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RENORMALIZATION IN THE COULOMB GAUGE AND ORDER PARAMETER FOR CONFINEMENT IN QCD

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Abstract

Renormalizability of the Coulomb gauge is studied in the phase space formalism, where one integrates over both the vector potential A, and its canonical momentum Π. The obvious divergences are regularized, although a proof that all diagrams are regularized is not attempted. The algebraic constraints on the renormalization constants are derived from the symmetries of the theory. In particular, a Ward identity is derived that holds at a fixed time t, and is an analog of Gauss's law in the BRST formalism. The familiar Zinn-Justin equation results when this identity is integrated over all t. As a consequence of this identity, $g^2D^{Ao,Ao}$ is a renormalization-group invariant, where $D^{Ao,Ao}$ is the time-time component of the gluon propagator. The contribution to the Wilson loop of the instantaneous part V(R) of $g^2D^{Ao,Ao}$ exponentiates. It is proposed that the string tension defined by $K_{coul} = \lim_{R\to\infty} CV(R)/R$ may serve as an order parameter for color confinement, where $C = (2N)^{-1}(N^2-1)$ for SU(N) gauge theory. A further consequence of the above-mentioned Ward identity, is that the fourier transform $V(\mathbf{k})$ of V(R) has the product form $V(\mathbf{k}) = [\mathbf{k}^2D^{C,C^*}(\mathbf{k})]^2\, L(\mathbf{k})$, where $D^{C,C^*}(\mathbf{k})$ is the ghost propagator, and $L(\mathbf{k})$ is a correlation function of longitudinal gluons. This exact equation combines with a previous analysis of the Gribov problem according to which $\mathbf{k}^2D^{C,C^*}(\mathbf{k})$ diverges at $\mathbf{k} = 0$, to provide a scenario for confinement.

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1. Introduction

Why renormalize in the Coulomb gauge? In a perturbative expansion of gauge-invariant quantities, the Coulomb gauge should agree order by order with the familiar Lorentz-covariant gauges, so it might seem that nothing much new is to be learned. On the contrary, it is found for non-Abelian gauge theories in the Coulomb gauge, that the time-time component of the gluon propagator is a renormalization-group invariant,

$$g^2 D^{Ao,Ao} = g_r^2 D_r^{Ao,Ao}, \tag{1.1}$$

where g is the coupling constant, $D^{Ao,Ao}$ is the time-time component of the gluon propagator, and the subscript r refers to renormalized quantities. Thus in the Coulomb gauge, $g^2 D^{Ao,Ao}$ is independent of the cut-off and of the renormalization mass, and depends only on physical masses, whereas this is not true in covariant gauges.

The hamiltonian for non-Abelian gauge theory in the Coulomb gauge has been known for some time in its continuum version [1], and the lattice version has been found recently [2]. The existence of a lattice Coulomb hamiltonian suggests that the Coulomb gauge ought to be renormalizable.

As originally suggested by Gribov [3], and substantiated recently by detailed arguments [2], the time-time component of the gluon propagator provides a long-range confining force, while the 3-dimensionally transverse propagator of the would-be physical gluons is suppressed at low momentum, reflecting the absence of gluons from the physical spectrum. These properties are a consequence of the restriction of the functional integral to the fundamental modular region, a region in configuration space that is free of Gribov copies. The renormalization properties found here support this picture of confinement.

Technically the most interesting aspect of renormalization theory in the Coulomb gauge is a Ward identity, $F_t(\Gamma) = 0$, where Γ is the quantum effective action. [See eq. (6.23) or (7.3) below.] This identity holds at fixed time t. It is an analog of Gauss's law in the BRST formalism which we call “the Gauss-BRST identity”. The integral of this identity over all t gives the Zinn-Justin equation, $\int dt\, F_t(\Gamma) = \Gamma*\Gamma = 0$, where $\Gamma*\Gamma$, is the left-hand side of the Zinn-Justin equation [written explicitly in eq. (A.1) below]. The Gauss-BRST identity assures renormalizability in the Coulomb gauge by fixing the form of the possible divergences, just as the Zinn-Justin equation does in covariant gauges. It also implies the identity

$$g^2 D^{A_0,A_0}(\mathbf{k}, k_0) = g^2\,[\mathbf{k}^2 D^{C,C^*}(\mathbf{k})]^2\, L(\mathbf{k}, k_0), \qquad (1.2)$$

where $D^{C,C^*}(\mathbf{k})$ is the (instantaneous) ghost propagator and $L(\mathbf{k}, k_0)$ the longitudinal gluon correlation function.

The organization of the article is as follows. In sect. 2, the problem of renormalization in the Coulomb gauge is posed and solved heuristically, by integrating out the non-physical degrees of freedom in the phase-space formalism so only the "physical" degrees of freedom remain namely, the 3-dimensionally transverse vector potential $\mathbf{A}^{tr}$, and its canonically conjugate field Π^{tr}. The Coulomb gauge in the second-order formalism contains apparently non-renormalizable terms such as the Faddeev-Popov determinant $\det(-D_i\nabla_i)$ whose perturbative expansion contains instantaneous closed fermi-ghost loops. Such unrenormalizable terms manifestly cancel in the phase-space formalism, in which both the A and Π fields are integrated over. The cancellation occurs because the Faddeev-Popov determinant $\det(-D_i\nabla_i)$ gets multiplied by the δ-function $\delta(D_i\Pi_i - \rho_{qu})$ that expresses the Gauss's law constraint, and which results from integrating out the A_0 field. (Here $D = D(A)$ is the gauge covariant derivative, ρ_{qu} is the quark color-charge density, and $i = 1, 2, 3$.) As a result, when one decomposes Π into its transverse and longitudinal parts, $\Pi_i = \Pi_i^{tr} - \nabla_i\Omega$, the integral over Ω gives

$$\int d\Omega\, \det(-D_i\nabla_i)\, \delta(-D_i\nabla_i\Omega + D_i\Pi_i^{tr} - \rho_{qu}) = 1. \qquad (1.3)$$

This eliminates the Faddeev-Popov determinant, and with it the apparently unrenormalizable terms. In the local BRST formalism, the instantaneous closed fermi-ghost loops cancel against instantaneous closed bose-ghost loops.

At the end of sect. 2, a local first-order BRST-invariant Coulomb-gauge action S is derived, eq. (2.18), on which the results of the present article are based.

In sect. 3, BRST-invariance of the local action is exhibited, and it is shown that this assures Lorentz invariance of the expectation values of gauge-invariant quantities. In sect. 4, the Feynman rules are derived from the first order action. Two supplementary rules are introduced which assign the value zero to ambiguous expressions that formally vanish. It is assumed that dimensional regularization suffices to regularize remaining divergences. In sect. 5, the divergent parts of all two-point functions are calculated to one-loop order. In sect. 6, it is shown that in the Coulomb gauge, the conserved BRST-charge has the simple form

$$Q_{BRST} = -\int d^3x\, C\cdot(D_i\Pi_i - \rho_{qu}) = -\int d^3x\, C\cdot\delta S/\delta A_0, \qquad (1.4)$$

where C is the ghost field, and $(D_i\Pi_i - \rho_{qu})$ is the left-hand side of Gauss's law. We call Q_{BRST} the Gauss-BRST charge. Use of the last expression for Q_{BRST} allows the symmetry transformation generated by this charge to be written in the form of the Gauss-BRST identity. This assures the non-renormalization of $gA_0 = g_rA_{r,0}$, where A_0 is the time-component of the vector potential in the external field (effective action) formalism. In sect. 7, elementary properties of the quantum effective action Γ are derived. The somewhat technical proof of algebraic renormalizability which logically follows sect. 7 is presented in Appendices A and. In Appendix A, the general form of divergences consistent with the Ward identity is derived, and in Appendix B, algebraic renormalizibility is proven recursively. At the end of Appendix B it is verified that to one-loop order the charge renormalization is the same in the Coulomb gauge as in covariant gauges. In sect. 8 the instantanteous part V(R) of $g^2D^{A_0,A_0}$ is introduced. We call it the color-Coulomb potential. The renormalization-group-invariant running coupling constant is defined by $g_c(\mathbf{k}/\Lambda_{QCD}) = \mathbf{k}^2V(\mathbf{k})$, where $V(\mathbf{k})$ is the fourier transform of V(R), and it is verified that the first two terms of its β-function are universal. In sect. 9, the Ward identity is used to solve for D^{A_0,A_0}, which yields (1.2) above and

$$V(\mathbf{k}) = g^2\, [\mathbf{k}^2\, D^{C,C^*}(\mathbf{k})]^2\, L(\mathbf{k}), \qquad (1.5)$$

where $L(\mathbf{k})$ is the instantaneous part of the longitudinal gluon correlation function given in (9.8) and (9.17). In sect. 10, it is shown that the contribution of $V(\mathbf{k})$ to the Wilson loop exponentiates, giving exp[-CTV(R)], where V(R) is the fourier transform of $V(\mathbf{k})$, and C is defined above. In sect. 11, the merits of the order-parameter $K_{coul} \equiv \lim_{R\to\infty} CV(R)/R$ are argued. In sect. 12, it is recalled that an analysis of the Gribov problem [2, 3] leads to the conclusion that $\mathbf{k}^2D^{C,C^*}(\mathbf{k})$ is singular at $\mathbf{k} = 0$, thus providing a long-range component to the renormalization-group invariant color-Coulomb potential V. Conclusions are presented in sect. 13.

2. The problem and its solution

In this section we first give a heuristic discussion of the problem and then derive the local, BRST-invariant phase-space action S.

When applied to the Coulomb gauge, the standard Faddeev-Popov procedure gives the functional integral in configuration space,

$$Z = \int dA \prod_t (\delta[\nabla_i A_i(t)] \det\{-\nabla_i D_i[A(t)]\}) \exp(-S_{cl}), \qquad (2.1)$$

where sources are temporarily suppressed, $D_i(A)$ is the spatial gauge-covariant derivative, $[D_i(A)C]^a \equiv \nabla_i C^a + gf^{abd}A_i^b C^d$, $S_{cl} \equiv 2^{-1}\int d^4x\ (E^2 + B^2)$, and

$$E_i^a \equiv F_{0i}^a = \partial_0 A_i^a - \nabla_i A_0^a + gf^{abc}A_0^b A_i^c \qquad (2.2a)$$

$$B_i^a \equiv F_{jk}^a = \partial_j A_k^a - \nabla_k A_j^a + gf^{abc}A_j^b A_k^c, \qquad (2.2b)$$

for i, j, and k cyclic. (All vectors here and below are 3-dimensional.) This expression may be written in local form

$$Z = \int dAdCdC^*d\lambda \exp(-S_{FP}), \qquad (2.3)$$

where

$$S_{FP} = \int d^4x \{ 2^{-1}(E^2 + B^2) + \nabla_i C^* \cdot D_i(A)C - \nabla_i\lambda \cdot A_i \}, \qquad (2.4)$$

and $\lambda = ib$, where b is a real field. Here and below, the center dot means contraction on color indices.

The problem with the configuration-space functional integral is that the closed ghost loops are not of renormalizable type. In lowest order, the ghost loop is given by

$$(2\pi)^{-4} \int dk_0 \int d^3k\ [(\mathbf{k} - \mathbf{p})^2\ \mathbf{k}^2]^{-1}, \qquad (2.5)$$

where $\mathbf{p}$ and $\mathbf{k}$ are spatial vectors. The integrand is independent of k_0, which is characteristic of an instantaneous closed ghost loop. The divergent integral $\int dk_0$ multiplies a non-polynomial function of $\mathbf{p}$. This divergence occurs in any number of dimensions and the integral is not regularized by dimensional regularization.

One suspects that this divergence cancels against some other term. To organize the cancellation systematically, we introduce the Gaussian identity

$$1 = N \int dP \exp[- (1/2)(P - iE)^2], \tag{2.6}$$

into the configuration-space functional integral, which converts it to the phase-space functional integral. To avoid a profusion of i's, we write $\Pi \equiv iP$, and obtain

$$Z = \int d\Pi dA_0 dA^{tr} \prod_t \{\det[-\nabla_i D_i(A^{tr})] \} \exp(-S_1), \tag{2.7}$$

where

$$S_1 \equiv \int d^4x \, (\Pi_i \cdot E_i \; - 2^{-1}\Pi^2 + 2^{-1}B^2), \tag{2.8}$$

$B = B(A^{tr})$, and $E_i = \partial_0 A_i^{tr} - D_i(A^{tr})A_0$. The superscript tr refers to 3-dimensionally transverse vectors $\nabla_i A_i^{tr} = 0$. The A_0 field appears linearly in the action, and not at all in the Faddeev-Popov determinant, so the dA_0 integration gives simply

$$Z = \int d\Pi dA^{tr} \prod_t [\det(-\nabla_i D_i) \, \delta(D_i \Pi_i)] \exp(-S_2), \tag{2.9}$$

where $D = D(A^{tr})$, and

$$S_2 \equiv \int d^4x \, (\Pi_i \cdot \partial A^{tr}_i/\partial t - 2^{-1}\Pi^2 + 2^{-1}B^2). \tag{2.10}$$

The appearance of $\delta(D_i\Pi_i)$ in the functional integral means that Gauss's law, $D_i\Pi_i = 0$, is satisfied at every time. We decompose Π_i into its transverse and longitudinal parts

$$\Pi_i = \Pi_i^{tr} - \nabla_i \Omega, \tag{2.11}$$

so $d\Pi = d\Pi^{tr} \, d\Omega$. Gauss's law reads

$$- D_i(A^{tr})\nabla_i\Omega = - gA_i^{tr} \times \Pi_i^{tr} \equiv \rho, \tag{2.12}$$

where we have introduced the notation $(A \times B)^a \equiv f^{abc}A^bB^c$. Here ρ^a is the part of the color-charge density that comes from the dynamical fields A^{tr} and Π^{tr}. (If quarks were present then there would be a quark contribution to ρ.) We solve this time-independent constraint for Ω,

$$\Omega = \Omega_{ph} \equiv [M(A^{tr})]^{-1} \rho. \tag{2.13}$$

We call Ω_{ph} the color-Coulomb field. The Green's function M^{-1} is the inverse of the Faddeev-Popov operator $M(A) \equiv -D_i(A)\nabla_i$, which is symmetric when A is transverse, $M(A^{tr}) = -D_i(A^{tr})\nabla_i = -\nabla_i D_i(A^{tr}) = M^{\dagger}(A^{tr})$.

The integral over $d\Omega$ eliminates the Faddeev-Popov determinant, as desired,

$$\int d\Omega \;\det[-\nabla_i D_i(A^{tr})]\; \delta[- D_i(A^{tr})\nabla_i\Omega + \rho]\; f(\Omega) = f(\Omega_{ph}). \tag{2.14}$$

We obtain

$$Z = \int d\Pi^{tr} dA^{tr} \exp(- S_{ph}), \tag{2.15}$$

where

$$S_{ph} \equiv \int d^4x \; \{ \; \Pi^{tr}{}_i \cdot \partial A^{tr}{}_i/\partial t - 2^{-1}[\; (\Pi^{tr})^2 + (\nabla\Omega_{ph})^2 \;] + 2^{-1}B^2 \; \}. \tag{2.16}$$

The functional integral is now expressed in terms of the dynamical fields A^{tr} and Π^{tr} for which the measure is Cartesian. This is the expression one would start with in canonical quantization.

The last expression for Z makes manifest the physical content of the functional integral. The dynamical fields A^{tr} and Π^{tr} propagate dynamically in time, and interact instantaneously at a distance by M^{-1} which occurs in $\Omega_{ph} = M^{-1}\rho$. However, because detM does not appear in the partition function Z, *there are no closed instantaneous loops*.

Unfortunately, this expression is only heuristic because S_{ph} is a non-local action (because of the appearance of $\Omega_{ph} = M^{-1}\rho$), which we don't know how to renormalize. To obtain a renormalizable theory, we start over again and introduce the Gaussian identity (2.6) into the functional integral (2.3) with the local Faddeev-Popov action (2.4). This gives

$$Z = \int d\Pi dA dC dC^* d\lambda \exp(-S). \tag{2.17}$$

Here S is the local phase-space action

$$S \equiv \int d^4x\, [\, \Pi_i \cdot E_i - 2^{-1}\Pi^2 + 2^{-1}B^2 + \nabla_i C^* \cdot D_i(A)C - \nabla_i \lambda \cdot A_i\,]. \quad (2.18)$$

The non-renormalizable instantaneous closed fermi-ghost loops now recur in the perturbative expansion but, according to the preceding discussion, they are canceled by corresponding instantaneous closed bose loops. We shall verify this cancellation explicitly.

The equations of motion of A_0 and Π_i that one derives from this action read

$$D_i \Pi_i = 0 \quad (2.19)$$

$$\Pi_i = E_i\, . \quad (2.20)$$

Thus despite the presence of ghost fields in the local Coulomb-gauge action (2.18), Gauss's law is satisfied at the level of equations of motion. This suggests that the Coulomb gauge is well-adapted for addressing the confinement problem, because one expects that Gauss's law is crucial to the long-range color-force. Indeed we shall find a Ward identity that expresses Gauss's law in the functional integral formalism, and which is essential for renormalizability.

3. BRST and Lorentz invariance

The Faddeev-Popov action is invariant under the BRST transformation

$$sA_\mu = D_\mu C \qquad sC = -\, 2^{-1}\, gC \times C$$

$$sC^* = \lambda \qquad s\lambda = 0. \quad (3.1)$$

Under this transformation the E and B fields transform covariantly, $sE = gE \times C$ and $sB = gB \times C$. Consequently if we assign to Π the BRST transformation

$$s\Pi = g\Pi \times C, \quad (3.2)$$

under which it remains nil-potent $s^2 = 0$, then the action (2.18) is BRST invariant

$$sS = 0. \quad (3.3)$$

and may be written

$$S = \int d^4x\, [\, - 2^{-1}(\Pi - E)^2 + 2^{-1}(E^2 + B^2) - s(\nabla_i C^* \cdot A_i)\,]. \qquad (3.4)$$

We must show that expectation values of gauge-invariant of gauge-invariant functions W(A) are Lorentz (Euclidean) invariant. As usual, A transforms as a vector field under Lorentz transformation, and C, C* and λ transform as scalars,

$$\delta_v A_\mu = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, A_\mu + v_{\mu,\lambda}\, A_\lambda,$$

$$\delta_v C = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, C$$

$$\delta_v C^* = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, C^*$$

$$\delta_v \lambda = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, \lambda. \qquad (3.5a)$$

Here $v_{\kappa,\lambda} = - v_{\lambda,\kappa}$ is an arbitrary infinitesimal Lorentz transformation. Consequently $(1/4)F_{\mu,\lambda}{}^2 = (1/2)(E^2 + B^2)$ is a Lorentz (Euclidean) scalar. We assign a Lorentz transformation law to Π_i which also makes the first term in (3.4) a Lorentz scalar. This is achieved by assigning a scalar transformation law to $\Pi_i' \equiv (\Pi_i - E_i)$ for each i = 1, 2, 3,

$$\delta_v \Pi_i' = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, \Pi_i'.$$

This gives, for $\Pi_i = \Pi_i' + E_i = \Pi_i' + F_{0,i}$,

$$\delta_v \Pi_i = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, (\Pi_i' + F_{0,i}) + v_{0,\mu}\, F_{\mu,i} + v_{i,\nu}\, F_{0,\nu},$$

$$\delta_v \Pi_i = v_{\kappa,\lambda}\, x_\kappa\, \partial_\lambda\, \Pi_i + v_{0,j}\, F_{j,i} + v_{i,k}\, F_{0,k}. \qquad (3.5b)$$

With this assignment, Lorentz transformations commute with the BRST transformation, $\delta_v s = s\delta_v$, and moreover the Lorentz variation of the action is BRST exact,

$$\delta_v S = s[\, - \int d^4x\;\; \delta_v(\nabla_i C^* \cdot A_i)\,] = s\psi. \qquad (3.6)$$

This is sufficient to assure that expectation values of gauge-invariant functions W(A) are Lorentz invariant. To prove this, we write

$$\langle W \rangle = N \int d\Phi \, W \exp(-S)$$

where Φ represents the set of all fields

$$\Phi \equiv (\Pi_i, A_i, A_0, C, C^*, \lambda). \qquad (3.7)$$

The change of variables $\Phi' = \Phi + \delta_v \Phi$ leaves the measure $d\Phi$ invariant, which gives

$$0 = \int d\Phi \, \delta_v \, [W \exp(-S)\,] = \int d\Phi \, (\delta_v W - W \, \delta_v S) \exp(-S)\,]$$

$$= \int d\Phi \, (\delta_v W - W \, s\, \psi) \exp(-S)\,] = \int d\Phi \, [\delta_v W - s(W\, \psi)\,] \exp(-S)\,],$$

where we have used the fact that the gauge-invariant quantity W(A) is BRST-exact, $sW = 0$. With $sS = 0$, this gives

$$0 = \int d\Phi \, \delta_v W \exp(-S), \qquad (3.8)$$

which shows that $\langle W \rangle$ is Lorentz-invariant.

4. Feynman rules

The zero-order action S_0 is the part of the action S, eq. (2.18), that is quadratic in the fields. We write $S_0 = (\Phi, \Gamma_0 \, \Phi)$, and Γ_0 is the matrix represented in momentum space by,

Γ_0	Π_j	A_j	A_0	λ
Π_i	$-\delta_{i,j}$	$i\, k_0\, \delta_{i,j}$	$-i\, k_i$	0
A_i	$-i\, k_0\, \delta_{i,j}$	$\delta_{i,j}\, k^2 - k_i\, k_j$	0	$-i\, k_i$
A_0	$i\, k_j$	0	0	0
λ	0	$i\, k_j$	0	0

The zero-order propagators used in Feynman rules are the elements of the inverse matrix

D_0	Π_j	A_j	A_0	λ
Π_i	$\frac{-\mathbf{k}^2 P_{i,j}^{tr}}{(k_0^2 + \mathbf{k}^2)}$	$\frac{i k_0 P_{i,j}^{tr}}{(k_0^2 + \mathbf{k}^2)}$	$-i k_i / \mathbf{k}^2$	0
A_i	$\frac{-i k_0 P_{i,j}^{tr}}{(k_0^2 + \mathbf{k}^2)}$	$\frac{P_{i,j}^{tr}}{(k_0^2 + \mathbf{k}^2)}$	0	$-i k_i / \mathbf{k}^2$
A_0	$i k_j / \mathbf{k}^2$	0	$1 / \mathbf{k}^2$	$-i k_0 / \mathbf{k}^2$
λ	0	$i k_j / \mathbf{k}^2$	$i k_0 / \mathbf{k}^2$	0

where i and j run from 1 to 3, $\mathbf{k}^2 = k_i k_i$, and $P_{i,j}^{tr} \equiv (\mathbf{k}^2 \delta_{i,j} - k_i k_j)/\mathbf{k}^2$ is the projector onto transverse 3-vectors. The fermi ghost C-C* propagator is given as usual by $1/\mathbf{k}^2$.

The elementary propagators and vertices are shown graphically in Fig. 1. The λ-field does not appear in any vertex and its propagators do not concern us further. The propagators that are independent of k_0 are instantaneous in time and are represented by dotted lines. These are $D_0^{\Pi i,Ao}$, $D_0^{Ao,Ao}$, and D_0^{C,C^*}. The remaining bose propagators satisfy $\lim_{k_0\to\infty} D(k_0, \mathbf{k}) = 0$, and thus do not contain any instantaneous part (as they would if there were a k_0^2 in the numerator). We call propagators that satisfy $\lim_{k_0\to\infty} D(k_0, \mathbf{k}) = 0$ "dynamical", and represent them by solid lines. These are $D_0^{\Pi i,\Pi j}$, $D_0^{\Pi i,Aj}$, and $D_0^{Ai,Aj}$. If one writes $\Pi_i = \Pi_i^{tr} - \nabla_i\Omega$, the Ω propagators are instantaneous and the Π^{tr} propagators are dynamical. So the instantaneous propagators are all time-like or 3-dimensionally longitudinal, and the dynamical propagators are 3-dimensionally transverse. There are no time-like momenta k_0 at the vertices.

The dangerous non-renormalizable diagrams contain closed instantaneous loops. For these loops have no time-like momenta in any denominator to provide convergence for the k_0 integration. Let us see what such loops consist of. Ghosts are coupled at the vertices $g\nabla_i C^*\cdot A_i \times C$, and the Π_i and A_0 fields are coupled at vertices, $g\Pi_i \cdot A_i \times A_0$, where

$\Pi_i = \Pi_i^{tr} - \nabla_i\Omega$ Thus instantaneous closed loops are either bose, consisting of A_0-Ω propagators, or fermi, consisting of C-C* propagators. These two loops precisely cancel each other. We have verified diagrammatically that in the Coulomb gauge, the role of the Faddeev-Popov determinant is to cancel unwanted instantaneous bose loops, leaving only instantaneous Coulomb exchanges. The separate expressions for the instantaneous loops are not finite before cancellation, and so, to avoid ambiguity, we complete the usual Feynman rules with
Supplemental rule 1: all diagrams that contain instantaneous bose or fermi loops (that formally cancel each other) are suppressed.

There are closed bose loops that are formed of a single dynamical Π_i-A_j propagator, with all other propagators in the loop being instantaneous. Let the Π_i-A_j propagator carry momentum k_μ, and write the loop integral (in d spatial dimensions) $\int dk_0 d^d k$. The integral over k_0 is given by $\int dk_0\, k_0\, (k_0^2 + \mathbf{k}^2)^{-1}$. It diverges logarithmically in any number of spatial dimensions, so it is not regularized by dimensional regularization, but is formally 0. We deal with this ambiguity by
Supplemental rule 2: all diagrams containing such loops are assigned the (formally correct) value 0.

We suppose without proof that the remaining graphs are made finite by dimensional regularization, and that the theory defined by with these supplementary rules obeys the Ward identities that will be derived below.

5. Calculation of proper two-point functions

In this section we shall calculate all the two-point functions to one-loop order. We do this by dimensional regularization, using the exponential representation of all denominators. For example the zeroth order A-A propagator is written

$$D_0^{A_i,A_j} = (\mathbf{k}^2\, \delta_{i,j} - k_i\, k_j)\, [\mathbf{k}^2\, (k_0^2 + \mathbf{k}^2)]^{-1}$$

$$= (\mathbf{k}^2\, \delta_{i,j} - k_i\, k_j) \int_0^\infty d\alpha d\beta\, \exp[\, - \alpha\mathbf{k}^2 - \beta(\, k_0^2 + \mathbf{k}^2)\,].$$

The loop momenta are then performed by Gaussian integration. As we are interested in renormalizability we shall write only the divergent parts.

The most intricate of the proper functions is $\Gamma^{A\text{-}A}$. The diagrams which contribute to it are shown in Fig. 2. Diagrams (c) and (d) are instantaneous bose and fermi loops which cancel each other and are suppressed by supplemental rule 1. The result for the divergent part is

$$\Gamma^{Ai,Aj} = \lambda_0\, \varepsilon^{-1}\ [\, (1/3)\, k_0^2\, \delta_{i,j} - (\mathbf{k}^2 \delta_{i,j} - k_i k_j)\,], \tag{5.1}$$

Here we have introduced the divergent constant

$$1/\varepsilon \equiv (4\text{-}D)^{-1}, \tag{5.2}$$

and the coupling parameter

$$\lambda_0 \equiv (8\pi^2)^{-1}\, N g^2, \tag{5.3}$$

where $D = d+1$ is the number of space-time dimensions. The diagrams which contribute to $\Gamma^{Ai,Ao}$ are shown in Fig. 3. Diagram (b) is suppressed by supplemental rule 2. The result is

$$\Gamma^{Ai,Ao} = \lambda_0\, \varepsilon^{-1}\, (-\, 1/3)\, k_0\, k_i. \tag{5.4}$$

The diagrams which contribute to $\Gamma^{Ao,Ao}$ are shown in Fig. 4. The result is

$$\Gamma^{Ao,Ao} = \lambda_0\, \varepsilon^{-1}\, (1/3)\, \mathbf{k}^2\, . \tag{5.5}$$

These give the divergent part of the effective action quadratic in the vector potential,

$$\Gamma(A, A) = \lambda_0\, \varepsilon^{-1} (1/2) \int d^4x\, [\, (1/3)\, (\partial_0 A_i - \nabla_i A_0)^2 - (\nabla \times \mathbf{A})^2\,]. \tag{5.6}$$

Fig. 5 shows the diagram that contributes to $\Gamma^{\Pi i,Aj}$. It is suppressed by supplemental rule 2. Here and below we include the zero-order term [which vanishes for $\Gamma(A, A)$], and we have

$$\Gamma^{\Pi i,Aj} = i k_0\, \delta_{i,j}\, . \tag{5.7}$$

The diagram that contributes to $\Gamma^{\Pi i,Ao}$ are shown in Fig. 6, with the result

$$\Gamma^{\Pi i,Ao} = -\, \{\, 1 - (4/3)\, \lambda_0\, \varepsilon^{-1}\, \}\ i\, k_i\, . \tag{5.8}$$

We have

$$\Gamma(\Pi, A) = \int d^4x\, \Pi_i \{ \partial_0 A_i - [1 - (4/3)\lambda_0 \varepsilon^{-1}] \nabla_i A_0 \}. \tag{5.9}$$

The diagram which contributes to Γ^{Π_i,Π_j} is shown in Fig. 7. It gives

$$\Gamma^{\Pi_i,\Pi_j} = - [1 + (4/3)\lambda_0 \varepsilon^{-1}] \delta_{i,j}, \tag{5.10}$$

$$\Gamma(\Pi, \Pi) = \int d^4x\, [1 + (4/3)\lambda_0 \varepsilon^{-1}] (-1/2)\Pi_i \Pi_i . \tag{5.11}$$

The diagram which contributes to $\Gamma^{C^*,C}$ is shown in Fig. 8. It gives

$$\Gamma^{C^*,C} = [1 - (4/3)\lambda_0 \varepsilon^{-1}] \mathbf{k}^2, \tag{5.12}$$

$$\Gamma(C^*, C) = \int d^4x\, [1 - (4/3)\lambda_0 \varepsilon^{-1}] \nabla_i C^* \nabla_i C. \tag{5.13}$$

This completes the calculation of the quadratic part of the effective action Γ_{qu} to one loop order, keeping only the divergent part,

$$\Gamma_{qu} = \Gamma(A, A) + \Gamma(\Pi, A) + \Gamma(\Pi, \Pi) + \Gamma(C^*, C). \tag{5.14}$$

The symmetries of the theory place severe constraints on the divergent parts which we investigate next.

6. Ward identity generated by the Gauss-BRST charge

We introduce the sources for the partition function

$$Z = \int d\Phi \exp[- \Sigma + (\Phi, J)], \tag{6.1}$$

where Φ is the set of all elementary fields, as in (3.7), and

$$(\Phi, J) \equiv \int d^4x\, (A_\mu \cdot J_{A\mu} + \Pi_i \cdot J_{\Pi i} + C \cdot J_C + C^* \cdot J_{C^*} + \lambda \cdot J_\lambda). \tag{6.2}$$

Here the extended action Σ includes sources K_μ, L, and M_i, for the non-linear BRST transforms sA_μ, sC and $s\Pi_i$ namely,

$$\Sigma \equiv S + \int d^Dx \, [\, K_\mu \cdot D_\mu C + M_i \cdot (g\Pi_i \times C) + L \cdot (-g/2)\, C \times C \,] \,, \qquad (6.3)$$

where S is given in (2.18). We write $\Sigma = \int d^Dx \, \Lambda$, where the extended Lagrangian density is given by

$$\Lambda = \Pi_i \cdot E_i - 2^{-1}\Pi^2 + 2^{-1}B^2 + \nabla C^* \cdot D(A)C - \nabla\lambda \cdot A$$

$$+ K_\mu \cdot D_\mu C + M_i \cdot (g\Pi_i \times C) + L \cdot (-g/2) C \times C. \qquad (6.4)$$

It is BRST invariant, as is Σ,

$$s\Lambda = 0; \qquad s\Sigma = 0. \qquad (6.5)$$

To derive the Ward identity, consider the infinitesimal variation

$$\delta\Phi_\alpha = \varepsilon(x)\, s\Phi_\alpha, \qquad (6.6)$$

where $\varepsilon(x)$ is space-time dependent, and α is an index that runs over all components of all elementary fields. It reduces to the BRST transformation when ε is constant. Because Λ is BRST-invariant, its variation under (6.6) is given by

$$\delta\Lambda = (\partial_\mu \varepsilon)\, s\Phi_\alpha \cdot \partial\Lambda / \partial(\partial_\mu \Phi_\alpha) \equiv (\partial_\mu \varepsilon)\, j_\mu, \qquad (6.7)$$

which defines the conserved BRST-current j_μ. Here and henceforth, all derivatives with respect to fermionic fields are understood to be left derivatives. We have

$$\delta\Sigma = - \int d^Dx \, \varepsilon(x)\, \partial_\mu j_\mu, \qquad (6.8)$$

$$\delta(\Phi, J) \equiv \int d^Dx \, \varepsilon(x)\, (sA_\mu \cdot J_{A\mu} + s\Pi_i \cdot J_{\Pi i} + sC \cdot J_C + \lambda \cdot J_{C^*}). \qquad (6.9)$$

The change of variable $\Phi' = \Phi + \delta\Phi$ leaves the measure invariant, and we have, since $\varepsilon(x)$ is arbitrary,

$$0 = \int d\Phi \, (\, \partial_\mu j_\mu + sA_\mu \cdot J_{A\mu} + s\Pi_i \cdot J_{\Pi i} + sC \cdot J_C + \lambda \cdot J_{C^*} \,)$$

$$\times \exp[\, - \Sigma + (\Phi, J) \,], \qquad (6.10)$$

$$(J_{A\mu}\cdot\delta/\delta K_\mu + J_{\Pi i}\cdot\delta/\delta M_i + J_C\cdot\delta/\delta L - J_{C*}\cdot\delta/\delta J_\lambda)\, Z$$

$$= \int d\Phi \;\; \partial_\mu j_\mu \;\; \exp[\, - \Sigma + (\Phi, J)\,]. \qquad (6.11)$$

Instead of integrating this identity over all space-time, which would give the Zinn-Justin equation, we integrate over space only, with spatially periodic boundary conditions, and obtain

$$\int d^3x\, (J_{A\mu}\cdot\delta/\delta K_\mu + J_{\Pi i}\cdot\delta/\delta M_i + J_C\cdot\delta/\delta L - J_{C*}\cdot\delta/\delta J_\lambda)\, Z$$

$$= \partial_0 \int d\Phi \;\; Q \;\; \exp[\, - \Sigma + (\Phi, J)\,], \qquad (6.12)$$

where Q is the conserved BRST charge,

$$Q = \int d^3x \;\; s\Phi_i\cdot\partial\Lambda/\partial(\partial_0\Phi_i). \qquad (6.13)$$

The only time derivatives in Λ are contained in $\Pi_i\cdot E_i = \Pi_i\cdot(\partial_0 A_i - D_i A_0)$ and in $K_0 D_0 C$, and we have, after an integration by parts,

$$Q = \int d^3x\, [\, - C\cdot D_i\Pi_i + K_0\cdot(g/2)\, C\times C\,]\,. \qquad (6.14)$$

The first term in the integrand is the left-hand side of the Gauss's law multiplied by C, which is characteristic of the Coulomb gauge. We call Q the Gauss-BRST charge. Moreover A_0 also only appears in Λ in the terms $\Pi_i\cdot E_i = \Pi_i\cdot(\partial_0 A_i - D_i A_0)$ and $K_0\cdot D_0 C$, so

$$\delta\Sigma/\delta A_0 = D_i\Pi_i - gK_0\times C, \qquad (6.15)$$

and the Gauss-BRST charge has the simple form

$$Q = \int d^3x\, [\, - C\cdot\delta\Sigma/\delta A_0 + K_0\cdot\delta\Sigma/\delta L\,]\,. \qquad (6.16)$$

By (6.12), this gives the Ward identity satisfied by the partition function Z,

$$\int d^3x\, (J_{A\mu}\cdot\delta/\delta K_\mu + J_{\Pi i}\cdot\delta/\delta M_i + J_C\cdot\delta/\delta L - J_{C*}\cdot\delta/\delta J_\lambda)\, Z$$

$$= \partial_0 \int d^3x\, (\, J_{A0}\cdot\delta/\delta J_C - K_0\cdot\delta/\delta L\,)\, Z\,. \qquad (6.17)$$

(The sign of $J_{Ao}\cdot\delta/\delta J_C$ is correct because the left derivative of (Φ, J) with respect to J_C is taken.) We call this identity the Gauss-BRST identity. It is the functional analog of the operator statement that the BRST-symmetry transformation is generated by the Gauss-BRST charge [4]. Remarkably, only first functional derivatives of Z appear, so the unacceptably singular expression of Green's functions at coincident points is absent.

This identity may be expressed in various ways. Define the linear differential operator

$$P(\varepsilon) \equiv \int d^4x\, \varepsilon(t)\, (J_{A\mu}\cdot\delta/\delta K_\mu + J_{\Pi i}\cdot\delta/\delta M_i + J_C\cdot\delta/\delta L - J_{C^*}\cdot\delta/\delta J_\lambda)$$
$$+ \partial_0\varepsilon(t)\, (\, J_{Ao}\cdot\delta/\delta J_C - K_0\cdot\delta/\delta L\,), \qquad (6.18)$$

where $\varepsilon(t)$ is an ordinary function of t. Then the partition function Z and the generating functional of connected Green's functions $W = \ln Z$ satisfy

$$P(\varepsilon)Z = P(\varepsilon)W = 0 \qquad (6.19)$$

for all $\varepsilon(t)$. This is consistent without further restrictions on Z or W because $P(\varepsilon)$ satisfies

$$P(\varepsilon_1)\, P(\varepsilon_2) + P(\varepsilon_2)\, P(\varepsilon_1) = 0. \qquad (6.20)$$

This identity maintains the cancellation of instantaneous fermi and bose closed loops discussed heuristically in sect. 2.

We make a Legendre transformation from W(J) to the effective action $\Gamma(\Phi)$, which is the generating functional of proper functions,

$$\Gamma(\Phi) = (\Phi, J) - W(J), \qquad (6.21)$$

where

$$A_\mu = \delta W/\delta J_{A\mu} \qquad J_{A\mu} = \delta\Gamma/\delta A_\mu$$

$$\Pi_i = \delta W/\delta J_{\Pi i} \qquad J_{\Pi i} = \delta\Gamma/\delta \Pi_i$$

$$C = -\,\delta W/\delta J_C \qquad J_C = \delta\Gamma/\delta C$$

$$C^* = - \delta W/\delta J_{C^*} \qquad J_{C^*} = \delta\Gamma/\delta C^*$$

$$\lambda = \delta W/\delta J_\lambda \qquad J_\lambda = \delta\Gamma/\delta\lambda$$

$$\delta W/\delta K_\mu = - \delta\Gamma/\delta K_\mu$$

$$\delta W/\delta M_i = - \delta\Gamma/\delta M_i$$

$$\delta W/\delta L = - \delta\Gamma/\delta L, \tag{6.22}$$

and all fermionic derivatives are left derivatives. This gives the Gauss-BRST identity satisfied by the effective action, which is its most convenient expression,

$$\int d^3x\, (\delta\Gamma/\delta A_\mu \cdot \delta\Gamma/\delta K_\mu + \delta\Gamma/\delta C \cdot \delta\Gamma/\delta L + \delta\Gamma/\delta\Pi_i \cdot \delta\Gamma/\delta M_i + \delta\Gamma/\delta C^* \cdot \lambda)$$

$$= \partial_0 \int d^3x\, (\delta\Gamma/\delta A_0 \cdot C - K_0 \cdot \delta\Gamma/\delta L). \tag{6.23}$$

One may verify that this identity is satisfied by the local action Σ. It retains its form when gluons are minimally coupled to quarks, because the quark Lagrangian density L_{qu} satisfies $j_{qu,0} = s\psi \cdot \partial L_{qu}/\partial\partial_0\psi = - C \cdot \partial L_{qu}/\partial A_0$, where $j_{qu,0}$ is the quark contribution to the BRST current.

Just as global BRST-invariance is thought to encode the information contained in local gauge invariance, the Gauss-BRST identity appears to encode the information contained in Gauss's.

7. Elementary properties of the effective action

The dependence of Γ on λ and C^* is determined by their equations of motion, which is standard, except that only spatial derivatives are encountered. We have

$$0 = \int d\Phi\, \delta/\delta\lambda \exp[- \Sigma + (\Phi, J)],$$

$$= \int d\Phi\, (J_\lambda - \nabla_i A_i) (\exp[- \Sigma + (\Phi, J)]$$

$$= (J_\lambda - \nabla_i \delta/J_{Ai})\, Z,$$

or

$$\delta\Gamma/\delta\lambda - \nabla_i A_i = 0.$$

This equation has the solution

$$\Gamma = \int d^4x \, (- \lambda \cdot \nabla_i A_i \ ) + \Gamma_1, \tag{7.1}$$

where Γ_1 is independent of λ. Similarly we have

$$0 = \int d\Phi \, \delta/\delta C^* \exp[- \Sigma + (\Phi, J)],$$

$$= \int d\Phi \, (J_{C^*} + \nabla_i D_i C) \, (\exp[- \Sigma + (\Phi, J)]$$

$$= (J_{C^*} - \nabla_i \delta/\delta K_i) \, Z,$$

or

$$\delta\Gamma/\delta C^* + \nabla_i \delta\Gamma/\delta K_i \ = 0,$$

which gives the dependence of Γ on C^*,

$$\Gamma = \int d^4x \, (- \nabla_i A_i \, \lambda) + \Gamma^*(A_0, A_i, C, \Pi_i, K_0, K_i + \nabla_i C^*, L, M_i). \tag{7.2}$$

In terms of $\Gamma^*(A_0, A_i, C, \Pi_i, K_0, K_i, L, M_i)$, the Gauss-BRST identity reads

$$F_t(\Gamma^*) = 0, \tag{7.3a}$$

where we have introduced the notation

$$F_t(\Gamma^*) \equiv \int d^3x \, [\, (\delta\Gamma^*/\delta A_\mu \cdot \delta\Gamma^*/\delta K_\mu + \delta\Gamma^*/\delta C \cdot \delta\Gamma^*/\delta L + \delta\Gamma^*/\delta \Pi_i \cdot \delta\Gamma^*/\delta M_i \,) - \partial_0 \, (C \cdot \delta\Gamma^*/\delta A_0 - K_0 \cdot \delta\Gamma^*/\delta L \,) \,] \, . \tag{7.3b}$$

The subscript t indicates that the Ward identity holds at every fixed time t.

Another property of Γ^* that we will need later is that its dependence on the ghost field C is of the form

$$\Gamma^* = \int d^4x\, K_\mu D_\mu C + \Gamma_2(\nabla_i C). \qquad (7.4)$$

Thus, apart from the term shown explicitly, Γ^* depends on C only through its derivative with respect to x_i, for i = 1, 2, 3. This is the Coulomb-gauge analog of the well-known Landau-gauge property of factorization of external ghost momentum from every diagram. (It implies $Z_1' = 1$, where Z_1' is the ghost-ghost-gluon vertex renormalization constant.). In both gauges it follows from transversality of the gluon propagator [5].

Another general property, which Γ inherits from Σ, is invariance $\Gamma(\Psi) = \Gamma(\Psi')$ under the time-reversal transformation

$$\Psi'_\alpha(t, x) = \eta_\alpha\, \Psi_\alpha(-t, x) \qquad (7.5),$$

where Ψ_α represents all fields and sources, and η_α is a sign factor defined by $\eta_\alpha = 1$ for $\Psi_\alpha = (A_i, C, C^*, \lambda, K_i, L)$, and $\eta_\alpha = -1$ for $\Psi_\alpha = (A_0, \Pi_i, K_0, M_i)$.

It follows in particular that the proper two-point functions $\Gamma^{K_0C}(k_0, k_i)$ and $\Gamma^{\Pi_iC}(k_0, k_i)$ are odd in k_0, $\Gamma^{K_0C}(k_0, k_i) = -\Gamma^{K_0C}(-k_0, k_i)$, and $\Gamma^{\Pi_iC}(k_0, k_i) = -\Gamma^{\Pi_iC}(-k_0, k_i)$. On the other hand the diagrams which contribute to these two-point functions contain a continuous ghost line which enters and leaves the diagram, which consists of instantaneous propagators. If the external momentum k_μ is routed through these propagators, then the integrand is independent of k_0. Thus all loop corrections to these functions vanish, and we conclude by inspection of Σ, that

$$\Gamma^{K_0C}(k_0, k_i) = ik_0 \qquad \Gamma^{\Pi_iC}(k_0, k_i) = 0. \qquad (7.6)$$

The properties of Γ derived in this and the preceding section are applied to prove algebraic renormalizability of the Coulomb gauge in Appendices A and B, assuming that the dimensionally regularized theory, with supplementary rules 1 and 2, is well-defined and obeys the identities that we have derived.

8. Running coupling constant.

Because relation (B.19) between renormalized and unrenoralized fields mixes elementary, composite and derivative fields, partial derivativeswith respect to the elementary fields do not renormalize multiplicatively, in general, as one sees from (B.32). However the gluon propagator does, as we now show.

The unrenormalized and renormalized propagators (two-point functions) are given respectively by

$$[D^{-1}(\Phi)]_{x,\alpha;y,\beta} = \delta^2\Gamma(\Phi)/\delta\Phi_{x,\alpha}\delta\Phi_{y,\beta} \tag{8.1}$$

$$[D_r{}^{-1}(\Phi_r)]_{x,\alpha;y,\beta} = \delta^2\Gamma_r(\Phi_r)/\delta\Phi_{r,x,\alpha}\delta\Phi_{r,y,\beta}, \tag{8.2}$$

where $\Phi_\alpha = (A_\mu, \Pi_i, C, C^*)$ represents the set of components of all elementary fields. The two are related by

$$D_{x,y}{}^{A\mu,A\nu}(\Phi)] = \int d^4u\, d^4v\, \delta A_{x,\mu}/\delta\Phi_{r,u,\gamma}\, \delta A_{y,\beta}/\delta\Phi_{r,v,\varepsilon}\, D_{r;u,v;\gamma,\varepsilon}(\Phi). \tag{8.3}$$

By (B.18), this gives

$$D_{x,y}{}^{A\mu,A\nu}(\Phi) = Z_{A\mu}Z_{A\nu}\, D_{r;x,y}{}^{A\mu,A\nu}(\Phi), \tag{8.4}$$

so the gluon propagator does renormalize multiplicatively.

We now set $\mu = \nu = 0$, and obtain, using $Z_g Z_{Ao} = 1$, eq. (B.36),

$$g^2 D_{x,y}{}^{Ao,Ao}(\Phi) = g_r{}^2 D_{r;x,y}{}^{Ao,Ao}(\Phi). \tag{8.5}$$

Thus $g^2 D_{x,y}{}^{Ao,Ao}(\Phi)$ is a renormalization-group invariant. The propagator at physical values of the sources is obtained by setting all external fields to 0,

$$g^2 D_{x-y}{}^{A\mu,A\nu} = g^2 D_{x,y}{}^{A\mu,A\nu}(\Phi)|_{\Phi = K = L = M = 0}. \tag{8.6}$$

Its time-time component remains a renormalization-group invariant

$$g^2 D_x^{Ao,Ao} = g_r^2 D_{r;x}^{Ao,Ao}. \tag{8.7}$$

In this respect, the time-time component of the gluon propagator in the Coulomb gauge in QCD behaves like the 4-dimensionally transverse part of the photon propagator in a covariant gauge in QED.

Part of $g^2 D_x^{Ao,Ao}$ is instantaneous in position space

$$g^2 D_x^{Ao,Ao} = \delta(x_0) V(\mathbf{x}) + g^2 D'_x{}^{Ao,Ao}. \tag{8.8}$$

This part is separated out in momentum space by

$$V(\mathbf{k}) = \lim_{k_0 \to \infty} g^2 D^{Ao,Ao}(k_0, \mathbf{k}) = \lim_{k_0 \to \infty} g_r^2 D_r^{Ao,Ao}(k_0, \mathbf{k}), \tag{8.9}$$

where we have written $V(\mathbf{k})$ for the fourier transform of $V(\mathbf{x})$. We assume that this limit exists. This limit may simply give back the sum of all instantaneous diagrams that contribute to $g^2 D_x^{Ao,Ao}$, but we shall not attempt to prove this. Because $V(\mathbf{k})$ is a renormalization-group invariant, it is independent of the renormalization mass μ and depends only on $\mathbf{k}$ and Λ_{QCD}.

Thus, in the Coulomb gauge, we may define a renormalized running coupling constant that makes no reference to a regularization scheme, by

$$g_c^2(|\mathbf{k}|/\Lambda_{QCD}) \equiv \mathbf{k}^2 \, V(\mathbf{k}). \tag{8.10}$$

In renormalized perturbation theory it has the expansion

$$g_c^2(|\mathbf{k}|/\Lambda_{QCD}) = g_r^2(\mu/\Lambda_{QCD}) \, [\, 1 + \Sigma_{n=1}^{\infty} \, g_r^{2n}(\mu/\Lambda_{QCD}) \, f_n(|\mathbf{k}|/\mu) \,], \tag{8.11}$$

where μ is a renormalization mass in some scheme, for example the minimal subtraction scheme. We set $\mu = |\mathbf{k}|$, and obtain

$$g_c^2(|\mathbf{k}|/\Lambda_{QCD}) = g_r^2(|\mathbf{k}|/\Lambda_{QCD}) \, [\, 1 + \Sigma_{n=1}^{\infty} \, g_r^{2n}(|\mathbf{k}|/\Lambda_{QCD}) \, f_n(1) \,]. \tag{8.12}$$

Thus the change from $g_r(|\mathbf{k}|/\Lambda_{QCD})$ to $g_c(|\mathbf{k}|/\Lambda_{QCD})$ is a regular redefinition of the coupling constant, and so, as is well known [6], the β-functions of g_r and g_c have the same first two terms,

$$\beta(g_c) = |\mathbf{k}|\partial g_c/\partial|\mathbf{k}| = - b_0 g_c{}^3 - b_1 g_c{}^5 + \dots . \tag{8.13}$$

We have seen that the coupling constant renormalization in one-loop order is the same in the Coulomb gauge and in covariant gauges, so b_0 has the same value as in covariant gauges. We expect that this will also be true for b_1. At large $\mathbf{k}$, $g^2(|\mathbf{k}|/\Lambda_{QCD})$ has the well-known asymptotic form

$$g_c{}^2(|\mathbf{k}|/\Lambda_{QCD}) = \mathbf{k}^2\ V(\mathbf{k}) \sim [2b_0 \ln(|\mathbf{k}|/\Lambda_{QCD})]^{-1}. \tag{8.14}$$

9. Color-coulomb propagator from the Ward identity

The unrenormalized and renormalized effective actions both satisfy the same Gauss-BRST identity, eqs. (7.3) and (B.36). Therefore when we solve this identity for D^{A_0,A_0} the results hold both for renormalized and unrenormalized quantities, and we shall suppress the subscript r everywhere.

The Gauss-BRST identity holds term by term in a functional power series expansion in the fields. For the term which is linear in C and A_0, eq. (7.3) gives, after Fourier transform,

$$\int d\mathbf{k}\, dk_0\, dp_0 \,[\, \Gamma^{A_0A_0}(\mathbf{k}, k_0)A_0(\mathbf{k}, k_0)\,(ip_0)\, C(-\mathbf{k}, p_0)$$

$$+ \Gamma^{A_iA_0}(\mathbf{k}, k_0)A_0(\mathbf{k}, k_0)\Gamma^{K_iC}(-\mathbf{k}, p_0)\, C(-\mathbf{k}, p_0)\,]$$

$$= \int d\mathbf{k}\, dk_0\, dp_0\, (\, ik_0 + ip_0\,)\, \Gamma^{A_0A_0}(\mathbf{k}, k_0)A_0(\mathbf{k}, k_0)\, C(-\mathbf{k}, p_0),$$

or

$$\Gamma^{A_0A_0}(\mathbf{k}, k_0) = (\, ik_0\,)^{-1}\, \Gamma^{A_iA_0}(\mathbf{k}, k_0)\, \Gamma^{K_iC}(-\mathbf{k}, p_0),$$

where we have used (7.6). This identity requires that $\Gamma^{K_iC}(-\mathbf{k}, p_0)$ be independent of p_0, a property which may be verified diagrammatically by following the instantaneous ghost line that enters and leaves each diagram. We write $\Gamma^{K_iC}(-\mathbf{k})$ for $\Gamma^{K_iC}(-\mathbf{k}, p_0)$, and obtain

$$\Gamma^{A_0A_0}(\mathbf{k}, k_0) = (\, ik_0\,)^{-1}\, \Gamma^{A_iA_0}(\mathbf{k}, k_0)\, \Gamma^{K_iC}(-\mathbf{k}). \tag{9.1}$$

Apart from kinematic factors Γ^{AoAo} is expressed as the product $\Gamma^{AoAo} \sim \Gamma^{AiAo}\Gamma^{KiC}$. We simplify the last expression as follows. By rotational invariance we have

$$\Gamma^{K_iC}(\mathbf{k}) = i\, k_i\, f(\mathbf{k}), \tag{9.2}$$

where, by (7.4), $f(\mathbf{k}) = f(-\mathbf{k})$ is regular at $\mathbf{k} = 0$. Moreover, according to eq. (7.2), $\Gamma^{K_iC}(\mathbf{k})$ is related to $\Gamma^{C^*C}(\mathbf{k})$ by

$$\Gamma^{C^*C}(\mathbf{k}) = -\, ik_i\, \Gamma^{K_iC}(\mathbf{k}) = \mathbf{k}^2 f(\mathbf{k}),$$

$$\Gamma^{K_iC}(\mathbf{k}) = i\, k_i\, \Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2. \tag{9.3}$$

This gives

$$\Gamma^{AoAo}(\mathbf{k}, k_0) = [\, \Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2\,]\, (-k_0)^{-1}\, k_i\, \Gamma^{AiAo}(\mathbf{k}, k_0). \tag{9.4}$$

Thus, apart from kinematic factors $\Gamma^{AoAo}(\mathbf{k}, k_0)$ is expressed as the product

$$\Gamma^{AoAo}(\mathbf{k}, k_0) \sim \Gamma^{C^*C}(\mathbf{k})\, \Gamma^{AiAo}(\mathbf{k}, k_0)\,. \tag{9.5}$$

We return to the Gauss-BRST identity (7.3), and equate the terms that are linear in A_i and C. By repeating the above reasoning, we obtain

$$\Gamma^{AoAi}(\mathbf{k}, k_0) = [\, \Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2\,]\, (-k_0)^{-1}\, k_j\, \Gamma^{AjAi}(\mathbf{k}, k_0)\,. \tag{9.6}$$

and similarly, by equating terms that are linear in Π_i and C,

$$\Gamma^{Ao\Pi i}(\mathbf{k}, k_0) = [\, \Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2\,]\, (-\, k_0)^{-1}\, k_j\, \Gamma^{Aj\Pi i}(\mathbf{k}, k_0). \tag{9.7}$$

Finally, by combining (9.4) and (9.6) we obtain

$$\Gamma^{AoAo}(\mathbf{k}, k_0) = [\Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2]^2\, (k_0)^{-2}\, k_i\, k_j\, \Gamma^{AiAj}(\mathbf{k}, k_0)\,. \tag{9.8}$$

Thus each appearance of A_0 in a proper function gives a factor of $\Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2$. These relations are satisfied by the one-loop expressions of sect. 5.

The inverse, $D^{C^*C}(\mathbf{k}) = [\Gamma^{CC^*}(\mathbf{k})]^{-1}$, is the ghost or Faddeev-Popov propagator,

$$D^{C,C^*}(\mathbf{x}) \equiv (2\pi)^{-3} \int d^3k \exp(i\mathbf{k}\cdot\mathbf{x}) D^{C,C^*}(\mathbf{k}). \tag{9.9}$$

It is the expectation-value the Green's function $G(\mathbf{x}, \mathbf{y}; A)$ of the 3-dimensional Faddeev-Popov operator,

$$- \nabla_i D_i(A) G(\mathbf{x}, \mathbf{y}; A) = \delta(\mathbf{x} - \mathbf{y}), \tag{9.10}$$

namely,

$$\langle C(\mathbf{x}, x_0)\, C^*(\mathbf{y}, y_0) \rangle = \delta(x_0 - y_0)\, D^{C,C^*}(\mathbf{x} - \mathbf{y})$$

$$= \delta(x_0 - y_0) \langle G(\mathbf{x}, \mathbf{y}; A) \rangle. \tag{9.11}$$

The identities (9.6) - (9.8) imply a corresponding formula for $D^{AoAo}(\mathbf{k}, k_0)$, the time-time component of the gluon propagator. Because the propagator matrix mixes A_μ and Π_i, we must consider the inverse of the matrix $\Gamma^{\alpha,\beta}$, where α and β run over the field components of A_μ and Π_i. This matrix is reduced by taking longitudinal and transverse components. We write $A_i = A_i^{tr} - \nabla_i \sigma$, $\Pi_i = \Pi_i^{tr} - \nabla_i \Omega$, and we have

$$\Gamma^{\Omega,Ao}(\mathbf{k}, k_0) = ik_i\, \Gamma^{\Pi i,Ao} (\mathbf{k}, k_0),$$

$$\Gamma^{\Omega,\Omega}(\mathbf{k}, k_0) = k_i\, k_j\, \Gamma^{\Pi i,\Pi j}(\mathbf{k}, k_0)$$

$$\Gamma^{\sigma,Ao}(\mathbf{k}, k_0) = ik_i\, \Gamma^{Ai,Ao}(\mathbf{k}, k_0),$$

$$\Gamma^{\sigma,\sigma}(\mathbf{k}, k_0) = k_i\, k_j\, \Gamma^{Ai,Aj}(\mathbf{k}, k_0). \tag{9.12}$$

The propagator matrix $D^{\alpha,\beta}$ is the inverse of the matrix $\Gamma^{\alpha,\beta}$ that has the elements

$\Gamma^{\alpha,\beta}$	A_0	Ω	σ	λ
A_0	$\Gamma^{Ao,Ao}$	$\Gamma^{Ao,\Omega}$	$\Gamma^{Ao,\sigma}$	0
Ω	$\Gamma^{\Omega,Ao}$	$\Gamma^{\Omega,\Omega}$	$\Gamma^{\Omega,\sigma}$	0

σ	$\Gamma^{\sigma,Ao}$	$\Gamma^{\sigma,\Omega}$	$\Gamma^{\sigma,\sigma}$	$\mathbf{k}^2$
λ	0	0	$\mathbf{k}^2$	0

which gives

$$D^{Ao,Ao} = (-\Gamma^{\Omega,\Omega}) [\Gamma^{Ao,Ao} (-\Gamma^{\Omega,\Omega}) + (\Gamma^{Ao,\Omega})^2]^{-1}. \qquad (9.13)$$

(The quantity $(-\Gamma^{\Omega,\Omega})$ is positive in leading order.) In terms of these quantities, (9.7) and (9.8) read

$$\Gamma^{Ao,\Omega}(\mathbf{k}, k_0) = [\Gamma^{C^*,C}(\mathbf{k})/\mathbf{k}^2] \, i(k_0)^{-1} \, \Gamma^{\sigma,\Omega}(\mathbf{k}, k_0). \qquad (9.14)$$

$$\Gamma^{AoAo}(\mathbf{k}, k_0) = [\Gamma^{C^*C}(\mathbf{k})/\mathbf{k}^2]^2 \, (k_0)^{-2} \, \Gamma^{\sigma,\sigma}(\mathbf{k}, k_0) \, . \qquad (9.15)$$

and we obtain

$$D^{Ao,Ao}(\mathbf{k}, k_0) = [\mathbf{k}^2 D^{C,C^*}(\mathbf{k})]^2$$

$$\times \, (k_0)^2 (-\Gamma^{\Omega,\Omega}) [\Gamma^{\sigma,\sigma} (-\Gamma^{\Omega,\Omega}) + (\Gamma^{\sigma,\Omega})^2]^{-1}. \qquad (9.16)$$

We substitute this result into expressions (8.9) and (8.10) for the renormalization-group invariant running coupling constant, and obtain

$$g^2(\mathbf{k}/\Lambda_{QCD}) = \mathbf{k}^2 V(\mathbf{k}) = g_r^2 \, [\mathbf{k}^2 D_r^{C,C^*}(\mathbf{k})]^2 \, L(\mathbf{k}) \qquad (9.17a)$$

$$L(\mathbf{k}) \equiv \lim_{k_0\to\infty} \{ \mathbf{k}^2 (- \Gamma_r^{\Omega,\Omega}) (k_0)^2 [\Gamma_r^{\sigma,\sigma} (- \Gamma_r^{\Omega,\Omega}) + (\Gamma_r^{\sigma,\Omega})^2]^{-1} \}. \qquad (9.17b)$$

Here we have written the corresponding renormalized equations. Both $D^{Ao,Ao}(\mathbf{k}, k_0)$ and $g^2(\mathbf{k}/\Lambda_{QCD})$ are proportional to the square of the Faddeev-Popov propagator $[D_r^{C,C^*}(\mathbf{k})]^2$.

10. Exponentiation of the color-Coulomb potential†

We wish to evaluate the expectation-value $\langle I(gA_{qu}) \rangle$ of the Wilson loop

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† The exponentiation demonstrated in this section for instantaneous propagators was inspired by the corresponding result for planar diagrams, which was discovered by Martin Schaden. In both cases a ladder structure leads to exponentiation [7].

$$I(gA_{qu}) = \mathrm{tr}\{\ P[\exp(\int dx^{\mu} gA_{qu,\mu}{}^a t^a)]\ \}/\mathrm{tr}1, \qquad (10.1)$$

where g is the unrenormalized coupling constant, A_{qu} is the unrenormalized connection, P indicates path ordering, the t^a are a representation of the Lie algebra $[t^a, t^b] = f^{abc}t^c$, and the subscript 'qu' distinguishes the quantum (Euclidean) field A_{qu} from the external field, designated A, that is an argument of Γ.

To make use of the preceding results, we express the expectation-value in terms of the effective action,

$$\langle I(gA_{qu})\rangle = Z^{-1}\, I(g\delta/\delta J_A) Z|_{J=0} = I(g\delta W/\delta J_A + g\delta/\delta J_A)|_{J=0}$$

$$\langle I(gA_{qu})\rangle = I(gA_{x,\mu} + \int d^4y\ gD_{x,y}{}^{A\mu,\alpha}(\Phi)\delta/\delta\Phi_{y,\alpha})|_{\Phi=0}. \qquad (10.2)$$

In covariant gauges, it has been shown that the expectation-value (10.2) is finite [8], even though the arguments g and A_{qu} are unrenormalized. This happens because $\langle I(gA_{qu})\rangle$ is a renormalization-group invariant, and thus independent of the cut-off Λ. We shall not attempt such a demonstration in the Coulomb gauge. Instead we shall show that the contribution of the color-Coulomb potential exponentiates.

Call E the contribution to (10.2) that is the sum of all diagrams whose connected components consist entirely of gluon propagators. [Each exact gluon propagator $D^{A\mu,A\nu}(x, y)$ is attached to the external quark path at points x and y.] This is the contribution that is obtained from (10.2) when the dependence of $D(\Phi)$ on Φ is neglected,

$$E = I[\ gA_{x,\mu} + \int d^4y\ gD_{x-y}{}^{A\mu,A\nu}\delta/\delta A_{y,\nu}\ ]|_{A=0}$$

$$E = I[\ A_{x,\mu} + \int d^4y\ g^2 D_{x-y}{}^{A\mu,A\nu}\delta/\delta A_{y,\nu}\ ]|_{A=0}. \qquad (10.3)$$

This formula states that E results from expanding $I(A_\mu)$ in powers of A_μ, replacing products of A's by the sum all possible pairings of the A's, where each pairing is given the value $g^2 D_{x-y}{}^{A\mu,A\nu}$.

We restrict our consideration to a rectangular Wilson loop, of size T×R, that is aligned along the time axis. Call E_{inst} the contribution to E that results when all gluon propagators are replaced by their instantaneous parts. For a rectangular Wilson loop aligned along the time axis, the instantaneous part of $D_{x-y}{}^{A\mu,A\nu}$ contributes only when x and y lie on the vertical parts of the path, so also $\mu = \nu = 0$. We make the substitution

$$g^2 D_{x-y}{}^{A\mu,A\nu} \rightarrow \delta(x_0-y_0)V(\mathbf{x}-\mathbf{y})\delta_{\mu,0}\delta_{\nu,0} \, . \tag{10.4}$$

and obtain

$$E_{inst} = I[\, A_{x,\mu} + \int d^4y \; \delta(x_0-y_0)V(\mathbf{x}-\mathbf{y})\delta/\delta A_{y,o} \,]|_{A=0}. \tag{10.5}$$

Here $V(\mathbf{x})$ is the renormalization-group invariant color-Coulomb potential defined in (8.9).

The instantaneous parts have the ladder structure shown in Fig. 9. We expand the exponential (10.1) in powers of A. Only even powers contribute to (10.5). For a rectangular Wilson loop of dimension R×T, this gives loop integrals of the form

$$\int_0^T dx_0\, dy_0\, du_0\, dv_0 \ldots$$

$$\times \operatorname{tr} P[\delta(x_0-y_0)V(R)\; t_x{}^a\, t_y{}^a\; \delta(u_0-v_0)V(R)\; t_u{}^b\, t_v{}^b \ldots] \, , \tag{10.6}$$

where P represents path ordering around the loop, and the t's are ordered according to their space-time label. Suppose that $x_0 = y_0 \leq u_0 = v_0$ etc. The t^c's are placed in nested order by the path ordering, so contracted pairs are successively adjacent $\ldots t^b\, t^a\, t^a\, t^b \ldots = \ldots t^b\, (-C)\, t^b \ldots = \ldots C^2 \ldots$, etc., where $C > 0$ is a Casimir invariant. [It has the value $C = (2N)^{-1}(N^2-1)$ in the fundamental representation of SU(N).] Consequently the integral (10.6) has the value $2^{n/2}[-CTV(R)\,]^{n/2}$, where n is even. The factor $2^{n/2}$ appears because both x and y can be on either side of the loop, and similarly for u and v, etc. There are $n![(n/2)!(2!)^{n/2}]^{-1}$ pairings, with overall coefficient $1/n!$, and we obtain

$$E_{inst} = \exp[-\, CTV(R)]. \tag{10.7}$$

We conclude that the contribution of the renormalization-group invariant color-Coulomb potential exponentiates just as it does in Abelian gauge theory.

The Coulomb gauge is a "physical gauge", in the sense that intermediate states in the time direction are purely physical. Therefore, if the theory has a mass gap, then the diagrams with non-instantaneous contributions, which we have neglected in calculating E_{inst}, are all short-range, and E_{inst} is the sum of all the purely long-range diagrams that contribute to the Wilson loop. This raises the question whether E_{inst} is a good estimate of E.

In the presence of dynamical quarks, the Wilson loop does not obey an area law because a dynamical quark pair is created from the vacuum to form a pair of mesons, each consisting of one external quark and one dynamical quark. Such diagrams are not included among the ladder diagrams that contribute to E_{inst}. For similar reasons, in pure gluodynamics, the Wilson loop does not obey an area law if the external quarks are in the adjoint representation. In both of these cases, one should not expect E_{inst} to be a good estimate for E. However the breakdown of the vacuum cannot occur in pure gluodynamics, i. e. in the absence of dynamical quarks, when the Wilson loop is in the fundamental representation. We conjecture that in this case, $E_{inst} = \exp[-C_f V(R) T]$, with $C_f = (2N)^{-1}(N^2-1)$, does give the correct area law or, in other words, that in pure gluodynamics the string tension is correctly given by

$$K = \lim_{R\to\infty} C_f \, V(R)/R. \tag{10.8}$$

11. Color-Coulomb potential as an order parameter for confinement

If the color-Coulomb potential rises linearly, a string tension may be defined by

$$K_{coul} \equiv \lim_{R\to\infty} V(R)/R. \tag{11.1}$$

We would like to propose that a non-zero value of K_{coul} as the signal for color confinement. This has the virtue of simplicity because V(R) is a gluonic two-point function.

There is also considerable self-consistency to this proposal. Note first that V(R) depends only on the physical mass scale, as required (see sect. 8). Note also that if the Coulomb gauge were defined with the "time" axis aligned along the other side of the Wilson loop we would have obtained $E_{inst} = \exp[-CRV(T)]$. Thus only a linearly rising potential at large R is consistent with (Euclidean) Lorentz invariance under 90^o rotations. (This is related to Seiler's theorem that the exponent in the Wilson loop cannot grow faster than the area [9].)

As was noted at the end of the last section, $CK_{coul}R \neq 0$ does not correspond to the physical interaction energy of an external quark pair in the adjoint representation nor, probably, in the presence of dynamical quarks (This happens because a gluon or dynamical quark pair is created from the vacuum, which screens the color, thus preventing an area law.) This does not mean that $K_{coul} \neq 0$ is a bad order parameter for color confinement in these cases. On the contrary, color is confined in these cases. Moreover we may understand that it is energetically favorable for the dynamical quark pair to be created from the vacuum, thus preventing an area law, precisely because the color-Coulomb potential rises linearly at large distance. Thus K_{coul} may serve as an order parameter for color confinement precisely when the Wilson loop fails to do so.

12. Confinement and the Gribov problem

So far we have not discussed the Gribov ambiguity that affects the Coulomb gauge. In the regularized form of the theory provided by the Wilson lattice, this ambiguity may be resolved, in principle at least, by restricting the functional integral over A^{tr} to the fundamental modular region (FMR). As originally proposed by Gribov [3], it has been found [2, 10], that the restriction to the FMR leads precisely to a divergence of $\mathbf{k}^2 D^{C,C^*}(\mathbf{k})$ at $\mathbf{k} = 0$. Recall that we found in sect. 9 that $g^2(\mathbf{k}/\Lambda_{QCD}) = \mathbf{k}^2 V(\mathbf{k})$ contains $[\mathbf{k}^2 D^{C,C^*}(\mathbf{k})]^2$ as a factor.

An intuitive idea of why the restriction to the FMR leads to a singularity of $\mathbf{k}^2 D^{C,C^*}(\mathbf{k})$ at $\mathbf{k} = 0$ is as follows. The argument should properly be made in the context of the lattice regularization, as is done in [2], but we give here a continuum version. Gribov showed that at the nonperturbative level, the transversality condition $\nabla_i \cdot A_i = 0$ does not uniquely fix the gauge. A unique way to fix the gauge [11] is to choose as representative on each gauge orbit that configuration $\mathbf{A}$ which makes the Hilbert norm the *absolute* minimum with respect to all local gauge transformations g(x), so

$$||\mathbf{A}||^2 \equiv \int d^4x\, \mathbf{A}^2 \leq ||\mathbf{A}^g||^2 \text{ for all } g, \tag{12.1}$$

where $\mathbf{A}^g$ is the gauge transform of $\mathbf{A}(x)$ by $g(x)$. The fundamental modular region Λ is the set of these absolute minima,

$$\Lambda \equiv \{\mathbf{A}: ||\mathbf{A}||^2 \leq ||\mathbf{A}^g||^2 \text{ for all } g\}. \qquad (12.2)$$

Under fairly general conditions the minimum (12.1) exists [12], and is unique, apart from points on the boundary of Λ which must be identified topologically [13].

An absolute minimum is also a local minimum. To see what this implies, write $g(x) = \exp[\omega(x)]$ and expand in powers of ω. For every $\mathbf{A}$ in the fundamental modular region Λ, the inequality

$$||\mathbf{A}^g||^2 - ||\mathbf{A}||^2 = -2(\nabla_i A_i, \omega) + (\omega, -D_i(\mathbf{A})\nabla_i \omega) + \ldots \geq 0 \qquad (12.3)$$

holds for all ω. This implies that $\mathbf{A}$ is transverse, $\nabla_i A_i = 0$, so we get back the Coulomb gauge condition in the new, well-defined gauge which we call the "minimal Coulomb gauge". *In addition* we find that the Faddeev-Popov operator is positive, $M(\mathbf{A}) = -D_i(\mathbf{A})\nabla_i \geq 0$, for all $\mathbf{A}$ in Λ. Thus all (non-trivial) eigenvalues of $M(\mathbf{A})$ are positive in the interior of Λ, so $M^{-1}(\mathbf{A})$ is well-defined there. However the fundamental modular region Λ is bounded in all directions [11], and its boundary contains points where $M^{-1}(\mathbf{A})$ is singular [13]. When the functional integral defined in lattice gauge theory is restricted to Λ, then, in the limit of large lattice volume, it is found [2] that entropy favors these points sufficiently that $\mathbf{k}^2 D^{C,C^*}(\mathbf{k})$ diverges at $\mathbf{k} = 0$. (Recall that $D^{C,C^*}(\mathbf{k})$ is the fourier transform of $\langle [M^{-1}(\mathbf{A})]_{\mathbf{x},\mathbf{0}} \rangle$.) Indeed, the condition that $\mathbf{k}^2 D^{C,C^*}(\mathbf{k})$ diverge at $\mathbf{k} = 0$ is mathematically equivalent to so-called "the horizon condition" [2]. This condition assures that the functional integral be cut-off at the boundary of the fundamental modular region Λ. That the probability gets concentrated where $M^{-1}(\mathbf{A})$ is singular is also supported by numerical studies [14].

It has been argued [15] and confirmed in models [16], that Gribov copies come in pairs that give equal and opposite contributions for gauge-invariant quantities, so one need not in fact cut off the functional integral at the boundary of Λ at all (at the cost however of having a Euclidean weight that is not positive). This does not alter the conclusion that $\mathbf{k}^2 D^{C,C^*}(\mathbf{k})$ diverges at $\mathbf{k} = 0$ in the minimal Coulomb gauge. The results of the present article hold to all orders of perturbation theory, and it is expected that they hold also non-perturbatively in the minimal Coulomb gauge.

With the conclusion that $\mathbf{k}^2 D^{C,C*}(\mathbf{k})$ is singular at $\mathbf{k} = 0$, the elements of a confining theory are in place. We have argued that the color-Coulomb potential V(R) at large R, or its fourier transform $V(\mathbf{k})$ at low $\mathbf{k}$, provides an order parameter for confinement, and we have shown that $V(\mathbf{k})$ contains $[\mathbf{k}^2 D^{C,C*}(\mathbf{k})]^2$ as a factor. Confinement arises because this factor diverges at low $\mathbf{k}$ as a result of the restriction of the functional integral to the fundamental modular region.

This argument may be turned around. Suppose that QCD is a confining theory, as expected, and that V(R) provides an order parameter for confinement. Then its fourier transform $V(\mathbf{k})$ is singular at $\mathbf{k} = 0$. In the preceding section it was proven that $V(\mathbf{k})$ is of product form, $V(\mathbf{k}) = [\mathbf{k}^2 D^{C,C*}(\mathbf{k})]^2 L(\mathbf{k})$. It follows that at least one of its factors is singular, in agreement with the conclusion that the restriction to the FMR makes $\mathbf{k}^2 D^{C,C*}(\mathbf{k})$ singular.

13. Conclusion

A complete proof of renormalizability, which would require showing that all divergences are controlled, has not been attempted here. However we have controlled the obvious divergences in the first-order or phase-space formalism by the supplementary Feynman rules 1 and 2, and we have verified that the algebraic conditions for renormalization are satisfied. Essential to the renormalization program is the Gauss-BRST identity (6.23) or (7.3), which holds at a fixed time t, and which is an analog of Gauss's law in the BRST formalism. This identity has the consequence that $g^2 D^{Ao,Ao} = g_r^2 D_r^{Ao,Ao}$, is a renormalization-group invariant, as is the instantaneous part of the latter quantity, which we call the color-Coulomb potential V(R). The contribution of V(R) to the Wilson loop exponentiates. We have proposed the string tension $K_{coul} = \lim_{R\to\infty} CV(R)/R$ as an order parameter for color confinement, where C is a Casimir invariant. A remarkable consequence of the Gauss-BRST identity, is that the fourier transform $V(\mathbf{k})$ of V(R) has the product form $V(\mathbf{k}) = [\mathbf{k}^2 D^{C,C*}(\mathbf{k})]^2 L(\mathbf{k})$, where $D^{C,C*}(\mathbf{k})$ is the ghost propagator, and $L(\mathbf{k})$ is a correlation function of longitudinal gluons.

A correct treatment of the Gribov ambiguity requires that the functional integral be restricted to the fundamental modular region. The study of this restriction has shown that $\mathbf{k}^2 D^{C,C*}(\mathbf{k})$ is singular at $\mathbf{k} = 0$ [2, 10]. When this result is combined with the exact identity $V(\mathbf{k}) =$

$[\mathbf{k}^2 D^{C,C*}(\mathbf{k})]^2 L(\mathbf{k})$ derived in the present article, one obtains a long-range, color-Coulomb potential. Thus the previous results on the resolution of the Gribov ambiguity, and the present conclusions from the Gauss-BRST identity in the Coulomb gauge, together provide a rather complete scenario for understanding confinement in the Coulomb gauge. It is expected that this identity will allow a recent calculation of the quark-pair potential [17] to be improved.

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Appendix A: General form of divergences consistent with Ward identity

In Appendices A and B we shall demonstrate recursive renormalizability, assuming that the theory defined with the supplementary rules and dimensionally regularized is finite and obeys the Ward identities derived in sects. 6 and 7. We call this "algebraic renormalizability".

As a first step,
we shall determine the form of possible divergences. We first determine the form consistent with the Zinn-Justin equation, obtained by integrating (7.3) over all time,

$$\int d^Dx\, (\delta\Gamma^*/\delta A_\mu \cdot \delta\Gamma^*/\delta K_\mu + \delta\Gamma^*/\delta C \cdot \delta\Gamma^*/\delta L + \delta\Gamma^*/\delta\Pi_i \cdot \delta\Gamma^*/\delta M_i) = 0. \quad \text{(A.1)}$$

This condition is familiar and the method well known in covariant gauges [18]. We expand Γ^* in a loop expansion

$$\Gamma^* = \Sigma_{n=0}^{\infty}\, \Gamma^{*(n)}, \quad \text{(A.2)}$$

where

$$\Gamma^{*(0)} = \Sigma^* \quad \text{(A.3)}$$

$$\Sigma^* = \int d^Dx\, [\, (1/2)\,(E^2 + B^2) - (1/2)(\Pi\ - E)^2$$
$$+ K_\mu \cdot D_\mu C + M_i \cdot (g\Pi_i \times C) + L \cdot (-g/2) C \times C. \quad \text{(A.4)}$$

by (6.4) and (7.2). The n'th order condition is satisfied separately by the finite and divergent parts. We assume that (A.1) is satisfied exactly for the divergent parts by the n-th order expression for Γ^*, as will be assured by conditions (B.26) and (B.27) below. We obtain for the (n+1)-th order divergent part of Γ^* the linear equation

$$\sigma\Gamma_{div}^{(n+1)} = 0, \tag{A.5}$$

where the operator σ, defined by

$$\sigma \equiv \int d^4x\, (\delta\Sigma^*/\delta K_\mu \cdot \delta/\delta A_\mu + \delta\Sigma^*/\delta A_\mu \cdot \delta/\delta K_\mu$$

$$+ \delta\Sigma^*/\delta L \cdot \delta/\delta C + \delta\Gamma^*/\delta C \cdot \delta/\delta L$$

$$+ \delta\Sigma^*/\delta M_i \cdot \delta/\delta\Pi_i + \delta\Gamma^*/\delta\Pi_i \cdot \delta/\delta M_i) , \tag{A.6}$$

is nil-potent $\sigma^2 = 0$. It acts on the fundamental fields just like s, namely $\sigma A_\mu = D_\mu C$, $\sigma C = (-g/2)C\times C$, and $\sigma\Pi_i = g\Pi_i\times C$, so $\sigma E_i = gE_i\times C$.

The cohomological problem (A.5) has the well known solution

$$\Gamma_{div}^{(n+1)} = R_{inv}(A) + \sigma Q, \tag{A.7}$$

where $R_{inv}(A)$ is a gauge-invariant function of A. The divergent terms are assumed to be local in the fields. Making use of locality, conservation of ghost number, rotational invariance and dimensional analysis, we obtain

$$\Gamma_{div}^{(n+1)} = \int d^4x\, [\, c_1E^2 + c_2\, B^2 + \sigma(c_3K_i\cdot A_i + c_4K_0\cdot A_0 + c_5\Lambda\cdot C$$

$$+ c_6M_i\cdot\Pi_i + c_7M_i\cdot E_i + c_8M_i\cdot\nabla_iA_0 + c_9M_i\cdot\partial_0A_i)\,], \tag{A.8}$$

$$\equiv \Sigma_{a=1}^{9}\, c_a\, S_a\, . \tag{A.9}$$

where the c_a's are divergent constants. We have not assumed Lorentz (Euclidean) invariance in determining the possible independent constants.

We assume that the divergent parts coming from lower order terms also satisfy the Ward identity (7.3) exactly, as is assured by (B.26) and (B.27), below, so that $\Gamma_{div}^{(n+1)}$ satisfies the linear equation

$$f_t\, (\Gamma_{div}^{(n+1)}) = 0, \tag{A.10}$$

where the operator f_t, defined by

$$f_t \equiv \int d^3x \, [\, (\, \delta\Sigma^*/\delta K_\mu \cdot \delta/\delta A_\mu + \delta\Sigma^*/\delta A_\mu \cdot \delta/\delta K_\mu + \delta\Sigma^*/\delta L \cdot \delta/\delta C$$

$$+ \delta\Sigma^*/\delta C \cdot \delta/\delta L + \delta\Sigma^*/\delta M_i \cdot \delta/\delta \Pi_i + \delta\Sigma^*/\delta \Pi_i \cdot \delta/\delta M_i \,)$$

$$- \partial_0 \; (\, C \cdot \delta/\delta A_0 - K_0 \cdot \delta/\delta L \,) \,], \qquad \text{(A.11)}$$

is a linearized version of F_t, eq (7.3). If one substitutes (A.8) into this equation, one obtains

$$c_5 = - c_4; \qquad c_8 = c_9 = 0, \text{(A.12)}$$

whereas c_1, c_2, c_3, c_6, and c_7 are arbitrary. Explicit calculation gives

$$S_4 - S_5 = \int d^4x \, [\, A_0 \cdot (\, D_i \Pi_i - g K_0 \times C \,) + K_i \cdot D_i C$$

$$+ M_i \cdot (g \Pi_i \times C) - (1/2) L \cdot g C \times C) \,]. \qquad \text{(A.13)}$$

Because this expression is not of the form (7.4), which asserts that the loop corrections to Γ^* depend on C only through ∇C, we obtain

$$c_4 = c_5 = 0. \qquad \text{(A.14)}$$

We conclude that $\Gamma_{div}^{(n+1)}$ is of the form

$$\Gamma_{div}^{(n+1)} = \int d^4x \, [\, c_1 E^2 + c_2 \, B^2 + \sigma (c_3 K_i \cdot A_i + c_6 M_i \cdot \Pi_i + c_7 M_i \cdot E_i) \,], \qquad \text{(A.15)}$$

where the coefficients are arbitrary. The vanishing of c_4, the coefficient of $\sigma K_0 \cdot A_0$, will lead to the important result that gA_0 is a renormalization-group invariant.

The quantities with non-zero coefficients are

$$S_3 = \sigma \int d^4x \, K_i \cdot A_i = \int d^4x \, (\, F_{ij} \cdot D_i A_j + \Pi_i \cdot D_0 A_i - K_i \cdot \nabla_i C \,) \qquad \text{(A.16)}$$

$$S_6 = \sigma \int d^4x \, M_i \cdot \Pi_i = \int d^4x \, (\, - \Pi^2 + \Pi_i \cdot E_i \,) \qquad \text{(A.17)}$$

$$S_7 = \sigma \int d^4x \;\; M_i{\cdot}E_i \;\; = \int d^4x \; (- \Pi_i{\cdot}E_i + E^2) \, . \qquad (A.18)$$

One may compare the quadratic part of this general expression for $\Gamma_{div}{}^{(n)}$ with the explicit expressions for $\Gamma_{div}{}^{(1)}$ of sec. 5, with $K_i \leftrightarrow \nabla_i C^*$, in accordance with (7.2). This determines all the coefficients (c_3 is in fact overdetermined), with the result

$$c_1 = c_2 = (-11/6) \; \lambda_0/\varepsilon \qquad (A.19)$$

$$c_3 = (4/3) \; \lambda_0/\varepsilon \qquad (A.20)$$

$$c_6 = (2/3) \; \lambda_0/\varepsilon \qquad (A.21)$$

$$c_7 = 2 \; \lambda_0/\varepsilon \, , \qquad (A.22)$$

where $\lambda_0 = (8\pi^2)^{-1}Ng^2$, and $\varepsilon = 4\text{-}D$. We have

$$\Gamma_{div}{}^{(1)} = \lambda_0/\varepsilon \int d^4x \; \{ \; (-11/6) \; (E^2 + B^2)$$

$$+ \; \sigma \; [\; (4/3) \; K_i{\cdot}A_i + (2/3) \; M_i{\cdot}\Pi_i + 2 \; M_i{\cdot}E_i) \;] \; \} . \qquad (A.23)$$

This gives the complete one-loop divergence structure. The Ward identity determines the cubic and quartic terms from the quadratic terms.

As a bonus, we observe that the term which is not σ-exact, is Lorentz-invariant: the coefficients of E^2 and B^2 are equal $c_1 = c_2$. One expects that holds in every order, for otherwise the expectation-value of a Wilson-loop would not be Lorentz invariant, as it must be, as we have seen in sec. 3.

Appendix B: Recursive proof of renormalizability

We wish to show that the divergent expression (A.15) with

$$c_1 = c_2 \qquad (B.1)$$

namely

$$\Gamma_{div}{}^{(n)} = \int d^4x \; [\; c_1(E^2 + B^2) + \sigma(c_3K_i{\cdot}A_i + c_6M_i{\cdot}\Pi_i + c_7M_i{\cdot}E_i) \;], \qquad (B.2)$$

can be canceled by a renormalization of the quantities that appear in $\Gamma^{(0)} = \Sigma^*$, given in eq. (A.4). For this purpose we must express $\Gamma_{div}{}^{(n)}$ as a derivative of Σ^*.

Let H be the Euler differential operator

$$H \equiv -g\partial/\partial g + \int d^4x\, A_\mu \cdot \delta/\delta A_\mu \,. \qquad \text{(B.3)}$$

By simple power counting we have

$$H\Sigma^* = \int d^4x\, [\, (E^2 + B^2) + (\Pi_i - E_i)\cdot E_i - M_i\cdot(g\Pi_i \times C) - L\cdot(-g/2)C\times C\,]$$

$$= \int d^4x\, [\, (E^2 + B^2) - \sigma\, M_i\cdot E_i - M_i\cdot\delta\Sigma^*/\delta M_i - L\cdot\delta\Sigma^*/\delta L\,], \qquad \text{(B.4)}$$

by (A.18), which gives

$$\int d^4x\, (E^2 + B^2) = [H + \int d^4x\, (M_i\cdot\delta/\delta M_i + L\cdot\delta/\delta L)]\, \Sigma^* + \int d^4x\, \sigma\, M_i\cdot E_i. \qquad \text{(B.5)}$$

This allows us to write

$$\Gamma_{div}{}^{(n)} = \Gamma_1 + \Gamma_2 + \Gamma_3 \qquad \text{(B.6)}$$

where

$$\Gamma_1 \equiv c_1\, [\, -g\partial/\partial g + \int d^4x\, A_\mu\cdot\delta/\delta A_\mu + M_i\cdot\delta/\delta M_i + L\cdot\delta/\delta L\,)\,]\, \Sigma^* \qquad \text{(B.7)}$$

and

$$\Gamma_2 \equiv \sigma \int d^4x\, (c_3 K_i\cdot A_i + c_6 M_i\cdot\Pi_i\,),$$

$$\Gamma_2 = \int d^4x\, [\, c_3\, (A_i\cdot\delta\Sigma^*/\delta A_i - K_i\cdot\delta\Sigma^*/\delta K_i) + c_6(\Pi_i\cdot\delta\Sigma^*/\delta\Pi_i - M_i\cdot\delta\Sigma^*/\delta M_i)\,], \qquad \text{(B.8)}$$

are of the desired form, and

$$\Gamma_3 \equiv (c_1 + c_7)\, \sigma \int d^4x\, M_i \cdot E_i \,. \qquad \text{(B.9)}$$

There remains to cast Γ_3 into the desired form. This is a novel term that leads to mixing of the Π and E fields under renormalization. We have

$$\sigma\, (E_i \cdot M_i) = g(E_i \times C) \cdot M_i + E_i \cdot \delta\Sigma^*/\delta\Pi_i \,.$$

To proceed, we use the equation of motion of the Π field,

$$E_i = \delta\Sigma^*/\delta\Pi_i + \Pi_i + gM_i \times C, \qquad \text{(B.10)}$$

and obtain

$$\sigma\, (E_i \cdot M_i) = g(M_i \times C) \cdot (\delta\Sigma^*/\delta\Pi_i + \Pi_i + gM_i \times C) + E_i \cdot \delta\Sigma^*/\delta\Pi_i \,.$$

$$\sigma\, (E_i \cdot M_i) = [\, (E_i + gM_i \times C) \cdot \delta/\delta\Pi_i - M_i \cdot \delta/\delta M_i + g(M_i \times M_i) \cdot \delta/\delta L \,]\, \Sigma^*. \qquad \text{(B.11)}$$

where we have used $C \times (M_i \times C) = (-1/2) M_i \times (C \times C)$. This gives

$$\Gamma_{div}^{(n)} = - c_1\, g\partial\Sigma^*/\partial g + \int d^4x\, \{\, c_1 A_0 \cdot \delta/\delta A_0 + (c_1 + c_3) A_i \cdot \delta/\delta A_i$$
$$+ [c_6 \Pi_i + (c_1 + c_7)(E_i + gM_i \times C)] \cdot \delta/\delta\Pi_i - c_3 K_i \cdot \delta/\delta K_i$$
$$- (\, c_6 + c_7)\, M_i \cdot \delta/\delta M_i + [c_1 L + (c_1 + c_7) g(M_i \times M_i)] \cdot \delta/\delta L \,\}\Sigma^*$$

.

(B.12)

This shows that $\Gamma_{div}^{(n)}$ may be canceled by a renormalization of quantities in the zero-order action Σ^*.

Observe that A_0 and g renormalize oppositely. This will lead to the notable result that

$$gA_0 = g_r A_{r,0}, \qquad \text{(B.13)}$$

is a renormalization-group invariant.

Ghost number conservation is expressed by the identity

$$[\, C\cdot\delta/\delta C - K_\mu\cdot\delta/\delta K_\mu - M_i\cdot\delta/\delta M_i - 2L\cdot\delta/\delta L \,]\, \Sigma^* = 0, \qquad \text{(B.14)}$$

which may be added to the right-hand side of (B.12) with an arbitrary coefficient. A convenient choice is c_1, giving

$$\Gamma_{div}{}^{(n)} = - c_1\, g\partial\Sigma^*/\partial g + \int d^4x\, \{\, c_1(A_0\cdot\delta/\delta A_0 + C\cdot\delta/\delta C - K_0\cdot\delta/\delta K_0)$$

$$+ (c_1+c_3)\,(\, A_i\cdot\delta/\delta A_i - K_i\cdot\delta/\delta K_i) + [c_6\Pi_i + (c_1+c_7)(E_i + gM_i\times C)]\cdot\delta/\delta\Pi_i$$

$$- (c_1+c_6+c_7)\, M_i\cdot\delta/\delta M_i + [- c_1L + (c_1+c_7)g(M_i\times M_i)]\cdot\delta/\delta L\, \}\Sigma^* \qquad \text{(B.15)}$$

so

$$gC = g_rC_r, \qquad \text{(B.16)}$$

will also be a renormalization-group invariant. There are four independent renormalization constants, c_1, c_3, c_6 and c_7, whose value is not determined by symmetries.

Cancellation of the divergences by renormalization requires an exquisitely Baroque mixing of fields. We introduce renormalized quantities according to the following scheme

$$g = Z_g g_r \qquad \text{(B.17)}$$

$$A_0 = Z_{A_0}\, A_{r,0}$$

$$A_i = Z_{\mathbf{A}}\, A_{r,i}$$

$$C = Z_C C_r$$

$$C^* = Z_{C^*}\, C^*{}_r,$$

$$K_i = Z_{\mathbf{K}} K_{r,i}$$

$$K_0 = Z_{K_0} K_{r,0}$$

$$M_i = Z_M M_{r,i} \qquad \text{(B.18)}$$

$$\Pi_i = Z_\Pi \Pi_{r,i}$$

$$+ Z_\Pi' (Z_A \partial_0 A_{r,i} - Z_{Ao} \nabla_i A_{r,0} + Z_A g_r A_{r,0}\times A_{r,i} + Z_M g_r M_{r,i}\times C_r)$$

$$L = Z_L L_r + Z_L' Z_M^2 Z_g g_r(M_{r,i}\times M_{r,i}). \qquad (B.19)$$

The Π_i and L fields mix under renormalization with composite fields made of elementary fields and of the source M_i, and with derivatives of the elementary fields. The mixing equations may be written more simply

$$\Pi_i = Z_\Pi \Pi_{r,i} + Z_\Pi' (E_i + g M_i\times C)$$

$$L = Z_L L_r + Z_L' g(M_i\times M_i). \qquad (B.20)$$

The four independent renormalization constants may be chosen to be Z_{Ao}, Z_A, Z_Π, and Z_Π'. With reference to (B.15), one sees that if they are defined recursively by

$$Z_{Ao}^{(n+1)} = Z_{Ao}^{(n)} - c_1$$

$$Z_A^{(n+1)} = Z_A^{(n)} - c_1 - c_3$$

$$Z_\Pi^{(n+1)} = Z_\Pi^{(n)} - c_6$$

$$Z_\Pi'^{(n+1)} = Z_\Pi'^{(n)} - c_1 - c_7, \qquad (B.21)$$

and if the remaining normalization constants satisfy

$$Z_g^{(n+1)} = Z_g^{(n)} + c_1 + O(n+2)$$

$$Z_C^{(n+1)} = = Z_C^{(n)} - c_1 + O(n+2)$$

$$Z_K^{(n+1)} = Z_{C*}^{(n)} = Z_{C*}^{(n)} + c_1 + c_3 + O(n+2)$$

$$Z_{Ko}^{(n+1)} = Z_{Ko}^{(n)} + c_1 + O(n+2)$$

$$Z_L^{(n+1)} = Z_L^{(n)} + c_1 + O(n+2)$$

$$Z_L'^{(n+1)} = Z_L'^{(n)} - c_1 - c_7 + O(n+2)$$

$$Z_M^{(n+1)} = Z_M^{(n)} + c_1 + c_6 + c_7 + O(n+2), \qquad (B.22)$$

then

$$\Sigma^*(\Psi_r^{(n+1)}, g_r^{(n+1)}) = \Sigma^*(\Psi_r^{(n)}, g_r^{(n)}) - \Gamma_{div}^{(n+1)}(\Psi_r^{(n)}, g_r^{(n)})$$

$$+ O(n+2) \qquad \text{(B.23)}$$

provides the counter terms which cancel the divergences in order n+1. Here we have written relations (B.17) - (B.19) in order n as

$$g^{(n)} = Z^{(n)} g_r \qquad \text{(B.24)}$$

$$\Psi_\alpha^{(n)} = \Psi_\alpha(Z^{(n)}, \Psi_r, g_r), \qquad \text{(B.25)}$$

and α is an index that runs over all components of all fields and sources, $\Psi_\alpha = (A_0, A_i, C, \Pi_i, K_0, K_i, L, M_i,)$.

For the recursive proof we impose the conditions

$$1 = Z_g Z_{Ao} = Z_g Z_C = Z_{\mathbf{A}} Z_{C*} = Z_{\mathbf{A}} Z_{\mathbf{K}} = Z_{Ao} Z_{Ko} = Z_C Z_L \qquad \text{(B.26)}$$

$$Z_\Pi Z_M = 1 - Z_\Pi', \qquad Z_\Pi' = Z_\Pi Z_M Z_L' \qquad \text{(B.27)}$$

which are obviously consistent with (B.22). We have set

$$Z_{C*} = Z_{\mathbf{K}} \qquad \text{(B.28)}$$

to be consistent with (7.2). Conditions (B.26) and (B.27) are determined by requiring that, in each order n, the unrenormalized action regarded as a function of the renormalized fields,

$$\Sigma_r^{*(n)}(\Psi_r, g_r) \equiv \Sigma^*(\Psi^{(n)}, g^{(n)}), \text{(B.29)}$$

satisfy the Ward identity (7.3) *exactly*

$$F_t(\Sigma_r^{*(n)}) = 0. \qquad \text{(B.30)}$$

To prove that (B.30) is satisfied when (B.26) and (B.27) hold, one writes

$$F_t(\Sigma_r^*) = \int d^3x \, [\, (\, \delta\Sigma_r^*/\delta A_{r,\mu} \cdot \delta\Sigma_r^*/\delta K_{r,\mu} + \delta\Sigma_r^*/\delta C_r \cdot \delta\Sigma_r^*/\delta L_r$$

$$+ \delta\Sigma_r^*/\delta\Pi_i \cdot \Sigma_r^*/\delta M_{r,i}) - \partial_0 (C \cdot \delta\Sigma_r^*/\delta A_{r,0} - K_{r,0} \cdot \delta\Sigma_r^*/\delta L_r)] , \qquad \text{(B.31)}$$

and uses

$$\delta/\delta A_{r,0} = Z_{A_0} [\delta/\delta A_0 + Z_\Pi' D_i(g\mathbf{A}) \delta/\delta\Pi_i]$$

$$\delta/\delta A_{r,i} = Z_{\mathbf{A}} [\delta/\delta A_i - Z_\Pi' D_i(gA_0) \delta/\delta\Pi_i]$$

$$\delta/\delta C_r = Z_C [\delta/\delta C + Z_\Pi' g\, M_i \times \delta/\delta\Pi_i]$$

$$\delta/\delta M_{r,i} = Z_M [\delta/\delta M_i + 2\, Z_L' g\, M_i \times \delta/\delta L + Z_\Pi' g C \times \delta/\delta\Pi_i] , \qquad \text{(B.32)}$$

as follows from the change of variables (B.18) and (B.19). The other partial derivatives renormalize multiplicative.

The recursive renormalization proceeds as follows. It is supposed that there is a set of $Z^{(n)}$'s such that (B.30) is satisfied exactly and such that the renormalized action given by

$$\Gamma_r(\Psi_r) \equiv \Gamma(\Psi) = \Sigma_{m=0}{}^{n}\, \Gamma^{(m)}(\Psi^{(n)}, g^{(n)}), \qquad \text{(B.33)}$$

is finite to loop-order n, and satisfies the Ward identity to loop-order n,

$$F_t(\Gamma_r) = O(n+1). \qquad \text{(B.34)}$$

Then the Ward identity $F_t(\Sigma_r^{*(n+1)}) = 0$ is satisfied *exactly* by $\Sigma_r^{*(n+1)}$. Moreover

$$\Gamma_r(\Psi_r) \equiv \Gamma(\Psi) = \Sigma_{m=0}{}^{n+1}\, \Gamma^{(m)}(\Psi^{(n+1)}, g^{(n+1)}) \qquad \text{(B.35)}$$

is finite to order n+1, because conditions (B.26) and (B.27) are consistent with (B.21) and (B.22). Consequently the renormalized Ward identity (B.30) is satisfied to order n+1. This completes the recursive proof of renormalizability.

We have also proven that the finite renormalized effective action $\Gamma_r(\Psi_r)$ satisfies the Ward identity

$$F_t(\Gamma_r) = 0. \qquad \text{(B.36)}$$

To one-loop order, we have, by (A.19) to (A.22),

$Z_g^{(1)} = 1 - (11/6)\, \lambda_0/\varepsilon$

$Z_{A_0}^{(1)} = Z_C^{(1)} = 1 + (11/6)\, \lambda_0/\varepsilon$

$Z_A^{(1)} = 1 + (1/2)\lambda_0/\varepsilon$

$Z_\Pi^{(1)} = 1 - (2/3)\, \lambda_0/\varepsilon$

$Z_\Pi'^{(1)} = (-1/6)\, \lambda_0/\varepsilon$

$Z_{C^*}^{(1)} = Z_K^{(1)} = 1 - (1/2)\, \lambda_0/\varepsilon$

$Z_L^{(1)} = 1 - (11/6)\, \lambda_0/\varepsilon$

$Z_L'^{(1)} = -\, (1/6)\, \lambda_0/\varepsilon$

$Z_M^{(1)} = 1 + (5/6)\, \lambda_0/\varepsilon\ ,$ (B.37)

where $\lambda_0 = (8\pi^2)^{-1}\, Ng^2$, and $\varepsilon = 4\text{-}D$. The coupling renormalization constant Z_g is the same as in covariant gauges.

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Figure Captions

Fig. 1. (a) Elementary propagators (b) Elementary vertices
Fig. 2. Diagrams that contribute to $\Gamma^{Ai,Aj}$
Fig. 3. Diagrams that contribute to $\Gamma^{Ai,Ao}$
Fig. 4. Diagrams that contribute to $\Gamma^{Ao,Ao}$
Fig. 5. Diagram that contributes to $\Gamma^{\Pi i,Aj}$
Fig. 6. Diagram that contributes to $\Gamma^{\Pi i,Ao}$
Fig. 7. Diagram that contributes to $\Gamma^{\Pi i,Aj}$
Fig. 8. Diagram that contributes to $\Gamma^{C^*,C}$
Fig. 9. Graph consisting of instantaneous gluon propagators

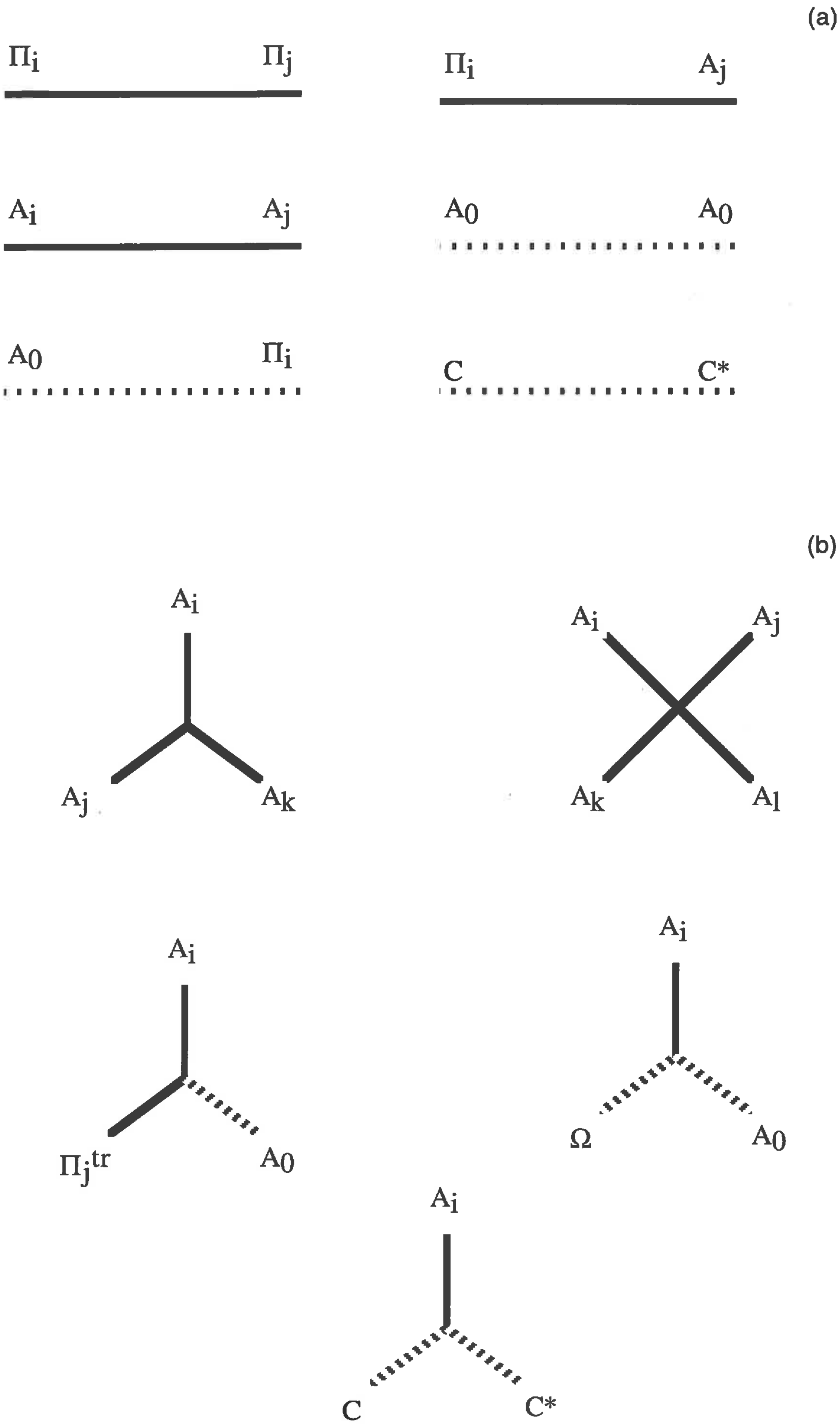
(a)
Π_i Π_j
Π_i A_j
A_i A_j
A_0 A_0
A_0 Π_i
C C*
(b)
A_i A_j A_k
A_i A_j A_k A_l
A_i Π_j^{tr} A_0
A_i Ω A_0
A_i C C*

Fig. 1

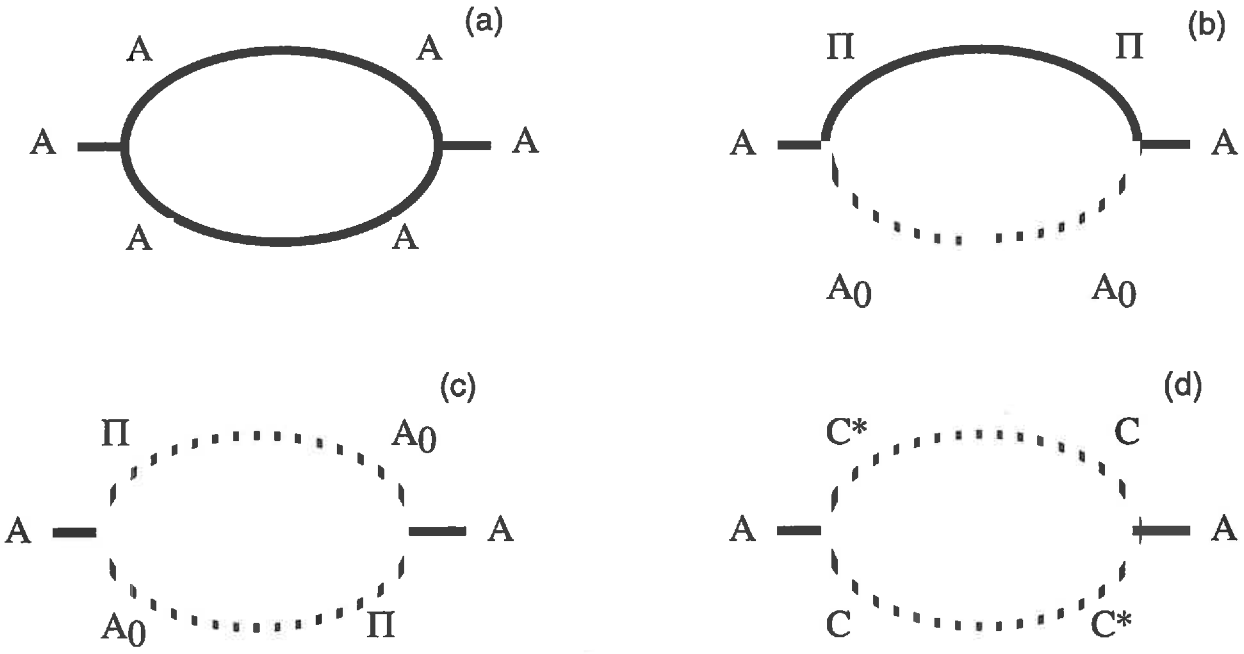

Fig. 2.

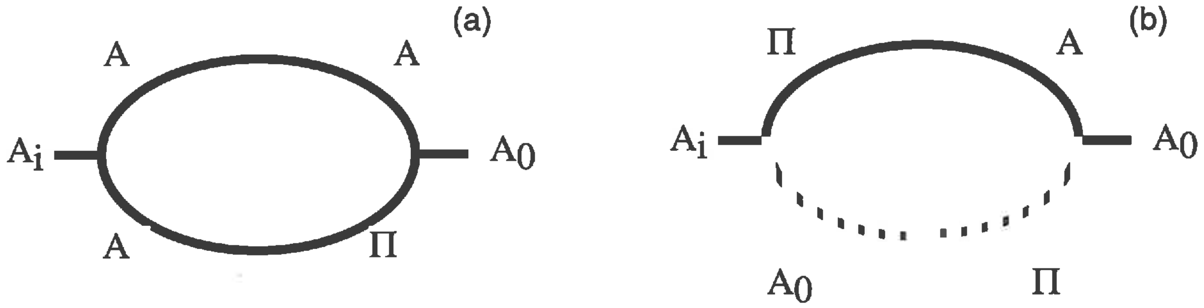

Fig. 3.

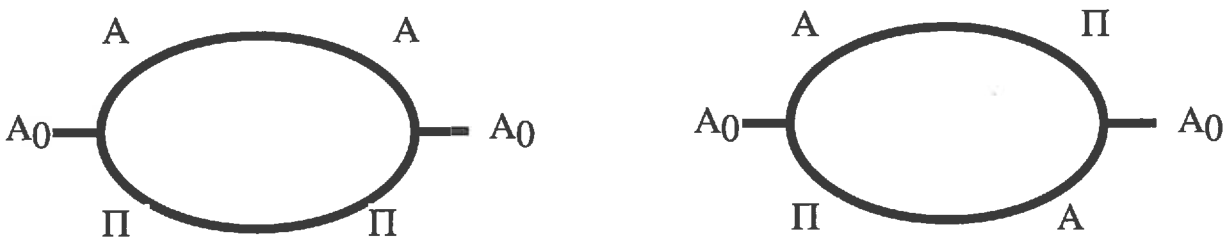

Fig. 4.

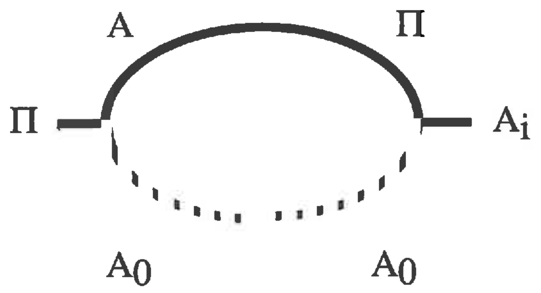

Fig. 5.

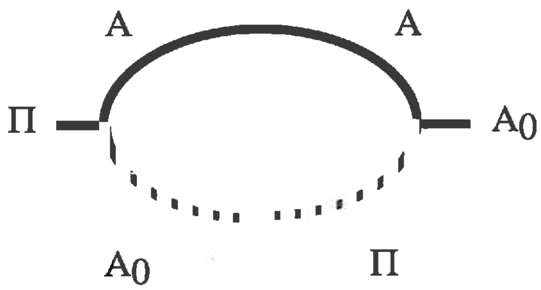

Fig. 6

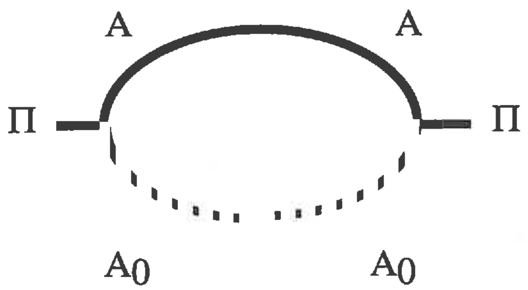

Fig. 7.

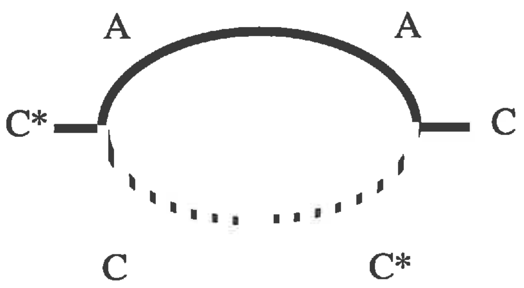

Fig. 8.

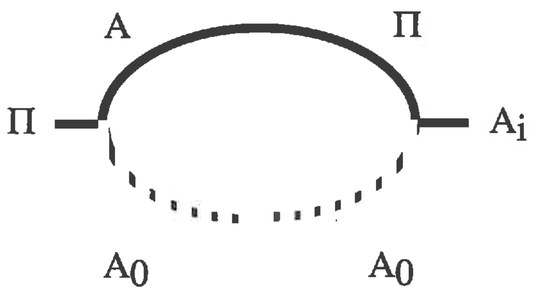

Fig. 5.

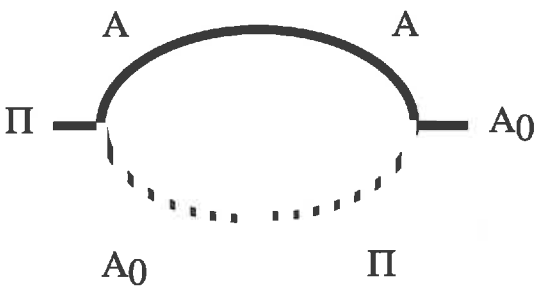

Fig. 6

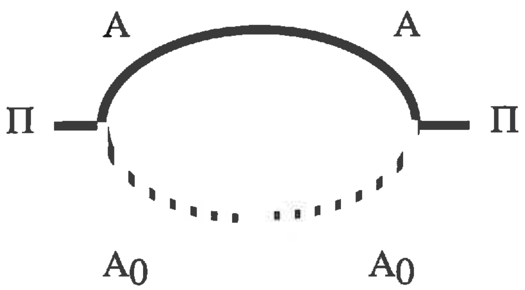

Fig. 7.

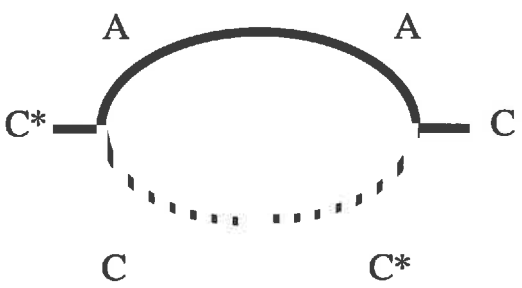

Fig. 8.

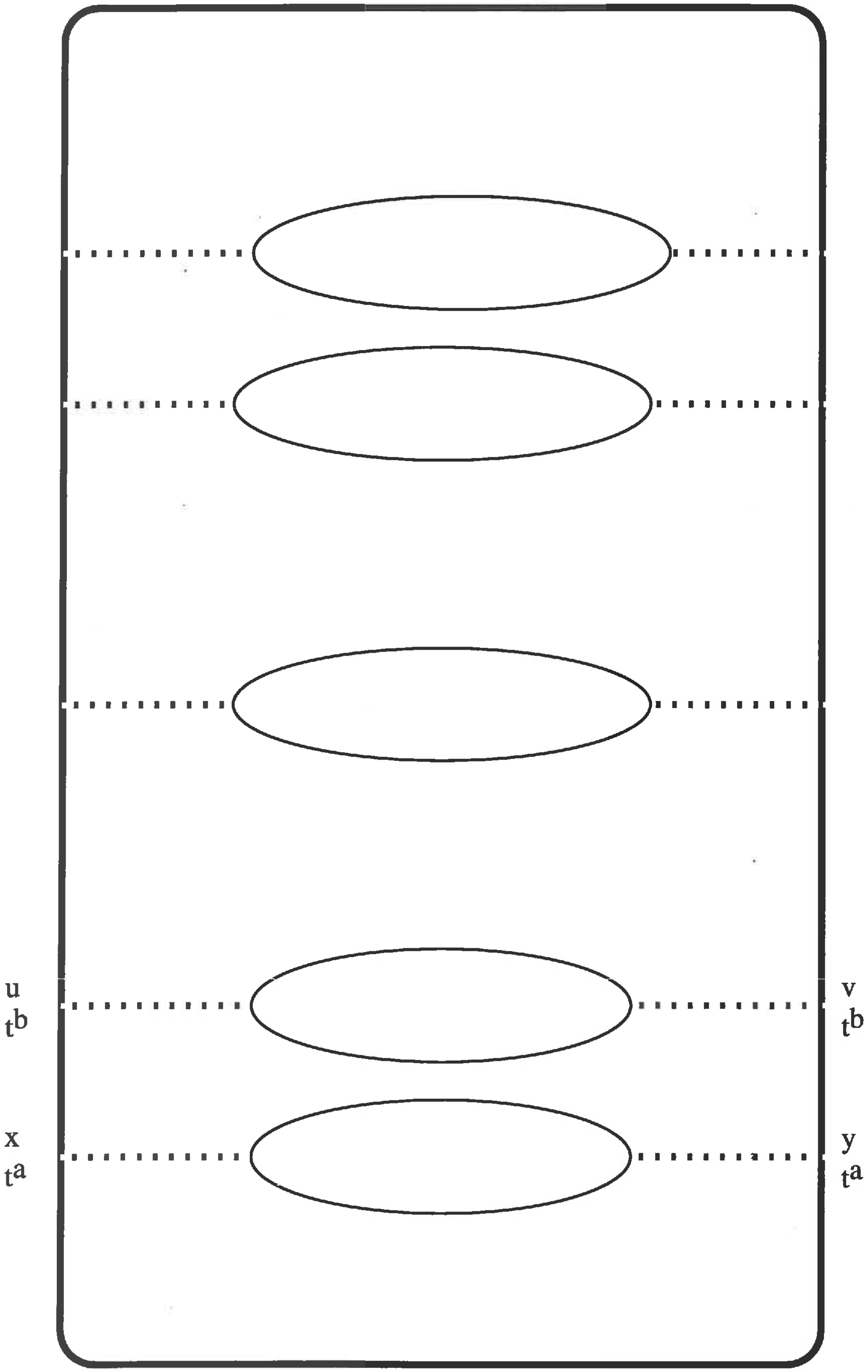

Fig. 9