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Rational Maps, Monopoles and Skyrmions

Conor J. Houghton†, Nicholas S. Manton† and Paul M. Sutcliffe‡

† *Department of Applied Mathematics and Theoretical Physics*
University of Cambridge, Silver St., Cambridge CB3 9EW, England.
Email C.J.Houghton@damtp.cam.ac.uk
Email N.S.Manton@damtp.cam.ac.uk

‡ *Institute of Mathematics, University of Kent at Canterbury,*
Canterbury CT2 7NF, England.
Email P.M.Sutcliffe@ukc.ac.uk

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Abstract

We discuss the similarities between BPS monopoles and Skyrmions, and point to an underlying connection in terms of rational maps between Riemann spheres. This involves the introduction of a new ansatz for Skyrme fields. We use this to construct good approximations to several known Skyrmions, including all the minimal energy configurations up to baryon number nine, and some new solutions such as a baryon number seventeen Skyrme field with the truncated icosahedron structure of a buckyball.

The new approach is also used to understand the low-lying vibrational modes of Skyrmions, which are required for quantization. Along the way we discover an interesting Morse function on the space of rational maps which may be of use in understanding the Sen forms on the monopole moduli spaces.

1 Introduction

There is considerable evidence that there is rather a close connection between $SU(2)$ BPS monopoles and Skyrmions, despite their obvious differences.

Recall that BPS monopoles are minimal energy classical solutions in a Yang-Mills-Higgs theory. They satisfy the Bogomolny equation

$$B_i = D_i\Phi \tag{1.1}$$

where B_i is the magnetic part of the $SU(2)$ Yang-Mills field tensor, and $D_i\Phi$ is the covariant derivative of an adjoint Higgs field. There is a $(4N-1)$-dimensional moduli space of gauge inequivalent solutions with monopole number N, all with the same energy $4\pi N$. Among these solutions are some special ones of rather high symmetry, representing N coalesced single monopoles.

The Skyrme model is a nonlinear theory of pions, with an $SU(2)$ valued scalar field $U(\mathbf{x}, t)$, the Skyrme field, satisfying the boundary condition $U \to 1$ as $|\mathbf{x}| \to \infty$. Static fields obey the equation

$$\partial_i(R_i - \frac{1}{4}[R_j, [R_j, R_i]]) = 0 \tag{1.2}$$

where R_i is the $su(2)$ valued current $R_i = (\partial_i U)U^{-1}$. Such fields are stationary points (either minima or saddle points) of the energy function

$$E = \int \{-\frac{1}{2}\mathrm{Tr}(R_iR_i) - \frac{1}{16}\mathrm{Tr}([R_i, R_j][R_i, R_j])\}\ d^3\mathbf{x}. \tag{1.3}$$

Associated with a Skyrme field is a topological integer, the baryon number B, defined as the degree of the map $U : \mathbb{R}^3 \mapsto SU(2)$. This is calculated at a given time, but is time independent. It is well defined because of the boundary condition at infinity.

Solutions of Skyrme equation (1.2) are known for several values of B, but they can only be obtained numerically. Many of these solutions are stable, and probably represent the global minimum of the energy for given B. We shall refer to the solutions believed to be of lowest energy for each B as Skyrmions. Some saddle-point solutions are also known.

All known solutions appear to be isolated and their only moduli are the obvious ones associated with the large symmetry group of the equation and boundary condition. This symmetry group is nine-dimensional. It consists of translations and rotations in $\mathbb{R}^3$ and the $SO(3)$ isospin transformations $U \mapsto \mathcal{O}U\mathcal{O}^{-1}$ where $\mathcal{O}$ is a constant element of $SU(2)$. Generic solutions therefore have nine moduli, although solutions with axial or spherical symmetry have fewer.

It has been found that many solutions of the Skyrme equation, and particularly those of low energy, look rather like monopoles, with the baryon number B being identified with the monopole number N. The fields are not really the same, but the energy density has equivalent symmetries and approximately the same spatial distribution. The history of the discovery of these solutions is perhaps worth recalling. First Skyrme [31] found the spherically symmetric $B = 1$ Skyrmion. Later Prasad & Sommerfield [26] found the analytic form of the $N = 1$ monopole, which is also spherically symmetric. Bogomolny [7] then showed that minimal energy monopoles should satisfy (1.1), and that the Prasad-Sommerfield solution is the unique spherically symmetric solution of (1.1). Next, and with difficulty, an axially symmetric $N = 2$ monopole was discovered by Ward [34], and after that it was found that the $B = 2$ Skyrmion is axially symmetric [20, 25, 32]. (There are spherically symmetric solutions of (1.2) for all B, but for $B > 1$ they have rather high energy.)

Next, a substantial numerical search for Skyrmion solutions was undertaken by Braaten et al. [8], and minimal energy solutions up to $B = 6$ were constructed. (Their solution for $B = 6$ was rather inaccurate, and its symmetry was wrongly identified.) Surprisingly, the $B = 3$ solution has tetrahedral symmetry T_d, and the $B = 4$ solution has octahedral symmetry O_h. The $B = 5$ (and $B = 6$) solutions have lower symmetry. These findings suggested that monopoles with similar symmetries might exist, and indeed they do.

Hitchin et al. [13] established the existence of an $N = 3$ monopole with tetrahedral symmetry and an $N = 4$ monopole with octahedral symmetry. These solutions are unique, up to the action of the Euclidean group. All other BPS monopole solutions with these monopole numbers (other than the axially symmetric configurations) have lower symmetry. Hitchin et al. also showed that no monopoles with icosahedral symmetry are possible with $N \leq 6$ (although it appeared at first that one with $N = 6$ might be possible). Subsequently, Houghton & Sutcliffe [15] found an $N = 5$ monopole with octahedral symmetry, and an $N = 7$ monopole with icosahedral symmetry.

The search was on for further solutions of the Skyrme equation. There is a $B = 5$ solution with octahedral symmetry, but it has slightly higher energy than the minimal energy solution [5]. More importantly, Battye & Sutcliffe [4] established that the $B = 7$ Skyrmion has icosahedral symmetry. Battye & Sutcliffe [4] have recently found all Skyrmions up to $B = 9$. The $B = 6$ and $B = 8$ solutions have the relatively low symmetries D_{4d} and D_{6d} respectively, but the $B = 9$ solution has tetrahedral symmetry. The results in this paper imply that monopoles with these symmetries, for $N = 6, 8$ and 9 respectively, exist too, but little is known about them.

Pictures of all these Skyrmion solutions can be found in ref. [4]. Qualitatively, they are like the pictures in Fig. 1, whose significance we will explain later.

As we mentioned earlier, Skyrmion solutions are isolated, but it is physically interesting to study the small oscillation vibrations around them. The vibrational modes of the axisymmetric $B = 2$ Skyrmion and the octahedrally symmetric $B = 4$ Skyrmion have recently been studied by Barnes et al. [1, 2]. The frequencies and degeneracies of these modes have been calculated, and the way the Skyrmion vibrates can be visualized. At least for these two examples, the lowest frequency modes can clearly be identified with the deformations of the moduli of the corresponding symmetric monopole. For example, for the octahedrally symmetric $N = 4$ monopole there are fifteen moduli in all, but six of these are associated with the Euclidean group. The nine remaining moduli can be identified with the nine lowest frequency vibrational modes of the $B = 4$ Skyrmion.

How can we understand all these results? The aim of this paper is to point to an explanation in terms of rational maps. A rational map is a holomorphic function from

$S^2 \mapsto S^2$. If we treat each S^2 as a Riemann sphere, the first having coordinate z, a rational map of degree N is a function $R : S^2 \mapsto S^2$ where

$$R(z) = \frac{p(z)}{q(z)} \tag{1.4}$$

and p and q are polynomials of degree at most N. At least one of p and q must have degree precisely N, and p and q must have no common factors (ie. no common roots).

Rational maps were introduced into the theory of monopoles by Donaldson [9]. Indeed Donaldson showed that there is a one-to-one correspondence between maps of degree N (with the basing condition $R(z) \to 0$ as $z \to \infty$) and N-monopoles. Donaldson's work, following Hitchin [11, 12], relies on a choice of direction in $\mathbb{R}^3$, and this is not helpful in the present context.

A new relationship between monopoles and rational maps has recently been established by Jarvis [18] (following a suggestion of Atiyah). This requires the choice of an origin, and is much better adapted for studying fields invariant under a subgroup of the group of rotations about the origin. The Jarvis map is obtained by considering Hitchin's equation

$$(D_r - i\Phi)s = 0 \tag{1.5}$$

along each radial line from the origin to infinity. Here D_r is the covariant derivative in the radial direction and Φ is the Higgs field. s is an auxiliary complex doublet field transforming via the fundamental representation of the gauge group $SU(2)$. Because Φ is asymptotically conjugate to $\mathrm{diag}(\frac{i}{2}, -\frac{i}{2})$, equation (1.5) has, up to a constant multiple, just one solution which decays asymptotically as $r \to \infty$. Let $\begin{pmatrix} s_1(r) \\ s_2(r) \end{pmatrix}$ be this solution and $\begin{pmatrix} s_1(0) \\ s_2(0) \end{pmatrix}$ its value at the origin. Because of the arbitrariness of the constant multiple, it is only the ratio $R = s_1(0)/s_2(0)$ that is interesting. Now a particular radial line is labelled by its direction, regarded as a point z on the Riemann sphere. R depends holomorphically on the direction z, so we write $R(z)$. The reason R is holomorphic is that the complex covariant derivative in the angular direction, $D_{\bar{z}}$, commutes with the operator $D_r - i\Phi$, because of the Bogomolny equation (1.1). It can be shown that the degree of R is equal to the monopole number N, and hence R is rational of degree N. There is one remaining ambiguity in $R(z)$. If we carry out a gauge transformation then $R(z)$ is replaced by its Möbius transformation by an $SU(2)$ matrix

$$R(z) \mapsto \frac{\alpha R(z) + \beta}{-\bar{\beta} R(z) + \bar{\alpha}} \tag{1.6}$$

with $|\alpha|^2 + |\beta|^2 = 1$. The $SU(2)$ matrix here, $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, is the gauge transformation matrix evaluated at the origin, and it acts globally on $R(z)$, that is, the same matrix occurs for all z.

Thus the moduli space of rational maps $R(z)$ that Jarvis associates with N-monopoles is the complete $(4N+2)$-dimensional space of unbased rational maps of degree N. For each monopole there is a map which is uniquely defined up to an $SU(2)$ Möbius transformation. Moreover, each rational map arises from some monopole. Jarvis shows how, in principle, one may directly reconstruct the monopole from the rational map. This algorithm could be implemented numerically, and currently work is in progress to achieve this. Note that this construction still requires the solution of a partial differential equation in three-dimensional space so the computational gain is small. However, its advantage over a direct numerical solution of the Bogomolny equation is that the selection of a given monopole solution can be made precise via the rational map input, which is easy to obtain. This contrasts with an existing numerical construction [14], where the computational gain is great, since only ordinary differential equations need to be solved, but where the input is more difficult to obtain since it consists of Nahm data which can only be found after the solution of a nonlinear system of matrix differential equations.

The naturalness of the Jarvis construction means that a monopole invariant under a subgroup G of the spatial rotation group $SO(3)$ will have an associated map $R(z)$ which is G-invariant (up to Möbius transformations), and conversely, if we find a G-invariant map of a given degree N then there is an N-monopole with symmetry G. If we find the complete set of rational maps invariant under G, then the corresponding set of G-invariant monopoles will form a geodesic submanifold of the monopole moduli space. In particular, if, for some G, the set of maps is one-dimensional, then the corresponding monopoles lie on a geodesic in the moduli space. Using the geodesic approximation to monopole motion [23], we obtain, usually, an example of monopole scattering with G-invariance.

An important quantity associated with a rational map $R(z) = p(z)/q(z)$ is the Wronskian

$$W(z) = p'(z)q(z) - q'(z)p(z) \tag{1.7}$$

or more precisely, the zeros of W, which are the branch points of the map. If R is of degree N, then generically, W is a polynomial of degree $2N-2$. The zeros of W are invariant under any Möbius transformation of R, which replaces p by $\alpha p + \beta q$ and q by $\gamma q + \delta p$ and hence simply multiplies W by $(\alpha\gamma - \beta\delta)$. Occasionally, W is a polynomial of degree less than $2N-2$, but one then interprets the missing zeros as being at $z = \infty$. The symmetries of the map R, and hence of the N-monopole which corresponds to it, are captured by the symmetries of the Wronskian W. Sometimes W has more symmetry than the rational map R, and we shall see examples of this.

Monopoles with given symmetries have been constructed before, for example, the $N=7$ monopole with icosahedral symmetry Y_h. But the construction depended on a careful study of Nahm's equation, and the existence of the solution was not known in advance. The Nahm equation approach has only been successfully applied in relatively simple cases. It is much easier to classify rational maps with given symmetries, and we shall give a number of examples later. This establishes the existence of monopoles with these symmetries, but we have not constructed all the solutions, even numerically!

2 Skyrme Fields from Rational Maps

The understanding of monopoles in terms of rational maps suggests that one might understand a range of Skyrmion solutions using rational maps. Rational maps are maps from $S^2 \mapsto S^2$, whereas Skyrmions are maps from $\mathbb{R}^3 \mapsto S^3$. A rather naive idea, which we find works quite well, is to identify the domain S^2 with concentric spheres in $\mathbb{R}^3$, and the target S^2 with spheres of latitude on S^3. This leads to a new ansatz for Skyrme fields.

It is convenient to use Cartesian notation to present the ansatz. Recall that via stereographic projection, the complex coordinate z on a sphere can be identified with conventional polar coordinates by $z = \tan(\theta/2)e^{i\varphi}$. Equivalently, the point z corresponds to the unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}(2\Re(z), 2\Im(z), 1-|z|^2). \tag{2.1}$$

Similarly the value of the rational map $R(z)$ is associated with the unit vector

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}(2\Re(R), 2\Im(R), 1-|R|^2). \tag{2.2}$$

Let us denote a point in $\mathbb{R}^3$ by its coordinates (r, z) where r is the radial distance from the origin and z specifies the direction from the origin. Our ansatz for the Skyrme field depends on a rational map $R(z)$ and a radial profile function $f(r)$. The ansatz is

$$U(r,z) = \exp(if(r)\,\hat{\mathbf{n}}_R \cdot \boldsymbol{\sigma}) \tag{2.3}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices. For this to be well-defined at the origin, $f(0) = k\pi$, for some integer k. The boundary value $U=1$ at $r=\infty$ requires that $f(\infty)=0$. It is straightforward to verify that the baryon number of this field is $B = Nk$, where N is the degree of R. In the remainder of this paper we shall only consider the case $k=1$, and then $B=N$. Note that an $SU(2)$ Möbius transformation on the target S^2 of the rational map corresponds to a rotation of $\hat{\mathbf{n}}_R$, and hence to an isospin rotation of the Skyrme field.

In the case $N=1$, the basic map is $R(z)=z$, and (2.3) reduces to Skyrme's hedgehog field

$$U(r,\theta,\varphi) = \cos f + i \sin f(\sin\theta\cos\varphi\ \sigma_1 + \sin\theta\sin\varphi\ \sigma_2 + \cos\theta\ \sigma_3). \tag{2.4}$$

The simplest case beyond this, with $N=2$, is $R=z^2$, which gives an ansatz rather different from that tried in ref. [35] for the $B=2$ Skyrmion. We shall return to this case in more detail in Section 3.

An attractive feature of the ansatz (2.3) is that it leads to a simple energy expression which can be minimized with respect to the rational map R and the profile function f to obtain close approximations to several known Skyrmion solutions. Starting with these approximations is an efficient method to find new exact solutions, although we shall not pursue this application here. To calculate the energy of a field of the form (2.3) we exploit an interpretation of the Skyrme energy function given in ref. [24].

As in nonlinear elasticity theory, the energy density of a Skyrme field depends on the local stretching associated with the map $U : \mathbb{R}^3 \mapsto S^3$. The Riemannian geometry of $\mathbb{R}^3$

(flat) and of S^3 (a unit radius 3-sphere) are necessary to define this stretching. Consider the strain tensor at a point in $\mathbb{R}^3$

$$D_{ij} = -\frac{1}{2}\mathrm{Tr}(R_i R_j) = -\frac{1}{2}\mathrm{Tr}((\partial_i U U^{-1})(\partial_j U U^{-1})). \tag{2.5}$$

This is symmetric, and positive semi-definite as R_i is antihermitian. Let its eigenvalues be λ_1^2, λ_2^2 and λ_3^2. The Skyrme energy can be reexpressed as

$$E = \int (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2)\, d^3\mathbf{x}, \tag{2.6}$$

and the baryon density as $\lambda_1\lambda_2\lambda_3/2\pi^2$. For the ansatz (2.3), the strain in the radial direction is orthogonal to the strain in the angular directions. Moreover, because $R(z)$ is conformal, the angular strains are isotropic. If we identify λ_1^2 with the radial strain and λ_2^2 and λ_3^2 with the angular strains, we can easily compute that

$$\lambda_1 = f'(r), \qquad \lambda_2 = \lambda_3 = \frac{\sin f}{r}\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|. \tag{2.7}$$

Therefore the energy is

$$E = \int \Big[f'^2 + 2(f'^2 + 1)\frac{\sin^2 f}{r^2}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2 + \frac{\sin^4 f}{r^4}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 \Big] \frac{2i\, dz d\bar{z} r^2 dr}{(1+|z|^2)^2}, \tag{2.8}$$

where $2i\, dz d\bar{z}/(1+|z|^2)^2$ is equivalent to the usual area element on a 2-sphere $\sin\theta d\theta d\varphi$. Now the part of the integrand

$$\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2 \frac{2i\, dz d\bar{z}}{(1+|z|^2)^2} \tag{2.9}$$

is precisely the pull-back of the area form $2i\, dR d\bar{R}/(1+|R|^2)^2$ on the target sphere of the rational map R; therefore its integral is 4π times the degree N of R. So the energy simplifies to

$$E = 4\pi \int \left(r^2 f'^2 + 2N(f'^2+1)\sin^2 f + \mathcal{I}\frac{\sin^4 f}{r^2} \right) dr \tag{2.10}$$

where $\mathcal{I}$ denotes the integral

$$\mathcal{I} = \frac{1}{4\pi}\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 \frac{2i\, dz d\bar{z}}{(1+|z|^2)^2}. \tag{2.11}$$

$\mathcal{I}$ depends only on the rational map R, and, as we explain in Section 5, it is an interesting function on the space of rational maps.

To minimize E, for maps of a given degree N, one should first minimize $\mathcal{I}$ over all maps of degree N. Then, the profile function f minimizing the energy (2.10) may be

found by solving a second order differential equation with N and $\mathcal{I}$ as parameters. In practice, we have considered rational maps of a given symmetric form, with symmetries corresponding to a known Skyrmion solution (or monopole). If these maps still contain a few free parameters, we have minimized $\mathcal{I}$ with respect to these (using an appropriate search algorithm). Then, the minimizing profile function f is determined by first bijectively mapping the radial coordinate r onto the unit interval, and then discretizing the energy functional using symmetric finite differences and finally obtaining the minimizing grid values by applying a conjugate gradient algorithm. This procedure seems appropriate for all baryon numbers up to $B = 9$, where the Skyrmion solutions all have considerable symmetry, but for some higher values of B one will have to consider quite general maps as the Skyrmions probably have very little symmetry.

Detailed examples of rational maps with various degrees and symmetries will be described in the next Section, and we shall compare the result of minimizing E for these maps with the energies of the numerically determined exact Skyrmion solutions.

Note the following pair of inequalities associated with the expression (2.10) for the energy E. The elementary inequality

$$\left(\int 1\, dS\right)\left(\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 dS\right) \geq \left(\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2 dS\right)^2, \tag{2.12}$$

where $dS = 2i\, dz d\bar{z}/(1+|z|^2)^2$, implies that $\mathcal{I} \geq N^2$. Next, by a Bogomolny-type argument, we see that

$$E = 4\pi \int \left((rf' + \sqrt{\mathcal{I}}\frac{\sin^2 f}{r})^2 + 2N(f'+1)^2\sin^2 f - 2(2N+\sqrt{\mathcal{I}})f'\sin^2 f \right) dr \tag{2.13}$$

so

$$E \geq 4\pi(2N+\sqrt{\mathcal{I}})\int_0^\infty (-2f'\sin^2 f)\, dr = 4\pi(2N+\sqrt{\mathcal{I}})\Big[-f + \frac{1}{2}\sin 2f\Big]_0^\infty \tag{2.14}$$

and so, if $f(0) = \pi$ and $f(\infty) = 0$,

$$E \geq 4\pi^2(2N+\sqrt{\mathcal{I}}). \tag{2.15}$$

Combined with the earlier inequality for $\mathcal{I}$, we recover the usual Fadeev-Bogomolny bound $E \geq 12\pi^2 N$. The bound (2.15) is stronger than this, for fields of the form we are considering, but there is no reason to think that true solutions of the Skyrme equation are constrained by this bound.

We conclude this Section by observing that the zeros of the Wronskian $W(z)$ of a rational map $R(z)$ give interesting information about the shape of the Skyrme field which is constructed from R using our ansatz (2.3). Where W is zero, the derivative dR/dz is zero, so the strain eigenvalues in the angular directions, λ_2 and λ_3, vanish. The baryon density, being proportional to $\lambda_1\lambda_2\lambda_3$, vanishes along the entire radial line in the direction specified by any zero of W. The energy density will also be low along such a radial line, since there will only be the contribution λ_1^2 from the radial strain eigenvalue. The Skyrme field baryon density contours will therefore look like a polyhedron with holes in the directions given by the zeros of W, and there will be $2N-2$ such holes. This structure is seen in all the plots shown in Fig. 1, for example, the $B=7$ Skyrmion having 12 holes arranged at the face centres of a dodecahedron.

3 Symmetric Rational Maps and Skyrmions

In this Section, we present the detailed form of certain symmetric rational maps of degrees one to nine, and also of degrees eleven and seventeen. Using our ansatz (2.3) we turn these rational maps into Skyrme fields with baryon number equal to the degree of the map. In each case, except degree eleven, we determine the parameters of the rational map that minimize its contribution to the energy, and then find the profile function $f(r)$ which minimizes the energy function (2.10) using the method explained earlier. In Fig. 2 we plot these profile functions for baryon numbers one to nine and also seventeen. The size increases with increasing baryon number, corresponding to a shift to the right of the profile function, hence they need not be labelled individually. In Table 1 we present the values of the energy of the resulting Skyrme field; these can be compared with the bound (2.15), and also with the energy of the corresponding numerically known exact Skyrmion solution. All numerical values for the energies quoted in this Section are the real energies divided by $12\pi^2 B$, and hence close to unity. In Fig. 1 we plot a surface of constant baryon density for several of our computed Skyrme fields. It should be noted that the value of the baryon density on the surface shown is not the same in each case, so the scale of these pictures should not be used to infer information on the size. However, the size can be deduced from the profile plots in Fig. 2. The data on the energies and shapes show that the Skyrme fields we obtain closely approximate the true Skyrmions.

The symmetries we impose are not chosen systematically; they are motivated by the symmetries of the known Skyrmion solutions (for $B \le 9$) [4]. For $B=11$ and $B=17$ we impose icosahedral symmetry, which is a possibility in both cases, and which has been conjectured as the symmetry of the $B=17$ Skyrmion.

Although most of the approximate solutions we find using our ansatz are minima of the energy, some are saddle points. True saddle point solutions of the Skyrme equation are often the minimal energy configurations having a particular symmetry not possessed by the Skyrmion solution.

A rational map, $R: S^2 \mapsto S^2$, is invariant or symmetric under a subgroup $G \subset SO(3)$ if there is a set of Möbius transformation pairs $\{g, D_g\}$ with $g \in G$ acting on the domain S^2 and D_g acting on the target S^2, such that

$$R(g(z)) = D_g R(z). \tag{3.1}$$

The transformations D_g should represent G in the sense that $D_{g_1} D_{g_2} = D_{g_1 g_2}$. Both g and D_g will in practice be $SU(2)$ matrices. For example, $g(z)$ can be expressed as $g(z) =$

$(\alpha z+\beta)/(-\bar\beta z+\bar\alpha)$ with $|\alpha|^2+|\beta|^2=1$. Replacing (α,β) by $(-\alpha,-\beta)$ has no effect, so g is effectively in $SO(3)$. The same is true for D_g.

Some of our rational maps possess an additional reflection or inversion symmetry. The transformation $z \mapsto \bar z$ is a reflection, whereas $z \mapsto -1/\bar z$ is the antipodal map on S^2, or inversion. We shall deal with reflection and inversion on a case by case basis.

The detailed form of our maps will depend on choices of the orientation of axes, both in the domain S^2 and target S^2. Our choice is made to simplify our maps as far as possible, but equivalent maps, differently oriented, are sometimes advantageous.

It is helpful to identify the Cartesian axes with certain directions specified by values of z. The formula (2.1) for a unit vector associated with z implies that the positive x_3-axis is in the direction $z=0$, the positive x_1-axis corresponds to $z=1$ and the positive x_2-axis corresponds to $z=i$.

$N=1$

The hedgehog map is $R(z)=z$. It is fully $O(3)$ invariant, since $R(g(z))=g(z)$ for any $g \in SU(2)$ and $R(-1/\bar z) = -1/\bar R(z)$. It gives the standard exact hedgehog Skyrmion solution with the usual profile $f(r)$, and with energy $E=1.232$. The map $R(z)=z$ is also the Jarvis rational map of a monopole centred at the origin.

$N=2$

A general degree two map is of the form

$$R(z) = \frac{\alpha z^2+\beta z+\gamma}{\lambda z^2+\mu z+\nu}. \tag{3.2}$$

Let us impose the two $\mathbb{Z}_2$ symmetries $z \mapsto -z$ and $z \mapsto 1/z$ which generate the viergruppe of 180° rotations about all three Cartesian axes. The conditions

$$R(-z) = R(z) \qquad \text{and} \qquad R(1/z) = 1/R(z) \tag{3.3}$$

restrict R to the form

$$R(z) = \frac{z^2-a}{-az^2+1}. \tag{3.4}$$

By a target space Möbius transformation, we can bring a to lie in the interval $-1 \le a \le 1$, with the map degenerating at the endpoints. Further, a 90° rotation, $z \mapsto iz$, reverses the sign of a. The maps (3.4) have three reflection symmetries in the Cartesian axes, which are manifest when a is real. For example, $R(\bar z) = \bar R(z)$ when a is real.

The Jarvis map of any centred and suitably oriented $N=2$ monopole is of this form. When $a=0$ the rational map has the additional symmetry $R(e^{i\chi}z) = e^{2i\chi}R(z)$; it is the Jarvis map of the axially symmetric $N=2$ monopole. The maps (3.4), with $-1<a<1$, parametrize a geodesic in the monopole moduli space, along which two monopoles scatter by 90° symmetrically from the x_1-axis to the x_2-axis. As $a \to \pm 1$ the monopoles separate to infinity.

If we use the maps (3.4) in our ansatz for the Skyrme field, we find the integral $\mathcal{I}$ increases monotonically to infinity as a increases from 0 to 1. For $a = 0$, $\mathcal{I} = \pi + 8/3$ and after determining the profile $f(r)$ in this case we obtain $E = 1.208$. So the Skyrme field based on the map $R(z) = z^2$ has the same symmetry as the $B = 2$ Skyrmion and energy just 3% higher (see Table 1). A baryon density plot for this configuration is shown in Fig. 1a.

One might consider imposing $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in other ways than (3.3) (eg. $R(-z) = -R(z)$) but this leads to maps which are equivalent but differently oriented.

$N = 3$

We recall that there is a unique tetrahedrally symmetric $N = 3$ monopole, and that the $B = 3$ Skyrmion has the same symmetry. There is also an axially symmetric, toroidal monopole (as for all $N > 1$), and a saddle point solution of the Skyrme equations with this shape.

A subset of the degree three rational maps which allows for both these solutions and a smooth interpolation between them is the subset with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, realized by the requirements

$$R(-z) = -R(z) \qquad \text{and} \qquad R(1/z) = 1/R(z). \tag{3.5}$$

The first condition implies that the numerator of R is even in z and the denominator is odd, or vice versa. These two possibilities are related by an $SU(2)$ Möbius transformation, so we choose the former and ignore the latter. Imposing the second condition as well gives us maps of the form

$$R(z) = \frac{\sqrt{3}az^2 - 1}{z(z^2 - \sqrt{3}a)} \tag{3.6}$$

with a complex. The inclusion of the $\sqrt{3}$ factor is a convenience. The parameter space of these maps should be thought of as a Riemann sphere with complex coordinate a. The rational map degenerates for three values of a, namely $a = \infty$, $a = \pm 1/\sqrt{3}$.

There is a further reflection symmetry $R(\bar{z}) = \bar{R}(z)$ if a is real. Together with the rotational symmetries, this implies reflection symmetry in all three Cartesian axes. A slightly subtler symmetry occurs if a is imaginary. The full symmetry group becomes D_{2d}, where the extra generator is a rotation by 90° about the x_3-axis combined with the reflection $x_3 \mapsto -x_3$. On the z-sphere the generator is $z \mapsto i/\bar{z}$, and $R(i/\bar{z}) = i/\bar{R}(z)$ if a is imaginary.

Tetrahedral symmetry is obtained by imposing the further symmetry

$$R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1} \tag{3.7}$$

which is satisfied by (3.6) if $a = \pm i$. Note that $z \mapsto (iz+1)/(-iz+1)$ sends $0 \mapsto 1 \mapsto i \mapsto 0$ and hence generates the 120° rotation cyclically permuting the Cartesian axes.

Finally, there is axial symmetry about the x_3-axis when $a = 0$, since then $R(z) = -1/z^3$. There is also axial symmetry when $a = \pm\sqrt{3}$. These further solutions are related to the first by 120° rotations that take the x_3-axis to the x_1-axis and x_2-axis.

The a-sphere, with the special points we have discussed, is sketched in Fig. 3. The Jarvis maps of the form (3.6) parametrize a geodesic submanifold of the $N = 3$ monopole moduli space. One particular geodesic is the great circle segment $-1/\sqrt{3} < a < 1/\sqrt{3}$. This describes 90° scattering of monopoles, with two single monopoles scattering from the x_1-axis to the x_2-axis, the third monopole remaining at the origin. This type of geodesic was previously described by Bielawski [6] and Houghton & Sutcliffe [17]. A second geodesic is the great circle $-\infty < ia < \infty$, which passes through both tetrahedra and one of the tori. This is the twisted line scattering described in [16]. Dynamical simulations of the Skyrme equation have revealed that remarkably similar scattering processes also occur for Skyrmions [3, 5].

Using (3.6) in the rational map ansatz for Skyrme fields, we find that on the a-sphere, the angular integral $\mathcal{I}$ has just two types of stationary point. There are minima at the tetrahedral points $\pm i$, where $\mathcal{I} = 13.58$, and there are saddle points at the tori $a = 0, \pm\sqrt{3}$, where $\mathcal{I} = 18.67$. $\mathcal{I}$ diverges as the degenerate points are approached. Using the tetrahedral map and solving for the profile $f(r)$, we find an approximation to the tetrahedral Skyrmion with energy $E = 1.184$ (see Fig. 1b). Similarly, using the toroidal map and again solving for the profile $f(r)$, we find an approximation to the toroidal saddle point solution of the Skyrme equations, with energy $E = 1.256$.

In addition to the tetrahedral and toroidal solutions of the Skyrme equation, there is a pretzel, or figure eight shaped solution, discovered in an approximate form by Walet [33]. This is a saddle point and slightly lower in energy than the torus. Its existence has been confirmed using a full field simulation and its energy computed to be $E = 1.164$ [5]. One might expect, based on symmetry, that this solution could be described approximately with our rational map ansatz. It would occur for a map of type (3.6), with a in the range $0 < a < 1/\sqrt{3}$. However, no saddle point occurs in this range. So the pretzel solution is not accessible with the rational map ansatz, and this appears to be because it is a configuration of three Skyrmions in a line, whereas the rational map ansatz appears to work best for shell-like structures, where all the baryon density is concentrated at roughly the same distance from the origin.

It is interesting to look at the Wronskian of maps of the form (3.6). Recall that $W = p'q - q'p$, where p and q are the numerator and denominator. Calculating, we find

$$W(z) = -\sqrt{3}a(z^4 + \sqrt{3}(a - a^{-1})z^2 + 1). \tag{3.8}$$

Note that for $a = \pm i$, W is proportional to a tetrahedral Klein polynomial [19]. If $a = 1$, W has square symmetry, but the rational map does not have as much symmetry as this.

$N = 4$

The minimal energy $B = 4$ Skyrmion has octahedral symmetry, and there is a unique octahedrally symmetric $N = 4$ monopole. The octahedrally symmetric rational map of degree four can be embedded in a one parameter family of tetrahedrally symmetric maps

$$R(z) = c\frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1} \tag{3.9}$$

where c is real. The numerator and denominator are tetrahedrally symmetric Klein polynomials, so R is invariant up to a constant factor under any transformation in the tetrahedral group.

For $c = 1$ there is octahedral symmetry. The extra generator is a 90° rotation about the x_3-axis, $z \mapsto iz$. Clearly

$$R(iz) = 1/R(z) \tag{3.10}$$

when $c = 1$. More generally, this 90° rotation replaces c by $1/c$. There is a geodesic motion of monopoles, with tetrahedral symmetry throughout, in which four single monopoles approach on the vertices of a contracting tetrahedron, and recede on the vertices of an expanding tetrahedron dual to the first [14]. This just corresponds to c running from 0 to ∞. Octahedral symmetry occurs at the moment of closest approach.

Using (3.9) in the Skyrme field ansatz, we find that the minimal energy occurs at $c = 1$, with the value $E = 1.137$. This is quite close to the energy of the $B = 4$ Skyrmion $E = 1.116$, and almost the same as the energy of the best $B = 4$ instanton generated Skyrme field [22] which has $E = 1.132$.

The Wronskian of the map (3.9) is proportional to $z(z^4 - 1)$ for all values of c. This is the face polynomial of a cube, with faces in the directions $0, 1, i, -1, -i, \infty$ (i.e. the directions of the Cartesian axes). We understand from this why the baryon density vanishes in these directions, and hence why the Skyrmion has a cubic shape, with its energy concentrated on the vertices and edges of the cube (see Fig. 1c).

$N = 5$

The $B = 5$ Skyrmion of minimal energy has symmetry D_{2d}, which is somewhat surprising. An octahedrally symmetric solution exists but has higher energy [5]. There is a family of rational maps with two real parameters, with the generic map having D_{2d} symmetry, but having higher symmetry at special parameter values.

The family of maps is

$$R(z) = \frac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1} \tag{3.11}$$

with a and b real. The two generators of the D_{2d} symmetry are realized as

$$R(i/\bar{z}) = i/\bar{R}(z) \quad \text{and} \quad R(-z) = -R(z). \tag{3.12}$$

Additional symmetry occurs if $b = 0$; $R(z)$ then has D_4 symmetry, the symmetry of a square. There is octahedral symmetry if, in addition, $a = -5$. This value ensures the 120° rotational symmetry

$$R\Big(\frac{iz + 1}{-iz + 1}\Big) = \frac{iR(z) + 1}{-iR(z) + 1}. \tag{3.13}$$

The map $R(z) = z(z^4 - 5)/(-5z^4 + 1)$ has Wronskian

$$W(z) = -5(z^8 + 14z^4 + 1) \tag{3.14}$$

which is proportional to the face polynomial of an octahedron.

Using the maps (3.11) in the Skyrme field ansatz gives a structure which is a polyhedron with eight faces. In the special case $b = 0, a = -5$, this polyhedron is an octahedron, and the angular integral is $\mathcal{I} = 52.05$, whereas performing a numerical search over the parameters a and b we find that $\mathcal{I}$ is minimized when $a = 3.07$, $b = 3.94$, taking the value $\mathcal{I} = 35.75$. This is consistent with the structure and symmetry of the known $B = 5$ Skyrmion, which is that of a polyhedron made from four pentagons and four quadrilaterals. Minimizing over the profile function we find a Skyrme field with energy $E = 1.147$ (see Fig. 1d). There is a saddle point at the octahedral parameter values, where $E = 1.232$ (see Fig. 1j). There is a further, higher saddle point at $a = b = 0$, where the map (3.11) simplifies to $R(z) = z^5$, and gives a toroidal Skyrme field.

$N = 6$ and $N = 8$

The Skyrmions with $B = 6$ and $B = 8$ both have extended cyclic symmetry. It is straightforward to find rational maps with these symmetries, and use them with our ansatz.

For $B = 6$, the desired symmetry is D_{4d}, generated by $z \mapsto iz$ and $z \mapsto 1/z$. The rational maps

$$R(z) = \frac{z^4 + a}{z^2(az^4 + 1)} \tag{3.15}$$

have this symmetry, since $R(iz) = -R(z)$ and $R(1/z) = 1/R(z)$. With these maps in our ansatz, the minimal energy occurs at $a = 0.16$, when $E = 1.137$. The Skyrme field has a polyhedral shape consisting of a ring of eight pentagons capped by squares above and below (see Fig. 1e).

For $B = 8$, the symmetry is D_{6d}, generated by $z \mapsto e^{i\pi/3}z$ and $z \mapsto i/z$. The rational maps

$$R(z) = \frac{z^6 - a}{z^2(az^6 + 1)} \tag{3.16}$$

have the required symmetry. This time the minimal energy Skyrme field obtained using these maps has $E = 1.118$ when $a = 0.14$. The polyhedral shape is now a ring of 12 pentagons capped by hexagons above and below (see Fig. 1g).

$N = 7$

In a sense, the $N = 7$ case is similar to the cases $N = 6$ and $N = 8$, but the Skyrmion has dodecahedral shape. A dodecahedron is a ring of ten pentagons capped by pentagons above and below.

Among the rational maps with D_{5d} symmetry

$$R(z) = \frac{z^5 - a}{z^2(az^5 + 1)} \tag{3.17}$$

the one with icosahedral symmetry has $a = -3$. The Wronskian is then proportional to the face polynomial of a dodecahedron, namely $z(z^{10} + 11z^5 - 1)$.

In a different orientation, the icosahedrally symmetric map is

$$R(z) = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1} \tag{3.18}$$

which has a similar structure to the octahedrally symmetric $N = 4$ map ((3.9) with $c = 1$).

We have found it interesting to investigate the dodecahedron in yet another orientation, where tetrahedral symmetry is manifest. There is a one parameter family of degree seven maps with symmetry T. The family is

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)} \tag{3.19}$$

where b is complex. It is easy to verify that

$$R(-z) = -R(z), \quad R(1/z) = -1/R(z) \quad \text{and} \quad R\Big(\frac{iz+1}{-iz+1}\Big) = \frac{iR(z)+1}{-iR(z)+1}. \tag{3.20}$$

For real b, the symmetry extends to T_d and for b imaginary it extends to T_h. When $b = 0$ there is octahedral symmetry, and when $b = \pm 7/\sqrt{5}$ there is icosahedral symmetry Y_h. Using (3.19) in our ansatz, we have found the minimal energy at $b = \pm 7/\sqrt{5}$, which gives a dodecahedral Skyrme field, with energy $E = 1.107$ (see Fig. 1f). There is a saddle point at $b = 0$ with a cubic shape.

The imaginary b-axis represents an interesting dynamical process. If we regard the rational maps as Jarvis maps of $N = 7$ monopoles, then motion along the imaginary b-axis is a geodesic in which there is a single monopole at the origin, and six monopoles approaching it along the positive and negative Cartesian axes. They pass successively through a dodecahedron, a cube and a second dodecahedron (rotated by 90° relative to the first) before separating into a configuration similar to the incoming one (again rotated by 90°, which affects the shape but not the positions of the monopoles). A similar motion is possible with seven Skyrmions, but one must allow for the varying potential energy. The energy of seven separated Skyrmions is greater either than the cube or dodecahedron. If the energy is sufficient, an oscillatory motion between the two dodecahedra can occur,

through the cubic configuration. At higher energy there can be a 7-Skyrmion scattering process going through all these configurations.

So far we have used fairly *ad hoc* methods to obtain our symmetric rational maps. However, this approach becomes cumbersome for higher degrees and it is more efficient to use a systematic algorithm. By constructing symmetric rational maps we are in effect computing bases for two-dimensional representations of finite subgroups of $SU(2)$, for which classical group theory can be employed. In the Appendix we describe the construction of symmetric rational maps from this more systematic point of view. This construction is illustrated with the example above, tetrahedral symmetry for $N = 7$, and some further examples occur below.

$N = 9$

Imposing tetrahedral symmetry on degree nine maps we find, see Appendix, the one parameter family

$$R(z) = \frac{5i\sqrt{3}z^6 - 9z^4 + 3i\sqrt{3}z^2 + 1 + az^2(z^6 - i\sqrt{3}z^4 - z^2 + i\sqrt{3})}{z^3(-z^6 - 3i\sqrt{3}z^4 + 9z^2 - 5i\sqrt{3}) + az(-i\sqrt{3}z^6 + z^4 + i\sqrt{3}z^2 - 1)} \tag{3.21}$$

where a is real. This map is degenerate at the values $a = 1, -5, \pm\infty$. In terms of the corresponding 9-monopole configurations, the first of these degenerate values, $a = 1$, corresponds to a single monopole at the origin with eight monopoles on the vertices of a cube at infinity. The value $a = -5$ corresponds to four monopoles at infinity on the vertices of a tetrahedron, and $a = \pm\infty$ represents six monopoles at infinity on the vertices of an octahedron.

The angular integral $\mathcal{I}$ diverges as a degenerate map is approached, so it is clear from the above that the family of maps (3.21) contains at least three local minima for $\mathcal{I}$ as a function of a. In Fig. 4 we plot $\mathcal{I}$ as a function of a, from which it can be seen that the global minimum occurs in the middle interval $-5 < a < 1$. More explicitly, the minimum occurs when $a = -1.98$, and the energy of the resulting Skyrme field is $E = 1.123$. The Skyrme field has a polyhedral shape consisting of four hexagons centred on the vertices of a tetrahedron, linked by four triples of pentagons (see Fig. 1h).

$N = 17$

For general $B > 9$ the expected symmetries of the Skyrmion are not great enough to cut down the associated family of rational maps to just one or two parameters. Thus a minimization over a large family of rational maps probably has to be undertaken. However, given the complicated nature of such a space of rational maps (for example, recall the above one parameter family of degree nine maps, which contains spurious local minima that lie in disconnected sectors) this is a difficult numerical task.

Fortunately there are exceptional cases where we expect a highly symmetric configuration to occur. One of these is at $B = 17$, where it has been conjectured [4] that the Skyrmion has the icosahedrally symmetric, buckyball structure of carbon 60. An $N = 17$ rational map with symmetry Y_h is, see Appendix,

$$R(z) = \frac{17z^{15} - 187z^{10} + 119z^5 - 1}{z^2(z^{15} + 119z^{10} + 187z^5 + 17)}. \tag{3.22}$$

Using this map in our ansatz we obtain a Skyrme field with energy $E = 1.092$. The very low value for this energy supports the conjecture that Y_h is the symmetry of the minimal energy $B = 17$ configuration. The polyhedron does indeed have the buckyball form (see Fig. 1i), consisting of twelve pentagons, each surrounded by five hexagons, making a total of 32 polygons.

From Fig. 2 it can be seen that the buckyball Skyrmion is quite large, and furthermore the profile function is extremely flat for small r. This implies that there is a region inside the shell of the buckyball where the Skyrme field is close to the vacuum (in fact $U = -1$ but this is not important), possibly allowing smaller Skyrmions, for example $B = 4$, to sit inside the buckyball with little distortion. It would be interesting to investigate this further. It may shed some light on the shell structure of Skyrmions, which appears to be favoured over a crystal structure for the cases investigated so far, but presumably fails for sufficiently large B.

$N = 11$

We have already constructed several symmetric rational maps, such as the $N = 3$ example with axial symmetry and the $N = 5$ example with octahedral symmetry, which generate approximations to saddle point Skyrme fields. Although we have not computed the energy minimizing rational map of degree eleven, we can compute an interesting saddle point map which has icosahedral symmetry, see Appendix. Note that the existence of this map proves the existence of an icosahedrally symmetric $N = 11$ monopole, as conjectured in ref. [15]. The rational map is

$$R(z) = \frac{11z^{10} + 66z^5 - 1}{z(z^{10} + 66z^5 - 11)}. \tag{3.23}$$

The value of the angular integral for this map is $\mathcal{I} = 486.84$, which is very large; it is even greater than the value for the $N = 17$ map given above (see Table 1). This indicates that the minimal energy $B = 11$ Skyrmion will not have Y_h symmetry. Computing the energy we find $E = 1.406$, which is considerably higher than that of eleven well-separated $B = 1$ Skyrmions. This icosahedral configuration is shown in Fig. 1k.

B	$\mathcal{I}$	APPROX	TRUE	SYM
1	1.00	1.232	1.232	$O(3)$
2	5.81	1.208	1.171	$O(2) \times \mathbb{Z}_2$
3	13.58	1.184	1.143	T_d
4	20.65	1.137	1.116	O_h
5	35.75	1.147	1.116	D_{2d}
6	50.76	1.137	1.109	D_{4d}
7	60.87	1.107	1.099	Y_h
8	85.63	1.118	1.100	D_{6d}
9	112.83	1.123	1.099	T_d
17	367.41	1.092	1.073	Y_h
3*	18.67	1.256	1.191	$O(2) \times \mathbb{Z}_2$
5*	52.05	1.232	1.138	O_h
11*	486.84	1.406	1.158	Y_h

Table 1 : Comparison between the energies of approximate Skyrmions generated from rational maps, and the energies of true Skyrmions. The table gives the value of the angular integral $\mathcal{I}$, and the associated Skyrme field energy (APPROX), together with the energy of the true solution (TRUE), as determined in refs. [4, 5], and the symmetry (SYM) of the corresponding Skyrme field. A $*$ denotes a saddle point configuration.

4 Rational Maps and Skyrmion Vibrations

Through our ansatz for Skyrme fields in terms of rational maps we have found approximations to several minimal energy Skyrmions of various baryon numbers. It is natural to guess that varying the rational map parameters will correspond to distortions of the Skyrmions into some of their vibrational modes. It is interesting to investigate this, as Barnes et al. [1, 2] have recently used a numerical simulation of the Skyrme equation to study the spectrum of vibrations around the $B = 2$ and $B = 4$ Skyrmions. We can interpret some of the qualitative features of their results in terms of rational maps, and can predict what happens in some examples not yet analysed.

We consider first the vibrations of the $B = 4$ Skyrmion with octahedral symmetry. This Skyrmion has nine zero modes corresponding to translations, rotations and isospin rotations. There are nine low-lying vibrational modes, with frequencies somewhat less than the pion mass. These modes lie in multiplets transforming under certain irreducible representations of the octahedral group O. In increasing order of frequency, these representations are E^O, A_2^O, F_2^O, F_2^O (in the notation of ref. [10]), respectively of dimensions two, one, three and three. Barnes et al. have presented pictures of the Skyrmion distortion for these modes of vibration. The next mode is the breather mode (a vibration of the scale size) which is invariant under the octahedral group, and some higher frequency modes have been identified, separate from the continuum of pion field vibrations.

It is the modes below the breather which can be identified with variations of the rational map parameters. Recall that the rational map of degree four with octahedral symmetry is

$$R_0(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \tag{4.1}$$

The general variation of this map, in which we preserve the leading coefficient of the numerator as 1 as a normalization, is

$$R(z) = \frac{z^4 + \alpha z^3 + (2\sqrt{3}i + \beta)z^2 + \gamma z + 1 + \delta}{(1+\lambda)z^4 + \mu z^3 + (-2\sqrt{3}i + \nu)z^2 + \sigma z + 1 + \tau} \tag{4.2}$$

where $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \sigma, \tau$ are small complex numbers. We now calculate the effect of the transformations of the octahedral group leaving R_0 fixed. For example, the 90° rotation, represented by the transformation $R(z) \mapsto 1/R(iz)$ leaves R_0 fixed, but transforms the general map $R(z)$ to

$$\tilde{R}(z) = \frac{(1+\lambda)z^4 - i\mu z^3 + (2\sqrt{3}i - \nu)z^2 + i\sigma z + 1 + \tau}{z^4 - i\alpha z^3 - (2\sqrt{3}i + \beta)z^2 + i\gamma z + 1 + \delta}. \tag{4.3}$$

Normalizing this by dividing top and bottom by $1+\lambda$, and ignoring quadratic and smaller terms in the small parameters, we get

$$\tilde{R}(z) = \frac{z^4 - i\mu z^3 + (2\sqrt{3}i - \nu - 2\sqrt{3}i\lambda)z^2 + i\sigma z + 1 + \tau - \lambda}{(1-\lambda)z^4 - i\alpha z^3 + (-2\sqrt{3}i - \beta + 2\sqrt{3}i\lambda)z^2 + i\gamma z + 1 + \delta - \lambda}. \tag{4.4}$$

Therefore the transformation acts linearly on the nine parameters $\alpha, .., \tau$ via a 9×9 matrix that can be read off from this expression. The only contribution to the trace of the 9×9 matrix is the -1 associated with the replacement of λ by $-\lambda$ in the leading term of the denominator. So the character χ of the 90° rotation in this representation is -1.

We really need to consider this representation as a real eighteen-dimensional one, so the character above becomes $\chi = -2$. From now on we shall work with real representations.

Similar calculations for the elements of each conjugacy class of the octahedral group give the characters listed in Table 2, where I is the identity, C_4 denotes a 90° rotation and C_4^2 is the square of this, C_3 denotes a 120° rotation and C_2 a rotation by 180° which is not the square of a 90° rotation.

Class	Character χ
I	18
$6C_4$	-2
$3C_4^2$	2
$8C_3$	0
$6C_2$	-2

Table 2 : Characters of the group O acting on the real eighteen-dimensional parameter space of deformations of the octahedral degree four rational map.

The character table of O tells us that this eighteen-dimensional representation splits into the irreducible components $2A_2^O + 2E^O + 2F_1^O + 2F_2^O$.

To find which of the irreducible representations correspond to true vibrations we need to remove those corresponding to zero modes. First we need to remove the representation associated with $SU(2)$ Möbius transformations of $R_0(z)$ which correspond to isospin rotations of the Skyrme field. So we consider the infinitesimal deformations

$$R_0(z) \mapsto \frac{(1+i\epsilon)R_0(z) + \epsilon'}{-\bar{\epsilon}' R_0(z) + (1 - i\epsilon)} \tag{4.5}$$

where ϵ is real, and ϵ' complex. Under the transformations of the octahedral group the characters are $\chi(I) = 3$, $\chi(C_4) = -1$, $\chi(C_4^2) = 3$, $\chi(C_3) = 0$, $\chi(C_2) = -1$, so these parameter variations transform as $A_2^O + E^O$. Similarly, the parameter variations which correspond to translations and rotations transform under the octahedral group as $F_1^O + F_1^O$. From the above eighteen-dimensional representation we therefore subtract $A_2^O + E^O + F_1^O + F_1^O$ to obtain the representation of the true vibrations, which has the irreducible components $A_2^O + E^O + F_2^O + F_2^O$, and is nine-dimensional. These irreducible representations are precisely the ones found by Barnes et al. for the low-lying Skyrmion vibrations.

Barnes et al., in their calculations of the vibrations of the $B = 2$ toroidal Skyrmion[2], found just one doubly degenerate mode of vibration of low frequency (below the breather), and it corresponds to the deformation of the rational map (3.4) as a varies away from zero (corresponding to the separation mode for two monopoles).

We have done a similar analysis for the vibrational modes of the $B = 3$ tetrahedral Skyrmion. From the rational map parametrization we predict that there are five low-lying modes, transforming as $E^T + F^T$ of the tetrahedral group T_d. This result slightly disagrees with Walet's [33] estimate of the vibrations using the instanton approximation of Skyrme fields. Although Walet found the lowest modes to be in an $E^T + F^T$, he also found a second triplet of modes just below the breather. Our results suggest that this second triplet should really have a higher frequency, but this must be checked using the exact solution and its vibrations.

Since the $B = 7$ Skyrmion has Y symmetry, its vibrational modes also fall into large degenerate multiplets. The rational maps involved have degree seven and it is useful to simplify the calculation by adopting the representation theory perspective of the Appendix.

A degree N rational map $R = (p_0, q_0)$ is G-symmetric when p_0 and q_0 span a two-dimensional representation of G inside $\underline{N+1}$. This means that acting with $g \in G$ on (x, y) has the effect of transforming (p_0, q_0) by some 2×2 matrix D_g. Put another way, the g transformation of (x, y) followed by the D_g^{-1} transformation of the rational map leaves (p_0, q_0) unchanged. To find the transformation properties of the vibrations, a general (p, q) is transformed in this way.

We know how a general homogeneous polynomial transforms under G; it is in the representation $\underline{N+1}|_G$. We also know the D_g representation; it is the two-dimensional representation in $\underline{N+1}|_G$ corresponding to R. The D_g^{-1} representation E can be calculated from this. Transforming p and q under $\underline{N+1}|_G$ and then under E is a $\underline{N+1}|_G \times E$ transformation of (p, q), where (p, q) is regarded as a $(2N+2)$-dimensional vector. Thus, to find the transformation properties of the vibrations we decompose $\underline{N+1}|_G \times E$ into irreducible representations of G.

In the $B = 7$ case

$$\underline{8}|_Y = E_2^{\prime Y} + I^{\prime Y} . \tag{4.6}$$

The icosahedral Skyrmion corresponds to $E_2^{\prime Y}$. That is the representation of the D_g's mentioned above. All elements of Y lie in the same conjugacy class as their inverses, so the D_g^{-1} representation is also $E_2^{\prime Y}$. Each character of $\underline{8}|_Y \times E_2^{\prime Y}$ is obtained by multiplying the corresponding one for $\underline{8}|_Y$ with that for $E_2^{\prime Y}$. These are listed in Table 3.

Class	$\underline{8}\|_Y$	$E_2^{\prime Y}$	$\underline{8}\|_Y \times E_2^{\prime Y}$
I	8	2	16
$12C_5$	$-1/2-\sqrt{5}/2$	$1/2-\sqrt{5}/2$	1
$12C_5^2$	$1/2-\sqrt{5}/2$	$-1/2-\sqrt{5}/2$	1
$20C_3$	1	1	1
$15C_2$	0	0	0

Table 3 : Characters for representations of Y associated with vibrations of the $B = 7$ Skyrmion.

Knowing the characters, we find the decomposition

$$\underline{8}|_Y \times E_2^{\prime Y} = A^Y + F_1^Y + F_2^Y + G^Y + H^Y . \tag{4.7}$$

There are copies of this decomposition corresponding to real variations and to imaginary variations. This means the variations around the $B = 7$ Skyrmion transform as $2A^Y + 2F_1^Y + 2F_2^Y + 2G^Y + 2H^Y$. The $2A^Y$ are the trivial variations caused by multiplying the icosahedral p_0 and q_0 by the same constant. The vector representation of the icosahedral group is F_1^Y, so translations and rotations account for $2F_1^Y$, and Möbius transformations account for an F_2^Y. The representation of the true vibrations therefore has irreducible components $F_2^Y + 2G^Y + 2H^Y$, with degeneracies three, four, four, five and five, respectively.

5 Morse Function on Monopole Moduli Spaces

The Skyrme field ansatz (2.3), using a rational map $R(z)$, leads to a contribution to the Skyrme energy given by

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i \, dz d\bar{z}}{(1+|z|^2)^2} . \tag{5.1}$$

Now we may regard $\mathcal{I}$ simply as a function on the space of rational maps of any given degree, N. If we also identify rational maps with monopoles, via the Jarvis construction, $\mathcal{I}$ becomes a function on the N-monopole moduli space. $\mathcal{I}$ respects some, but not all,

the natural symmetries of the monopole moduli space. $\mathcal{I}$ is invariant under rotations of the target S^2, hence descends to the usual $(4N-1)$-dimensional moduli space $\mathcal{M}_N$. It is also invariant under rotations of the domain S^2, hence is unchanged when the monopole configuration is rotated. $\mathcal{I}$ is not, however, invariant under a translation of the monopole configuration in $\mathbf{R}^3$.

It appears that $\mathcal{I}$ is a "proper" Morse function, that is, the set of rational maps, and hence monopoles, for which $\mathcal{I}$ has any particular finite value is compact. We have not verified this in general. It is necessary to prove that $\mathcal{I}$ tends to infinity whenever the rational map degenerates. We have seen this happen in several cases mentioned in Section 4. Such a degeneracy corresponds to one or more monopoles moving off to infinity.

We have calculated one special case analytically. Consider the rational maps $R(z) = cz$. The phase of c is unimportant, so let c be real and positive. R degenerates if either c becomes zero or infinite. Since R has degree one, it is the Jarvis map of a single monopole, centred, in fact, at $(0, 0, 2\log c)$. The integral $\mathcal{I}$ reduces to

$$\mathcal{I} = 2c^4 \int_0^\infty \frac{\rho(1+\rho^2)^2 \, d\rho}{(1+c^2\rho^2)^4} = \frac{1}{3}(c^2 + 1 + 1/c^2) . \tag{5.2}$$

So $\mathcal{I}$ indeed diverges if $c \to 0$ or $c \to \infty$. The minimal value is $\mathcal{I} = 1$ when $c = 1$, as expected. For a rational map of the form $R = (z-a)/(z-b)$, the integral again diverges as b approaches a; this is equivalent, by a Möbius transformation, to the example $R(z) = cz$ with $c \to \infty$. Generally, one may expect $\mathcal{I}$ to diverge whenever a zero and a pole of R come together.

Having a proper Morse function $\mathcal{I}$ defined on the monopole moduli space helps us understand the topology of the moduli space. We have investigated the 3-monopole moduli space in this way. The stationary points of $\mathcal{I}$ on $\mathcal{M}_3$ consist of a number of orbits of the rotation group $SO(3)$. Among the D_2 symmetric maps of the form (3.6) we found just two types of stationary point for $\mathcal{I}$. Assuming that $\mathcal{I}$ has no further types of stationary point, then on $\mathcal{M}_3$, $\mathcal{I}$ has two stationary orbits. One is the set of $N = 3$ tori (centred at the origin). This is a two-dimensional orbit. Each torus is a saddle point, with two independent unstable modes (related by rotations about the symmetry axis). The unstable manifold of this orbit (suitably completed) is therefore a 4-cycle. By symmetry, the unstable manifold includes all the rational maps (3.6) with a lying on the great circle segment $0 \le ia < 1$. The unstable manifold therefore consists of the orbits under $SO(3)$ of all the rational maps of the form (3.6), with a in this interval. The other stationary orbit is the set of tetrahedra (again centred at the origin), which is the orbit of minima. This orbit is three-dimensional, and completes the 4-cycle. We have tried to visualize this 4-cycle as a smooth submanifold of $\mathcal{M}_3$, but have found this difficult in the neighbourhood of the tetrahedra.

A 4-cycle is the basic non-trivial compact homology cycle which is predicted by the calculations of Segal & Selby [28]. It would be interesting if the Sen 4-form, representing a bound state of three monopoles [29], were concentrated around the particular 4-cycle we have found.

These calculations suggest that further investigation of $\mathcal{I}$ as a Morse function on $\mathcal{M}_N$ would be worthwhile.

6 Conclusion

We have introduced a new ansatz for Skyrme fields, based on rational maps. This allows us to construct good approximations to several Skyrmions and helps us understand the similarities which have been observed between Skyrmions and BPS monopoles. A certain black hole with hair has states with a remarkably similar structure to Skyrmions, also related to rational maps [27]. Thus it appears that a whole class of solitonic objects in three space dimensions may be understood via the kind of rational map approach which we employ here.

We have used our ansatz to study the low-lying vibrational modes of Skyrmions. For the $B = 2$ and $B = 4$ Skyrmions, our results agree qualitatively with those obtained numerically, and we can predict the structure of the vibrational spectrum for other cases, in particular $B = 3$ and $B = 7$.

Finally, the relationship between monopoles and Skyrmions has led us to an interesting Morse function on the monopole moduli spaces which may be of use in understanding the homology of the moduli spaces and thus predictions made by duality.

Acknowledgements

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Appendix Systematic calculation of symmetric maps

For low degrees, symmetric rational maps may be constructed by explicitly performing the group transformations on a general rational map and deriving constraints on the coefficients. For higher degrees and for larger groups it is useful to employ the theory of group representations in the construction of the symmetric rational maps. In this Appendix such a construction will be described, and applied to the example of degree seven maps with tetrahedral symmetry.

To construct symmetric rational maps it is convenient to employ homogeneous projective coordinates x and y on the Riemann sphere, rather than the inhomogeneous $z = x/y$ employed earlier. A rational map is a map from Riemann sphere to Riemann sphere of the form

$$R(x,y) = (p(x,y), q(x,y)) \tag{A1}$$

where p and q are homogeneous polynomials. In the (x, y) coordinates, an $SO(3)$ rotation in space by θ about the direction of the unit vector (n_1, n_2, n_3) is realized by the $SU(2)$ transformation $\exp\left(i\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right)$, whose action on the Riemann sphere is

$$\begin{aligned} x \mapsto x' &= (d+ic)x - (b-ia)y \\ y \mapsto y' &= (b+ia)x + (d-ic)y \end{aligned} \tag{A2}$$

where $a = n_1 \sin\frac{\theta}{2}$, $b = n_2 \sin\frac{\theta}{2}$, $c = n_3 \sin\frac{\theta}{2}$ and $d = \cos\frac{\theta}{2}$. Furthermore, for our purposes, two rational maps are equivalent if they can be mapped into each other by an $SU(2)$ transformation of the target sphere, that is by a transformation of p and q of the form (A2). A rational map is symmetric under some finite group $G \subset SU(2)$ if G transformations of x and y map it into an equivalent map.

A degree N homogeneous polynomial is a polynomial of the form

$$p(x,y) = \sum_{i=0}^{N} a_i x^i y^{N-i}. \tag{A3}$$

Under $SU(2)$ transformations (A2) of x and y the space of degree N homogeneous polynomials transforms under the unique irreducible $(N+1)$-dimensional representation of $SU(2)$: $\underline{N+1}$. This $\underline{N+1}$ is also a representation of any finite subgroup G of $SU(2)$, generally reducible. It is easy to calculate its decomposition into irreducible representations, because, in $\underline{N+1}$, the element $\exp\left(i\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right)$ has character

$$\frac{\sin\left(\frac{N+1}{2}\right)\theta}{\sin\frac{\theta}{2}} \tag{A4}$$

for any $\mathbf{n}$. There are tables of these reductions given in, for example, ref. [21].

Suppose two degree N homogeneous polynomials $p(x, y)$ and $q(x, y)$ lie in the same two-dimensional representation of G; then, G transformations of x and y will result in $GL(2,\mathbb{C})$ transformations of $(p(x, y), q(x, y))$. If, further, $p(x, y)$ and $q(x, y)$ are orthonormal as vectors in the $\underline{N+1}$ carrier space, then, projectively, the G action on x and y results only in $SU(2)$ transformations of $(p(x, y), q(x, y))$. Therefore, the rational map $R(x, y) = (p(x, y), q(x, y))$ is G symmetric.

This means that there is a systematic way of deciding whether there are G symmetric maps of some degree N. The representation $\underline{N+1}$ is decomposed into irreducible representations of G. If

$$\underline{N+1}|_G = E + \text{other irreducible representations of } G, \tag{A5}$$

where E is a two-dimensional irreducible representation of G, and if the basis polynomials for E have no common factor, then there is a G symmetric degree N map. If they have a common factor then the resulting rational map has lower degree. This occurs when the E in $\underline{N+1}$ is the product of lower degree polynomials; this is illustrated with an example below. It might also happen that

$$\underline{N+1}|_G = A_1 + A_2 + \text{other irreducible representations of } G, \tag{A6}$$

where A_1 and A_2 are one-dimensional representations of G. In this case there is a one parameter family of G symmetric rational maps: if $p(x,y)$ is in A_1 and $q(x,y)$ is in A_2 then the family

$$R(x,y) = (ap(x,y), q(x,y)) \tag{A7}$$

is G symmetric.

The example of tetrahedral symmetry for degree seven is now discussed. Let us consider the representation $\underline{8}$. Under restriction to T

$$\underline{8}|_T = 2E'^T + G'^T, \tag{A8}$$

that is, two two-dimensional irreducible representations of T occur in the decomposition of $\underline{8}$. Furthermore, there is an arbitrariness in the decomposition

$$2E'^T = E'^T + E'^T, \tag{A9}$$

and this allows a one parameter family of tetrahedrally symmetric rational maps to be constructed.

The tetrahedral group is both a subgroup of the octahedral group O and a subgroup of the icosahedral group Y. We can decompose $\underline{8}$ as a representation of Y and of O. We find

$$\underline{8}|_O = E_1'^O + E_2'^O + G'^O, \tag{A10}$$
$$\underline{8}|_Y = E_2'^Y + I'^Y. \tag{A11}$$

We can decompose these representations further by restriction to T

$$\begin{aligned} E_1'^O|_T &= E'^T, \\ E_2'^O|_T &= E'^T, \\ G'^O|_T &= G'^T \end{aligned} \tag{A12}$$

and

$$\begin{aligned} E_2'^Y|_T &= E'^T, \\ I'^Y|_T &= E'^T + G'^T. \end{aligned} \tag{A13}$$

In this way, we see that T has two identical two-dimensional irreducible representations in $\underline{8}$. O has two as well but they are different and Y only has one. The carrier spaces of these representations are two-dimensional subspaces of the carrier space of $\underline{8}$, a space which is realised as degree seven homogeneous polynomials. The symmetric rational maps we wish to calculate are constructed from the bases of the two-dimensional spaces.

There are simple and venerable methods for calculating such bases explicitly. They are explained in Serre's book [30]. Consider U, a reducible representation of a group G,

$$\begin{aligned} G &\to GL(U) \\ g &\mapsto \rho(g), \end{aligned} \tag{A14}$$

which decomposes into irreducible representations V_i,

$$\begin{aligned} U &= V_1 + \ldots + V_1 + V_2 + \ldots + V_2 + \ldots\ldots + V_h + \ldots + V_h \\ &= W_1 + \ldots + W_h \end{aligned} \tag{A15}$$

where

$$W_i = V_i + V_i + \ldots + V_i. \tag{A16}$$

If the irreducible representation V_i has character $\chi_i(g)$ for $g \in G$, and $n_i = \dim W_i$, then

$$P_i = \frac{n_i}{|G|}\sum_{g\in G}\chi_i(g)^*\rho(g) \tag{A17}$$

is the projection operator

$$P_i : U \to W_i. \tag{A18}$$

Using MAPLE these projection operators can be calculated.

Since E'^T appears twice in $\underline{8}|_T$, projection onto E'^T gives a four-dimensional space. To work out a basis for this space, the projection operator

$$P : \underline{8} \to 2E'^T \tag{A19}$$

must be calculated using (A17). The $T \subset SU(2)$ transformations of (x,y) are first calculated explicitly. In the orientation where each edge of the tetrahedron has its midpoint on a Cartesian axis, the C_2 element about the x_3-axis has $c = -1$ and $a = b = d = 0$ and hence

$$\begin{aligned} x' &= -ix \\ y' &= iy. \end{aligned} \tag{A20}$$

The C_3 element about the $x_1 = x_2 = x_3$ axis has $a = b = c = d = 1/2$ and hence

$$\begin{aligned} x' &= \frac{1+i}{2}x + \frac{1-i}{2}y \\ y' &= -\frac{1+i}{2}x + \frac{1-i}{2}y. \end{aligned} \tag{A21}$$

These two generate T, so we can calculate expressions for the (x,y) transformations for all 24 elements of T. Using MAPLE, we calculate the effects of these transformations on degree seven polynomials, hence determining the 8×8 matrices $\rho(g)$ for each element $g \in T$, and hence, using (A17), the projection operator P. The resulting polynomials in the image of P are

$$\begin{aligned} p_1(x,y) &= -7x^4y^3 - y^7, \\ p_2(x,y) &= x^7 + 7x^3y^4, \\ p_3(x,y) &= x^6y - x^2y^5, \\ p_4(x,y) &= x^5y^2 - xy^6. \end{aligned} \tag{A22}$$

This particular basis is chosen because it is convenient for what follows.

From (A12) it follows that there lie in this four-dimensional space two different representations of the octahedral group O. In the chosen orientation, O is generated by T and the C_4 rotation around the x_3-axis:

$$x' = \frac{1+i}{\sqrt{2}}x \qquad \text{(A23)}$$
$$y' = \frac{1-i}{\sqrt{2}}y$$

and so the projection operators for $E_1'^O$ and $E_2'^O$ can be calculated. It is found that $p_1(x,y)$ and $p_2(x,y)$ are a basis for $E_1'^O$ and $p_3(x,y)$ and $p_4(x,y)$ are a basis for $E_2'^O$. The rational map

$$R(x,y) = (p_1(x,y), p_2(x,y)) \qquad \text{(A24)}$$

is therefore octahedrally symmetric. However $p_3(x,y)$ and $p_4(x,y)$ have a common factor and the corresponding rational map is spurious; it is not of degree seven. This is not surprising. The one-dimensional representation A_2^O in $\underline{7}|_O = A_2^O + F_1^O + F_2^O$ has basis $x^5y - xy^5$, the two-dimensional representation $\underline{2}|_O = E_1'^O$ has basis x, y, and $A_2^O \times E_1'^O = E_2'^O$.

Recall that T is also a subgroup of Y. In fact, for our choice of orientation for the tetrahedral group, there are two possible icosahedral groups with it as a subgroup. The group Y is generated by T and a C_5 element. The two choices of Y correspond to adding a C_5 rotation about the radial line passing through $(-1,0,\tau)$ or about the line passing through $(1,0,\tau)$, where $\tau = (1+\sqrt{5})/2$. The two possibilities are related by a rotation by 90° about the x_3-axis. The $E_2'^Y$ has basis $p_1(x,y) \pm (7/\sqrt{5})p_3(x,y)$ and $p_2(x,y) \pm (7/\sqrt{5})p_4(x,y)$; the sign depends on the choice of C_5 element.

Let us now consider the decomposition of $2E'^T$ into $E'^T + E'^T$. Luckily, such decompositions are discussed in [30] where the following construction is presented. We have, generally, some reducible representation U, where, as in (A15),

$$U = W + \text{other irreducible representations of } G \qquad \text{(A25)}$$

and W is the sum of m identical irreducible representations V,

$$W = mV. \qquad \text{(A26)}$$

Let $n = \dim V$ (in our example $n = 2$). In V each $g \in G$ is represented by an $n \times n$ matrix, say $r(g)$. From these the projection operators

$$P_{\alpha\beta} = \frac{n}{|G|}\sum_{g\in G} r_{\alpha\beta}(g^{-1})\rho(g) \qquad \text{(A27)}$$

are calculated. Here, α, β are simply the matrix indices of r. Now $P_{\alpha\alpha}$ projects onto an m-dimensional space we will call Ω_α, and W can be expressed as the direct sum

$$W = \Omega_1 + \Omega_2 + \ldots + \Omega_n. \qquad \text{(A28)}$$

Furthermore, the map $P_{\beta\alpha}$ is an isomorphism from Ω_α to Ω_β and vanishes on all Ω_γ for $\gamma \neq \alpha$. If $(\omega_1, \omega_2, \ldots, \omega_m)$ is a basis for Ω_1 then the space spanned by

$$Y_\nu = (\omega_\nu, P_{21}(\omega_\nu), P_{31}(\omega_\nu), \ldots, P_{n1}(\omega_\nu)). \qquad \text{(A29)}$$

is isomorphic to V and

$$W = Y_1 + Y_2 + \ldots + Y_m \qquad \text{(A30)}$$

is a decomposition of W of the form (A26). Choosing a particular decomposition is equivalent to choosing a particular basis $(\omega_1, \omega_2, \ldots, \omega_m)$ for the space Ω_1.

In the example we are considering, $W = 2E'^T$. This space is spanned by the polynomials (A22). Using MAPLE the projection operators P_{11} and P_{21} are constructed. It is found that the space $P_{11} : W \to \Omega_1$ is spanned by p_1 and p_3. Choosing a vector $p_1 + bp_3$ in this space defines a particular $E'^T \subset 2E'^T$. Using P_{21} we derive from this the one-parameter family of tetrahedrally symmetric rational maps

$$R(x,y) = (p_1 + bp_3, p_2 + bp_4), \qquad \text{(A31)}$$

or in inhomogenous coordinates

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)} \qquad \text{(A32)}$$

where b is complex. For real b, the symmetry extends to T_d and for b imaginary it extends to T_h. For $b = 0$, there is octahedral symmetry O_h and for $b = \pm 7/\sqrt{5}$ there is icosahedral symmetry Y_h.

We have used similar methods to calculate icosahedrally symmetric maps for degrees eleven and seventeen and to calculate tetrahedrally symmetric maps of degree nine. In the two icosahedral cases there is a single symmetric rational map

$$\underline{12}|_Y = E_2'^Y + G'^Y + I'^Y, \qquad \text{(A33)}$$
$$\underline{18}|_Y = E_2'^Y + G'^Y + 2I'^Y,$$

and to construct the map we need only calculate a basis for $E_2'^Y$ in each case. For degree nine

$$\underline{10}|_T = E'^T + 2G'^T. \qquad \text{(A34)}$$

The representation G'^T is a sum of two two-dimensional irreducible representations of T. Because they are complex conjugate representations they are amalgamated under the name G'^T in the standard nomenclature. If we write $G'^T = E_1'^T + E_2'^T$ then

$$\underline{10}|_T = E'^T + 2E_1'^T + 2E_2'^T \qquad \text{(A35)}$$

and a one parameter family of symmetric rational maps can be constructed from $2E_1'^T$. The corresponding family constructed from $2E_2'^T$ is related by inversion. The representation E'^T does not give a genuine degree nine map.

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Figure Captions

Fig. 1: Surfaces of constant baryon density for the following Skyrme fields:
a) $B = 2$ torus
b) $B = 3$ tetrahedron
c) $B = 4$ cube
d) $B = 5$ with D_{2d} symmetry
e) $B = 6$ with D_{4d} symmetry
f) $B = 7$ dodecahedron
g) $B = 8$ with D_{6d} symmetry
h) $B = 9$ with tetrahedral symmetry
i) $B = 17$ buckyball
j) $B = 5$ octahedron
k) $B = 11$ icosahedron

Fig. 2: The profile functions $f(r)$ for baryon numbers one to nine and also seventeen.

Fig. 3: The a-sphere parametrizing the degree three rational maps (3.6). Crosses denote degenerate maps, dots denote toroidal maps and triangles denote the tetrahedral maps.

Fig. 4: The integral $\mathcal{I}$ for the family of degree nine maps (3.21).

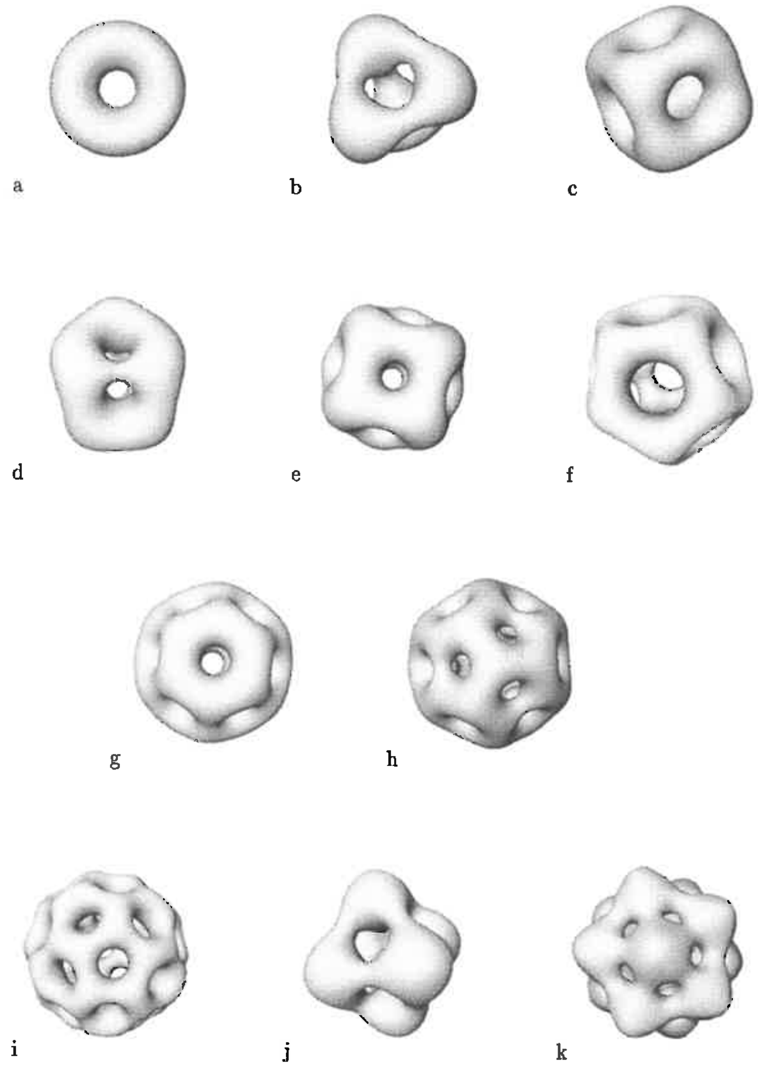

Figure 1

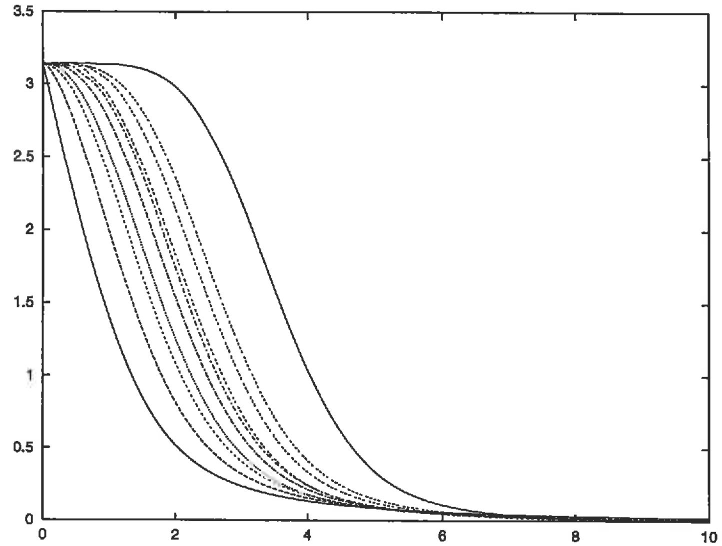

Figure 2

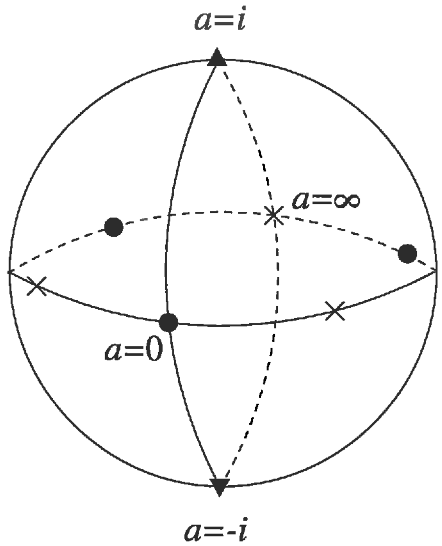

Figure 3

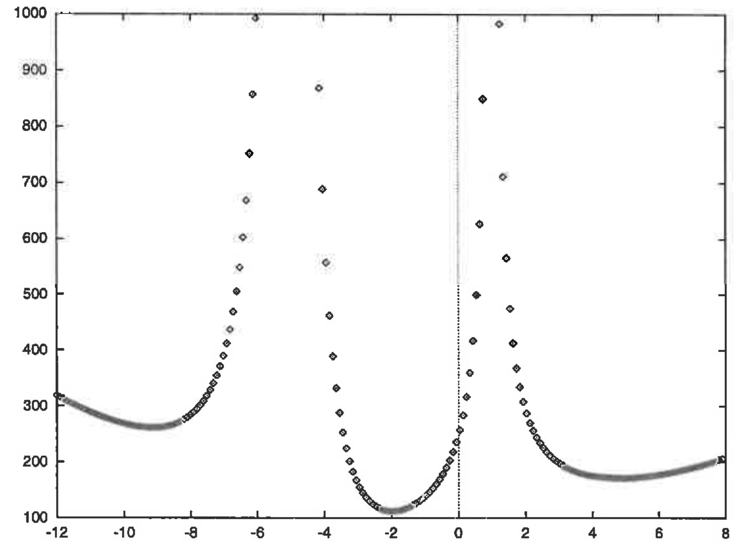

Figure 4