ON SPECIAL PIECES IN THE UNIPOTENT VARIETY

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ABSTRACT. This article is the result of experiments performed using computer programs written in the GAP language [17]. We describe an algorithm which computes a set of rational functions attached to a finite Coxeter group W. Conjecturally, these rational functions should be polynomials, and in the case where W is the Weyl group of a Chevalley group G defined over $\mathbb{F}_q$, the values of our polynomials at q should give the number of $\mathbb{F}_q$-rational points of Lusztig's special pieces [14] in the unipotent variety of G. The algorithm even works for complex reflection groups. We give a number of examples which show, in particular, that our conjecture is true for all types except possibly B_n and D_n.

1. INTRODUCTION

Let G be a connected reductive algebraic group defined over some algebraically closed field k. Let X_G be the partially ordered set of unipotent classes of G, where we write $C \leq C'$ if and only if C lies in the Zariski closure of C'. Following Spaltenstein [20] and Lusztig [14], we can define a partition of X_G into so-called *special pieces*. To do this, we first have to recall some facts about the Springer correspondence and special characters of Weyl groups.

Let W be the Weyl group of G (with respect to some maximal torus). The Springer correspondence associates with each irreducible character of W a pair (C, ψ), where $C \in X_G$ and ψ is an irreducible character of the group of components of the centralizer of an element in C. This correspondence is injective but, in general, not surjective. However, it is known that all pairs $(C, 1)$ arise in this way. Given $C \in X_G$ we denote by ϕ_C the irreducible character of W such that ϕ_C corresponds to $(C, 1)$.

Now recall from [9], [11, Chap. 4] that the irreducible characters of W are partitioned into families and that each family contains a unique *special character*. It is known that all special characters are of the form ϕ_C for some $C \in X_G$ (see [11, (13.1)] in good characteristic and [7, Theorem 2.1] in bad characteristic). A unipotent class $C \in X_G$ is called *special* if the character ϕ_C is special.

The required partition of X_G is now defined as follows. Each piece of this partition is a union of some unipotent classes of G. Two unipotent classes $C, C' \in X_G$ belong to the same piece if and only if $\phi_C, \phi_{C'}$ belong to the same family of characters of W. Since each family contains a unique special character and each special character is of the form ϕ_C for some class C, we see that each piece of X_G contains a unique special unipotent class. These pieces are called the *special pieces* of X_G. One of the main results of [14] asserts that a special piece consists precisely of the unique special unipotent class C in it and all unipotent classes in the closure of C which are not contained in the closure of any strictly smaller special unipotent class.

Now let us assume that k is an algebraic closure of the finite field $\mathbb{F}_q$ (where q is a power of some prime p) and that G has a split $\mathbb{F}_q$-rational structure, with corresponding Frobenius map F. Then all unipotent classes are F-stable. It is known that for each unipotent class C there exists a polynomial $f_C \in \mathbb{Q}[u]$ (where u is an indeterminate) such that $|C^{F^s}| = f_C(q^s)$ for all $s \geq 1$. Note, however, that the classification of unipotent classes is different for different primes p. Nevertheless, Lusztig has shown the following surprising result in [14]:

Theorem 1.1 (Lusztig [14]). *Let W be a finite Weyl group. Then there exists a collection of polynomials $\{f_\phi\} \subseteq \mathbb{Z}[u]$, one for each special character ϕ of W, such that the following hold: whenever G is a connected reductive algebraic group with Weyl group W and $F\colon G \to G$ is a Frobenius map corresponding to some split $\mathbb{F}_q$-rational structure on G (for some prime power q), then $|C^F| = f_\phi(q)$ where C is the special piece corresponding to ϕ.*

Lusztig's proof is case by case, using some very elaborate counting arguments. This paper arose from an attempt to find a more conceptual proof. We propose a general algorithm for computing the polynomials f_ϕ. This algorithm even works for complex reflection groups. Several examples of computations will be given. The algorithm was found by experimentation, using programs written in CHEVIE [6] or GAP[17].

2. The algorithm

We will describe an algorithm, which takes as input a finite Coxeter system (W, S) and returns a list of polynomials, one for each special character of W. This algorithm is a variant of that for computing Green functions, as explained in Shoji [19].

First, we need to recall the basic definitions of the a-invariants and the b-invariants of the irreducible characters of W (cf. [9]).

Let V be a real vector space and $W \subset \mathrm{GL}(V)$ the standard geometric realization of W, where the elements in S are reflections (see [3, Chap. V, §4]). Let u be an indeterminate; we define

$$P_W := \prod_{i=1}^{|S|} \frac{u^{d_i}-1}{u-1} \qquad \text{where } d_1, d_2, \ldots \text{ are the degrees of } W.$$

Let $\mathrm{CF}(W)$ be the space of $\mathbb{R}$-valued class functions on W, and let $R\colon \mathrm{CF}(W) \to \mathbb{R}(u)$ be the map defined by

$$R(f) := P_W(u-1)^{|S|} \frac{1}{|W|} \sum_{w \in W} \frac{\varepsilon(w) f(w)}{\det(u \cdot \mathrm{id}_V - w)} \qquad \text{for } f \in \mathrm{CF}(W),$$

where ε denotes the sign character. It is known that we have in fact $R(f) \in \mathbb{R}[u]$. If $\phi \in \mathrm{Irr}(W)$, then $R(\phi)$ is called the *fake degree* of ϕ. The *b-invariant* of ϕ is defined as the largest $r \geq 0$ such that u^r divides $R(\phi)$ or, equivalently, as the smallest $r \geq 0$ such that ϕ occurs with non-zero multiplicity in the character of the r-th symmetric power of the W-module V.

We define a matrix $\Omega = (\omega_{\phi,\phi'})_{\phi,\phi' \in \mathrm{Irr}(W)}$ by

$$\omega_{\phi,\phi'} = u^N R(\phi \otimes \phi' \otimes \varepsilon)$$

where N is the number of reflections in W. We shall need the following result:

Lemma 2.1 (Lusztig). *For any $\phi, \phi' \in \mathrm{Irr}(W)$, we have*

$$\omega_{\phi,\phi'} = \delta_{\phi,\phi'} u^{2N} + \textit{ linear combination of strictly smaller powers of } u.$$

(Here, $\delta_{\phi,\phi'}$ is the Kronecker symbol.) Consequently, the determinant of any principal minor of Ω is non-zero.

Proof. Let us write $\phi \otimes \phi' \otimes \varepsilon = \sum_{\phi''} c_{\phi''}\phi''$ where the sum is over all $\phi'' \in \mathrm{Irr}(W)$ and $c_{\phi''}$ are non-negative integers. It is clear that $c_\varepsilon = \delta_{\phi,\phi'}$. Hence

$$\omega_{\phi,\phi'} = u^N \delta_{\phi,\phi'} R(\varepsilon) + \sum_{\phi'' \neq \varepsilon} c_{\phi''} u^N R(\phi'').$$

Now we have $R(\varepsilon) = u^N$, and $R(\phi'')$ is a polynomial in u of degree $< N$ if $\phi'' \neq \varepsilon$. This proves the first statement. Now consider a principal minor of Ω of size k. The diagonal entries of that minor are all monic polynomials of degree $2N$, and the off-diagonal entries are polynomials in u of degree strictly smaller than $2N$. This implies that the determinant of that minor is a monic polynomial in u of degree $2Nk$; in particular, it is non-zero. □

To define the a-invariants, we need the notion of the generic degree of an irreducible character of W. These are defined in terms of the 1-parameter generic Iwahori–Hecke algebra H associated with (W, S). This is an associative algebra over the field $\mathbb{R}(u^{1/2})$ (where $u^{1/2}$ is an indeterminate), with a basis $\{T_w \mid w \in W\}$ such that the following relations hold:

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{if } l(ww') = l(w) + l(w'), \\ T_s^2 &= uT_1 + (u-1)T_s && \text{for } s \in S. \end{aligned}$$

It is known that the algebra H is split semisimple (see [10, 8, 1]) and that the values of the irreducible characters of H at basis elements T_w lie in $\mathbb{R}[u^{1/2}]$. By a deformation argument, we have in fact a bijection between the irreducible characters of H and those of W. If ϕ is an irreducible character of W, we denote by ϕ_u the corresponding character of H; this correspondence is uniquely determined by the condition that $\theta(\phi_u(T_w)) = \phi(w)$ for all $w \in W$, where $\theta\colon \mathbb{R}[u^{1/2}] \to \mathbb{R}$, $u^{1/2} \mapsto 1$. The algebra H carries a symmetrizing trace $\tau\colon H \to \mathbb{R}(u)$ given by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. A specialization argument shows that every irreducible character of H appears in τ with non-zero multiplicity. The generic degrees D_ϕ associated with the irreducible characters ϕ of W can now be defined by the equation:

$$\tau = \sum_{\phi \in \mathrm{Irr}(W)} \frac{P_W}{D_\phi} \phi_u.$$

By [2, 8, 9, 1], it is known that $D_\phi \in \mathbb{R}[u]$ for all $\phi \in \mathrm{Irr}(W)$. The a-invariant of ϕ is defined to be the largest $s \geq 0$ such that u^s divides the polynomial D_ϕ. We always have $a_\phi \leq b_\phi$, and ϕ is called *special* if we have equality.

We define a preorder on $\mathrm{Irr}(W)$ by the condition that $\phi \leq \phi'$ if and only if $a_\phi \geq a_{\phi'}$. The equivalence relation associated with this preorder will be denoted by $\phi \sim \phi'$. Thus, we have $\phi \sim \phi'$ if and only if $a_\phi = a_{\phi'}$. The following result and its proof yield the promised algorithm.

Proposition 2.2. *There exist unique elements $p_{\phi,\phi'} \in \mathbb{R}(u)$ and $\lambda_{\phi,\phi'} \in \mathbb{R}(u)$, where $\phi, \phi' \in \mathrm{Irr}(W)$, such that the following holds:*

$$\begin{aligned} \lambda_{\phi,\phi'} &= 0 && \textit{unless } \phi \sim \phi' \\ p_{\phi,\phi'} &= 0 && \textit{unless } \phi > \phi' \textit{ or } \phi = \phi' \\ p_{\phi,\phi} &= u^{a_\phi} && \textit{for all } \phi \\ \sum_{\phi_1,\phi_1' \in \mathrm{Irr}(W)} p_{\phi,\phi_1}\lambda_{\phi_1,\phi_1'}p_{\phi',\phi_1'} &= \omega_{\phi,\phi'} && \textit{for all } \phi, \phi' \end{aligned}$$

The uniqueness is clear. We prove the existence by describing an algorithm for solving the above system of equations. Choose a total ordering on $\mathrm{Irr}(W)$ compatible with the preorder $\geq$ and define matrices of unknowns $P = (p_{\phi,\phi'})$ and $\Lambda = (\lambda_{\phi,\phi'})$. Then the above system of equations says that $P\Lambda P^{tr} = \Omega$. Moreover, Λ is a block diagonal matrix, with blocks corresponding to the equivalences classes under $\sim$, and P is a block lower triangular matrix with diagonal blocks consisting of identity matrices multiplied by u^{a_ϕ}. Assume we have r blocks, of sizes $n_1, \ldots, n_r$ and with corresponding a-values $a_1, \ldots, a_r$; partitioning P, Λ, Ω into blocks, the above matrix equation has the form

$$\begin{bmatrix} I_1 & 0 & \cdots & 0 \\ P_{2,1} & I_2 & & \vdots \\ \vdots & & \ddots & 0 \\ P_{r,1} & \cdots & P_{r,r-1} & I_r \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_r \end{bmatrix} \begin{bmatrix} I_1 & P_{2,1}^{tr} & \cdots & P_{r,1}^{tr} \\ 0 & I_2 & & \vdots \\ \vdots & & \ddots & P_{r,r-1}^{tr} \\ 0 & \cdots & 0 & I_r \end{bmatrix} = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,r} \\ \Omega_{2,1} & & & \vdots \\ \vdots & & & \Omega_{r-1,r} \\ \Omega_{r,1} & \cdots & \Omega_{r,r-1} & \Omega_{r,r} \end{bmatrix}$$

where $I_i = u^{a_i}\mathrm{id}_{n_i}$. We can solve this system recursively as follows.

We begin with the first block column. We have $I_1\Lambda_1 I_1 = \Omega_{1,1}$, which determines Λ_1. For $i > 1$ we have $P_{i,1}\Lambda_1 I_1 = \Omega_{i,1}$. By Lemma 2.1, we know that $\det \Omega_{1,1} \neq 0$. Hence Λ_1 is invertible, and we can determine $P_{i,1}$. Now consider the j-th block column, where $j > 1$. Assume that the first $j-1$ block columns of P and the first $j-1$ diagonal blocks of Λ have already been determined. We have an equation

$$I_j\Lambda_j I_j + P_{j,j-1}\Lambda_{j-1}P_{j,j-1}^{tr} + \cdots + P_{j,1}\Lambda_1 P_{j,1}^{tr} = \Omega_{j,j},$$

which can be solved uniquely for Λ_j. In particular, we have now determined all coefficients in P and Λ which belong to the first j blocks. We consider the subsystem of equations made up of these blocks; this subsystem looks like the original system written in matrix form above, with r replaced by j. By Lemma 2.1, the right hand side has a non-zero determinant. Hence so have the blocks $\Lambda_1, \ldots, \Lambda_j$. Now we can determine the coefficients of P in the j-column: for $i > j$, we have an equation

$$P_{i,j}\Lambda_j I_j + P_{i,j-1}\Lambda_{j-1}P_{j,j-1}^{tr} + \cdots + P_{i,1}\Lambda_1 P_{j,1}^{tr} = \Omega_{i,j}.$$

Since Λ_j is invertible, $P_{i,j}$ is determined. Continuing in this way, the above system of equations is solved.

Remark 2.3. Lusztig has described in [12, §24] a similar algorithm for the computation of (generalized) Green functions of finite reductive groups. But in that case, it is known in advance that solutions exist (since the equations came from orthogonality relations for Green functions). In our case, we had done some experiments in GAP (see Prop. 2.8 and the examples below), and there it always turned out that solutions exist. Lusztig pointed out that in order to prove this in general, it is necessary to use Lemma 2.1, which he kindly communicated to us.

Remark 2.4. Instead of the preorder $\leq$ defined above, we could have also used any refinement of it such that the equivalence classes are precisely the families of $\mathrm{Irr}(W)$ (in the sense of [9]). Since the a-function is constant on families, this would just yield a finer partition of $\mathrm{Irr}(W)$, but otherwise the algorithm would be the same. But is not clear that the result would also be the same; for this it would be required that the following condition is satisfied:

$$\text{We have } \lambda_{\phi,\phi'} = 0 \text{ unless } \phi, \phi' \text{ belong to the same family.} \qquad (*)$$

In all examples that we computed, this condition turns out to be satisfied.

Similarly to [12, Theorem 24.8], we expect that the above algorithm actually yields polynomials:

Conjecture 2.5. We have $p_{\phi,\phi'} \in \mathbb{R}[u]$ and $\lambda_{\phi,\phi'} \in \mathbb{R}[u]$ for all $\phi, \phi' \in \mathrm{Irr}(W)$.

To each irreducible character ϕ of W, we can associate a rational function $f_\phi \in \mathbb{R}(u)$ by $f_\phi = \lambda_{\phi,\phi}$. We expect that the rational functions associated with the special characters will be of particular importance:

Conjecture 2.6. We have $\sum_\phi f_\phi = u^{2N}$, where the sum is over all special characters ϕ of W.

Conjecture 2.7. Assume that W is a Weyl group, and let G be a connected reductive algebraic group such that W is the Weyl group of G with respect to some maximal torus. Assume, moreover, that $F\colon G \to G$ is a Frobenius map corresponding to some split $\mathbb{F}_q$-rational structure on G (where q is some prime power). Let ϕ be a special character of W and let C be the corresponding special piece of the unipotent variety of G. Then we have $|C^F| = f_\phi(q)$.

Proposition 2.8. *The above three conjectures are true if (W, S) is irreducible of type A_n (any $n \geq 1$), G_2, F_4, E_6, E_7, or E_8. Moreover, condition $(*)$ in Remark 2.4 holds in these cases.*

Proof. If (W, S) is of exceptional type, we have used an implementation of our algorithm in GAP [17] and CHEVIE [6] to compute explicitly all elements $p_{\phi,\phi'}$ and $\lambda_{\phi,\phi'}$. By inspection, Conjectures 2.5 and 2.6, and condition $(*)$ are verified. Moreover, we are indebted to Frank Lübeck for comparing the results of our algorithm with his data files containing the existing tables (due to Shoji, Mizuno) for unipotent classes in exceptional Chevalley groups, and thus also verifying Conjecture 2.7.

Finally, let (W, S) be of type A_{n-1} so that the corresponding Chevalley group is $G = \mathrm{GL}_n$. In this case, all irreducible characters of W are special, and the special pieces are just the unipotent classes of G. Hence it would be sufficient to show that our algorithm produces the same result as that for computing the Green functions of G; see [19]. In the latter algorithm, we have to consider a system of matrix equations $Q\Lambda' Q^{tr} = \Omega$ where the matrices Q, Λ' satisfy similar requirements as in Prop. 2.2 but they are partitioned into blocks of size 1 (since all characters are special). Thus, by the uniqueness of solutions, it is enough to show that Q, Λ' are automatically partitioned into blocks as required by our algorithm. This is clear for Λ' (since this is a diagonal matrix). As far as Q is concerned, we must show that if $a_\phi = a_{\phi'}$ (for $\phi \neq \phi'$) then $q_{\phi,\phi'} = 0$. Assume, if possible, that this is not the case. Let C, C' be unipotent classes in G such that $\phi = \phi_C$, $\phi' = \phi_{C'}$. The condition $\phi \neq \phi'$ implies $C \neq C'$. Now it is known that $q_{\phi,\phi'} = 0$ unless C is contained in the closure of C' (see [19, Sect. 5]). But if $a_\phi = a_{\phi'}$, then $\dim C = \dim C'$ and hence $C = C'$, a contradiction. □

We have also checked that the conjectures are true for all Weyl groups of classical types of low rank. In the case of non-crystallographic finite Coxeter groups, the algorithm yields the following results:

Lemma 2.9. *Let $W = \langle s, t\rangle$ be a dihedral group such that st has order $m \geq 3$. Then there are precisely three special characters, namely the trivial character 1_W, the sign character ε, and the character ρ of the standard reflection representation. The associated polynomials are given as follows.*

ϕ	b_ϕ	f_ϕ
ε	m	1
ρ	1	$(u^{m-2}+1)(u^m-1)$
1_W	0	$u^{m-2}(u^2-1)(u^m-1)$

The sum of these three polynomials is u^{2m}, as it should be.

Proof. We solve the system of equations defining P and Λ along the lines of the proof of Proposition 2.2. We label the irreducible characters of W such that the first is the sign character, the second the reflection character and the last the trivial character. This ordering is compatible with the preorder introduced above. Then P, Λ have the shapes

$$P = \begin{pmatrix} u^m & 0 & 0 \\ p & uI_{k-2} & 0 \\ p_{k1} & q^t & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & \lambda_{kk} \end{pmatrix},$$

with $p = (p_{21}, \dots, p_{k-1,1})^t$, $q = (p_{k2}, \dots, p_{k,k-1})^t$, while

$$\Omega = \begin{pmatrix} u^{2m} & u^m R^t & u^m \\ u^m R & \Omega' & u^m \tilde{R} \\ u^m & u^m \tilde{R}^t & u^{2m} \end{pmatrix},$$

with $R = (R(\phi_2), \dots, R(\phi_{k-1}))^t$, $\tilde{R} = (R(\phi_2 \otimes \varepsilon), \dots, R(\phi_{k-1} \otimes \varepsilon))^t$. The upper left 2 by 2 block of $P\Lambda P^{tr}$ equals

$$\begin{pmatrix} u^{2m}\lambda_{11} & u^m \lambda_{11} p_{21} \\ u^m \lambda_{11} p_{21} & p_{21}^2 \lambda_{11} + u^2 \lambda_{22} \end{pmatrix}$$

while the upper leftmost part of Ω equals

$$\begin{pmatrix} u^{2m} & u^m(u^{m-1} + u) \\ u^m(u^{m-1} + u) & u^m(1 + u^m + u^{m-2} + u^2) \end{pmatrix}$$

by the definition of ω_{ij}. The assertion on the first and second line of the table follows.

More generally, this leads to the equations $p = R$, $p_{k1} = 1$ and then

$$RR^t + u^2 \Lambda' = \Omega', \quad R + u\Lambda' q = u^m \tilde{R}, \quad 1 + q^t \Lambda' q + \lambda_{kk} = u^{2m}.$$

Clearly this determines $\Lambda' = u^{-2}(\Omega' - RR^t)$. Inserting this into the next equation gives $R + u^{-1}(\Omega' - RR^t)q = u^m \tilde{R}$. We claim that $q = (1, 0, \dots, 0)^t$ is a solution to this. Then the last equation simplifies to $1 + \lambda_{22} + \lambda_{kk} = u^{2m}$ and the lemma is proved. Thus it remains to check that

$$R(\phi) + u^{m-1} R(\rho \otimes \phi \otimes \varepsilon) - u^{-1} R(\rho) R(\phi) = u^m R(\phi \otimes \varepsilon)$$

for all irreducible characters ϕ of W lying in the same family as ρ, i.e., different from 1 and ε. This is an easy exercise. □

Example 2.10. Let (W, S) be a Coxeter system of non-crystallographic type H_3 or H_4. Using an implementation of the above algorithm in GAP [17] and CHEVIE [6], we find the following polynomials f_ϕ corresponding to special characters ϕ. We label an irreducible characters of W by a pair (m, e), where m denotes the degree and e is the b-invariant. To abbreviate notation, we write $[i] = u^i - 1$.

H_3	f_ϕ
$\phi_{1,15}$	1
$\phi_{3,6}$	$(u^8+u^4+1)[10]$
$\phi_{5,5}$	$u^4[6][10]$
$\phi_{4,3}$	$u^8[6][10]$
$\phi_{5,2}$	$u^{10}[6][10]$
$\phi_{3,1}$	$u^{10}[2][6][10]$
$\phi_{1,0}$	$u^{12}[2][6][10]$

H_4	f_ϕ
$\phi_{1,60}$	1
$\phi_{4,31}$	$(u^{28}+u^{18}+u^{10}+1)[30]$
$\phi_{9,22}$	$u^{10}(u^{16}+u^8+1)[20][30]$
$\phi_{16,18}$	$u^{28}(u^6+1)[20][30]$
$\phi_{25,16}$	$u^{26}[12][20][30]$
$\phi_{36,15}$	$u^{38}(u^{10}+1)[12][30]$
$\phi_{24,6}$	$u^{34}(u^{12}+u^4+1)[12][20][30]$
$\phi_{36,5}$	$u^{48}[12][20][30]$
$\phi_{25,4}$	$u^{48}[2][12][20][30]$
$\phi_{16,3}$	$u^{50}[2][12][20][30]$
$\phi_{9,2}$	$u^{52}[2][12][20][30]$
$\phi_{4,1}$	$u^{54}[2][12][20][30]$
$\phi_{1,0}$	$u^{56}[2][12][20][30]$

The sum of these polynomials is u^{30} and u^{120}, respectively. Moreover, we have checked that condition $(*)$ in Remark 2.4 is satisfied.

We think that the above polynomials for non-crystallographic finite Coxeter groups are those on whose existence was speculated in [14, (6.10)]. For this note that the first, second and last polynomial in each case coincides with the value predicted in loc. cit.

Remark 2.11. In [14, (6.10)] Lusztig gives a formula for the size of the special piece corresponding to the special character $\rho\otimes\varepsilon$. Namely, let $d_1,\dots,d_l$ be the degrees of W, $m_1,\dots,m_l$ the coexponents (see for example [16]). Then we should have

$$f_{\rho\otimes\varepsilon} = (u^h-1)\sum_{i=1}^{l} u^{m_i-1}$$

where $h=\max\{d_1,\dots,d_l\}$ is the Coxeter number of W. A short calculation shows that this is the result given by our algorithm if and only if

$$R(\rho\otimes\rho) = R(\rho)\,(R(\rho)+u^{-1}-u^{h-1}).$$

This can be checked for the irreducible finite Coxeter groups. Unfortunately we do not see an a priori proof of this formula. (The fake degree of the antisymmetric square $\Lambda^2(\rho)$ was computed in [16, Cor. 3.2].)

3. An extension

Recall that our algorithm is a variant of that for computing Green functions. Now the latter admits a generalization to the computation of generalized Green functions; see [12, §24]. Lusztig suggested that our algorithm should admit a similar generalization.

What we have to do is to consider another Coxeter system (W_1, S_1) such that S is a subset of S_1 and the relations for W are determined from those in W_1 by the scheme explained in [13, (1.3)]. The choice of W_1 is subject to the requirement that the parabolic subgroup of W_1 generated by $S_1 \setminus S$ should admit a "cuspidal unipotent character" (see [13, (2.4)]) and hence a cuspidal family of characters in the sense of [11, (8.1)]. We then consider essentially a similar system of equations as before, but with some modifications taking into account the presence of W_1.

We define a new matrix $\tilde{\Omega} = (\tilde{\omega}_{\phi,\phi'})_{\phi,\phi' \in \mathrm{Irr}(W)}$ by

$$\tilde{\omega}_{\phi,\phi'} = u^{N_1-N} u^{|S|-|S_1|}(u-1)^{|S_1|-|S|} \frac{P_{W_1}}{P_W} \omega_{\phi,\phi'},$$

where N_1 is the number of reflections in W_1 and P_{W_1} is defined in terms of the degrees of W_1. (This is analogous to the definition in [12, (24.3.4)].)

We also have to modify the a-invariants attached to the irreducible characters of W. The pair (W, W_1) determines a function $f \colon S \to \{1, 2, \ldots\}$ such that $f(s) = f(t)$ whenever $s, t \in S$ are conjugate in W (see [13, 2.4(b)]). We consider the generic Iwahori–Hecke algebra H^f defined in a similar way as before, but now the quadratic relations read:

$$T_s^2 = u^{f(s)} T_1 + (u^{f(s)} - 1) T_s \qquad \text{for } s \in S.$$

Again, we have corresponding generic degrees D_ϕ^f (which are not necessarily polynomials!). The new a-invariants are now defined by

$$a_\phi^f = a_0 + (\text{order of the pole at } u = 0 \text{ of } D_\phi^f),$$

where a_0 is the (usual) a-invariant of the characters belonging to the cuspidal family of characters of the parabolic subgroup of W_1 generated by $S_1 \setminus S$.

Taking these data, we can formulate an analogous version of Prop. 2.2, and one might expect that Conjecture 2.5 still holds. We have checked that this is in fact true for all (W, W_1) where W_1 is a finite Coxeter group of exceptional type. Note that no new cases arise for W_1 of type A_{n-1}.

Example 3.1. Let (W_1, S_1) be of type H_4. According to [13, §3.3], we have three possibilities such that the requirements for the above setting are satisfied: (W, S) of type $\emptyset$, A_1 or $I_2(10)$. The first case is trivial; let us consider the other two possibilities.

If (W, S) is of type A_1, the function f takes value 15, and we have $a_0 = 3$. The modified a-invariants of the sign and the trivial character are 15 and 0, respectively. The matrix Λ consists of two 1×1-blocks with entries $u^{20}[12][20][30]$ and $u^{50}[12][20][30]$. We have

$$P = \begin{bmatrix} u^{18} & \cdot \\ u^{17} & u^3 \end{bmatrix}.$$

If (W,S) is of type $I_2(10)$, the function f takes values $1,5$, and we have $a_0=1$. The modified a-invariants are given by

ϕ	$\phi_{1,10}$	$\phi'_{1,5}$	$\phi_{2,1}$	$\phi_{2,2}$	$\phi_{2,3}$	$\phi_{2,4}$	$\phi''_{1,5}$	$\phi_{1,0}$
a^f_ϕ	31	22	6	6	6	6	2	1

The matrix Λ has 5 blocks, of sizes $1,1,4,1,1$: the entries are $u^6(u^{10}+1)[12][30]$, $u^{14}[12][20][30]$,

$$\begin{bmatrix} u^{46}[12][20][30] & u^{45}[12][20][30] & u^{44}[12][20][30] & u^{43}[12][20][30] \\ u^{45}[12][20][30] & u^{46}[12][20][30] & u^{45}[12][20][30] & u^{44}[12][20][30] \\ u^{44}[12][20][30] & u^{45}[12][20][30] & u^{46}[12][20][30] & u^{45}[12][20][30] \\ u^{43}[12][20][30] & u^{44}[12][20][30] & u^{45}[12][20][30] & u^{46}[12][20][30] \end{bmatrix},$$

and $u^{52}[2][12][30][30]$, $u^{54}[2][12][20][30]$. We have

$$P=\begin{bmatrix} u^{31} & . & . & . & . & . & . & . \\ u^{26} & u^{22} & . & . & . & . & . & . \\ u^{30}+u^{22} & u^{18} & u^6 & . & . & . & . & . \\ u^{29}+u^{23} & u^{19} & . & u^6 & . & . & . & . \\ u^{28}+u^{24} & u^{20} & . & . & u^6 & . & . & . \\ u^{27}+u^{25} & u^{21} & . & . & . & u^6 & . & . \\ u^{26} & . & . & . & . & u^5 & u^2 & . \\ u^{21} & u^{17} & u^5 & . & . & . & . & u \end{bmatrix}$$

In particular, we see that all entries in these matrices are polynomials.

4. Complex reflection groups

Let now V be a complex vector space and $W\subset \mathrm{GL}(V)$ be a finite group generated by pseudo-reflections. In order to describe a generalization of the algorithm put forward in the previous section to W we mimic the approach in the real case.

First note that the following definition of $R\colon \mathrm{CF}(W)\to\mathbb{C}[u]$,

$$R(f):=P_W(u-1)^{\dim(V)}\frac{1}{|W|}\sum_{w\in W}\frac{\det_V(w)f(w)}{\det_V(u\cdot \mathrm{id}_V-w^{-1})}\qquad \text{for } f\in \mathrm{CF}(W),$$

where $\det_V$ denotes the determinant character of W on V, makes sense for complex reflection groups and generalizes the definition of R in Section 2. We let N^* be the number of pseudo-reflections in W and define a matrix Ω by $\omega_{\phi,\phi'}:=u^{N^*}R(\phi\otimes\phi'\otimes\det_V)$ as in Section 2.

To define the a-invariant of an irreducible character of W, we now work with $H=H(W,u)$, the cyclotomic Hecke algebra for W over $\mathbb{C}[u,u^{-1}]$ with one single parameter u (see [4]). Let K be a sufficiently large extension of $\mathbb{C}(u)$ and H_K the algebra obtained by extending scalars from $\mathbb{C}[u,u^{-1}]$ to K. A deformation argument shows again that we have a bijection, $\phi\leftrightarrow\phi_u$, between the irreducible characters of W and those of H_K. The definition of generic degrees is more subtle in the present situation: it is conjectured in

[4] (and has now been proved for all but finitely many irreducible W) that H carries a canonical symmetrizing form $\tau\colon H \to \mathbb{C}[u,u^{-1}]$, which in particular vanishes on all elements of a suitable basis (except the identity element), and which specializes to the usual trace form on the group ring of W. Hence, in a similar way as before, we see that every irreducible character of H_K appears in τ with non-zero multiplicity, and we may define generic degrees by the equation:

$$\tau = \sum_{\phi \in \mathrm{Irr}(W)} \frac{P_W}{D_\phi} \phi_u.$$

Assume that W is irreducible and generated by $\dim V = n$ reflections of order 2. Then it is expected that D_ϕ is a polynomial in $\mathbb{C}[u]$. We can then define a_ϕ to be the precise power of u dividing D_ϕ. A character $\phi \in \mathrm{Irr}(W)$ is called *special* if a_ϕ is also the precise power of u dividing $R(\phi)$.

Let W be an irreducible complex reflection group satisfying the assumptions made above. Then either W is real, or $W = G(e,e,n)$ for some $e \geq 3$, $n \geq 3$, (here, the special characters have been identified in [15, Lemma 5.16]), or W is one of the primitive complex reflection groups G_i, $i \in \{24, 27, 29, 33, 34\}$ in the notation of Shephard and Todd [18]. For such W the algorithm put forward in Section 2 still makes perfect sense. We believe that the analogues of Conjectures 2.5 and 2.6 remain valid in this more general situation.

Example 4.1. We have used an implementation of the algorithm in GAP [17] and CHEVIE [6] to verify the conjectures on all the primitive complex reflection groups G_i, $i \in \{24, 27, 29, 33, 34\}$. The a-values of the irreducible characters of these groups were determined in [5] as a consequence of the determination of unipotent degrees. Our algorithm yields polynomial entries for P and Λ. The diagonal entries of Λ corresponding to special characters (the lengths of the special pieces) are collected in the subsequent tables. Their sum equals u^{42}, u^{90}, u^{80}, u^{90}, u^{252} respectively. Note also that for each of the complex reflection groups above, the size of the second special piece is again given by the formula in Remark 2.11. Here, the special irreducible characters are labeled by pairs (m, e), where m denotes the degree and e is the a-invariant.

G_{24}	
$\phi_{1,21}$	1
$\phi_{3,8}$	$(u^{10}+u^8+1)[14]$
$\phi_{7,6}$	$u^8(u^2+1)[6][14]$
$\phi_{8,4}$	$u^{12}(u^2+1)[6][14]$
$\phi_{7,3}$	$u^{12}[4][6][14]$
$\phi_{3,1}$	$u^{14}[4][6][14]$
$\phi_{1,0}$	$u^{18}[4][6][14]$

G_{27}	
$\phi_{1,45}$	1
$\phi_{3,16}$	$(u^{24}+u^{18}+1)[30]$
$\phi_{10,12}$	$u^{18}(u^{6}+1)[12][30]$
$\phi_{9,9}$	$u^{30}[12][30]$
$\phi_{15,8}$	$u^{30}[12][30]$
$\phi_{8,6}$	$u^{30}[6][12][30]$
$\phi_{15,5}$	$u^{30}[6][12][30]$
$\phi_{9,4}$	0
$\phi_{10,3}$	$u^{36}[6][12][30]$
$\phi_{3,1}$	0
$\phi_{1,0}$	$u^{42}[6][12][30]$

G_{29}	
$\phi_{1,40}$	1
$\phi_{4,21}$	$(u^{16}+u^{12}+u^{8}+1)[20]$
$\phi_{10,18}$	$u^{8}(u^{4}+1)[12][20]$
$\phi_{16,13}$	$2u^{16}(u^{4}+1)[12][20]$
$\phi_{15,12}$	$u^{24}[12][20]$
$\phi_{15,12}$	$u^{16}[8][12][20]$
$\phi_{20,9}$	$2u^{20}[8][12][20]$
$\phi_{24,6}$	$u^{24}(u^{4}+1)[8][12][20]$
$\phi_{20,5}$	$u^{28}[8][12][20]$
$\phi_{15,4}$	$u^{28}[4][8][12][20]$
$\phi_{15,4}$	$u^{32}[8][12][20]$
$\phi_{16,3}$	0
$\phi_{10,2}$	$u^{32}[4][8][12][20]$
$\phi_{4,1}$	0
$\phi_{1,0}$	$u^{36}[4][8][12][20]$

G_{33}	
$\phi_{1,45}$	1
$\phi_{5,28}$	$(u^{14}+u^{12}+u^{8}+u^{6}+1)[18]$
$\phi_{15,23}$	$u^{6}(u^{10}+u^{8}+u^{6}+u^{4}+u^{2}+1)[10][18]$
$\phi_{30,18}$	$u^{10}(u^{16}+u^{14}+2u^{12}+2u^{10}+u^{8}+u^{6}+u^{4}-1)[10][18]$
$\phi_{30,13}$	$u^{14}(u^{8}+2u^{6}+u^{4}+u^{2}+1)[10][12][18]$
$\phi_{15,12}$	$u^{26}[10][12][18]$
$\phi_{81,11}$	$u^{22}(u^{6}+2u^{4}+2u^{2}+1)[10][12][18]$
$\phi_{60,10}$	$u^{22}(u^{2}+1)[6][10][12][18]$
$\phi_{45,10}$	$u^{24}[4][10][18][12]$
$\phi_{15,9}$	$u^{28}[4][10][18][12]$
$\phi_{64,8}$	$u^{26}(u^{2}+1)[6][10][12][18]$
$\phi_{60,7}$	$u^{28}(u^{2}+1)[6][10][12][18]$
$\phi_{45,7}$	$u^{26}(u^{2}+1)[6][10][12][18]$
$\phi_{81,6}$	$u^{28}[4][6][10][12][18]$
$\phi_{30,4}$	$u^{28}(u^{2}+1)[4][6][10][12][18]$
$\phi_{30,3}$	$u^{36}(u^{2}+1)[6][10][12][18]$
$\phi_{15,2}$	$u^{36}[4][6][10][12][18]$
$\phi_{5,1}$	0
$\phi_{1,0}$	$u^{40}[4][6][10][12][18]$

G_{34}	
$\phi_{1,126}$	1
$\phi_{6,85}$	$(u^{36}+u^{30}+u^{12}+u^{24}+u^{18}+1)[42]$
$\phi_{21,68}$	$u^{12}(u^{30}+2u^{24}+2u^{18}+2u^{12}+u^{6}+1)[30][42]$
$\phi_{56,57}$	$u^{24}(u^{42}+2u^{36}+3u^{30}+3u^{24}+u^{18}-u^{6}-1)[30][42]$
$\phi_{105,46}$	$u^{36}(2u^{24}+4u^{18}+4u^{12}+3u^{6}+1)[24][30][42]$
$\phi_{70,45}$	$u^{66}[24][30][42]$
$\phi_{126,41}$	$u^{42}(u^{12}+2u^{6}+1)[18][24][30][42]$
$\phi_{315,36}$	$u^{54}(u^{30}+3u^{24}+4u^{18}+2u^{12}-1)[24][30][42]$
$\phi_{420,31}$	$u^{60}(3u^{12}+3u^{6}+1)[18][24][30][42]$
$\phi_{210,30}$	$u^{78}[18][24][30][42]$
$\phi_{384,29}$	$u^{72}(u^{6}+1)[18][24][30][42]$
$\phi_{315,28}$	$u^{66}(u^{12}+u^{6}+1)[18][24][30][42]$
$\phi_{560,27}$	$u^{66}(u^{6}+1)[12][18][24][30][42]$
$\phi_{729,24}$	$u^{78}(u^{12}+3u^{6}+2)[18][24][30][42]$
$\phi_{840,23}$	$u^{72}(u^{6}+1)[12][18][24][30][42]$
$\phi_{630,23}$	$u^{78}[12][18][24][30][42]$
$\phi_{896,21}$	$u^{84}[12][18][24][30][42]$
$\phi_{630,20}$	$u^{84}[12][18][24][30][42]$
$\phi_{840,19}$	$u^{78}(2u^{6}+1)[12][18][24][30][42]$
$\phi_{560,18}$	$u^{96}(u^{6}+1)[18][24][30][42]$
$\phi_{1280,15}$	$u^{84}(u^{12}+2u^{6}-1)[12][18][24][30][42]$
$\phi_{630,14}$	$u^{96}[12][18][24][30][42]$
$\phi_{840,13}$	$2u^{96}[12][18][24][30][42]$
$\phi_{896,12}$	$u^{96}[6][12][18][24][30][42]$
$\phi_{210,12}$	$u^{102}[12][18][24][30][42]$
$\phi_{840,11}$	$u^{102}[12][18][24][30][42]$
$\phi_{630,11}$	$u^{96}[6][12][18][24][30][42]$
$\phi_{729,10}$	0
$\phi_{315,10}$	0
$\phi_{560,9}$	$u^{102}[6][12][18][24][30][42]$
$\phi_{70,9}$	$u^{108}[12][18][24][30][42]$
$\phi_{384,8}$	0
$\phi_{420,7}$	$u^{102}[6][12][18][24][30][42]$
$\phi_{315,6}$	$u^{108}[6][12][18][24][30][42]$
$\phi_{126,5}$	0
$\phi_{105,4}$	$u^{108}[6][12][18][24][30][42]$
$\phi_{56,3}$	$u^{114}[6][12][18][24][30][42]$
$\phi_{21,2}$	0
$\phi_{6,1}$	0
$\phi_{1,0}$	$u^{120}[6][12][18][24][30][42]$

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