

IRREDUCIBILITY OF ALTERNATING AND SYMMETRIC SQUARES

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We investigate the question when the alternating or symmetric square of an absolutely irreducible projective representation of a non-abelian simple group G is again irreducible. The knowledge of such representations is of importance in the description of the maximal subgroups of simple classical groups of Lie type. We obtain complete results for G an alternating group and for G a projective special linear group when the given representation is in non-defining characteristic. For the proof we exhibit a linear composition factor in the socle of the restriction to a large subgroup of the alternating or symmetric square of a given projective representation V . Assuming irreducibility this shows that the dimension of V has to be very small. A good knowledge of projective representations of small dimension allows to rule out these cases as well.

1. INTRODUCTION

Let $R = R(\ell^f)$ be a finite classical group of Lie type. Let $G < R$ be a quasi-simple subgroup acting absolutely irreducibly on the natural module of R , not of Lie type in characteristic ℓ . We study those cases where G has the same number of composition factors on the adjoint module for R as R itself. These embeddings are of importance in the determination of maximal subgroups of the finite classical groups of Lie type.

Let V be the natural module for a classical group R . We will write $\tilde{\Lambda}^2(V)$, $\tilde{\Sigma}^2(V)$ respectively $\tilde{A}(V)$ for the largest irreducible R -sub-quotient of $\Lambda^2(V)$, $\text{Sym}^2(V)$, $V \otimes V^*$. In Table 1.1 we recall the dimension of $\tilde{X}(V)$ for certain choices (R, X) with $X \in \{\Lambda, \Sigma^2, A\}$ (see [3], resp. [1, (8.9), (9.6) and (11.6)]).

In this paper we study quasi-simple subgroups G of classical groups R which act irreducibly on V as well as on $\tilde{X}(V)$ with X as in Table 1.1. We say that V is of plus-type (for G) if it carries a G -invariant quadratic form, and that V is of minus-type if it carries a bilinear alternating form but no quadratic form. Thus if V is of plus-type for G , we have to consider both $\tilde{\Lambda}^2(V)$ and $\tilde{\Sigma}^2(V)$, if V is of minus-type, we consider $\tilde{\Sigma}^2(V)$, and $\tilde{\Lambda}^2(V)$ if $\ell = 2$, and if V carries no G -invariant non-degenerate form, we have to consider $\tilde{A}(V)$. Previous results in this direction were obtained by the first author for special linear groups [8], and by the second author for arbitrary quasi-simple groups in the case $\ell = 0$ and $X = A$ [9]. We hope to pursue this investigation for other types of quasi-simple groups G in the future.

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1.1. Choices for R and X .

R	\tilde{X}	$\dim(\tilde{X}(V))$	condition
SL_m	\tilde{A}	$m^2 - 1$	$\ell \nmid m$
	\tilde{A}	$m^2 - 2$	$\ell \mid m$
Sp_m	$\tilde{\Sigma}^2$	$\frac{1}{2}m(m+1)$	ℓ odd
	$\tilde{\Sigma}^2$	$\frac{1}{2}m(m-1) - 1$	$\ell = 2, m \equiv 2 \pmod{4}$
	$\tilde{\Sigma}^2$	$\frac{1}{2}m(m-1) - 2$	$\ell = 2, m \equiv 0 \pmod{4}$
SO_m	$\tilde{\Lambda}^2$	$\frac{1}{2}m(m-1)$	ℓ odd
	$\tilde{\Lambda}^2$	$\frac{1}{2}m(m-1) - 1$	$\ell = 2, m \equiv 2 \pmod{4}$
	$\tilde{\Lambda}^2$	$\frac{1}{2}m(m-1) - 2$	$\ell = 2, m \equiv 0 \pmod{4}$
	$\tilde{\Sigma}^2$	$\frac{1}{2}m(m+1) - 1$	$\ell \nmid m$
	$\tilde{\Sigma}^2$	$\frac{1}{2}m(m+1) - 2$	$\ell \mid m$

2. THE ALTERNATING GROUPS

For a field F and a partition $\lambda \vdash n$ of n let S^λ denote the Specht module for the symmetric group \mathfrak{S}_n over F . It is irreducible if the characteristic of F is prime to $n!$. We first collect some results on the constituents of alternating and symmetric squares of small Specht modules.

Lemma 2.1. *For all $n \geq 9$ we have*

$$\begin{aligned} \Lambda^2(S^{n-1,1}) &= S^{n-2,1^2}, & \Sigma^2(S^{n-1,1}) &= S^n + S^{n-1,1} + S^{n-2,2}, \\ \Lambda^2(S^{n-2,2}) &= S^{n-2,1^2} + S^{n-3,2,1} + S^{n-3,1^3} + S^{n-4,3,1}, \\ \Sigma^2(S^{n-2,2}) &= S^n + S^{n-1,1} + 2S^{n-2,2} + S^{n-3,3} + S^{n-3,2,1} + S^{n-4,4} + S^{n-4,2^2} \\ \Lambda^2(S^{n-2,1^2}) &= S^{n-2,1^2} + S^{n-3,2,1} + S^{n-3,1^3} + S^{n-4,2,1^2} \\ \Sigma^2(S^{n-2,1^2}) &= S^n + S^{n-1,1} + 2S^{n-2,2} + S^{n-3,3} + S^{n-3,2,1} + S^{n-4,2^2} + S^{n-4,1^4}. \end{aligned}$$

Proof. It is well-known that the character χ_λ of any Specht module S^λ can be written as a linear combination of permutation characters $1_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}$ on Young subgroups \mathfrak{S}_μ for partitions $\mu \supseteq \lambda$. The decomposition of $1_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} \otimes 1_{\mathfrak{S}_\nu}^{\mathfrak{S}_n}$ where the largest part μ_1 of μ is at least $n-2$ can easily be done using the Mackey formula for tensor products. The formulas for the decomposition of alternating and symmetric squares are then proved by induction. We give the details for the simple case of $S^{n-1,1}$. By the Mackey formula we have

$$1_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \otimes 1_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = 1_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} + 1_{\mathfrak{S}_{n-2}}^{\mathfrak{S}_n},$$

hence since $1_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = S^n + S^{n-1}$ we obtain

$$(2.2) \quad S^{n-1,1} \otimes S^{n-1,1} = S^n + S^{n-1,1} + S^{n-2,2} + S^{n-2,1^2}.$$

The branching rule yields

$$\Lambda^2(S^{n-1,1})|_{\mathfrak{S}_{n-1}} = \Lambda^2(S^{n-2,1} + S^{n-1}) = \Lambda^2(S^{n-2,1}) + S^{n-2,1} = S^{n-3,1^2} + S^{n-2,1},$$

the last equality by the induction hypothesis, starting from $n = 4$ where the assertion is easy to check. Now conversely the branching rule shows that $\Lambda^2(S^{n-1,1})$ can only have been $S^{n-2,1^2}$. Equation (2.2) then also determines the decomposition of the symmetric square. The other cases can be handled similarly. \square

For any field F of characteristic ℓ and any ℓ -regular partition $\lambda \vdash n$ we denote by D^λ the ℓ -modular irreducible $F\mathfrak{S}_n$ -module indexed by λ . The decomposition of the Specht modules S^λ into irreducible constituents D^μ is known for small partitions. We will need the following assertions:

$$\begin{aligned} S^{n-1,1} &= D^{n-1,1} + \delta_1 D^n, \\ S^{n-2,2} &= D^{n-2,2} + \delta_2 D^{n-1,1} + \delta_3 D^n, \\ S^{n-2,1^2} &= D^{n-2,1^2} + \delta_1 D^{n-1,1} \quad \text{if } \ell \text{ is odd,} \end{aligned}$$

where

$$\begin{aligned} \delta_1 &:= \begin{cases} 1 & \text{if } \ell | n, \\ 0 & \text{else,} \end{cases} & \delta_2 &:= \begin{cases} 1 & \text{if } \ell | n - 2, \\ 0 & \text{else,} \end{cases} \\ \delta_3 &:= \begin{cases} 1 & \text{if } 2 \neq \ell | n - 1, \text{ or } \ell = 2, n \equiv 1, 2 \pmod{4}, \\ 0 & \text{else} \end{cases} \end{aligned}$$

(see [5, Th. 24.1 and 24.15]). Finally, we need some information on irreducible modules of small dimension. For $k \geq 1$ we define the set $R_n(k)$ to consist of those irreducible \mathfrak{S}_n -modules D for which there exists an ℓ -regular partition $\lambda \vdash n$ with $\lambda_1 \geq n - k$ such that either $D \cong D^\lambda$ or $D \cong S^{1^n} \otimes D^\lambda$. The following is a slight strengthening of a result of James [6, Th. 7] (where the bound $(n-1)(n-2)/2$ was given):

Proposition 2.3. *Let F be a field of characteristic $\ell \geq 0$. Let $n \geq 12$ if $\ell \neq 2$, respectively $n \geq 17$ if $\ell = 2$. Then any irreducible $F\mathfrak{S}_n$ -module D satisfies one of*

$$D \in R_n(2) \quad \text{or} \quad \dim(D) > (n-2)(n-3).$$

Proof. Let first $\ell \neq 2$. Define $f(n) := (n-2)(n-3)$ for $n \geq 13$ and $f(12) := 88$. Then $2f(n) > f(n+2)$ for all $n \geq 12$. Also, from the known decomposition matrices [5] it follows that for $n \in \{12, 13\}$ and F of characteristic different from 2 the irreducible $F\mathfrak{S}_n$ -modules D either lie in $R_n(2)$ or satisfy $\dim(D) > f(n)$. (\mathfrak{S}_{12} has an 89-dimensional 5-modular irreducible module not in $R_n(2)$.) Moreover, by [6, Table 1] we have $\dim(D) > f(n)$ for all $D \in R_n(4) \setminus R_n(2)$ and $n \geq 12$. But then by Lemma 4 in [6] the first part follows.

If F has characteristic 2 we define $f(15) = f(16) = 126$, $f(n) = (n-2)(n-3)$ for $n \geq 17$. Then the same argument can be used since for $n \in \{15, 16\}$ the irreducible $F\mathfrak{S}_n$ -modules D either lie in $R_n(2)$ or satisfy $\dim(D) > 126$. \square

A corresponding result for the projective representations of the alternating group was proved by Wagner [11]:

Proposition 2.4 (Wagner). *Let F be a field of characteristic $\ell \neq 2$, $n \geq 8$, and V an absolutely irreducible faithful $F[2.\mathcal{A}_n]$ -module. Then $\dim(V)$ is divisible by $2^{\lfloor \frac{n-s-1}{2} \rfloor}$ where s is the number of 1's in the 2-adic expansion of n .*

Lemma 2.5. *Let G be a covering group of the alternating group \mathfrak{A}_n , $n \geq 8$, V a faithful FG -module with F of characteristic $\ell \geq 0$. Then $\tilde{X}(V)$, for $\tilde{X} \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$, contains one of the constituents of the permutation character $1_{\mathfrak{A}_{n-4}}^{\mathfrak{A}_n}$.*

Proof. Let first V be an absolutely irreducible faithful \mathfrak{A}_n -module and $\ell \neq 2$. We consider the restriction of V to a natural subgroup $H = \mathfrak{A}_4$ and write V_1, V_2, V_3 for the non-trivial isotypic components of V under the elementary abelian Sylow 2-subgroup of H . Since $\ell \neq 2$ these are direct summands of V which are cyclically permuted by the elements of order 3 in H . Thus V_1, V_2, V_3 are equivalent modules for the centralizer \mathfrak{A}_{n-4} of H . Hence $X(V)$ contains at least three trivial \mathfrak{A}_{n-4} -composition factors in the socle. By our above considerations this implies that $\tilde{X}(V)$ contains a constituent of the permutation module $1_{\mathfrak{A}_{n-4}}^{\mathfrak{A}_n}$ as claimed.

In the case $\ell = 2$ we let $\sigma_1, \sigma_2 \in G = \mathfrak{A}_n$ be 3-cycles which generate an \mathfrak{A}_4 . Let V_ω, V_{ω^2} denote the non-trivial eigenspaces of σ_1 on V . These are modules for the centralizer \mathfrak{A}_{n-3} of σ_1 . Let W_1, \dots, W_k denote the composition factors in the socle of the restriction of V_ω to the centralizer \mathfrak{A}_{n-4} of $H := \langle \sigma_1, \sigma_2 \rangle$. Assume first that V is not self-dual, so $X = A$. If $k \geq 2$, or if $k = 1$ and W_1 is equivalent to the socle of V_{ω^2} , $X(V)$ contains at least four trivial \mathfrak{A}_{n-4} -composition factors in the socle. If the socle of $V_\omega \oplus V_{\omega^2}$ consists of two non-equivalent \mathfrak{A}_{n-4} -modules then W_1 has to be σ_2 -invariant, so V_ω is G -invariant, which is a contradiction. So now assume that V is self-dual. If one of the W_i is self-dual, or if two of them are equivalent, or if $k \geq 3$ then $X(V)$ has sufficiently many trivial \mathfrak{A}_{n-4} -composition factors in the socle and $\tilde{X}(V)$ contains a constituent of $1_{\mathfrak{A}_{n-4}}^{\mathfrak{A}_n}$. If $k = 1$ and W_1 is not self-dual then V_ω is $\langle \sigma_1, \sigma_2, \mathfrak{A}_{n-3} \rangle = G$ -invariant, so $V = V_\omega$, which is a contradiction. Thus finally assume that $k = 2$ and W_1, W_2 are non-equivalent and not self-dual. Then the conjugate of W_1 by σ_2 is another composition factor of the socle of V as \mathfrak{A}_{n-4} -module. Since V_ω cannot be G -invariant, W_2 is dual to W_1 so that again $X(V)$ has three trivial \mathfrak{A}_{n-4} -composition factors in the socle.

Finally, assume that $G = \tilde{\mathfrak{A}}_n = 2.\mathfrak{A}_n$ and $\ell \neq 2$. Here we restrict to a cyclic subgroup generated by an element σ of order 4 in a natural subgroup $H = \tilde{\mathfrak{A}}_4$ and denote by V_i, V_{-i} the eigenspaces of σ for the primitive fourth roots of unity. These are direct summands of V which are interchanged by elements in the Sylow 2-subgroup of H , hence they are equivalent modules for the centralizer $\tilde{\mathfrak{A}}_{n-4}$ of H . Thus $X(V)$ contains at least one trivial composition factor in the socle. If V is of plus-type, then so are the V_i as $\tilde{\mathfrak{A}}_{n-4}$ -modules, and $\Sigma^2(V)$ contains at least three trivial composition factors in the socle. Thus we obtain that $\tilde{X}(V)$ contains a constituent of $1_{\tilde{\mathfrak{A}}_{n-4}}^{\tilde{\mathfrak{A}}_n}$ as claimed. \square

Proposition 2.6. *Let G be a covering group of the alternating group \mathfrak{A}_n , $n \geq 7$, V an absolutely irreducible faithful FG -module with F of characteristic $\ell \geq 0$. Then $\tilde{X}(V)$ is reducible for $\tilde{X} \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$ unless $(G, \ell, \dim(V), \tilde{X})$ are as in Table 2.7.*

Proof. The Brauer character tables for $2.\mathfrak{A}_n$, $n \leq 13$, are known (see [5]) and the assertion can be checked, so we may assume $n \geq 14$.

We first consider irreducible representations V of \mathfrak{A}_n . Then $\tilde{X}(V)$ is a constituent of $W := 1_{\mathfrak{A}_{n-4}}^{\mathfrak{A}_n}$ by Lemma 2.5. The ordinary constituents of W are Specht modules S^λ for partitions $\lambda \vdash n$ with $\lambda_1 \geq n - 4$. The largest degree of such a constituent is

2.7. Examples in alternating groups \mathfrak{A}_n , $n \geq 7$.

G	ℓ	$\dim(V)$	$\tilde{\Lambda}^2$	$\tilde{\Sigma}^2$	\tilde{A}
\mathfrak{A}_n	$\ell \nmid n$	$n - 1$	\times		
\mathfrak{A}_n	ℓn	$n - 2$	\times	\times	
$3.\mathfrak{A}_7$	5	3			\times
\mathfrak{A}_7	2	4			\times
$2.\mathfrak{A}_7$	7	4		\times	
$2.\mathfrak{A}_7$	$\neq 2, 7$	4			\times
$6.\mathfrak{A}_7$	$\neq 2, 3$	6			\times
\mathfrak{A}_8	2	4			\times
$2.\mathfrak{A}_8$	$\neq 2$	8		\times	
\mathfrak{A}_9	2	8	\times		
$2.\mathfrak{A}_9$	3	8		\times	
$2.\mathfrak{A}_9$	$\neq 2, 3$	8	\times	\times	
$2.\mathfrak{A}_{10}$	5	8	\times	\times	

$\dim(S^{n-4,2,1^2}) = n(n-2)(n-3)(n-5)/8$. On the other hand by Prop. 2.3 either $\dim(V) > (n-2)(n-3)$ or $V \in R_n(2)$, or $n \leq 16$ and $\ell = 2$. Since

$$\frac{1}{8}n(n-2)(n-3)(n-5) < \frac{1}{8}(n-2)(n-3)((n-2)(n-3) - 2) - 2$$

it follows that either V is a constituent of the restriction to \mathfrak{A}_n of some D^λ with $\lambda_1 \geq n-2$, or $n \leq 16$, $\ell = 2$. For these modules the assertion easily follows from Lemma 2.1 and the decompositions into irreducibles of S^λ collected above. If $\ell = 2$, $n \in \{14, 15, 16\}$, the degrees of the constituents of $1_{\mathfrak{A}_{n-4}}^{\mathfrak{A}_n}$ can be computed explicitly using [4]; they are collected in Table 2.8. The degrees in line 3 correspond to the examples in Table 2.7. The only other entries of the form $m^2 - \epsilon$, $m(m-1)/2 - \epsilon$ for integral m and $\epsilon \in \{1, 2\}$ are

$$208 = 21 \cdot 20/2 - 2, \quad 560 = 34 \cdot 33/2 - 1, \quad 624 = 25^2 - 1.$$

But \mathfrak{A}_{14} does not have 2-modular irreducible representations of degree 21 or 34, nor does \mathfrak{A}_{15} have such of degree 25, as restriction to \mathfrak{A}_{13} shows.

2.8. Dimensions of some \mathfrak{S}_n -modules D^λ , $\ell = 2$.

λ	$n = 14$	$n = 15$	$n = 16$
n	1	1	1
$n - 1, 1$	12	14	14
$n - 2, 2$	64	90	90
$n - 3, 3$	208	336	336
$n - 3, 2, 1$	560	624	896
$n - 4, 4$	364	910	910
$n - 4, 3, 1$	1300	1300	2548

In the case of faithful $2.\mathfrak{A}_n$ -modules Lemma 2.5 and the result of Wagner (Prop. 2.4) give a contradiction unless $n \in \{14, 15\}$, $\dim(V) \in \{32, 64\}$. We first consider $n = 15$. If $\ell \neq 13$ the smallest dimension of a faithful $2.\mathfrak{S}_{13}$ -module is 64, while for $\ell = 13$ the smallest degree of a faithful $2.\mathfrak{A}_{12}$ -module is 32. Thus restriction to these subgroups shows that in fact $\dim(V) \geq 64$ if $n = 15$. Now from [5, Th. 24.1 and 24.15] and Table 1 in [6] it follows that no irreducible constituent of W has the same degree as $\tilde{X}(V)$. So finally assume that $n = 14$. In any characteristic, any faithful irreducible $2.\mathfrak{A}_{11}$ -module of dimension less than 100 has dimension 16, and there are at most two inequivalent ones. Thus if $\dim(V) = 64$ we obtain enough trivial $2.\mathfrak{A}_{11}$ -composition factors in the socle of $X(V)$ to conclude that $\tilde{X}(V)$ is a constituent of $1_{2_{11}^{14}}$. This contradicts $\dim(V) = 64$. Thus $\dim(V) = 32$. Again from [6, Table 1] and [5, Th. 24.1 and 24.15] we deduce $\ell = 3$. From the known tables it can be checked that the smallest faithful $2.\mathfrak{A}_{13}$ -modules (of dimension 32) are of minus-type, while the 32-dimensional faithful $2.\mathfrak{S}_{12}$ -module is of plus-type. So V does not carry a non-degenerate form. The tensor product of a 32-dimensional faithful $2.\mathfrak{A}_{13}$ -module with itself contains a constituent of degree 66, so $\dim(\tilde{A}(V)) \leq 924$, which gives the final contradiction. \square

3. THE LINEAR GROUPS

Partial results in the case of linear groups had already been obtained in [8]. It turns out that the methods used there extend to our more general situation.

Proposition 3.1. *Let G be a covering group of $L_2(q)$ or $L_3(q)$, ℓ a prime with $\gcd(q, \ell) = 1$ or $\ell = 0$ and V an absolutely irreducible faithful FG -module. Then $X(V)$ is reducible for $X \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$ unless $(G, \ell, \dim(V), X)$ are as in Table 3.3.*

Proof. Let first $S = L_2(q)$. The cases $q \leq 13$ can be treated using the tables in the modular Atlas [7]. Hence assume that $q \geq 16$. Then, by [10] the minimal degree of a faithful irreducible projective representation of $L_2(q)$ in characteristic $\ell \nmid q$ is $(q-1)/\gcd(2, q-1)$. On the other hand, the largest degree of a complex projective irreducible character for S is $q+1$. For the case of the alternating square, irreducibility of $X(V)$ gives the inequality $\frac{1}{8}(q-1)(q-3) - 2 \leq q+1$ which is not satisfied for $q \geq 16$. For $\Sigma^2(V)$ and $A(V)$ the bounds restrict q even more.

The groups $L_3(q)$ for $q \leq 4$ can be handled using [7]. So assume now that $S = L_3(q)$, $q \geq 5$. Then the minimal degree of a faithful irreducible projective representation of S is at least $q^2 - 1$ [10], while the largest degree of a complex projective irreducible character for S is $(q^2 + q + 1)(q + 1)$. The irreducibility of the alternating square thus forces the inequality

$$\frac{1}{2}(q^2 - 1)(q^2 - 2) - 2 \leq (q^2 + q + 1)(q + 1)$$

which does not hold for $q \geq 5$. \square

Proposition 3.3. *Let G be a covering group of $L_n(q)$, $n \geq 4$, $q \neq 2$, ℓ a prime with $\gcd(q, \ell) = 1$ or $\ell = 0$ and V an absolutely irreducible faithful FG -module. Then $\tilde{X}(V)$ is reducible for $\tilde{X} \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$ unless $(G, \ell, \dim(V), \tilde{X})$ are as in Table 3.2.*

Proof. Using the modular Atlas to treat $L_4(3)$ we may assume that V is a module for $G = \mathrm{SL}_n(q)$, and $q \geq 4$ for $n = 4$. We now essentially follow the argument in [8].

3.2. Examples in linear groups.

G	ℓ	$\dim(V)$	$\tilde{\Lambda}^2$	$\tilde{\Sigma}^2$	\tilde{A}
$2.L_2(4)$	$\neq 2, 5$	2		\times	
$L_2(4)$	$\neq 2, 5$	3	\times	\times	
$L_2(7)$	$\neq 2, 7$	3			\times
$3.L_2(9)$	$\neq 3$	3			\times
$L_2(9)$	2	4	\times		
$2.L_2(9)$	$\neq 2, 3$	4		\times	
$L_2(9)$	$\neq 2, 3$	5	\times		
$L_2(13)$	2	6	\times		
$4_2.L_3(4)$	3	4			\times
$2.L_3(4)$	3	6	\times	\times	
$6.L_3(4)$	$\neq 2, 3$	6			\times
$4_1.L_3(4)$	$\neq 2$	8			\times
$2.L_3(4)$	7	10	\times		
$L_4(2)$	$\neq 2$	7	\times		

Let P_1 denote the stabilizer in G of a 1-dimensional subspace of the natural module. Thus P_1 is a maximal parabolic subgroup of G , with Levi decomposition $P_1 = U_1:G_1$ where the Levi factor $G_1 = \mathrm{GL}_{n-1}(q)$ acts on the elementary abelian group U_1 of order q^{n-1} as it does on its natural module. Upon restriction to U_1 the module V splits as

$$V|_{U_1} = \bigoplus_{\lambda \in \mathrm{Hom}(U_1, F^\times)} V_\lambda.$$

Moreover G_1 permutes transitively the spaces V_λ for $\lambda \neq 1$.

We fix $1 \neq \lambda \in \mathrm{Hom}(U_1, F^\times)$ and denote by P_2 its stabilizer in P_1 . Thus $P_2 = U_1:U_2:G_2$ where $U_2:G_2$ is the stabilizer of λ in $G_1 = \mathrm{GL}_{n-1}(q)$, $|U_2| = q^{n-2}$ and $G_2 \cong \mathrm{GL}_{n-2}(q)$. Let $\langle \lambda \rangle$ denote the set of $q-1$ non-trivial elements of $\mathrm{Hom}(U_1, F^\times)$ stabilized by P_2 . We consider two cases.

Case 1: First assume that U_2 acts nontrivially on V_λ . We write

$$V_\lambda|_{U_2} = \bigoplus_{\mu \in \mathrm{Hom}(U_2, F^\times)} V_{\lambda, \mu}.$$

By our assumption $V_{\lambda, \mu} \neq 0$ for some (and hence all) $\mu \neq 1$. The stabilizer P_3 in P_2 of such a $V_{\lambda, \mu}$ is of the shape $U_1:U_2:U_3:\mathrm{GL}_{n-3}(q)$, with U_3 elementary abelian of order q^{n-3} . Let $\langle \mu \rangle$ denote the set of non-trivial elements in the centralizer in $\mathrm{Hom}(U_2, F^\times)$ of $\mathrm{GL}_{n-3}(q)$, of order $q-1$, and

$$V_{\lambda, \langle \mu \rangle} := \bigoplus_{\mu' \in \langle \mu \rangle} V_{\lambda, \mu'}.$$

Thus the stabilizer H_3 of $V_{\lambda, \langle \mu \rangle}$ is of the form $U_1:U_2:U_3:(\mathrm{GL}_{n-3}(q) \times (q-1))$. Now $W := X(V_{\lambda, \langle \mu \rangle})$ is a non-trivial H_3 -invariant subspace of $X(V)$. Assuming that $\tilde{X}(V)$

is irreducible this gives the upper bound

$$\dim(\tilde{X}(V)) \leq [G : H_3] \dim(W) \leq (q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1) \dim(\tilde{X}(V_{\lambda, \mu})).$$

Since $|\text{Hom}(U_i, F^\times)| = |U_i| = q^{n-i}$ we have the lower bound $\dim(V) \geq (q^{n-1} - 1) \times (q^{n-2} - 1) d_{\lambda, \mu}$ where $d_{\lambda, \mu} := \dim(V_{\lambda, \mu})$, so

$$(3.4) \quad \dim(\tilde{X}(V)) \geq \frac{1}{2}(q^{n-1} - 1)(q^{n-2} - 1) d_{\lambda, \mu} ((q^{n-1} - 1)(q^{n-2} - 1) d_{\lambda, \mu} - 1) - 2.$$

These two inequalities yield a contradiction for all (n, q) .

Case 2: Thus we may assume that U_2 acts trivially on V_λ .

Case 2a: First we treat the subcase that $d_\lambda := \dim(V_\lambda) \geq 2$. The subspace $W := X(V_\lambda)$ of $X(V)$ is P_2 -invariant by definition and has trivial U_2 -action by assumption. Since the kernel $\tilde{U}_1 < U_1$ of λ (of order q^{n-2}) acts trivially on W , this is a $\tilde{P}_2 := \tilde{U}_1 : U_2 : G_2$ -submodule of $X(V)$. Our assumptions force $W \neq 0$, so if $\tilde{X}(V)$ is irreducible it must be a constituent of $W_{\tilde{P}_2}^G$. Now \tilde{P}_2 lies in a maximal parabolic subgroup P of G of the form $\tilde{U}_1 : U_2 : (\text{SL}_2(q) \circ \text{SL}_{n-2}(q)) : (q-1)$. Since any irreducible complex character of $\text{GL}_2(q)$ has degree at most $q+1$ we see that $\tilde{X}(V)$ is the constituent of the induced of a P -module of dimension at most $(q+1) \dim(W)$. This gives the bound

$$\dim(\tilde{X}(V)) \leq \frac{(q^n - 1)(q^{n-1} - 1)}{(q-1)^2} \dim(X(V_\lambda)).$$

On the other hand clearly $\dim(V) \geq (q^{n-1} - 1) d_\lambda$, so for example

$$(3.5) \quad \dim(\tilde{\Lambda}^2(V)) \geq \frac{1}{2}(q^{n-1} - 1) d_\lambda ((q^{n-1} - 1) d_\lambda - 1) - 2$$

in the case $X = \Lambda^2$. Comparing these bounds we obtain a contradiction unless $q = 3$, $d_\lambda \leq 3$, $X = \Sigma^2$. In the latter case we may assume $n \geq 5$. Then the derived group of G_2 has to act trivially on V_λ , hence V_λ has only linear G_2 -constituents. Replacing V_λ by one of these, the previous argument goes through. So $\tilde{X}(V)$ can not have been irreducible.

Case 2b: Now assume that $d_\lambda = 1$. Then the same argument as before can be applied to the submodules $W := V_\lambda \otimes V_{\lambda'}$, where $\lambda' \in \text{Hom}(U_1, F^\times)$ different from λ and from 1 is stabilized by P_2 , yielding the bound

$$\dim(\tilde{X}(V)) \leq \frac{(q^n - 1)(q^{n-1} - 1)}{(q-1)^2}.$$

By [10] we have $\dim(V) \geq (q^n - 1)/(q-1) - n \geq q^{n-1} + q^{n-2}$ for $(n, q) \neq \{(4, 3)\}$. The ensuing inequality is not satisfied for $n \geq 4$. This completes the proof. \square

To treat the linear groups over \mathbb{F}_2 we introduce some more notation. For a partition $\lambda \vdash n$ let χ_λ denote the (complex irreducible) unipotent character of $\text{SL}_n(q)$ indexed by λ . For a Levi subgroup L of $G = \text{SL}_n(q)$ we write R_L^G (respectively $*R_L^G$) for the operation of Harish-Chandra induction (restriction).

Lemma 3.6. *Let $n \geq 4$. The unipotent character $\chi_{n-2,1^2}$ of $\mathrm{SL}_n(2)$ is a constituent of both $\Lambda^2(\chi_{n-1,1})$ and $\Sigma^2(\chi_{n-1,1})$.*

Proof. We proceed by induction on n . The assertion is true for $\mathrm{SL}_4(2) \cong \mathfrak{A}_8$. Let $P = U:L$ be the maximal parabolic subgroup of $G = \mathrm{SL}_n(2)$ with $L = \mathrm{GL}_{n-1}(2)$ and denote by V a CG -module affording the character $\chi_{n-1,1}$. Harish-Chandra induction and restriction of unipotent characters χ_λ of $\mathrm{SL}_n(q)$ behave precisely as ordinary induction and restriction of Specht modules S^λ for the symmetric group \mathfrak{S}_n . Thus $V|_P$ contains an L -module with character $\chi_{n-2,1}$ on the centralizer V_1 of U . Now $X(V)|_P = X(V|_P)$ for $X \in \{\Lambda^2, \Sigma^2\}$, hence by the inductive assumption we have that $X(V)|_P$ contains a submodule with character $\chi_{n-3,1^2}$ on the centralizer $X(V)_1$ of U . In other words $\langle *R_L^G(X(\chi_{n-1,1})), \chi_{n-3,1^2} \rangle \neq 0$. By the adjointness of Harish-Chandra induction and restriction this implies that $X(\chi_{n-1,1})$ contains a constituent of $R_L^G(\chi_{n-3,1^2})$. By the Littlewood-Richardson rule we have

$$R_L^G(\chi_{n-3,1^2}) = \chi_{n-2,1^2} + \chi_{n-3,2,1} + \chi_{n-3,1^3}.$$

But the degrees

$$\begin{aligned} \chi_{n-3,2,1}(1) &= (2^n - 1)(2^n - 16)(2^{n-2} - 1)/7, \\ \chi_{n-3,1^3}(1) &= (2^n - 2)(2^n - 4)(2^n - 8)/21, \end{aligned}$$

of the second and third possible constituent of $X(V)$ are larger than $\dim(X(V))$. This shows the assertion. \square

Proposition 3.7. *Let G be a covering group of $\mathrm{L}_n(2)$, $n \geq 5$, $\ell \neq 2$ and V an absolutely irreducible faithful FG -module. Then $\tilde{X}(V)$ is reducible for $\tilde{X} \in \{\tilde{\Lambda}^2, \tilde{\Sigma}^2, \tilde{A}\}$.*

Proof. As above we may assume that V is a module for $G = \mathrm{SL}_n(q)$. We follow the proof of the preceding proposition. The argument in Case 1 goes through unless $d_{\lambda,\mu} := \dim(V_{\lambda,\mu}) = 1$ and $X = \Lambda^2$, when $W = 0$. In this case let $0 \neq \mu' \in \mathrm{Hom}(U_2, F^\times)$ be different from μ . Then $W := V_{\lambda,\mu} \otimes V_{\lambda,\mu'}$ is a 1-dimensional $P_4 := U_1:U_2:U_3:U_4:\mathrm{GL}_{n-4}(2)$ -module. The maximal dimension of an irreducible $\mathrm{GL}_4(2) \cong \mathfrak{A}_8$ -module is 70, so all constituents of the induction of W from P_4 to a maximal parabolic subgroup of type $[2^{4n-16}] : (\mathrm{GL}_4(2) \circ \mathrm{GL}_{n-4}(2))$ have dimension at most 70. This forces

$$\dim(\Lambda^2(V)) \leq 70 \frac{(2^n - 1)(2^{n-1} - 1)(2^{n-2} - 1)(2^{n-3} - 1)}{(2^2 - 1)(2^3 - 1)(2^4 - 1)}$$

which is a contradiction to the lower bound (3.4).

In Case 2, when U_2 acts trivially on V_λ , let $P_3 = U_1:U_2:U_3:\mathrm{GL}_n(2)$ denote the stabilizer of a pair λ, λ' of non-trivial elements of $\mathrm{Hom}(U_1, F^\times)$. We write $V_\lambda = \bigoplus_\mu V_{\lambda,\mu}$ for the decomposition of V_λ into isotypic components for U_3 . Now the submodule $\bigoplus_\mu (V_{\lambda,\mu} \otimes V_{\lambda',\mu})$ of $X(V)$ has trivial $U_1:U_2:U_3:\mathrm{GL}_n(2)$ -action. Thus $\tilde{X}(V)$ is a constituent of the permutation module for this subgroup. Since the largest dimension of an irreducible $\mathrm{GL}_3(2)$ -character is 8, we obtain the estimate

$$\dim(\tilde{X}(V)) \leq \frac{(2^n - 1)(2^{n-1} - 1)(2^{n-2} - 1)}{21}.$$

On the other hand, if $GL_{n-2}(2)$ acts non-trivially on V_λ , by [10] we get the lower bound

$$\dim(V) \geq (2^{n-1} - 1)(2^{n-2} - n + 1)$$

if $n \geq 7$, respectively $\dim(V) \geq 217$ if $n = 6$, $\dim(V) \geq 30$ if $n = 5$. This yields a contradiction. Hence V_λ cannot be a faithful $GL_{n-2}(2)$ -module, and replacing V_λ by a 1-dimensional submodule in its socle yields a contradiction to the upper bound (3.5) unless $d_\lambda = 1$ and $X = \Lambda$. Since G is generated by two conjugates of U_1 we must have $\dim(V_1) \leq \dim(V)/2$, so $\dim(V) \leq 2(2^{n-1} - 1) = 2^n - 2$.

By [2, Th. 1.1] this implies that V is contained in the ℓ -modular reduction of a characteristic 0 module with character $\chi_{n-1,1}$ and $\chi_{n-1,1}(1) - 1 \leq \dim(V) \leq \chi_{n-1,1}(1)$. But now the dimension of the submodule of $X(\chi_{n-1,1})$ exhibited in Lemma 3.6 is too small for $\tilde{X}(V)$ to be irreducible. This final contradiction completes the proof. \square

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