Newton Institute NI97036

On multi-particle entanglement

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October 1997

Abstract

We build, using group-theoretic methods, a general framework for approaching multi-particle entanglement. As far as entanglement is concerned, two states of n spin-1/2 particles are equivalent if they are on the same orbit of the group of local rotations $(U(2)^n)$. We give a method for finding the number of parameters needed to describe inequivalent n spin-1/2 particles states. We also describe how entanglement of states on a given orbit may be characterized by the stability group of the action of the group of local rotations on any point on the orbit.

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1 Introduction

Discovered in 1964 by J. Bell [1], the existence of non-local correlations among remote quantum systems is one of the most fascinating quantum phenomena. But while for long time these correlations were considered more as a curiosity, recently they have found a large range of applications, forming the very bases of quantum communication and quantum computation; obviously the interest in better understanding these correlations has increased dramatically.

Traditionally, starting with Bell, the example which has been most studied was that of non-local correlations between *two* remote quantum particles. However, it is now clear that the correlations among more than two remote particles present novel and highly nontrivial aspects compared to two-particles entanglement. (See for example the correlations generated by the GHZ state [2].) Nevertheless, very little is known yet about multi-particle entanglement. It is the aim of the present paper to take a few steps towards understanding the general structure of multi-particle entanglement.

The key element in our approach is to note that two states which can be transformed one into another by *local* operations (unitary transformations) are equivalent as far as their non-local properties are concerned. This leads us to investigate the properties of the Hilbert space of n spin-1/2 particles under local unitary transformations.

We find the following picture emerging.

- Each particular state Ψ belongs to an equivalence class comprised of by all states which can be obtained from Ψ by acting on it with local unitary operators; all states in a class are equivalent as far as non-locality is concerned. Obviously, the Hilbert space of states decomposes completely into equivalence classes, or "orbits", under the action of the group of local unitary transformations.

- An arbitrary state Ψ of n spin 1/2 particles is described by 2^n complex parameters. Some of these parameters (or functions of them) specify the equivalence class to which Ψ belongs. These parameters (or functions) are obviously *invariants* under local transformations. The reminder describe where Ψ is situated inside the equivalence class - they do change under local transformations.

Incidentally, two-particles entanglement is technically so much simpler to study than multi-particles entanglement because there is a simple way to identify the invariant parameters - the Schmidt decomposition. Indeed, let $e_i \otimes e_j$, $i, j = 1, 2$ be some arbitrary base vectors in the Hilbert space of the two particles then a general state of two particles is given by

$$\Psi = \sum_{i,j} \alpha_{ij} e_i \otimes e_j. \tag{1}$$

However, by choosing some appropriate base vectors for each particle, the double sum in (1) can be reduced to a single sum

$$\Psi = \sum_i \beta_i f_i \otimes f_i. \tag{2}$$

The Schmidt coefficients are manifestly invariant under local transformations. Indeed, local unitary transformations can only change the Schmidt base vectors, but not the Schmidt coefficients.

$$\Psi = \sum_i \beta_i f_i \otimes f_i \rightarrow \Psi' = \sum_i \beta_i f_i' \otimes f_i'. \tag{3}$$

As it is well-known, for multi-particle states in general there exists no similar simple decomposition [3]. What can one than do? Given our above analysis of multi-particle entanglement, it is now clear that instead of simply trying to find something which *formally* resembles the Schmidt decomposition, we should try to follow its spirit, not its form. That is, to try and find a representation which separates local and non-local parameters.

As an important result, we find that for large n, most of the parameters describe non-local properties. This is opposite to the case of small n - for two spins, out of the 8 real parameters which describe a generic (unnormalized) state, only 1, the unique independent Schmidt coefficient, has non-local significance.

- Finally, we note that in the case of two-particle entanglement some of the states are, in some sense, special. Such states are the direct-products and the singlet-like states. We show that the special nature of these states is determined by their invariance properties. Namely, for these special states there are more local actions which leave them unchanged than in the case of generic states. For example in the case of a singlet

$$\Psi = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1) \tag{4}$$

where e_1 and e_2 represent spin polarized "up" or "down" say, along the z axis, identical rotations of the two spins leave the state unchanged.

Furthermore, such enhanced invariance properties are in fact common for all states in an equivalence class, and thus characterize the class itself. To find the "special" equivalence classes, we have therefore to study their invariance properties. We argue that these "special" classes describe fundamentally different types of entanglement while a generic class represents a combination of different types of entanglement.

Group-theoretically, the situation is the following. The space of states of n spin $1/2$ particles is the n-fold tensor product $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$, and the group of local transformations is the n-fold product $U(2)^n = U(2) \times ... \times U(2)$, (each copy of $U(2)$ acting on a different spin, i.e. on the corresponding copy of $\mathbb{C}^2$). The equivalence classes are *orbits* under the action of the local transformations group. Hence, the space of orbits is

$$\frac{\mathbb{C}^{2^n}}{U(2) \times ... \times U(2)}; \tag{5}$$

this is the main mathematical object we are investigating.

The number of parameters needed to describe the position of Ψ on its orbit is the *dimension* of the orbit. Not all orbits have the same dimension. As noted above, there are "special" orbits - singular orbits- which have higher invariance, i.e. lower dimension.

The total number of parameters (2^n complex parameters $= 2^{n+1}$ real parameters) describing the space of states minus the number of parameters describing a generic orbit (the dimension of the orbit), gives the number of parameters describing the location of the orbit in the space of orbits, i.e. the number of parameters describing the non-local properties of the states.

2 The number of parameters needed to describe inequivalent states

In this section we are interested in finding out how many parameters are needed to describe the space of orbits of the action of $U(2)^n$ on the space of states, i.e. the number of parameters which describes inequivalent states. To do this it will be convenient to find the (real) dimension of a general orbit; the number of parameters is then found by subtracting this number from 2^{n+1}.

A lower bound on this number can be obtained by a simple argument of counting parameters. Each of the n copies of the local unitary group $U(2)$ is described by 4 real parameters. Thus there can be no more than $4n$ parameters describing local properties of the states, and hence at least $2^{n+1}-4n$ non-local parameters (i.e., invariants under local transformations)[3].

One can immediately see that for large n almost all parameters have non-local significance.

The above bound is, in general, not satisfied. The reason is that not all $4n$ parameters describing the local transformations lead to independent effects. For example, equally changing the phase of all states of any particular spin has the same effect as changing the phases of any other. Hence, at least, the group of local transformations reduces from $U(2)^n$ to $U(1) \times SU(2)^n$ which has dimension $3n+1$. This leads to a better lower bound on the number of non-local parameters of $2^{n+1} - (3n+1)$.

This is, however, not the end of the story. We will find below that the number of parameters describing independent local transformations may be fewer (and correspondingly, the number of non-local parameters larger).

2.1 Dimension of a general orbit

To find the dimension of a general orbit it is simplest to work infinitesimally. Thus, in general, associated to the action of each element of a Lie algebra of a Lie group K which acts on a space V there is a vector field: take an

[3]For convenience we always consider non-normalized states, and thus the norm also appears as one of the invariant parameters.

element T of a basis for the Lie algebra, the action of the group element $k = \exp i\epsilon T \in K$ on an element $v \in V$ induces an action on functions from V to $\mathbb{C}$; and the vector field, X_T, associated to the Lie algebra element T is found by differentiating:

$$X_T f(v) \stackrel{\text{def}}{=} \frac{\partial}{\partial \epsilon} f(e^{i\epsilon T} v)|_{\epsilon=0}. \tag{6}$$

The linear span of vector fields at the point v associated with the whole Lie algebra forms the tangent space to the orbit at the point v and so the number of linearly independent vector fields at this point gives the dimension of the orbit.

2.2 A single spin

The case $n = 1$ helps to illustrate the general formalism. The space of states has real dimension four (complex dimension two). It is also clear that the action of a unitary operator on a vector cannot change its norm, so that the dimension of the space of orbits must be at least one (in fact we will soon see that it is precisely one). However, the group $U(2)$ has dimension four so that the set of vector fields associated to an arbitrary basis for the Lie algebra cannot be linearly independent.

In the representation of $U(2)$ acting on $\mathbb{C}^2$ a convenient Hermitian basis for the Lie algebra is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Now take an element

$$\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \tag{8}$$

and consider the infinitesimal change under a transformation in the direction σ_x:

$$\delta\Psi = i\epsilon\sigma_x\Psi = i\epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi = \begin{pmatrix} i\epsilon\beta \\ i\epsilon\alpha \end{pmatrix}. \tag{9}$$

So that under a group transformation close to the identity,

$$\Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \Psi + \delta\Psi = \begin{pmatrix} \alpha + i\epsilon\beta \\ \beta + i\epsilon\alpha \end{pmatrix}. \tag{10}$$

We now write everything in terms of real variables:

$$\alpha = c_1 + id_1; \quad \beta = c_2 + id_2. \tag{11}$$

so that

$$\Psi = \begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix} \quad \text{and} \quad \delta\Psi = \epsilon \begin{pmatrix} -d_2 \\ c_2 \\ -d_1 \\ c_1 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix}. \tag{12}$$

Thus there is an induced action on a function $f(v) = f(c_1, d_1, c_2, d_2)$:

$$f(c_1, d_1, c_2, d_2) \mapsto f(c_1 - \epsilon d_2, d_1 + \epsilon c_2, c_2 - \epsilon d_1, d_2 + \epsilon c_1). \tag{13}$$

Differentiating with respect to ϵ we find:

$$\left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} = \left(-d_2\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial d_1} - d_1\frac{\partial}{\partial c_2} + c_1\frac{\partial}{\partial d_2}\right) f. \tag{14}$$

We write the vector field associated to this Lie algebra element σ_x as

$$\left(-d_2\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial d_1} - d_1\frac{\partial}{\partial c_2} + c_1\frac{\partial}{\partial d_2}\right) = u_x.\nabla \quad \text{where} \quad u_x = \begin{pmatrix} -d_2 \\ c_2 \\ -d_1 \\ c_1 \end{pmatrix}. \tag{15}$$

In a similar way we may find the vectors u_y, u_z and u_1 associated to transformations by σ_y, σ_z and 1_2:

$$u_y = \begin{pmatrix} c_2 \\ d_2 \\ -c_1 \\ -d_1 \end{pmatrix} \quad u_z = \begin{pmatrix} -d_1 \\ c_1 \\ d_2 \\ -c_2 \end{pmatrix} \quad u_1 = \begin{pmatrix} -d_1 \\ c_1 \\ -d_2 \\ c_2 \end{pmatrix}. \tag{16}$$

It is not too difficult to check that only three of these four vectors are linearly independent. Indeed

$$2(d_1d_2+c_1c_2)u_x+2(c_1d_2-d_1c_2)u_y+(c_1^2+d_1^2-c_2^2-d_2^2)u_z-(c_1^2+d_1^2+c_2^2+d_2^2)u_1 = 0. \tag{17}$$

Thus the dimension of the orbit is three and so there is one parameter (the norm) which describes the different orbits.

2.3 Two spins

In a similar way we may analyze the case of two spins. A general vector may be written

$$\Psi = \sum_{i,j=1}^{2} \alpha_{ij} e_i \otimes e_j = \sum_{i,j=1}^{2} (c_{ij} + i d_{ij}) e_i \otimes e_j, \tag{18}$$

where $\{e_1, e_2\}$ is a general basis of $\mathbb{C}^2$. In the representation of $U(2)^2$ on $\mathbb{C}^4$ we may use the following basis for the eight Lie algebra elements:

$$\sigma_x \otimes 1_2,\ \sigma_y \otimes 1_2,\ \sigma_z \otimes 1_2,\ 1_2 \otimes 1_2, 1_2 \otimes \sigma_x,\ 1_2 \otimes \sigma_y,\ 1_2 \otimes \sigma_z,\ 1_2 \otimes 1_2. \tag{19}$$

One sees that the element $1_2 \otimes 1_2$ appears twice, so that in fact there are only seven different Lie algebra elements to consider. If we choose the following order for the coordinates of the eight dimensional real vector space:

$$(c_{11}, d_{11}, c_{12}, d_{12}, c_{21}, d_{21}, c_{22}, d_{22}), \tag{20}$$

then the derivative operator is

$$\nabla_8 = (\frac{\partial}{\partial c_{11}}, \frac{\partial}{\partial d_{11}}, \frac{\partial}{\partial c_{12}}, \frac{\partial}{\partial d_{12}}, \frac{\partial}{\partial c_{21}}, \frac{\partial}{\partial d_{21}}, \frac{\partial}{\partial c_{22}}, \frac{\partial}{\partial d_{22}}), \tag{21}$$

and the vector fields are all of the form $u.\nabla_8$. The vectors $\{u_x^{(1)}, u_y^{(1)}, u_z^{(1)}, u_{\text{one}}, u_x^{(2)}, u_y^{(2)}, u_z^{(2)}\}$ associated to the Lie algebra elements $\{\sigma_x \otimes 1_2,\ \sigma_y \otimes 1_2,\ \sigma_z \otimes 1_2,\ 1_2 \otimes 1_2, 1_2 \otimes \sigma_x,\ 1_2 \otimes \sigma_y,\ 1_2 \otimes \sigma_z\}$ respectively (the superscript on u refers the component in the tensor product, the subscript the Lie algebra element) are

$$\begin{aligned}
u_x^{(1)} &= (-d_{21}, c_{21}, -d_{22}, c_{22}, -d_{11}, c_{11}, -d_{12}, c_{12})^T, \\
u_y^{(1)} &= (c_{21}, d_{21}, c_{22}, d_{22}, -c_{11}, -d_{11}, -c_{12}, -d_{12})^T, \\
u_z^{(1)} &= (-d_{11}, c_{11}, -d_{12}, c_{12}, d_{21}, -c_{21}, d_{22}, -c_{22})^T, \\
u_{\text{one}} &= (-d_{11}, c_{11}, -d_{12}, c_{12}, -d_{21}, c_{21}, -d_{22}, c_{22})^T, \\
u_x^{(2)} &= (-d_{12}, c_{12}, -d_{11}, c_{11}, -d_{22}, c_{22}, -d_{21}, c_{21})^T, \\
u_y^{(2)} &= (c_{12}, d_{12}, -c_{11}, -d_{11}, c_{22}, d_{22}, -c_{21}, -d_{21})^T, \\
u_z^{(2)} &= (-d_{11}, c_{11}, d_{12}, -c_{12}, -d_{21}, c_{21}, d_{22}, -c_{22})^T.
\end{aligned} \tag{22}$$

It may be shown that only six of these vectors are linearly independent for general values of the c_{ij} and d_{ij}. Thus the dimension of the generic orbit is six and therefore the number of parameters describing the different orbits is two. This confirms the well-known result that any state of two spins is equivalent, under local rotations, to one of the form

$$N(\cos\phi\ e_1 \otimes e_1\ +\ \sin\phi\ e_2 \otimes e_2). \tag{23}$$

2.4 Three spins

A computation similar to the one in the above subsections shows that in the case of 3 spin 1/2 particles the dimension of a generic orbit is 10, and hence the number of real non-local parameters (including the norm) is 6 $(= 2^{3+1} - 10)$.

It is interesting to note that in this case *all* the $3 \times 3 + 1 = 10$ parameters describing the local transformations $U(1) \times SU(2)^3$ are actually independent.

By brute force one can show that any 3 spin 1/2 particle state is equivalent, up to local transformations to [4]

$$\begin{aligned} N\cos\alpha e_1 \otimes (\cos\beta e_1 \otimes e_1 + \sin\beta e_2 \otimes e_2) + \\ N\sin\alpha\cos\gamma e_2 \otimes (\sin\beta e_1 \otimes e_1 - \cos\beta e_2 \otimes e_2) + \\ N\sin\alpha\sin\gamma e_2 \otimes (\cos\delta e_1 \otimes e_2 + e^{i\eta}\sin\delta e_2 \otimes e_1). \end{aligned} \tag{24}$$

A systematic way of finding the invariants is given in the next section.

3 Invariants

For some purposes one might wish to know whether or not two states are on the same orbit, i.e. are equivalent. In principle one can take the ideas of the previous section further to find invariants of the orbits. For consider any function on the space of states. If it is invariant under the action of the group then in particular it is invariant under infinitesimal group transformations. Thus it must be annihilated by the vector fields associated to the infinitesimal group transformations. Therefore in order to find a set of infinitesimal invariants one has to solve a set of simultaneous partial differential equations;

[4]This result was found independently by J. Schlienz [5]

the number of such equations is the number of linearly independent vectors associated with the Lie algebra, as in the previous section.

If we label the Lie algebra elements of local transformations $\{T_i\}$, $i = 1...3n+1$ (corresponding to the local transformations group $U(1) \times SU(2)^n$, see section 2) then the vector fields X_{T_i} are derived as in eq. (6) and an invariant function satisfies

$$X_{T_i} f = 0, \qquad i = 1...3n+1, \tag{25}$$

a set of $3n+1$ simultaneous linear partial differential equations. The method of characteristics allows one to solve the problem in principle, subject to being able to perform the integrals which arise. Unfortunately, one can easily see that the problem becomes very difficult, even for two spins, for in this case one has to solve six simultaneous partial differential equations[5].

It may turn out to be more profitable to realize that one can write down a series of *polynomial* expressions which are manifestly invariant under the local actions. We will first show a few examples and then discuss the general case.

3.1 Examples

In the case of one spin, with general state

$$\Psi = \sum_{i=1}^{2} \alpha_i e_i, \tag{26}$$

one can easily see that the expression

$$\sum_{i=1}^{2} \alpha_i \alpha_i^* \tag{27}$$

(i.e. the norm of the state) is invariant under local unitary transformations.

In the case of two spins, with general state $\Psi = \sum_{i,j=1}^{2} \alpha_{ij} e_i \otimes e_j$, the norm of the state is invariant and given by a similar expression:

$$I_1 = \sum_{i,i_1,j,j_1=1}^{2} \alpha_{ij}\alpha^*_{i_1 j_1}\delta_{ii_1}\delta_{jj_1} \quad = \quad \sum_{i,j=1}^{2} \alpha_{ij}\alpha^*_{ij}. \tag{28}$$

[5]As we saw in section 2.3 only six of the seven vector fields are linear independent in this case.

There is, however a second, quartic, expression which is functionally independent of I_1 which is also clearly invariant, since the indices have been contracted with the invariant tensor δ:

$$I_2 = \sum_1^2 \alpha_{ik}\alpha^*_{i_1 m}\alpha_{jm_1}\alpha^*_{j_1 k_1}\delta_{ii_1}\delta_{jj_1}\delta_{kk_1}\delta_{mm_1}$$
$$= \sum_1^2 \alpha_{ik}\alpha^*_{im}\alpha_{jm}\alpha^*_{jk} = \text{Trace}\left(\left(\alpha\alpha^\dagger\right)^2\right). \tag{29}$$

In the familiar form of the Schmidt coefficients eq. (23)

$$I_1 = N^2$$
$$I_2 = N^4(\cos^4\phi + \sin^4\phi). \tag{30}$$

Since we know that in the case of two spins there can only be two invariants, any further invariants must be able to be written in terms of I_1 and I_2. For example, consider

$$I_3 = \sum_{i,j,k,m,n,p=1}^2 \alpha_{ik}\alpha^*_{im}\alpha_{jn}\alpha^*_{jk}\alpha_{pm}\alpha^*_{pn} = \text{Trace}\left(\left(\alpha\alpha^\dagger\right)^3\right). \tag{31}$$

By noting, for example, that the 2×2 matrix $\alpha\alpha^\dagger$ is hermitian and satisfies a quadratic equation (by the Cayley Hamilton theorem), one may show that

$$I_3 = \frac{1}{2}\left(3I_1 I_2 - I_1^3\right). \tag{32}$$

In a similar way one may see that all higher order invariants are of the form

$$I_N = \text{Trace}\left(\left(\alpha\alpha^\dagger\right)^N\right), \quad N \geq 3 \tag{33}$$

and are expressible in terms of I_1 and I_2.

3.2 General case

A generic state of n spin $1/2$ particles can be written as

$$\Psi = \sum_{i_1,i_2,..i_n=1}^2 \alpha_{i_1 i_2 \ldots i_n} e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}.$$

Then a general polynomial expression in the coefficients is

$$\sum c_{i_1 \dots k_n \dots} \alpha_{i_1 i_2 \dots i_n} \alpha_{j_1 j_2 \dots j_n} \dots \alpha^*_{k_1 k_2 \dots k_n} \dots . \tag{34}$$

If the polynomial (34) has equal numbers of α and α^* and all the indexes of α are contracted with those of α^*, each index being contracted with an index located on the same slot (i.e. if $c_{i_1 \dots k_n \dots}$ are appropriate products of δ's) then the polynomial is manifestly invariant.

For example, in the case of three spins with generic state

$$\Psi = \sum_{i,j,k=1}^{2} \alpha_{ijk} e_i \otimes e_j \otimes e_k, \tag{35}$$

there is one quadratic invariant, the norm, there are the quartic invariants (in addition to the square of the norm),

$$\begin{aligned} J_1 &= \sum_1^2 \alpha_{ijk} \alpha^*_{ijm} \alpha_{pqm} \alpha^*_{pqk} \\ J_2 &= \sum_1^2 \alpha_{ikj} \alpha^*_{imj} \alpha_{pmq} \alpha^*_{pkq} \\ J_3 &= \sum_1^2 \alpha_{kij} \alpha^*_{mij} \alpha_{mpq} \alpha^*_{kpq}, \end{aligned} \tag{36}$$

and so on, the different invariants arising by contracting indices in different ways.

Furthermore, one can prove that *all* invariant polynomials are constructed in this way. The proof of this theorem (not given here) is based on the fact that all polynomial functions of k vectors in $\mathbb{C}^2$, invariant under $U(2)$ are polynomials in the inner-product of the vectors[6].

A key issue is the following. There are infinitely many polynomial invariants. We need to be able to construct from these polynomials a complete set of functionally independent invariants for arbitrary numbers of spins. Fortunately, given any set of polynomials, there is a algorithmic procedure of determining the relations between them using the theory of Grobner bases [7]. Thus there is a systematic way of constructing sufficient independent invariants to classify states; indeed the procedure applies to more general groups and representations than $U(2)$ acting on $\mathbb{C}^2$. Firstly determine the number N_I of independent invariants using the ideas of the previous section. Then determine the number of independent invariants at lowest order (in the case of $U(2)^n$ acting on $\otimes^n \mathbb{C}^2$ there was just one, the norm, of the form $\alpha\alpha^*$). If this number is less than N_I, construct the invariants at the next order and see which of these are functionally independent of each other and the ones previously constructed. The procedure continues until N_I are found. There is a simple formula, the Molien formula, for the generating function of the number of linearly independent invariants at each order which may well simplify the task[7].

4 Orbit Types

As discussed in the introduction, a further important question that the group theoretic approach allows one to address is what types of entanglement can occur. One can do this by recalling that by definition any group G acts transitively on an orbit O and thus an orbit may be written as

$$O = G/H \tag{37}$$

where H is the stability group of any point on the orbit. Thus the space of states of n-spins, $\mathbb{C}^{2^n}$ breaks up into orbits each of which is characterized by its stability group. Each stability group is a subgroup of $U(2)^n$, so the issue is then to find which subgroups occur as stability groups. A generic orbit will have a certain stability group, but there are also special cases are where an orbit has a larger symmetry group. If we denote by H_Ψ the invariance group of the state Ψ, we will see that states with "maximal" symmetry are particularly interesting. By states of "maximal" symmetry, we mean those states Ψ for which there are no others which have an invariance group which contain H_Ψ as a proper subgroup.

One systematic way to analyze the space of states, in principle, is to use the infinitesimal methods of section 2. Consider the 2-spin case. We found that of the eight generators of $U(2)^2$, only six were linearly independent for generic states so that generic orbits have a two dimensional invariance group. However there will be some values of the parameters describing the states for which the number of linearly independent vectors is smaller than six. Finding these points is a problem in linear algebra. Unfortunately the complexity of the calculation seems to make it impractical.

An alternative approach is to make use of the fact that every stability group is a subgroup of $U(2)^n$. One can make a list of subgroups of $U(2)^n$ and check which subgroups occur as stability groups. Goursat's theorem [4] gives a complete characterization of subgroups of any direct product of two groups and this enables one, in principle, to produce this list. The complete set of subgroups, even of $U(2)\times U(2)$ is considerable, once all discrete subgroups are taken into account. However, the example below shows that much progress in understanding the space of states can be made by considering only continuous subgroups in the first instance.

As an example, consider a (fairly general) three-spin state of the form

$$\Psi = ae_1 \otimes e_1 \otimes e_1 + be_2 \otimes e_2 \otimes e_2 + ce_1 \otimes e_1 \otimes e_2 + de_2 \otimes e_1 \otimes e_1. \qquad (38)$$

In order to find whether this state is invariant under any continuous (connected) group, it suffices to check whether it is annihilated by any Lie algebra element. As mentioned in section 2, since each copy of $U(2)$ in the group $U(2)^3$ contains a $U(1)$ subgroup corresponding to changing the global phase of the state, it suffices to consider $SU(2)^3 \times U(1)$; thus the phase is counted only once. The most general Lie algebra element in this case is

$$\begin{aligned} T &= \alpha_1(\sigma_x)_1 + \alpha_2(\sigma_x)_2 + \alpha_3(\sigma_x)_3 + \beta_1(\sigma_y)_1 + \beta_2(\sigma_y)_2 + \beta_3(\sigma_y)_3 \\ &\qquad + \gamma_1(\sigma_z)_1 + \gamma_2(\sigma_z)_2 + \gamma_3(\sigma_z)_3 + \delta 1_8, \end{aligned} \qquad (39)$$

where

$$(\sigma_x)_1 = \sigma_x \otimes 1_2 \otimes 1_2; \quad (\sigma_x)_2 = 1_2 \otimes \sigma_x \otimes 1_2 \quad \text{etc.} \qquad (40)$$

and 1_8 is the identity element

$$1_8 = 1_2 \otimes 1_2 \otimes 1_2. \qquad (41)$$

By direct calculation one can check that if a, b, c and d are all non-zero, then the state is not annihilated by any non-zero Lie algebra element so that the state is not invariant under any continuous (connected) group.

The special cases, where the state does have an invariance group, are interesting, however: consider first the case $a = 0$. If b, c and d are all non-zero then we find that the state is annihilated by the Lie algebra element with $\gamma_1 = -\gamma_2 = \gamma_3 = \delta$ with all other coefficients in T being zero; i.e. the state is invariant under $U(1)$.

If however $a = b = 0$ and c and d are non-zero with $|c| \neq |d|$, then we find that invariance is further enhanced and the state is invariant under $U(1)^2$. If $|c| = |d|$, the state has yet further symmetry, namely $U(1) \times SU(2)$ and one notices that the state is of the form a singlet with respect to particles 1 and 3 tensor product with a vector for particle 2; we write this as $singlet_{13} \otimes vector_2$. The invariance group $U(1) \times SU(2)$ arises since a singlet is invariant under a (diagonal) $SU(2)$ and the state $vector_2$ is invariant under $U(1)$. The invariance group of the state cannot be increased by choosing special (non-zero) values of c and d so a state of the form $singlet_{13} \otimes vector_2$ has maximal symmetry.

If $a = b = 0$ and one of c or d are also zero, we find the symmetry is also enhanced with respect to the case where c and d are non-zero: in this case the symmetry is $U(1)^3$ and such a state also has maximal symmetry in the sense that no state has symmetry group of which this is a subset. The state is of the form $w_1 \otimes w_2 \otimes w_3$ (i.e. it is homogeneous). In the case that $a = b = c = 0$ the generators may be taken to be $(\sigma_z)_1 + 1_8$, $(\sigma_z)_2 - 1_8$ and $(\sigma_z)_3 - 1_8$, for example.

One also finds a similar structure among the states with $a = 0$ and $c = 0$ or $a = 0$ and $d = 0$, namely invariance group of $U(1)^2$ in unless the state is one of the special ones with maximal symmetry namely either homogeneous with invariance $U(1)^3$, or of the form $singlet \otimes vector$ with invariance $SU(2) \times U(1)$.

The cases of the sets of states with $b = 0$ or $c = 0$ have similar structure to those with $a = 0$. The case of $d = 0$ is different, however.

If $d = 0$ and a, b and c are all non-zero, one calculates that the state is annihilated by $(\sigma_z)_1 - (\sigma_z)_2$ only; the state is invariant under $U(1)$. If $d = 0$ and $a = 0$ but b and c are non-zero, the invariance is enhanced to $U(1)^2$, in general or $SU(2) \times U(1)$ when $|b| = |c|$ in which case the state is of the form $singlet_{12} \otimes vector_3$, a state of maximal symmetry. When $d = 0$ and $b = 0$ but a and c are non-zero, the invariance is enhanced to $U(1)^3$; the state is homogeneous.

Perhaps the most interesting case is when $d = 0$ and $c = 0$ but a and b are non-zero, in which case one finds, for all values of a and b, that the state is invariant under $U(1)^2$. However although there are a number of states with this symmetry, thought of as an abstract group, as described above, the way that the group acts on the states is quite different in the case $d = c = 0$ than for example $d = a = 0$. In the case $d = c = 0$, the generators are $(\sigma_z)_1 - (\sigma_z)_2$ and $(\sigma_z)_2 - (\sigma_z)_3$; corresponding to correlation between spins 1 and 2 and between 2 and 3. In the case of $d = a = 0$, the invariance group arises since any vector in $\mathbb{C}^2$ is invariant under $U(1)$ and a generic two particle state is also invariant under $U(1)$.

Amongst those states with $d = c = 0$, there are some which larger symmetry groups than $U(1)^2$. If $a = 0$ or $b = 0$, then the invariance group is $U(1)^3$; the state is homogeneous. However the case $a = b$, while not having further continuous symmetry is picked out by the fact that only this state has a discrete symmetry of Z_2 corresponding to the operation of simultaneously flipping all spins. This is the famous GHZ [2] state.

5 Conclusion

In this paper we have started to build a general framework for understanding multi-particles entanglement. Obviously we have taken just a few steps here, and there are far more questions still open than answered. For example, it is known that in case of two-particle entanglement, to get a deeper understanding of entanglement one needs to take into account not only local unitary transformations but also measurements and classical communication between the two observers situated near the two particles. Also one has to consider actions taken on a large number of copies of the state Ψ and not only on a single copy as considered here. Nevertheless, it is clear that any "measure of entanglement" for multi-particles must be a function of the invariants described here.

Acknowledgments
We thank Graeme Segal and Tony Sudbery for very useful discussions. We are also very grateful to the Leverhulme and Newton Trusts for the financial support given to NL.

References

[1] J. Bell, *Physics* **1** (1964) 195.

[2] D.B. Greenberger, M. Horne and A.Zeilinger, *Physics Today*, **42** (1993) 22.

[3] A. Peres, quant-ph/9504006 and references therein.

[4] J.L. Alperin and R.B. Bell, *Groups and Representations,* Springer-Verlag, New York, 1995.

[5] J. Schlienz, PhD Thesis.

[6] M. Spivack, *A Comprehensive Introduction to Differential Geometry, Volume V, 2nd Edition,* Publish or Perish, Houston, 1979.

[7] see for example D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms,* Springer-Verlag, New York, 1992, or B. Sturmfels, *Algorithms in Invariant Theory,* Springer-Verlag, Vienna, 1993.