QMW-PH-97-12
hep-th/9704021

Intrinsic Geometry of D-Branes

M. Abou Zeid and C. M. Hull

Physics Department, Queen Mary and Westfield College, Mile End Road, London E1 4NS, U. K.

and

Isaac Newton Institute, 20 Clarkson Road, Cambridge CB3 0EH, U. K.

April 1997

Abstract

We obtain forms of Born-Infeld and D-brane actions that are quadratic in derivatives of X and linear in $F_{\mu\nu}$ by introducing an auxiliary 'metric' which has both symmetric and anti-symmetric parts, generalising the simplification of the Nambu-Goto action for p-branes using a symmetric metric. The abelian gauge field appears as a Lagrange multiplier, and solving the constraint gives the dual form of the n dimensional action with an $n-3$ form gauge field instead of a vector gauge field. We construct the dual action explicitly, including cases which could not be covered previously. The generalisation to supersymmetric D-brane actions with local fermionic symmetry is also discussed.

1 Actions

The Nambu-Goto action for a p-brane with $p = n - 1$ is

$$S_{NG} = -T_p \int d^n\sigma \sqrt{-\det\left(G_{\mu\nu}\right)}, \tag{1}$$

where T_p is the p-brane tension and

$$G_{\mu\nu} = G_{ij}\partial_\mu X^i \partial_\nu X^j \tag{2}$$

is the world-volume metric induced by the spacetime metric G_{ij}. The non-linear form of the action (1) is inconvenient for many purposes. However, introducing an intrinsic worldvolume metric $g_{\mu\nu}$ allows one to write down the equivalent action [1, 2, 3]

$$S_P = -\frac{1}{2}T'_p \int d^n\sigma \sqrt{-g}\left[g^{\mu\nu}G_{\mu\nu} - (n-2)\Lambda\right], \tag{3}$$

where $g \equiv \det\left(g_{\mu\nu}\right)$ and Λ is a constant. The metric $g_{\mu\nu}$ is an auxiliary field which can be eliminated using its equation of motion to recover action (1). The constants T_p and T'_p are related by

$$T'_p = \Lambda^{\frac{n}{2}-1} T_p. \tag{4}$$

This form of the action is much more convenient for many purposes, as it is quadratic in ∂X.

The Born-Infeld action for a vector field A_μ in an n-dimensional space-time with metric $G_{\mu\nu}$ is

$$S_{BI} = -T_p \int d^n\sigma \sqrt{-\det\left(G_{\mu\nu} + F_{\mu\nu}\right)}, \tag{5}$$

where $F = dA$ is the Maxwell field strength. A related $(n-1)$-brane action is

$$S_{DBI} = -T_p \int d^n\sigma \sqrt{-\det\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)}, \tag{6}$$

where $G_{\mu\nu}$ is the induced metric (2) and $\mathcal{F}_{\mu\nu}$ is the antisymmetric tensor field

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} - B_{\mu\nu} \tag{7}$$

with $B_{\mu\nu}$ the pull-back of a space-time 2-form gauge field B,

$$B_{\mu\nu} = B_{ij}\partial_\mu X^i \partial_\nu X^j. \tag{8}$$

The action (6) is closely related to the D-brane action, which has been the subject of much recent work [5, 6, 7, 8, 9, 10, 11, 12, 16, 17] and the Born-Infeld action (5) can be thought of as a special case of this, but with a different interpretation of $G_{\mu\nu}$. However, just as in the case of the action (5), the non-linearity of (6) makes it rather difficult to study. In particular, dualising the action (6) has proved rather difficult in this approach, and has only been achieved for $n \leq 5$ [8, 9, 13]. Clearly, an action analogous to (3) for this case would be very useful, and it is the aim of this paper to propose and study just such an action.

The key is to introduce a 'non-symmetric metric', in the form of an auxiliary world-volume tensor field

$$k_{\mu\nu} \equiv g_{\mu\nu} + b_{\mu\nu} \tag{9}$$

with both a symmetric part $g_{\mu\nu}$ and an antisymmetric part $b_{\mu\nu}$.[1] The action which is classically equivalent to (6) is

$$S = -\frac{1}{2}T'_p \int d^n\sigma \sqrt{-k}\left[(k^{-1})^{\mu\nu}\left(G_{\mu\nu} + \mathcal{F}_{\mu\nu}\right) - (n-2)\Lambda\right], \tag{10}$$

where $k \equiv \det(k_{\mu\nu})$; the inverse metric $(k^{-1})^{\mu\nu}$ satisfies

$$(k^{-1})^{\mu\nu}k_{\nu\rho} = \delta^{\mu}{}_{\rho}. \tag{11}$$

Such an action was proposed for Born-Infeld theory in [4]. For $n \neq 2$, the $k_{\mu\nu}$ field equation implies

$$G_{\mu\nu} + \mathcal{F}_{\mu\nu} = \Lambda k_{\nu\mu} \tag{12}$$

and substituting back into (10) yields the Born-Infeld-type action (6) where the constants T_p, T'_p are related as in eq. (4). For $n = 2$, the action (10) is invariant under the generalised Weyl transformation

$$k_{\mu\nu} \to \omega(\sigma)k_{\mu\nu} \tag{13}$$

and the $k_{\mu\nu}$ field equation implies

$$G_{\mu\nu} + \mathcal{F}_{\mu\nu} = \Omega k_{\nu\mu} \tag{14}$$

for some conformal factor Ω.

The action (10) is *linear* in $(G_{\mu\nu} + \mathcal{F}_{\mu\nu})$ and so is much easier to analyse than (6). In particular, it is linear in F, so that A_μ is a Lagrange multiplier imposing the constraint

$$\partial_\mu\left(\sqrt{-k}(k^{-1})^{[\mu\nu]}\right) = 0. \tag{15}$$

The general solution of this is $\sqrt{-k}(k^{-1})^{[\mu\nu]} = \tilde{H}^{\mu\nu}$, where

$$\tilde{H}^{\mu\nu} \equiv \frac{1}{(n-2)!}\epsilon^{\mu\nu\rho\gamma_1\ldots\gamma_{n-3}}\partial_{[\rho}\tilde{A}_{\gamma_1\ldots\gamma_{n-3}]}, \tag{16}$$

$\tilde{A}$ is an $n-3$ form and $\epsilon^{\mu\nu\rho\cdots}$ is the alternating tensor density. The anti-symmetric part of $k_{\mu\nu}$ can then in principle be solved for in terms of $\tilde{A}$, leaving a dual form of the action involving only the symmetric part of $k_{\mu\nu}$ and the dual potential $\tilde{A}$. To do this explicitly requires a judicious choice of variables, as we now show.

[1] Such 'metrics' have been used in alternative theories of gravitation; see e.g. [14, 15].

2 Dual Actions

Instead of introducing a tensor $k_{\mu\nu}$, we introduce a tensor density $\tilde{k}^{\mu\nu}$ with $\tilde{k} \equiv \det(\tilde{k}^{\mu\nu})$. For $n \neq 2$, the action

$$\tilde{S} = -\frac{1}{2} T'_p \int d^n\sigma \left[\tilde{k}^{\mu\nu} \left(G_{\mu\nu} - B_{\mu\nu} + F_{\mu\nu} \right) - (n-2)(-\tilde{k})^{\frac{1}{n-2}} \Lambda \right] \tag{17}$$

is equivalent to (10), as can be seen by defining a tensor $k_{\mu\nu}$ by $(k^{-1})^{\mu\nu} = (-\tilde{k})^{-\frac{1}{n-2}} \tilde{k}^{\mu\nu}$, so that

$$\tilde{k}^{\mu\nu} \equiv \sqrt{-k}(k^{-1})^{\mu\nu}. \tag{18}$$

Integrating out $\tilde{k}^{\mu\nu}$ yields the action (6) as before.

Integrating out the world-volume vector field A_μ from (17) gives $\partial_\mu \tilde{k}^{[\mu\nu]} = 0$ which is solved by $\tilde{k}^{[\mu\nu]} = \tilde{H}^{\mu\nu}$ where $\tilde{H}^{\mu\nu}$ is given in terms of an unconstrained $n-3$ form $\tilde{A}$ by (16), so that

$$\tilde{k}^{\mu\nu} = \tilde{g}^{\mu\nu} + \tilde{H}^{\mu\nu}, \tag{19}$$

where the symmetric tensor density $\tilde{g}^{\mu\nu}$ is defined by $\tilde{g}^{\mu\nu} \equiv \tilde{k}^{(\mu\nu)}$. The action (17) then becomes

$$S' = -\frac{1}{2} T'_p \int d^n\sigma \left[\left(\tilde{g}^{\mu\nu} + \tilde{H}^{\mu\nu} \right) \left(G_{\mu\nu} - B_{\mu\nu} \right) - (n-2)\Lambda \left(-\det\left[\tilde{g}^{\mu\nu} + \tilde{H}^{\mu\nu} \right] \right)^{\frac{1}{n-2}} \right]. \tag{20}$$

This is a dual form of the action in which A_μ has been replaced by $\tilde{A}$. It contains the auxiliary symmetric tensor density $\tilde{g}^{\mu\nu}$ which can in principle be integrated out; this can be done explicitly for low values of n, but is harder for general n.

We define a symmetric metric tensor $g_{\mu\nu}$ with inverse $g^{\mu\nu}$ by $g^{\mu\nu} = (-\tilde{g})^{-\frac{1}{n-2}} \tilde{g}^{\mu\nu}$ where $\tilde{g} = \det(\tilde{g}^{\mu\nu})$, so that

$$\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}, \tag{21}$$

where $g = \det(g_{\mu\nu})$, and an anti-symmetric tensor by

$$H^{\mu\nu} = \frac{1}{\sqrt{-g}} \tilde{H}^{\mu\nu}, \tag{22}$$

so that

$$\tilde{k}^{\mu\nu} = \sqrt{-g} \left(g^{\mu\nu} + H^{\mu\nu} \right) \tag{23}$$

and

$$\tilde{k} \equiv \det(\tilde{k}^{\mu\nu}) = -(-g)^{\frac{n}{2}-1} \Delta, \tag{24}$$

where

$$\Delta(g, H) \equiv \det\left(\delta_\mu{}^\nu + H_\mu{}^\nu \right) \tag{25}$$

and $H_\mu{}^\nu = g_{\mu\rho} H^{\rho\nu}$. Then the action (20) can be rewritten as

$$\tilde{S}_D = -\frac{1}{2} T'_p \int d^n\sigma \sqrt{-g} \left(g^{\mu\nu} G_{\mu\nu} + \Sigma \right), \tag{26}$$

where

$$\Sigma \equiv -(n-2)\Lambda\Delta^{\frac{1}{n-2}} - H^{\mu\nu}B_{\mu\nu}. \tag{27}$$

The action (26) is the dual form of action (17). Unfortunately, the metric dependence of Δ makes it hard to eliminate $g_{\mu\nu}$ from this action explicitly.

For $n=2$, the action (10) has the Weyl symmetry (13) and can be rewritten using a tensor density $\tilde{k}^{\mu\nu}$ as

$$\tilde{S}^2 = -\frac{1}{2}T_1\int d^2\sigma\left\{\tilde{k}^{\mu\nu}\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)+\lambda\left[\det\left(\tilde{k}^{\mu\nu}\right)+1\right]\right\}. \tag{28}$$

Integrating out λ yields the constraint

$$\tilde{k} = -1 \tag{29}$$

which is solved in $n=2$ dimensions by

$$\tilde{k}^{\mu\nu} \equiv \sqrt{-k}(k^{-1})^{\mu\nu}, \tag{30}$$

so that one recovers the original action (10). If instead one keeps the Lagrange multiplier and integrates out the world-volume vector A, one finds the constraint eq. (15) again. For $n=2$, this is solved by

$$\tilde{k}^{[\mu\nu]} = \epsilon^{\mu\nu}\Lambda, \tag{31}$$

where Λ is a constant. The dual action for $n=2$ is then

$$\tilde{S}^2_D = -\frac{1}{2}T_1\int d^2\sigma\left\{\left[\tilde{g}^{\mu\nu}+\Lambda\epsilon^{\mu\nu}\right]\left[G_{\mu\nu}-B_{\mu\nu}\right]+\lambda\left[\det\left(\tilde{g}^{\mu\nu}\right)+1+\Lambda^2\right]\right\} \tag{32}$$

where $\tilde{g}^{\mu\nu} = \tilde{k}^{(\mu\nu)}$ and we have used the identity

$$\det\left(\tilde{g}^{\mu\nu}+\epsilon^{\mu\nu}\Lambda\right) = \det\left(\tilde{g}^{\mu\nu}\right)+\Lambda^2. \tag{33}$$

Integrating out λ gives $\det\left(\tilde{g}^{\mu\nu}\right) = -1-\Lambda^2$, which is solved in terms of an unconstrained metric $g_{\mu\nu}$ by

$$\tilde{g}^{\mu\nu} = \sqrt{1+\Lambda^2}\sqrt{-g}g^{\mu\nu} \tag{34}$$

so that the action becomes

$$S^2_D = -\frac{1}{2}T_1\int d^2\sigma\left(\sqrt{1+\Lambda^2}\sqrt{-g}g^{\mu\nu}G_{\mu\nu}+\Lambda\epsilon^{\mu\nu}B_{\mu\nu}\right). \tag{35}$$

The metric can be eliminated from this to give the dual action of ref. [5, 6, 9]

$$S^2_D = -T_1\int d^2\sigma\left(\sqrt{1+\Lambda^2}\sqrt{-\det\left(G_{\mu\nu}\right)}+\frac{1}{2}\Lambda\epsilon^{\mu\nu}B_{\mu\nu}\right). \tag{36}$$

3 More Dual Actions

Consider actions given by the sum of (6) and some action $S_F = \int d^n\sigma f(F)$ which is algebraic in F; in the next section we will be interested in the example of D-brane actions which are of this form. Defining

$$N_{\mu\nu} \equiv G_{\mu\nu} - B_{\mu\nu}, \tag{37}$$

the action can be rewritten in first order form as

$$-\frac{1}{2}T'_p \int d^n\sigma \left\{ \sqrt{-k}\left[(k^{-1})^{\mu\nu}(N_{\mu\nu} + F_{\mu\nu}) - (n-2)\Lambda\right] + \frac{1}{2}\tilde{H}^{\mu\nu}\left(F_{\mu\nu} - 2\partial_{[\mu}A_{\nu]}\right) + f(F)\right\}. \tag{38}$$

Here the anti-symmetric tensor density $\tilde{H}^{\mu\nu}$ is a Lagrange multiplier imposing $F = dA$ and can be integrated out to regain the original action. Alternatively, integrating over A_μ imposes

$$\partial_\mu \tilde{H}^{\mu\nu} = 0, \tag{39}$$

which can be solved in terms of an $n-3$ form $\tilde{A}$ as before:

$$\tilde{H}^{\mu\nu} = \frac{1}{(n-2)!}\epsilon^{\mu\nu\rho\gamma_1\ldots\gamma_{n-3}}\partial_{[\rho}\tilde{A}_{\gamma_1\ldots\gamma_{n-3}]}. \tag{40}$$

Now F is an auxiliary 2-form occuring algebraically; we emphasize this by rewriting $F \to L$ so that the action is

$$-\frac{1}{2}T'_p \int d^n\sigma \left\{ \sqrt{-k}\left[(k^{-1})^{\mu\nu}(N_{\mu\nu} + L_{\mu\nu}) - (n-2)\Lambda\right] + \frac{1}{2}\tilde{H}^{\mu\nu}L_{\mu\nu} + f(L)\right\}. \tag{41}$$

The field equation for $L_{\mu\nu}$ is

$$\sqrt{-k}(k^{-1})^{[\mu\nu]} + \frac{1}{2}\tilde{H}^{\mu\nu} + \frac{\delta f}{\delta L_{\mu\nu}} = 0. \tag{42}$$

If $f = 0$, this can be used to recover the dual action (26) of the last section. More generally, if $f(L)$ is at most quartic in L, this can be solved to give an expression for $L_{\mu\nu}$ which can then be re-substituted in (41) to give a dual action analogous to (26). This is applicable to the D-brane actions considered in the next two sections, in which f is at most quartic for $p < 9$ branes.

Integrating out $k_{\mu\nu}$ from (41) gives

$$-T_p \int d^n\sigma \left\{ \sqrt{-\det(N_{\mu\nu} + L_{\mu\nu})} + \frac{1}{2}\tilde{H}^{\mu\nu}L_{\mu\nu} + \frac{T'_p}{T_p}f(L)\right\}. \tag{43}$$

If $f = 0$ and $n \leq 5$, the equation of motion for L can be solved explicitly and the solution substituted in (41) to get the dual action [9, 13]

$$S_D = -T_p \int d^n\sigma \left\{ \sqrt{-\det(G_{\mu\nu} + iK_{\mu\nu})} + \frac{1}{2}\tilde{H}^{\mu\nu}B_{\mu\nu}\right\}, \tag{44}$$

where

$$K_{\mu\nu} \equiv \frac{1}{\sqrt{-\det(G_{\mu\nu})}}G_{\mu\rho}G_{\nu\lambda}\tilde{H}^{\rho\lambda}. \tag{45}$$

This can in turn be linearised to give the equivalent action

$$-\frac{1}{2}T'_p \int d^n\sigma \left\{ \sqrt{-k}\left[(k^{-1})^{\mu\nu}(G_{\mu\nu} + iK_{\mu\nu}) - (n-2)\Lambda\right] + \frac{1}{2}\tilde{H}^{\mu\nu}B_{\mu\nu}\right\}. \tag{46}$$

4 D-Brane Actions

The bosonic part of the effective world-volume action for a D-brane in a type II supergravity background is [5, 7, 16]

$$S_1 = -T_p \int d^n\sigma e^{-\phi}\sqrt{-\det\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)} + T_p \int_{W_n} Ce^{\mathcal{F}}, \tag{47}$$

The first term is of the form (6) with an extra dependence on the dilaton field ϕ. The second term is a Wess-Zumino term and gives the coupling to the background Ramond-Ramond r-form gauge fields $C^{(r)}$ (where r is odd for type IIA and even for type IIB). The potentials $C^{(r)}$ for $r > 4$ are the duals of the potentials $C^{(8-r)}$. In (47), C is the formal sum [16]

$$C \equiv \sum_{r=0}^{9} C^{(r)}, \tag{48}$$

all forms in space-time are pulled back to the worldvolume of the brane W_n and it is understood that the n-form part of $Ce^{\mathcal{F}}$, which is $C^{(n)} + C^{(n-2)}\mathcal{F} + \frac{1}{2}C^{(n-4)}\mathcal{F}^2 + \ldots$, is selected. The case of the 9-form potential $C^{(9)}$ is special because its equation of motion implies that the dual of its field strength is a constant m. This constant will be taken to be zero here, so that $C^{(9)} = 0$; the more general situation will be discussed elsewhere [18].

Introducing $k_{\mu\nu}$, we obtain the classically equivalent D-brane action

$$S_1' = -\frac{1}{2}T_p' \int d^n\sigma\sqrt{-k}e^{-\phi}\left[(k^{-1})^{\mu\nu}\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right) - (n-2)\Lambda\right] + T_p\int_{W_n} Ce^{\mathcal{F}}. \tag{49}$$

The field equation for $k_{\mu\nu}$ is given in (12); substituting back into (49) yields (47). The action is of the form (38) (apart from the introduction of the dilaton) so that the dual action is (cf. (41))

$$\begin{aligned} &-\frac{1}{2}T_p' \int d^n\sigma \left\{\sqrt{-k}e^{-\phi}\left[(k^{-1})^{\mu\nu}\left(N_{\mu\nu}+L_{\mu\nu}\right) - (n-2)\Lambda\right] + \frac{1}{2}\tilde{H}^{\mu\nu}L_{\mu\nu}\right\} \\ &+T_p\int_{W_n} Ce^{L-B}. \end{aligned} \tag{50}$$

The potential $f(L) \sim Ce^{L-B}$ is a polynomial of order $[n/2]$ in L (i.e. the integer part of $n/2$), so that the field equation for $L_{\mu\nu}$ (42) is of order $[n/2]-1$ in L and so should be soluble explicitly for all $n \leq 10$. In particular, it is quadratic for $n \leq 8$, so that the dual action for p-branes with $p \leq 7$ can be obtained straightforwardly. This will be discussed elsewhere [18]; here we will consider only the case in which $C^{(n-4)} = C^{(n-6)} = C^{(n-8)} = 0$ so that the action is linear in F. Then A is a Lagrange multiplier imposing the constraint

$$\partial_\mu\left(\sqrt{-k}(k^{-1})^{[\mu\nu]}e^{-\phi} - \frac{2T_p}{T_p'}\frac{1}{(n-2)!}\epsilon^{\mu\nu\gamma_1\ldots\gamma_{n-2}}(C^{(n-2)})_{\gamma_1\ldots\gamma_{n-2}}\right) = 0. \tag{51}$$

The general solution of this constraint is

$$\sqrt{-k}(k^{-1})^{[\mu\nu]}e^{-\phi} = \tilde{H}^{\mu\nu} + \frac{2T_p}{T'_p}\frac{1}{(n-2)!}\epsilon^{\mu\nu\gamma_1\ldots\gamma_{n-2}}(C^{(n-2)})_{\gamma_1\ldots\gamma_{n-2}} \equiv \tilde{\mathcal{H}}^{\mu\nu}, \quad (52)$$

where $\tilde{H}^{\mu\nu}$ is given in terms of $\tilde{A}$ by (40).

To obtain the dual action, we first express (49) in terms of a density $\tilde{k}^{\mu\nu}$. For $n \neq 2$, this gives the equivalent action

$$\begin{aligned}\tilde{S}_1 &= -\frac{1}{2}T'_p\int d^n\sigma e^{-\phi}\left[\tilde{k}^{\mu\nu}\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)-(n-2)(-\tilde{k})^{\frac{1}{n-2}}\Lambda\right] \\ &\quad +T_p\int_{W_n} C^{(n)}+C^{(n-2)}\mathcal{F}. \end{aligned}\quad (53)$$

Integrating out $\tilde{k}^{\mu\nu}$ yields the action (47), while integrating out A gives

$$\tilde{k}^{\mu\nu} = \tilde{g}^{\mu\nu} + \tilde{\mathcal{H}}^{\mu\nu}, \quad (54)$$

where $\tilde{\mathcal{H}}^{\mu\nu}$ is given in terms of $\tilde{A}$ and $C^{(n-2)}$ by eq. (52), and $\tilde{g}^{\mu\nu}$ is a symmetric tensor density. The action (53) can be written in terms of tensors $g^{\mu\nu} = (-\tilde{g})^{-\frac{1}{n-2}}\tilde{g}^{\mu\nu}$ (with $\tilde{g} = \det(\tilde{g}^{\mu\nu})$) and $\mathcal{H}^{\mu\nu} = (-g)^{-\frac{1}{2}}\tilde{\mathcal{H}}^{\mu\nu}$ (with $g = \det(g_{\mu\nu})$) as

$$\begin{aligned}\tilde{S}_{1D} &= -\frac{1}{2}T'_p\int d^n\sigma e^{-\phi}\sqrt{-g}\left[\left(g^{\mu\nu}+\mathcal{H}^{\mu\nu}\right)N_{\mu\nu}-(n-2)\Omega^{\frac{1}{n-2}}\Lambda\right] \\ &\quad +T_p\int_{W_n} C^{(n)}-C^{(n-2)}B, \end{aligned}\quad (55)$$

where

$$\Omega \equiv \det\left(\delta_\mu{}^\nu + \mathcal{H}_\mu{}^\nu\right) \quad (56)$$

and $\mathcal{H}_\mu{}^\nu = (\tilde{g}^{-1})_{\mu\rho}\mathcal{H}^{\rho\nu}$. The action (55) is the dual form of action (49). Again, the dependence of Ω on the metric $g_{\mu\nu}$ makes the elimination of the latter from the action difficult, although possible in principle.

For $n = 2$, the action (49) still possesses the generalised Weyl symmetry (13) and can be rewritten in terms of a tensor density $\tilde{k}^{\mu\nu}$ as

$$\tilde{S}_1^2 = -\frac{1}{2}T_1\int d^2\sigma e^{-\phi}\left[\tilde{k}^{\mu\nu}\left(N_{\mu\nu}+F_{\mu\nu}\right)+\lambda\left(\tilde{k}+1\right)\right]+T_1\int_{W_2} C^{(2)}+C^{(0)}\mathcal{F}. \quad (57)$$

Integrating out λ yields the constraint $\tilde{k} = -1$, which is solved by eq. (30), so that one recovers the original action. Keeping the Lagrange multiplier and integrating out A one finds the constraint

$$\partial_\mu\left(e^{-\phi}\tilde{k}^{[\mu\nu]} - 2\epsilon^{\mu\nu}C^{(0)}\right) = 0, \quad (58)$$

which for $n = 2$ is solved by

$$e^{-\phi}\tilde{k}^{[\mu\nu]} = \epsilon^{\mu\nu}\mathcal{E}, \quad (59)$$

where

$$\mathcal{E} \equiv \tilde{\Lambda} + 2C^{(0)} \quad (60)$$

and $\tilde{\Lambda}$ is a constant. The dual action for $n = 2$ is then

$$\begin{aligned}\tilde{S}^2_{1D} &= -\frac{1}{2}T_1 \int d^2\sigma e^{-\phi} \left[\left(\tilde{g}^{\mu\nu} + e^{\phi}\epsilon^{\mu\nu}\mathcal{E} \right) N_{\mu\nu} + \lambda \left(\det \left(\tilde{g}^{\mu\nu} \right) + 1 + e^{2\phi}\mathcal{E}^2 \right) \right] \\ &\quad + T_1 \int_{W_2} C^{(2)} - C^{(0)} B, \end{aligned} \tag{61}$$

where $\tilde{g}^{\mu\nu} = \tilde{k}^{(\mu\nu)}$. Integrating out λ gives the dual action in the form

$$S^2_1 = -\frac{1}{2}T_1 \int d^2\sigma \sqrt{e^{-2\phi} + \mathcal{E}^2}\sqrt{-g} g^{\mu\nu} G_{\mu\nu} + T_1 \int_{W_2} C^{(2)} + (\mathcal{E} - C^{(0)})B. \tag{62}$$

Finally, integrating out $g_{\mu\nu}$ gives the action [9]

$$S^2_1 = -T_1 \int d^2\sigma \sqrt{e^{-2\phi} + \mathcal{E}^2}\sqrt{-\det \left(G_{\mu\nu} \right)} + T_1 \int_{W_2} C^{(2)} + (\mathcal{E} - C^{(0)})B. \tag{63}$$

5 Supersymmetric D-Brane Actions

The new actions discussed above can be extended to supersymmetric D-brane actions with local kappa symmetry equivalent to those presented in refs. [10, 17, 11, 12] at the classical level.

The (flat) superspace coordinates are the $D = 10$ space-time coordinates X^i and the Grassmann coordinates θ, which are space-time spinors and world-volume scalars. For the type IIA superstring (even p), θ is Majorana but not Weyl while in the IIB superstring there are two Majorana-Weyl spinors θ_α ($\alpha = 1, 2$) of the same chirality. The superspace (global) supersymmetry transformations are

$$\delta_\epsilon \theta = \epsilon \quad , \quad \delta_\epsilon X^i = \overline{\epsilon}\Gamma^i \theta. \tag{64}$$

The world-volume theory has global type IIA or type IIB super-Poincaré symmetry and is constructed using the supersymmetric one-forms $\partial_\mu \theta$ and

$$\Pi^i_\mu = \partial_\mu X^i - \overline{\theta}\Gamma^i \partial_\mu \theta. \tag{65}$$

The induced world-volume metric is

$$G_{\mu\nu} = G_{ij} \Pi^i_\mu \Pi^j_\nu. \tag{66}$$

The supersymmetric world-volume gauge field-strength two form $\mathcal{F}_{\mu\nu}$ is given by (7) for the following choice of the two form B [8]

$$B = -\overline{\theta}\Gamma_{11}\Gamma_i d\theta \left(dX^i + \frac{1}{2}\overline{\theta}\Gamma^i d\theta \right) \tag{67}$$

when p is even or the same formula with Γ_{11} replaced with the Pauli matrix τ_3 when p is odd. With the choice (67), $\delta_\epsilon B$ is an exact two-form and $\mathcal{F}$ is supersymmetric for an appropriate choice of $\delta_\epsilon A$ [12].

The effective world-volume action for a D-brane in flat superspace with constant dilaton is

$$S_1^{DBI} = -T_p \int d^n\sigma e^{-\phi}\sqrt{-\det\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)} + T_p \int_{W_n} Ce^{\mathcal{F}}, \tag{68}$$

which is formally of the same form as (47). Again C represents a complex of differential forms $C^{(r)}$, as in (48), but now the r-forms $C^{(r)}$ are the pull-backs of superspace forms $C^{(r)} = d\bar{\theta}T^{(r-2)}d\theta$ for certain $r-2$ forms $T^{(r-2)}$ given explicitly in [10, 17, 11, 12]. This action is supersymmetric and invariant under local kappa symmetry [10, 17, 11, 12].

A classically equivalent form of the D-brane action is given by

$$S_1^P = -\frac{1}{2}T_p' \int d^n\sigma e^{-\phi}\sqrt{-k}\left[(k^{-1})^{\mu\nu}\left(G_{\mu\nu}+\mathcal{F}_{\mu\nu}\right) - (n-2)\Lambda\right] + T_p \int_{W_n} Ce^{\mathcal{F}} \tag{69}$$

The action is of the same form as (49) and can be dualised to give (50) for $n \neq 2$ or (61) for $n = 2$.

Acknowledgements

We would like to thank John Schwarz for discussions. The work of M. A. is supported in part by the Overseas Research Scheme and in part by Queen Mary and Westfield College, London.

References

[1] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, Phys. Lett. **B103** (1981) 207

[2] L. Brink, P. Di Vecchia and P. S. Howe, *A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String*, Phys. Lett. **B65** (1976) 471

[3] P. S. Howe and R. W. Tucker, *A Locally Supersymmetric and Reparametrization Invariant Action for a Spinning Membrane*, J. Phys. **A10** (1977) L155

[4] U. Lindström, *First Order Actions for Gravitational Systems, Strings and Membranes*, Int. J. Mod. Phys. **A3** (1988) 2401

[5] C. Schmidhuber, *D-Brane Actions*, Nucl. Phys. **B467** (1996) 146

[6] S. P. de Alwis and K. Sato, *D-Strings and F-Strings from String Loops*, Phys. Rev. **D53** (1996) 7187

[7] M. Douglas, *Branes Within Branes*, hep-th/9512077

[8] P. K. Townsend, *D-Branes from M-Branes*, Phys. Lett. **B373** (1996) 68

[9] A. A. Tseytlin, *Self-Duality of Born-Infeld Action and Dirichlet 3-Brane of Type IIB Superstring Theory*, Nucl. Phys. **B469** (1996) 51

[10] M. Cederwall, A. von Gussich, B. E. W. Nilsson and A. Westerberg, *The Dirichlet Super-Three-Brane in Ten-Dimensional Type IIB Supergravity*, hep-th/9610148; M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet Super-p-Branes in Ten-Dimensional Type IIA and IIB Supergravity*, hep-th/9611159; M. Cederwall, *Aspects of D-Brane Actions*, hep-th/9612153

[11] M. Aganagic, C. Popescu and J. H. Schwarz, *D-Brane Actions with Local Kappa Symmetry*, hep-th/9610249

[12] M. Aganagic, C. Popescu and J. H. Schwarz, *Gauge-Invariant and Gauge-Fixed D-Brane Actions*, hep-th/9612080

[13] M. Aganagic, J. Park, C. Popescu and J. Schwarz, *Dual D-Brane Actions*, hep-th/9702133

[14] A. Einstein, Sitzungsberichte der Preussische Akademie der Wissenschaften (1925) 414; Ann. Math. **46** (1945) 578; Rev. Mod. Phys. **20** (1948) 35

[15] J. W. Moffat, *Nonsymmetric Gravitational Theory*, J. Math. Phys. **36** (1995) 3722

[16] M. B. Green, C. M. Hull and P. K. Townsend, *D-Brane Wess-Zumino Actions, T-Duality and the Cosmological Constant*, Phys. Lett. **B382** (1996) 65

[17] E. Bergshoeff and P. K. Townsend, *Super D-Branes*, hep-th/9611173

[18] M. Abou Zeid and C. M. Hull, to appear