable module category** $\text{StMod}(kG)$ of kG -modules has the kG -modules as its objects, and if M and N are two such modules, the set of morphisms between them is the set of equivalence classes

$$\underline{\text{Hom}}_{kG}(M, N) = \text{Hom}_{kG}(M, N) / \sim,$$

where two homomorphisms α and β from M to N are equivalent if their difference $\alpha - \beta$ factors through a projective module.

The full subcategory consisting of finitely generated modules is denoted by $\text{stmod}(kG)$.

Every kG -module M is the direct sum of a projective module and a module (called the **projective-free part** of M) that has no non-zero projective direct summands, and both of these are unique up to isomorphism. Two modules are ‘stably isomorphic’, i.e., isomorphic in $\text{StMod}(kG)$, if and only if their projective-free parts are isomorphic as modules. Hence there is a one-to-one correspondence between stable isomorphism classes and isomorphism classes of modules with no non-zero projective summands.

The stable module category is a triangulated category, where the shift functor is given by taking inverse syzygies: the shift of a module M is the cokernel $\Omega^{-1}M$ of the embedding of M into an injective module. The distinguished triangles come from short exact sequences of modules: if

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is a short exact sequence of modules, the extension class gives an element of

$$\text{Ext}^1(N, L) \cong \underline{\text{Hom}}(N, \Omega^{-1}L)$$

and hence a diagram

$$L \longrightarrow M \longrightarrow N \longrightarrow \Omega^{-1}L$$

in $\text{StMod}(kG)$. The distinguished triangles are the diagrams isomorphic to those arising in this way. So, informally, a distinguished triangle is a ‘short exact sequence up to projective summands’.

There are many *épaisse* subcategories of $\text{stmod}(kG)$ defined in a homological way, in terms of the theory of varieties for modules. A good exposition of this theory can be found in [3, Chapter 5], and we shall just recall some of the basic properties of these varieties.

The cohomology ring $H^*(G, k)$ is a finitely generated graded k -algebra which, modulo the ideal of nilpotent elements, is commutative, and so its maximal ideal spectrum $V_G(k)$ is an affine variety over k . Actually, since the cohomology ring is graded, it would be more natural to consider the associated projective variety, but we shall follow tradition by

using the affine variety. Given a finite-dimensional kG -module M , $\text{Ext}^*(M, M)$ is a graded module for $H^*(G, k)$, and so its annihilator is a graded ideal of $H^*(G, k)$ and so defines a closed homogeneous subvariety $V_G(M)$ of $V_G(k)$, called the ‘variety of the module M ’. The subvariety associated to the trivial module k is clearly the whole of $V_G(k)$, and so the notation is consistent (i.e., the two meanings of the notation $V_G(k)$ agree). A module M is projective if and only if $V_G(M) = \{0\}$, and if M and N are modules, then $V_G(M \oplus N)$ is precisely the union $V_G(M) \cup V_G(N)$, and hence the variety of a module is an invariant of its stable isomorphism class. The definition clearly makes sense even for infinitely generated modules, but it turns out that it is not the ‘correct’ definition in this case.

We can now define various subcategories of $\text{stmod}(kG)$.

Definition 3.2. Let W be a closed homogeneous subvariety of $V_G(k)$. Then $\mathcal{C}(W)$ is the full subcategory of $\text{stmod}(kG)$ consisting of the modules M such that $V_G(M)$ is contained in W .

More generally, if \mathcal{X} is a family of closed homogeneous subvarieties of $V_G(k)$ that is closed under taking finite unions and subvarieties, then $\mathcal{C}(\mathcal{X})$ is the full subcategory of $\text{stmod}(kG)$ consisting of the modules M such that $V_G(M) \in \mathcal{X}$.

For example, we could take \mathcal{X} to be the family of closed homogeneous subvarieties of $V_G(k)$ whose dimension is no more than some constant c . In this case $\mathcal{C}(\mathcal{X})$ is the category of modules with **complexity** at most c .

Many of the more elementary properties of these varieties can be summed up in the following simple statement.

Proposition 3.3. *With the notation of the last definition, $\mathcal{C}(W)$ and $\mathcal{C}(\mathcal{X})$ are épaisse subcategories of $\text{stmod}(kG)$.*

Another easy property that we shall need is the following.

Proposition 3.4. *If M^\vee is the dual of a finite-dimensional module M , then*

$$V_G(M^\vee) = V_G(M).$$

Perhaps the deepest and most important general property of these varieties is the following ‘Tensor Product Theorem’ of Carlson.

Theorem 3.5. *Let M and N be finitely generated kG -modules. Then*

$$V_G(M \otimes N) = V_G(M) \cap V_G(N).$$

3.2. Bousfield localization. Let \mathcal{C} be any épaisse subcategory of $\text{stmod}(kG)$. The first thing to do is to extend \mathcal{C} in a natural way to an épaisse subcategory of $\text{StMod}(kG)$. It turns out that all reasonable ways of doing this coincide. Let us give the definition that is easiest to state.

Definition 3.6. \mathcal{C}^\oplus is the smallest épaisse subcategory of $\text{StMod}(kG)$ that contains \mathcal{C} and which is closed under arbitrary direct sums (i.e., is closed under taking direct sums of possibly infinite sets of modules).

It is not immediately clear from this definition, but it is true, that \mathcal{C}^\oplus contains no finitely generated modules other than those that were already in \mathcal{C} .

There is another subcategory of $\text{StMod}(kG)$ associated with \mathcal{C} .

Definition 3.7. An object L of $\text{StMod}(kG)$ is said to be \mathcal{C} -local if $\underline{\text{Hom}}(C, L) = 0$ for every object C of \mathcal{C} .

It is easy to see that the \mathcal{C} -local objects form an épaisse subcategory of $\text{StMod}(kG)$. This subcategory is usually not of the form \mathcal{D}^\oplus for any épaisse subcategory \mathcal{D} of $\text{stmod}(kG)$.

The statement of Bousfield localization in this context is the following. A proof can be found in [21], although this is really just a special case of a more general theorem that goes back to Brown [11].

Theorem 3.8. *Let \mathcal{C} be an épaisse subcategory of $\text{stmod}(kG)$. For each kG -module M there is a distinguished triangle*

$$\mathcal{E}_{\mathcal{C}}(M) \longrightarrow M \longrightarrow \mathcal{F}_{\mathcal{C}}(M) \longrightarrow \Omega^{-1}\mathcal{E}_{\mathcal{C}}(M)$$

such that

- (a) $\mathcal{E}_{\mathcal{C}}(M)$ is in \mathcal{C}^\oplus , and
- (b) $\mathcal{F}_{\mathcal{C}}(M)$ is \mathcal{C} -local.

This triangle is functorial in M : in fact, it is unique up to isomorphism, being characterized by the properties (a) and (b).

Even if M is finite dimensional, the modules $\mathcal{E}_{\mathcal{C}}(M)$ and $\mathcal{F}_{\mathcal{C}}(M)$ are usually not, so this is why we are forced to consider infinite dimensional modules.

Applications of this theorem to the study of finite dimensional representation theory can be found in [21, 6, 4, 7]. Here we shall show how it can be used to classify the épaisse subcategories of $\text{stmod}(kG)$ when G is a p -group, a special case of a theorem proved in [7]. First we need to describe some of the properties of a generalization of the theory of varieties to infinite-dimensional modules due to Benson, Carlson and the author.

3.3. Varieties for arbitrary modules. As we mentioned when we discussed varieties for finite-dimensional modules in Section 3.1, the naive way of generalizing the theory to arbitrary modules does not work too well. In particular, it turns out that the Tensor Product Theorem 3.5 would no longer be true. This motivated a slightly more sophisticated generalization [6]. Here we shall not give the precise definition, but merely describe very briefly some of the more important properties.

To each kG -module M , possibly infinite-dimensional, we associate not a variety, but a family $\mathcal{V}_G(M)$ of non-zero closed homogeneous irreducible subvarieties of $V_G(k)$. Another way of thinking of such a family is as a subset of the projective prime ideal spectrum of $H^*(G, k)$.

If M is finite-dimensional, then $\mathcal{V}_G(M)$ consists of the set of all non-zero closed homogeneous irreducible subvarieties of $V_G(M)$. Most of the important properties of the ‘classical’ varieties generalize. In particular, a module M is projective if and only if $\mathcal{V}_G(M) = \emptyset$, and if M and N are modules, then

$$\mathcal{V}_G(M \otimes N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N),$$

so the Tensor Product Theorem generalizes.

3.4. The classification of épaisse subcategories of $\text{stmod}(kP)$. Suppose that $G = P$ is a p -group. In this case we can combine the theory of Bousfield localization with the theory of varieties for infinite-dimensional modules to prove the following classification theorem, which is Corollary 3.5 of [7]. Note that this is a theorem about finite-dimensional modules, although the proof makes heavy use of much larger modules.

Theorem 3.9. *Every épaisse subcategory of $\text{stmod}(kP)$ is of the form $\mathcal{C}(\mathcal{X})$ for some family \mathcal{X} of closed homogeneous subvarieties of $V_P(k)$ that is closed under taking finite unions and subvarieties.*

Proof. Let \mathcal{C} be an épaisse subcategory of $\text{stmod } kP$, and let \mathcal{X} be the set of all closed homogeneous subvarieties of $V_P(k)$ that occur as the variety of some object of \mathcal{C} . It is not hard to show that \mathcal{X} is closed under taking finite unions and subvarieties. We shall show that $\mathcal{C} = \mathcal{C}(\mathcal{X})$.

Clearly \mathcal{C} is contained in $\mathcal{C}(\mathcal{X})$, and to prove the other containment it is enough to show that if M is a finite-dimensional module and $V_P(M) = W$, then $\mathcal{C}(W)$ coincides with the smallest épaisse subcategory $\langle M \rangle$ of $\text{stmod}(kP)$ that contains M .

Let us simplify our notation by writing $\mathcal{E}_M(?)$ and $\mathcal{F}_M(?)$ for $\mathcal{E}_{\langle M \rangle}(?)$ and $\mathcal{F}_{\langle M \rangle}(?)$ and by writing $\mathcal{E}_W(?)$ and $\mathcal{F}_W(?)$ for $\mathcal{E}_{\mathcal{C}(W)}(?)$ and $\mathcal{F}_{\mathcal{C}(W)}(?)$.

Let N be a kP -module.

Since $\langle M \rangle$ is certainly contained in $\mathcal{C}(W)$, $\langle M \rangle^\oplus$ is contained in $\mathcal{C}(W)^\oplus$, and so $\mathcal{E}_M(N)$ is an object of $\mathcal{C}(W)^\oplus$.

Since $\mathcal{F}_M(N)$ is $\langle M \rangle$ -local,

$$0 = \underline{\text{Hom}}(M, \mathcal{F}_M(N)) = \underline{\text{Hom}}(k, M^\vee \otimes \mathcal{F}_M(N)).$$

Since P is a p -group, and so the trivial module k is the only simple kP -module, every non-zero object of $\text{StMod}(kP)$ has a non-zero map from k , and so $M^\vee \otimes \mathcal{F}_M(N)$ must be projective. Therefore

$$\emptyset = \mathcal{V}_P(M^\vee \otimes \mathcal{F}_M(N)) = \mathcal{V}_P(M) \cap \mathcal{V}_P(\mathcal{F}_M(N)).$$

But if C is in $\mathcal{C}(W)$, and so

$$\mathcal{V}_P(C) \subseteq \mathcal{V}_P(M),$$

then

$$\emptyset = \mathcal{V}_P(C) \cap \mathcal{V}_P(\mathcal{F}_M(N)) = \mathcal{V}_P(C^\vee \otimes \mathcal{F}_M(N)),$$

so $C^\vee \otimes \mathcal{F}_M(N)$ is projective, and

$$0 = \underline{\text{Hom}}(k, C^\vee \otimes \mathcal{F}_M(N)) = \underline{\text{Hom}}(C, \mathcal{F}_M(N)),$$

which proves that $\mathcal{F}_M(N)$ is $\mathcal{C}(W)$ -local.

The last two paragraphs prove that the distinguished triangle

$$\mathcal{E}_M(N) \longrightarrow N \longrightarrow \mathcal{F}_M(N) \longrightarrow \Omega^{-1}\mathcal{E}_M(N)$$

shares the characterizing properties of

$$\mathcal{E}_W(N) \longrightarrow N \longrightarrow \mathcal{F}_W(N) \longrightarrow \Omega^{-1}\mathcal{E}_W(N),$$

and so these two triangles are isomorphic.

But finally,

$$\begin{aligned} N \in \langle M \rangle &\Leftrightarrow \mathcal{F}_M(N) \cong 0 \\ &\Leftrightarrow \mathcal{F}_W(N) \cong 0 \\ &\Leftrightarrow N \in \mathcal{C}(W), \end{aligned}$$

proving that $\langle M \rangle$ and $\mathcal{C}(W)$ coincide, as required. \square

This theorem is not in general true for groups that are not p -groups. The general case is investigated in [7], although with only partial success. Even for relatively uncomplicated groups, such as $C_3 \times S_3$ (with $p = 3$), there are many épaisse subcategories and no plausible classification theorem seems visible.

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