Level spacings for quantum maps in genus zero [1]

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Abstract

We study the pair correlation function for a variety of completely integrable quantum maps in one degree of freedom. For simplicity we assume that the classical phase space M is $\mathbb{C}P^1 = S^2$ and that the classical map is a fixed-time map $expt\Xi_H$ of a Hamilton flow. The quantization is then a unitary N x N matrix $U_{t,N}$ and its pair correlation measure $\rho_{2,t}^{(N)}$ counts the number of eigenvalues in intervals of length comparable to the mean level spacing ($\sim 1/N$). The physicists' conjecture (Berry-Tabor conjecture) is that as $N \to \infty$, $\rho_{2,t}^{(N)}$ should converge to the pair correlation function $\rho_2^{POISSON} = \delta_o + 1$ of a Poisson process. We prove this on average in t if H is a perfect Morse function on $\mathbb{C}P^1$. Under some conditions on the second derivative we further prove that the variance from the mean tends to zero at a power law rate. It follows that for a slightly sparse sequence of Planck's constants, the quantum maps almost always have Poisson pair correlation functions.

Contents

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0 Introduction

In this paper we shall be concerned with the fine structure of the spectra of some completely integrable quantum maps in genus zero, that is, with quantizations of integrable symplectic maps χ on the Riemann sphere $M = (\mathbb{C}P^1, N dz \wedge d\bar{z})$. The quantum systems then consist of a sequence of unitary operators $\{U_{\chi,N}\}$ on a Hilbert space $\mathcal{H}_N$ of dimension N. For simplicity, we restrict attention to quantizations $U_{t,N}$ of Hamilton flows $\chi_t = expt\Xi_H$ where the Hamiltonian H has no separatrix levels. Our interest is in the semiclassical asymptotics ($N \to \infty$) of the pair correlation function $\rho_{2,t}^{(N)}$ and number variance $\Sigma_{2,t}(N, L)$ of the quantum systems. We first show that the time-averages of these objects tend to the Poisson limits

$$\frac{1}{b-a}\int_a^b \rho_{2,t}^N dt \to \rho_2^{POISSON} = \delta_o + 1, \qquad \frac{1}{b-a}\int_a^b \Sigma_{2,t}(N, L) dt \to L$$

as $N \to \infty$. This is consistent with the Berry-Tabor conjecture [B.T] that eigenvalues of completely integrable quantum systems behave like random numbers (waiting times of a Poisson process). However, it is only a weak test of the conjecture since the averaging process itself induces a good deal of the randomness. A much stronger test is whether the variance tends to zero. For special 2-parameter families of Hamiltonians $H_{\alpha,\beta} = \alpha\phi(\hat{I}) + \beta\hat{I}$ (see §2 for the definition) we show that the variance tends to zero at a power law rate. This implies that the individual systems are almost always Poisson along a slightly sparse subsequence of Planck constants.

Before describing the models and results more precisely, let us recall what the level spacings problems are about. In the quantization $H \to \hat{H}_N$ of Hamiltonians on compact phase spaces M of dimension $2f$, the 'Planck constant' is constrained to the values $h = 1/N$ and spectrum of $\hat{H}_N$ consist of $d_N \sim N^{\frac{1}{2}N}$ eigenvalues $\{\lambda_{N,j}\}$ in a bounded interval $[minH, maxH]$. Similarly, the spectrum of a quantum map $U_{\chi,N}$ consists of d_N eigenvalues $\{e^{i\theta_{N,j}}\}$ on the unit circle S^1. The density of states in degree N

$$d\rho_1^{(N)} = \frac{1}{d_N}\sum_{j=1}^{d_N}\delta(\lambda_{N,j}), \qquad \text{resp.} \quad d\rho_1^{(N)} = \frac{1}{d_N}\sum_{j=1}^{d_N}\delta(e^{i\theta_{N,j}})$$

has a well defined weak limit as $N \to \infty$ which may be calculated by standard methods of microlocal analysis (§2). According to the physicists, there also exist asymptotic patterns in the spectra on the much smaller length scale of the mean level spacing $\frac{1}{d_N}$ between consecutive eigenvalues. The pair correlation function ρ_2, for instance, is the limit distribution of spacings between all pairs of normalized eigenvalues $d_N\lambda_{Nj}$. The length scale $\frac{1}{d_N}$ is usually below the resolving power of micrlocal methods. Hence the problem of rigorously determining the limit , or even of determining whether it exists, has remained open for almost all quantum systems. The sole exceptions are the cases of almost all flat 2-tori [Sa.2] and Zoll surfaces [U.Z].

On the other hand, there exist numerous computer studies of eigenvalue spacings in the physics literature which indicate that limit PCFs often exist. The following conjectures give a rough guideline towards the expected shape of the level spacings statistics:

• When the classical system is generic chaotic, $\rho_2 = \rho_2^{GOE}$ where ρ_2^{GOE} is the limit expected PCF for NxN random matrices in the Gaussian orthogonal ensemble;

• When the classical system is generic completely integrable, $\rho_2 = \rho_2^{POISSON} := 1 + \delta_0$. That is, the normalized spacings between eigenvalues behave like waiting times of a Poisson process. The term δ_0 comes from the diagonal, while the term 1 reflects that any spacing between distinct pairs is as likely as any other.

These conjectures should not be taken too literally, and indeed cannot be since the term 'generic' is not precisely defined. Our main purpose in this article is to test the Poisson conjecture against Hamiltonian systems of one degree of freedom on the compact Kahler phase space $\mathbb{C}P^1$. Of course, they are necessarily completely integrable. It might also be suspected that quantized Hamilton flows in one degree of freedom are necessarily trivial, but this is not the case: as will be seen, toral completely integrable systems on $\mathbb{C}P^1$ are almost always Poisson along a slighty sparse subsequence of Planck constants. Moreover, it should be recalled that many of the model quantum chaotic systems such as kicked tops and rotors and cat maps take place in one degree of freedom and still defy rigorous analysis.

Now let us be more precise about the models we will study. In the usual Kahler quantization of $(\mathbb{C}P^1, Ndz\wedge d\bar{z})$, $\mathcal{H}_N$ may be identified with the space $\mathcal{P}_N$ of homogeneous holomorphic polynomials $f(z_1, z_2)$ of degree N on $\mathbb{C}^2$. A classial Hamiltonian $H \in C^\infty(\mathbb{C}P^1)$ resp. $C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$ is then quantized as a self-adjoint Toeplitz operator

$$\hat{H}^{(N)} := \Pi_N H \Pi_N : \mathcal{H}_N \to \mathcal{H}_N, \qquad \psi_{N,j} \to \Pi_N H \psi_{N,j}$$

where Π_N is the (Cauchy-Szego) orthogonal projection on $\mathcal{H}_N$ (§1). Hence the quantum Hamiltonian system amounts to the eigenvalue problem:

$$\hat{H}^{(N)}\phi_{N,j} = \lambda_{N,j}\phi_{N,j}, \qquad -||H||_\infty \leq \lambda_{N,1} \leq \lambda_{N,2} \leq \cdots \lambda_{N,N} \leq ||H||_\infty.$$

For a fixed value $h = \frac{1}{N}$ of the Planck constant, the distribution of normalized spacings between all possible pairs of eigenvalues of $\hat{H}^{(N)}$ is given by the Nth pair correlation function (or measure)

$$d\rho_2^{(N)}(x) = \frac{1}{N}\sum_{i,j=1}^{N} \delta(x - N(\lambda_{N,i} - \lambda_{N,j})).$$

Here, the eigenvalues are rescaled, $\lambda_{N,j} \to N\lambda_{N,j}$ to have unit mean level spacing, i.e. so that $N\lambda_{N,i+1} - N\lambda_{N,i} \sim 1$ on average

Our first result gives an explicit formula for the limit pair correlation function

$$d\rho_2 = lim_{N\to\infty} d\rho_2^{(N)}(x)$$

of a quantized Hamiltonian $\hat{H}^{(N)}$. It is of a similar nature to the pair correlation function for a Zoll Laplacian ([U.Z]) and involves dynamical invariants of the classical Hamiltonian flow $exp t\Xi_H$ generated by H on the classical phase space. Under some generic hypotheses (which will be stated precisely in §2), the formula is given by:

Theorem A *For the generic $H \in C^\infty(M)$, the limit pair correlation function for the system $\hat{H}^{(N)}$ is given by:*

$$\rho_2(f) = V\hat{f}(0) + \sum_{k\in\mathbb{Z}}\sum_{\nu=1}^{M}\sum_{j=1}^{N(\nu)} \int_{(c_\nu, c_{\nu+1})} \hat{f}(kT_j^\nu(E))T_j^\nu(E)^2 dE$$

where:
(i) $V = vol\{(z_1, z_2) \in M \times M : H(z_1) = H(z_2)\}$;
(ii) $\{c_\nu\}$ is the set of critical values of H;
(iii) For a regular value $E \in (c_\nu, c_{\nu+1}$, $H^{-1}(E)$ is a union of period orbits $\{\gamma_\nu^j\}_{j=1}^{N(\nu)}$ of $exp t\Xi_H$ and $T_j^\nu(E)$ is the minimal positive period of the jth component.

A more precise statement will be given in §2.

It follows that the pair correlation function of quantized Hamiltonians in one degree of freedom is quite deterministic. On the other hand, the eigenvalues of the associated quantized Hamiltonian flow are much more random.

Before describing the results, let us recall the definition of a quantum map and of its pair correlation function. Suppose that χ_o is a symplectic map of a compact symplectic manifold (M,ω). It is called *quantizable* if it can be lifted to a contact transformation χ of the prequantum S^1 bundle $\pi : (X,\alpha) \to (M,\omega)$, where $d\alpha = \pi^*\omega$. We are mainly interested here in Hamiltonian flows χ_t and these are always quantizable (§1). We then define the quantization of the map χ_o to be

$$U_\chi^{(N)} := \Pi_N \sigma_\chi T_\chi \Pi_N : \mathcal{H}_N \to \mathcal{H}_N$$

where T_χ is the translation operator by χ on $\mathcal{H}_N$ and $\sigma_\chi \in C^\infty(M)$ is the 'symbol', designed to make $U_\chi^{(N)}$ unitary. All of the usual quantum maps, e.g. 'cat maps' and kicker rotors can be obtained by this method [Z].

Since the eigenvalues lie on the unit circle, the rescaling to unit mean level spacing leads to the periodized pair correlation functions

$$d\rho_{2,N}^p(x) := \frac{1}{N}\sum_{i,j=1}^{N}\sum_{\ell\in\mathbb{Z}} \delta(x - N(\theta_{N,i} - \theta_{N,j} - \ell N) = \frac{1}{N^2}\sum_{i,j=1}^{N}\sum_{\ell\in\mathbb{Z}} \delta(\theta_{N,i} - \theta_{N,j} - 2\pi\ell - \frac{x}{N}).$$

The large N behaviour of $d\rho_{2,N}^p$ is quite 'random' in general because the rescaling destroys the Lagrangean nature of $U_\chi^{(N)}$. The question is whether there is some asymptotic pattern to the randomness.

Our focus in this paper is on the Hamiltonian flows generated by perfect Morse functions H on $S^2 = \mathbb{C}P^1$. The reason for restricting attention to $\mathbb{C}P^1$ is that it is the only symplectic surface carrying a Hamiltonian S^1 action (i.e. it is a 'toric variety'), namely the usual rotation of the sphere about an axis. The moment map is known as an action variable I. Any perfect Morse function may be written as a function $H = \phi(I)$ of a global action variable. Any toral action can be quantized and in particular I can be quantized as an operator $\hat{I}$ whose spectrum lies on a one dimensional 'lattice'. It follows that H is quantized as an operator of the form $\Phi(\hat{I})$ and its flow can be quantized as a unitary group of the form $U_t^{(N)} = \Pi_N e^{it\mathcal{N}\hat{H}^{(N)}}\Pi_N$, where $\mathcal{N}$ equals to N on $\mathcal{H}_N$. Hence the eigenangles have the form $tN\phi(\frac{j}{N})$ and the asymptotics of the PCF can be reduced to the study exponential sums of the form

$$S(N;\ell,t) = \sum_{j=1}^{N} e^{2\pi i N\ell\phi(\frac{j}{N})t}.$$

The Poisson conjecture is essentially that these exponential sums behave like random walks. It is too difficult to analyse the individual exponential sums, but we can successfully analyse some typical behaviour in families of such systems. The first result is about the mean behaviour as the t parameter varies.

Theorem B (a) *Suppose $H : M \to \mathbb{R}$ is a perfect Morse function on $\mathbb{C}P^1$. Then: the limit PCF $\rho_{2,t}$ for $U_{expt\Xi_H}^{(N)}$ and number variance $\Sigma_{2,t}(L)$ are Poisson on average in the sense:*

$$\lim_{N\to\infty}\frac{1}{b-a}\int_a^b \rho_{2,t}^{(N)}dt = \rho_2^{POISS} := 1 + \delta_0$$

and

$$\lim_{N\to\infty}\frac{1}{b-a}\int_a^b \Sigma_{2,t}^{(N)}(L)dt = \Sigma_2^{POISS}(L) := L$$

for any interval $[a,b]$ of $\mathbb{R}$.

This result applies to the case of linear Hamiltonians and their Hamilton flows, whose pair correlation functions are clearly not individually Poisson (see §3). For Poisson level spacings, we make some further assumptions on the Hamiltonian (or phase ϕ). Our main result concerns the mean and variance of a 2-parameter family of Hamiltonians:

Theorem B (b) *Let I denote an action variable on $\mathbb{C}P^1$ and let $H_{\alpha,\beta} = \alpha\phi(I) + \beta I$ with $|\phi''| > 0$. Denote by $\rho^N_{2;(t,\alpha,\beta)}$ the pair correlation measure for the quantum map $U_{(t,\alpha,\beta),N} = exp(itN\hat{H}_{(\alpha,\beta;N)})$. Then for any $t \neq 0$, any $T > 0$ and any $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C^\infty_o(\mathbb{R})$ we have*

$$\frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T}|\rho^N_{2;(t,\alpha,\beta)}(f) - \rho_2^{POISSON}(f)|^2 d\alpha d\beta = 0(\frac{logN}{N}).$$

Thus, the mean pair correlation function in the family is Poisson and the variance tends to zero at the rate $\frac{logN}{N}$. Following [Sa.2], we observe then that along the slightly sparse sequence of Planck constants $1/h = N(logN)^2$, the individual PCF's tend to Poisson for almost every (α,β) (cf. Corollary 5.1.2).

It would be interesting to study quantizations of Hamilton flows in the case where the Hamilton had saddle levels, as must happen if the genus is > 0. It would also be interesting to study completely integrable maps which are not Hamilton flows. We hope to extend our methods and results to these cases in the future.

1 Toeplitz quantization

We now review the basics of Toeplitz quantization on $\mathbb{C}P^1$. For futher background on Kahler quantization, we refer to [A] [G.S] [W]; for general Toeplitz quantization we refer to [B][B.G][Z].

Toeplitz quantization is a form of Kahler quantization, that is, of quantization of symplectic manifolds in the presence of a holomorphic structure. The basic idea is that the quantum system is the restriction of the classical system to holomorphic functions.

To be more precise, let (M,σ) be a compact Kahler manifold with integral symplectic form. Then there is a holomorphic line bundle $L \to M$ with connection 1-form α whose curvature equals σ. In Kahler quantization, the phase space (M,α) is quantized as the sequence of finite dimensional Hilbert spaces $\Gamma(L^{\otimes N})$ where Γ denotes the holomorphic sections. In Toeplitz quantization, these spaces are put together as the Hardy space $H^2(X)$ of CR functions on the unit circle bundle X in L^*.

Thus, the setting for Toeplitz quantization is a compact contact manifold (X,α) whose contact flow is periodic with orbit space M. Corresponding to the Kahler structure on M is a CR structure on X and in particular the Cauchy-Szego projector $\Pi : L^2(X) \to H^2(X)$. This projector has the microlocal properties of the orthogonal projection onto boundary values of holomorphic functions of a strictly pseudo-convex domain and is called a Toeplitz structure on X.

1.1 Toeplitz quantization in genus zero

In the case of $M = S^2 = \mathbb{C}P^1$, the contact manifold X and Toeplitz Π can be constructed explicitly.

We first recall that the holomorphic line bundles over $\mathbb{C}P^1$ are all powers of the hyperplane bundle H and hence $L = H^{\otimes k}$ for some $k \in \mathbb{Z}$. To determine the power we use that L is a positive line bundle of Chern class equal to the area form of S^2. It follows that $L = T^{1,0}S^2$, the holomorphic tangent bundle of S^2. The associated principal S^1 is the unit cotangent bundle S^*S^2 (relative to the standard metric). Hence $X = S^*S^2$.

To define a CR structure on S^*S^2, we represent it as the boundary of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^3$. Namely we consider the quadric cone

$$V = \{z \in \mathbb{C}^3 : z^2 = 0\}$$

with $z^2 = z_1^2 + z_2^2 + z_3^2$ and the plurisubharmonic function $\rho(z) = |z|^2/2 - 1$. Then the domain

$$\Omega = \{z \in V : \rho(z) \leq 0\}$$

is strictly pseudoconvex and its boundary

$$X = \{z \in \mathbb{C}^3 : z^2 = 0, |z|^2 = 2\}$$

may be identified with the unit tangent bundle of S^2: if $z = x + iy$, then $z \in X$ if and only if $|x|^2 = |y|^2 = 1, x \cdot y = 0$. Under the standard metric we may also identify the unit tangent bundle with S^*S^2. The contact form on X is by definition equal to $Im\bar{\partial}\rho = \frac{1}{2}(ydx - xdy)$. Since $x \cdot y = 0$ this is the usual action form $\alpha = y \cdot dx$.

The CR structure on $X = S^*S^2$ arising from its representation as $\partial\Omega$ is the most symmetric one on X. The group $O(3)$ acts holomorphically on V by $A(x + iy) = Ax + iAy$ and preserves ρ. Hence it acts by CR automorphisms on X. The geodesic flow also acts by CR automorphisms by $G^\theta z = e^{i\theta}z$ and commutes with the action of $O(3)$. It follows $S^1 \times O(3)$ acts by unitary transformations on the Hardy space $H^2(X)$ of boundary values of holomorphic functions on Ω and that the Cauchy-Szego projector $\Pi : L^2(X) \to H^2(X)$ commutes with the action. Under the S^1 action the Hardy space decomposes into weight spaces

$$H^2(X) = \bigoplus_{N=1}^{\infty} H^2(N). \tag{1}$$

The group $O(3)$ acts on $H^2(N)$ and as proved in [G], $H^2(N)$ is the irreducible representation of dimension $2N + 1$. In fact there is an isomorphism

$$f \to f^\# : V_N(S^2) \to H^2(N)$$

from the space V_N of Nth order spherical harmonics on S^2 to $H^2(N)$, obtained by extended f as a homogeneous harmonic polynomial on $\mathbb{R}^3$, analytically continuing it to $\mathbb{C}^3$ and restricting it to X.

An alternative approach is to view $X = SO(3)$ and pass to the double cover $SU(2)$. Then $L^2(SU(2)) = \oplus_{N=1}^{\infty} \mathcal{H} \otimes \mathcal{H}_N^*$ where $\{\mathcal{H}_N\}$ runs over the irreducibles of $SU(2)$. The CR structure is given by the lowering operator L_- for the right action. The Szego projector Π_N is then the orthogonal projection onto $\mathcal{H}_N \otimes \psi_N$ where ψ_N is the lowest weight vector in $\mathcal{H}_N^*$.

1.2 Quantum maps

Symplectic maps χ_o of $M = \mathbb{C}P^1$ may be quantized by the Toeplitz method as long as χ lifts to a contact transformation χ of (X, α). The Toeplitz quantization is then essentially the translation operator T_χ by χ compressed to the Hardy space $H^2(X)$. Since T_χ does not usually usually preserve $H^2(X)$, $\Pi T_\chi \Pi$ will not usually be unitary. But in [Z] it is shown that there always exists a canonically defined symbol σ_χ on M so that

$$U_\chi := \Pi \sigma_\chi T_\chi \Pi$$

is unitary. It automatically commutes with the central action, so is the direct sum of the finite unitary operators, $U_{\chi,N}$ on Θ_N. We define $U_{\chi,N}$ to be the quantization of χ_o with semiclassical parameter $1/N$. Its eigenvalues have the form

$$Sp(U_{\chi,N}) = \{e^{2\pi i\theta_{N,j}} : j = 1, \ldots, d_N\} \tag{2}$$

where $d_N = dim H_\Sigma^2(N) = N$.

Consider now the case of Hamilton flows $\chi_o^t = expt\Xi_H$ on M.

Proposition 1.2.1 *Hamilton flows are always quantizable.*

Proof What needs to be proved is that $expt\Xi_H$ always lifts to a contact flow $\phi^t : X \to X$. Equivalently that Ξ_H lifts to a contact vector field, say X_H. We prove this by lifting $expt\Xi_H$ to a homogeneous Hamilton flow $expt\bar{\Xi}_H$ on the symplectic cone

$$\Sigma := \{(x, r\alpha_x) : x \in X, r > 0\} \subset T^*X - 0.$$

Let us define the function

$$r : \Sigma \to \mathbb{R}^+, \qquad r(x, r\alpha_x) = r.$$

Thus, $\Sigma \cong X \times \mathbb{R}^+$, $X \cong \{r = 1\}$ and the $\mathbb{R}^+$ action is generated by the vector $\mathcal{R} = r\frac{\partial}{\partial r}$.

The natural symplectic structure ω on Σ is the restriction of the canonical symplectic structure ω_{T^*X} on T^*X, which is homogeneous of degree 1. Denoting by $\pi : X \to M$ the projection, we have:

$$\omega = r\pi^*\omega_M + dr \wedge \alpha. \tag{3}$$

The proof is simply that $\omega_{T^*X} = d\alpha_{T^*X}$ where α_{T^*X} is the action 1-form. This equation restricts to Σ where $\alpha_{T^*X} = r\alpha$. Taking the exterior derivative gives the formula.

Now return to $H \in C^\infty(M)$ and consider the Hamiltonian $\bar{H}(x, r) = r\pi^*H(x)$ on Σ. It is homogeneous of degree 1 so its Hamilton vector field $\bar{\Xi}_{\bar{H}} = \omega^{-1}(d\bar{H})$ is homogeneous of degree zero and then its Hamilton flow $expt\bar{\Xi}_{\bar{H}}$ is homogeneous of degree one. We claim that (i) the flow preserves X; and (ii) its restriction ϕ^t to X is a contact flow lifting $expt\Xi_H$.

Indeed, we have

$$\begin{aligned} \iota_{\bar{\Xi}_{\bar{H}}}\omega = d(rH) = rdH + Hdr &= r\iota_{\bar{\Xi}_{\bar{H}}}\omega_M + \iota_{\bar{\Xi}_{\bar{H}}} dr \wedge \alpha \\ &= r\iota_{\bar{\Xi}_{\bar{H}}}\omega_M - \alpha(\bar{\Xi}_{\bar{H}})dr + dr(\bar{\Xi}_{\bar{H}})\alpha. \end{aligned}$$

Since all terms except $dr(\bar{\Xi}_{\bar{H}})\alpha$ are dt-independent we must have $dr(\bar{\Xi}_{\bar{H}}) = 0$. It is then obvious that

$$-\alpha(\bar{\Xi}_{\bar{H}}) = H, \qquad \iota_{\bar{\Xi}_{\bar{H}}}\omega_M = dH.$$

The second equation says that $\bar{\Xi}_{\bar{H}}$ projects to Ξ_H, i.e $\bar{\Xi}_{\bar{H}}$ is a lift of Ξ_H. Since

$$\mathcal{L}_{\bar{\Xi}_{\bar{H}}}\alpha = \iota_{\bar{\Xi}_{\bar{H}}}d\alpha + d(\iota_{\bar{\Xi}_{\bar{H}}}\alpha) = dH - dH$$

we also see that $\bar{\Xi}_{\bar{H}}$ is a contact vector field (here, $\mathcal{L}$ is the Lie derivative). ∎

2 Density of States, pair correlation function and number variance

2.1 DOS

The limit DOS (density of states) of quantum Hamiltonians and quantum maps in the Toeplitz setting can be easily determined from the trace formulae of [BM.G] and indeed the calculation is carried out in [Z, Theorem A]. Let us recall the results.

In the case of Hamiltonians, the DOS in degree N is given by

$$d\rho_{1,N}(\lambda) := \frac{1}{N}\sum_{j=1}^{N}\delta(\lambda - \lambda_{N,j}). \tag{4}$$

The following proposition is proved in [B.G, Theorem 13.13]:

Proposition 2.1.1 *The limit DOS is given by*

$$\beta_o(f) = \int_M f(H)\omega \qquad (f \in C(\mathbb{R})).$$

In the case of quantum maps the DOS in degree N is given by

$$d\rho_{1,N}(z) := \frac{1}{N}\sum_{j=1}^{N}\delta(z - e^{2\pi i\theta_{N,i}}) \qquad z \in S^1. \tag{5}$$

The limit DOS β_o is determined in [Z, Theorem A] and depends on whether the classical map is periodic or aperiodic (i.e. the set of periodic points has measure zero).

Proposition 2.1.2 *Let χ be a symplectic map of (M,ω).*
(a) In the aperiodic case, $\beta = c_o d\theta$ where c_o is the constant $(\int_M \sigma d\mu)$ with σ the symbol of U_χ.
(b) If $\phi^k = id$, then β is a linear combination of delta functions at the kth roots of unity.

2.2 The pair correlation function and number variance

We recall here the definitions of the pair correlation function and number variance for quantum maps $U_{\chi,N}$ in one degree of freedom. Then $dim\mathcal{H}_N \sim N$ so the spectrum has the form $\mathrm{Sp}(U_{\chi,N}) = \{e^{i\theta_{Nj}} : j = 1,\dots,N\}$. The spectrum may be identified with the periodic sequence $\{\theta_{Nj} + 2\pi n : n \in \mathbb{Z}, j = 1,\dots,N\}$ and then rescaled to given a periodic sequence of period N and mean level spacing one: $\{N\theta_{Nj} + 2\pi nN : n \in \mathbb{Z}, j = 1,\dots,N\}$.

Definition 2.2.1 *The pair correlation function of level N of a quantum map in one degree of freedom is the distribution on $\mathbb{R}$ given by*

$$d\rho_{2,N}(x) := \frac{1}{N}\sum_{j,k=1}^{N}\sum_{n\in\mathbb{Z}}\delta(N(\theta_{Nj} - \theta_{Nj}) + 2\pi nN - x) \tag{6}$$

The limit pair correlation function is then:

$$d\rho_2^N(x) = w - \lim_{N\to\infty} d\rho_2^N(x).$$

We often write the integral $\int_{\mathbb{R}} f d\rho_2^N dx$ as

$$\rho_2^N(f) = \frac{1}{N}\sum_{j,k=1}^{N}\sum_{n\in\mathbb{Z}} f(N(\theta_{Nj} - \theta_{Nj}) + 2\pi nN).$$

By the Poisson summation formula an equivalent definition is:

$$\rho_2^N(f) = \frac{1}{N^2}\sum_{\ell\in\mathbb{Z}}\hat{f}(\frac{2\pi\ell}{N})\sum_{j,k=1}^{N} e^{it\ell(\theta_{N,j}-\theta_{N,k})} = \frac{1}{N^2}\sum_{\ell}\hat{f}(\frac{2\pi\ell}{N})|TrU_{\chi,N}^{\ell}|^2. \tag{7}$$

ΠThe number variance for a quantum map $U_{\chi,N}$ in one degree of freedom is defined as follows (cf. [Kea]): First, the density of the scaled eigenangles is given by

$$\rho_s(\theta) = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\delta(\theta - N\theta_{Nj}2\pi Nn) = \sum_{\ell\in\mathbb{Z}}[\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i\ell\theta_{N,j}}e^{-2\pi i\ell\theta/N}] = 1 + \frac{2}{N}\sum_{\ell=1}^{\infty}TrU_{\chi,N}^{\ell}e^{-2\pi i\ell\theta/N}. \tag{8}$$

Definition 2.2.2 *The* number variance *is defined by:*

$$\Sigma_2^{(N)}(L) = \frac{1}{N}\int_0^N |\int_{x-L/2}^{x+L/2} \rho_s(y)dy - L|^2 dx$$
$$= \frac{2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}sin^2(\frac{\pi\ell L}{N})|TrU^{\ell}_{\chi,N}|^2$$

.

We observe that $\Sigma_2^N(L)$ is similar to $\rho_2^N(f)$ for $\hat{f} = \frac{sinx}{x}$ except that the $\ell = 0$ term has been removed.

2.3 Asymptotics of traces and exponential sums

Before getting down to our specific models, let us make some general remarks about the exponential sums $S(N,\ell) := TrU^{\ell}_{\chi,N}$.

First, Toeplitz theory gives a complete asymptotic expansion for the traces $TrU^{\ell}_{\chi,N}$ as $N \to \infty$ (for fixed ℓ). To state the result, we will need some notation.

First, we recall that the S^1 action on X is denoted ϕ^{θ}. For each ℓ put

$$\Theta_{\chi,\ell} = \{\theta_j \mathrm{mod} 2\pi : \mathrm{Fix}(\phi^{\theta_j} \circ \chi) \neq \emptyset\}. \tag{9}$$

Assuming (as we will) that the maps have clean fixed point sets, the set $\Theta_{\chi,\ell}$ is finite and $\mathrm{Fix}(\phi^{\theta_j} \circ \chi)$ is a conic submanifold of Σ. We denote its dimension by e_j and its base $\mathrm{Fix}(\phi^{\theta_j} \circ \chi) \cap \mathrm{X}$ by $\mathrm{SFix}(\phi^{\theta_j} \circ \chi)$.

The trace asymptotics then have the form:

Proposition 2.3.1 *For each $\ell \neq 0$,*

$$TrU^{\ell}_{\chi,N} = \sum_{\theta_j \in \Theta_{\chi,\ell}} \sum_{r=0}^{\infty} a_{\ell,j,r} N^{\frac{e_j-1}{2}-r}$$

for certain coefficients $a_{\ell,j,r}$.

The angles θ_j thus play the role of actions in the semiclassical (Gutzwiller) trace formula. The leading coefficients $a_{\ell,j,o}$ may be described as the 'symplectic spinor traces' of the maps $\phi^{\theta_j} \circ \chi$. These traces are Toeplitz analogues of symplectic traces in the sense of [D.G][G.U] and have been discussed in [P.Z]. Namely, suppose χ_o is a quantizable symplectic map of (M,ω) and let χ be its lift as a homogeneous symplectic map of Σ. Suppose that it has a non-empty clean fixed point manifold $Fix(\chi)$. By [D.G] it carries a canonical density dV. By inserting the radial vector $\mathcal{R}$ of Σ (the generator of the dilation action) into dV we get a density $d\mu_{\chi}$ on the base $SFix(\chi)$. The symplectic trace is the canonical volume $\mu_{\chi}(SFix)$.

In the Toeplitz setting, the action of χ on the symplectic normal space $\Sigma^{\perp}$ must also be incorporated into the trace. Namely, associated to the Toeplitz structure is a positive definite Lagrangean sub-bundle Λ of $T\Sigma^{\perp} \otimes \mathbb{C}$ and a ground state e_{Λ}. We refer to [B.G, §11] for the definitions. Generally speaking, χ will take Λ to another Lagrangean-subbundle and e_{Λ} to another ground state $\chi_* e_{\Lambda}$. After trivializing the 'bundle of ground states' the map χ_* may be described as follows: the derivative $d\chi$ times a linear symplectic map $d\chi|_{\Sigma^{\perp}}$ on the symplectic normal bundle of Σ. The quantization of the normal space is a space of Schwartz functions and the quantization of $d\chi|_{\Sigma^{\perp}}$ is its image $\mathcal{M}(d\chi|_{\Sigma^{\perp}}$ under the metaplectic representation. Then $\chi_* = \mathcal{M}(d\chi|_{\Sigma^{\perp}}$. As in [Z], that in order for U_{χ} to be unitary, its principal symbol must equal $(\langle e_{\Lambda}, \chi_* e_{\Lambda}\rangle)^{-1}$

times the graph half-density. Since the symplectic spinor trace (as with the symplectic trace) is the symbolic trace of the quantization of χ, it is given by:

$$\tau(\chi) = \int_{SFix} (\langle e_\Lambda, \chi_* e_\Lambda \rangle)^{-1} d\mu_\chi. \tag{10}$$

Having discussed the ingredients in the above Proposition, we now sketch the proof (a small modification of [B.G, Theorem 12.9].

Proof Consider the Fourier series

$$\Upsilon_{\chi,\ell}(\theta) = \sum_{N=1}^{\infty} TrU^{\ell}_{\chi,N} e^{iN\theta} = TrU^{\ell}_{\chi} e^{i\theta\mathcal{N}}\Pi \tag{11}$$

By the composition theorem of [B.G], Υ_ℓ is a Lagrangean distribution on S^1 with singularities at the values $\theta_j \in \Theta_{\chi,\ell}$ and with singularity degrees beginning at $\frac{e_j}{2}$. Hence:

$$\Upsilon_{\chi,\ell} \cong \sum_{\theta_j \in \Theta_{\chi,\ell}} \sum_{r=0}^{\infty} a_{\theta_j,\ell,r} u_{\frac{e_j}{2}-r}(\theta - \theta_j) \tag{12}$$

where $u_m(\theta) = \sum_{N=0}^{\infty} N^{m-1} e^{iN\theta}$. Note that u_m is a periodic distribution with the same singularity at $\theta = 0$ as the homogeneous distribution $(\theta - \theta_j + i0)^{\frac{e_j}{2}-r}$.

The leading coefficients are given by the principal symbols of $\Upsilon_{\chi,\ell}(\theta)$ at the singularities. The recipe for these symbols is given in the composition theorem of [B.G] (and as in the Lagrangean case), it is essentially the integral of the principal symbol of $U_{\chi\circ\phi^{\theta_j}}$ over $SFix(\chi \circ \phi^{\theta_j}$ with respect to the canonical density (which comes from the half-density part of the symbol of $U_{\chi\circ\phi^{\theta_j}}$. As discussed above, the symplectic spinor factor of the symbol of $U_{\chi\circ\phi^{\theta_j}}$ is precisely what has been put into the symplectic spinor trace above. By matching Fourier series expansions one gets the expansion of the Proposition. ■

Examples Let us consider the form of the trace for quantum maps in one degree of freedom:

(a) Suppose that $\chi_t = exp t\Xi_H$ is the fixed time map of a Hamilton flow. The fixed point set then consists of a finite number of level sets $\{H = E_j(t)\}$, which must be periodic orbits or period t. Pick a base point m_j on each orbit and lift it to a point x_j lying over m_j in X. Then the lift of $\{H = E_j(t)\}$ to x_j is a curve which begins and ends on $\pi^{-1}(m_j)$. The difference in the initial and terminal angle is of course given by the holonomy with respect to α. This holonomy angle θ_j is independent of the choice of m_j and of x_j and $\Theta_{\chi,t}$ is the set of these holonomy angles. Then SFix $\chi_t \circ \phi^{\theta_j}$ is two dimensional and hence $e = 3$.

As will be seen below, $TrU^{\ell}_{\chi_t,N}$ is a classical exponential sum. Replacing it by its trace expansion above amounts to inverting it in the sense of the van der Corput method [G.K][H.1], i.e. applying Poisson summation and the method of stationary phase. Our experience is that this inversion does not simplify the pair correlation problem.

(b) Suppose next that χ_o has only isolated non-degenerate fixed point sets. Then χ fixes the entire fiber over each fixed point. The only singularity occurs at $\theta = 0$ and the dimension of Fixχ equals one. Hence $e = 2$.

Special case: Quantum cat maps These are the most familiar examples of quantum maps, so let us see what the above proposition says about them. In this case, the trace can be calculated exactly and equals the character of the finite metaplectic representations. For the exact calculation in Toeplitz setting, see [Z.1]. For the physics style calculation, see [Kea]. Here we do the calculation asymptotically.

First we observe that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is hyperbolic then it has non-degenerate fixed points at $(x, \xi) \in \mathbb{R}^2/\mathbb{Z}^2$ such that $g(x,\xi) \cong (x,\xi) mod \mathbb{Z}^2$, i.e. at the points $(g - I)^{-1}\mathbb{Z}^2$. The lifted map χ actually has no fixed points. However, $g \circ \phi^\theta$ has fixed points if and only if $[(g \cdot (x,\xi), e^{2\pi i(t+\theta)})] = [(x,\xi,\theta)]$, where the bracket denotes the equivalence in the quotient space. Since $g \cdot (x,\xi) = (x,\xi) + (m,n)$ for some $(m,n) \in \mathbb{Z}^2$ we get that $[(x,\xi)+(m,n), e^{2\pi i(t+\theta)})] = [(x,\xi,\theta)]$. But $[(x,\xi)+(m,n), e^{2\pi i(t+\theta)})] = [(x,\xi), e^{2\pi i(t+\theta)} e^{i\pi mn)} e^{i\pi\omega((x,\xi),(m,n))}]$. It follows that

$$\theta_{mn} = -\frac{1}{2}(mn + \omega((x,\xi),(m,n)).$$

Hence

$$\Theta_{g,\ell} = \{-\frac{1}{2}(mn + \omega((x,\xi),(m,n)) : g^\ell(x,\xi) = (x,\xi) + (m,n)\}$$

and

$$TrU_{g,N} = \frac{i^\sigma}{\sqrt{det(I-g)}} \sum_{(m,n)\in\mathbb{Z}^2/(I-g)\mathbb{Z}^2} e^{i\pi(mn+\omega((m,n),(1-g)(m,n))}.$$

3 PCF for Hamiltonians: Proof of Theorem A

As mentioned in the introduction, the pair correlation problem for quantized Hamiltonians is similar to that for Zoll surfaces as presented in [U.Z]. We can therefore follow the exposition in [U.Z, §3] to some degree and are sketchy on some common details.

There is no difference in this problem between the case of $M = \mathbb{C}P^1$ and the general case of symplectic surfaces. So in this section M can be any closed surface. However, we will make some generic simplifying assumptions on the Hamiltonian. The first one is

Assumption M: *$H : M \to \mathbb{R}$ is a Morse function.*

Let $\mathcal{E}$ denote the set of values of H and let $c_1 < c_2 < \ldots < c_{M+1}$ denote its set of critical values. Then the inverse image $H^{-1}(c_\nu, c_{\nu+1})$ consists of a finite number $N(\nu)$ of connected components X_j^ν each diffeomorphic to $(c_\nu, c_{\nu+1}) \times S^1$. Hence for $E \in (c_\nu, c_{\nu+1})$, $H^{-1}(E) \cap X_j^\nu$ consists of a periodic orbit $\gamma_j^\nu(E)$ of $expt\Xi_H$. Its minimal positive period will be denoted by $T_j^\nu(E)$.

For the sake of simplicity we will make a second assumption:

Assumption Q: *T_j^ν and T_k^ν are independent over $\mathbb{Q}$ if $j \neq k$.*

We can then state the formula for the pair correlation function:

Theorem 3.0.2 *With assumptions M and Q on H, the limit pair correlation function is given by*

$$\int_{\mathbb{R}} f(x) d\rho_2(x) = V\hat{f}(0) + \sum_{k\in\mathbb{Z}} \sum_{\nu=1}^{M} \sum_{j=1}^{N(\nu)} \int_{(c_\nu,c_{\nu+1})} \hat{f}(kT_j^\nu(E)) T_j^\nu(E)^2 dE$$

for any f such that $\hat{f} \in C_o^\infty(\mathbb{R})$.

Proof: To determine the asymptotics of the sequence $\rho_2^N(f)$ we form the generating function

$$\Upsilon_f(\theta) := \sum_{N=1}^{\infty} \rho_2^N(f) e^{iN\theta}.$$

We wish to show that Υ_f is a classical Hardy-Lagrangean distribution on S^1. The asympotics of $\rho_2^N(f)$ can then be determined from the singularity data of Υ_f.

To show that $\Upsilon_f(\theta)$ is a Lagrangean distribution, we will identify with the trace of a Toeplitz type Fourier Integral operator. The operator is defined as follows: We form the product manifold $X \times X$ and consider the product Szego projector

$$\Pi \otimes \Pi : L^2(X \times X) \to H^2(X) \otimes H^2(X)$$

and the diagonal projector

$$\Pi_{diag} := \bigoplus_{N=1}^{\infty} \Pi_N \otimes \Pi_N : L^2(X \times X) \to \bigoplus_{N=1}^{\infty} \Theta_N \otimes \Theta_N.$$

We next observe that

$$\rho_2^N(f) = Tr\Pi_N \otimes \Pi_N \int_{\mathbb{R}} \hat{f}(t) e^{itN(\hat{H}_N \otimes I - I \otimes \hat{H}_N)} dt.$$

Noting that N is the eigenvalue of the number operator $\mathcal{N}$ we can rewrite this in the form

$$Tr\Pi_N \otimes \Pi_N \int_{\mathbb{R}} \hat{f}(t) e^{it(\mathcal{N}\hat{H}_N \otimes I - I \otimes \mathcal{N}\hat{H}_N)} dt.$$

The generating function is then given by

$$\Upsilon_f(\theta) = \sum_{N=1}^{\infty} Tr e^{i\theta[\mathcal{N} \otimes I + I \otimes \mathcal{N}]} \Pi_N \otimes \Pi_N \int_{\mathbb{R}} \hat{f}(t) e^{it(\mathcal{N}\hat{H}_N \otimes I - I \otimes \mathcal{N}\hat{H}_N)} dt =$$

$$= Tr\Pi_{diag} e^{i\theta[\mathcal{N} \otimes I + I \otimes \mathcal{N}]} \int_{\mathbb{R}} \hat{f}(t) e^{it(\mathcal{N}\hat{H} \otimes I - I \otimes \mathcal{N}\hat{H})} dt.$$

We recall here that $\hat{H} = \Pi H \Pi$ where H is the pull back to X of the function so denoted on M. We now have to analyse each operator which occurs under the trace sign.

(a) Π_{diag}: This operator is the composition of the product Szego projector $\Pi \otimes \Pi$ with the full diagonal weight projection

$$P_{diag} : L^2(X \otimes X) \to L^2_N(X) \otimes L^2_N(X)$$

where $L^2_N(X)$ is the eigenspace of $\mathcal{N}$ of eigenvalue N.

From [G.S.2] it follows that P_{diag} is a Fourier Integral operator in the class $I^0(X^{(2)} \times X^{(2)}, \tilde{\Gamma})$ where $X^{(2)} = X \times X$ and where $\tilde{\Gamma}$ is the flow-out of the coisotropic cone

$$\tilde{\Theta} = \{(\zeta_1, \zeta_2) \in T^*(X^{(2)}) : H(\zeta_1) - H(\zeta_2) = 0\}.$$

That is, let $\Phi^t = exp t\Xi_H \times exp - t\Xi_H$ denote the Hamilton flow generated on $T^*(X^{(2)})$ by $H(\zeta_1) - H(\zeta_2)$. Then in a well-known way, the map

$$i_{\tilde{\Theta}} : \mathbb{R} \times \tilde{\Theta} \to T^*(X^{(2)}) \times T^*(X^{(2)}), \qquad (t, \zeta_1, \zeta_2) \to ((\zeta_1, \zeta_2), \Phi^t(\zeta_1, \zeta_2))$$

defines a Lagrange immersion with image equal (by definition) to the flow out $\tilde{\Gamma}$.

On the other hand $\Pi \otimes \Pi$ is the exterior tensor product of two Toeplitz (hence Hermite type Fourier Integral) operators. According to [B.G, Theorem 9.3], we therefore have

$$\Pi \otimes \Pi = \alpha + \beta$$

with

$$\alpha \in\in I^o(X^{(2)} \times X^{(2)}, \Sigma \times \Sigma)$$

and with $WF(\beta)$ contained in a small conic neighborhood $\mathcal{C}$ of $\Sigma \times 0 \cup 0 \times \Sigma$. Moreover, the symbol of $\Pi \otimes \Pi$ is given by

$$\sigma(\Pi \otimes \Pi) = \sigma(\Pi) \otimes \sigma(\Pi)$$

on $\Sigma \times \Sigma - \mathcal{C}$. Hence $\Pi \otimes \Pi$ is essentially a Toeplitz structure on the symplectic cone $\Sigma \times \Sigma \subset T^*(X^{(2)})$. The complication due to $\mathcal{C}$ will ultimately prove to be irrelevant in the analysis of the trace since it will not contribute to the singularities along the diagonal. Hence we can (and will) pretend that this component of $WF(\Pi \otimes \Pi)$ does not occur.

By the composition theorem for Hermite and ordinary Fourier Integral operators [BM, Theorem 7.5] it follows that (modulo the term $P_{diag} \circ \beta$)

$$\Pi_{diag} \in I^{o}(X^{(2)} \times X^{(2)}, \Gamma)$$

where Γ is the flowout Lagrangean in $\Sigma \times \Sigma$ for the co-isotropic subcone $\Theta := \tilde{\Theta} \cap \Sigma \subset \Sigma$. That is, the map

$$i_{\Theta} := i_{\tilde{\Theta}}|_{\mathbb{R} \times \Theta} : \mathbb{R} \times \Theta \to \Sigma \times \Sigma$$

is a Lagrange immersion with respect to the symplectic cone $\Sigma \times \Sigma$ and Γ is its image; of course it is only an isotropic immersion with respect to $T^*(X^{(2)} \times X^{(2)}$.

(b) $e^{i\theta[\mathcal{N} \otimes I + I \otimes \mathcal{N}]}$ This operator does not require a fancy analysis since $\mathcal{N}$ is simply the differentiation operator by the generator T of the contact flow. Hence $e^{i\theta[\mathcal{N} \otimes I + I \otimes \mathcal{N}]}$ is the translation operator $F(x, y) \to F(\phi^t(x), \phi^t(y))$ by $\phi^t \times \phi^t$ on $X^{(2)}$.

(c) $\Pi \otimes \Pi \int_{\mathbb{R}} \hat{f}(t) e^{it(\mathcal{N}\hat{H} \otimes I - I \otimes \mathcal{N}\hat{H})} dt$: Here we have inserted the factor $\Pi \otimes \Pi$, as we may, to simplify the discussion.

Since $[\mathcal{N}, \Pi] = 0 = [\mathcal{N}, \hat{H}]$, we may remove the projection Π from the exponent. Then the unitary under the integral is given by

$$e^{itH\mathcal{N}} \otimes e^{-itH\mathcal{N}}.$$

Each factor is the exponential of a pseudodifferential operator of real principal type and is therefore Fourier Integral. It follows again from the composition theorem [B.G, Theorem 7.5] that

$$\Pi e^{itH\mathcal{N}} \in I^{-\frac{1}{4}}(X \times X, C \cap \mathbb{R} \times \Sigma \times \Sigma), \qquad C := \{(t, \tau, x, \xi, y, \eta) : \tau + \sigma_{\mathcal{N}}(x, \xi) H(x) = 0, \psi^t(x, \xi) = (y, \eta)\}$$

where ψ^t is the Hamilton flow on T^*X generated by $\sigma_{\mathcal{N}}(x, \xi)H$. Note that $\sigma_{\mathcal{N}(x,\xi)} = \langle \xi, T \rangle$, which generates the lift of the central circle action on X to T^*X. Also, the Hamilton flow of H on T^*X is the two-fold lift of the Hamilton flow of H on M: first from M to X and then from X to T^*X. Since the Hamilton vector field of $\sigma_{\mathcal{N}(x,\xi)}H$ on T^*X is given by

$$\Xi_{\sigma_{\mathcal{N}(x,\xi)}H} = H \Xi_{\sigma_{\mathcal{N}(x,\xi)}} + \sigma_{\mathcal{N}(x,\xi)} \Xi_H$$

and the Lie bracket of the two terms is zero, we have

$$\psi^t = exptH\Xi_{\sigma_{\mathcal{N}(x,\xi)}} \circ expt\sigma_{\mathcal{N}(x,\xi)}\Xi_H.$$

The isotropic cone $C \cap \Sigma \times \Sigma$ can be parametrized by

$$i_{C_{\Sigma}} : \mathbb{R} \times \Sigma \to \mathbb{R} \times \Sigma \times \Sigma, \qquad (t, \zeta) \to (t, \sigma_{\mathcal{N}} H(\zeta), \zeta, \psi^t(\zeta)).$$

The symbol of $\Pi \otimes \Pi \int_{\mathbb{R}} \hat{f}(t) e^{it(\mathcal{N}\hat{H} \otimes I - I \otimes \mathcal{N}\hat{H})} dt$ may then be identified with the spinor $\hat{f}(t)|dt|^{\frac{1}{2}} \otimes \sigma_{\Pi}$.

Putting the above together we see that up to the factor of P_{diag} the operator under the trace is a Hermite Fourier Integral operator associated to the graph of $\phi^{\theta} \times id \circ \psi^t \times \psi^{-t}$ on $\Sigma \times \Sigma$. The effect of the P_{diag} factor is to reduce this torus action to the quotient of $\Sigma \times \Sigma$ by the diagonal contact flow. As discussed in [U.Z] and above in §2.3, the trace then has singularities at the angles θ_j for which the contact transformation $\phi^{\theta_j} \times id \circ \psi^t \times \psi^{-t}$ has a non-trivial fixed point set for some t. ?????

The fixed point set of the reduced flow has two components: that at $t = 0$, when all points of $\Theta_M := \{(m_1, m_2) : H(m_1) - H(m_2) = 0\}$ are fixed; and that for the interval of periods $t = T(E)$ of periodic orbits, when the corresponding level sets $\{(m_1, m_2) : H(m_1) = H(m_2) = E\}$ are fixed.

4 PCF for quantized perfect Hamiltonian flows on S^2: Proof of Theorem B

In this section we consider general Hamiltonians H on S^2 which are perfect Morse functions. Our purpose is to show that the pair correlation functions of their quantized Hamiltonian flows are Poisson on average.

4.1 The quantizations $U_{t,N}$ and $V_{t,N}$

There are two approaches to the quantization of a Hamiltonian flow $expt\Xi_H$: (i) by first exponentiating and then quantizing or (ii) by first quantizing and then exponentiating. The next proposition shows that the two procedures lead to the same result.

Let us define the two procedures precisely: First, we exponentiate H to get the flow $expt\Xi_H$. To quantize it, we use Proposition 2.3 to lift $expt\Xi_H$ to a contact transformation ϕ^t on X. Then $\Pi_N \chi_t \Pi_N$ is an elliptic Toeplitz operator, and by [Z, §3] there exists a canonical symbol $\sigma_t \in S^0(T^*(X))$ such that the $U_{\chi_t,N} := \Pi_N \sigma_t \chi_t \Pi_N$ is unitary.

On the other hand, we could quantize $H \to \hat{H}$ first and then exponentiate to get $V_{\chi_t,N} = \Pi_N e^{it\mathcal{N}\Pi H\Pi}\Pi_N$.

Proposition 4.1.1 $V_{t,N} = U_{t,N}$.

Proof: Observe that Π_N and $e^{it\mathcal{N}\Pi H\Pi}$ commute. Indeed, $\Pi_N = \Pi \circ \int_{S^1} Ad(e^{i\theta\mathcal{N}} e^{-iN\theta} d\theta$ and the averaging operator commutes with both Π and multiplication by H. It follows that

$$V_{t,N}^* V_{t,N} = \Pi_N e^{-it\mathcal{N}\Pi H\Pi}\Pi_N e^{it\mathcal{N}\Pi H\Pi}\Pi_N = \Pi_N e^{-it\mathcal{N}\Pi H\Pi} e^{it\mathcal{N}\Pi H\Pi}\Pi_N = \Pi_N.$$

Thus, $V_{t,N}$ is unitary on $H^2(N)$.

We then observe that $\Pi e^{it\mathcal{N}\Pi H\Pi}\Pi = \Pi e^{it\mathcal{N}H}\Pi$ is a unitary group of Fourier-Toeplitz integral operators with underlying canonical relation given by the lift of the Hamilton flow of H to the symplectic cone $\Sigma \subset T^*(X) = \{(x, \xi, r\alpha) : r \in \mathbb{R}^+\}$. Indeed, $\Pi\mathcal{N}H\Pi$ is a first order Toeplitz (pseudodifferential) operator with principal symbol rH. Hence its exponential is a Toeplitz Fourier integral operator with bicharacteristic flow equal to the Hamilton flow of rH on Σ. By Proposition 2.3, this Hamilton flow is the lift of $expt\Xi_H$ to Σ.

The theory of the finite Toeplitz operators $\Pi_N e^{it\mathcal{N}\Pi H\Pi}\Pi_N$ is the non-homogeneous analogue of the homogeneous theory. The only change is that the Lagrangean is non-homogeneous and equal to the graph of ϕ^t on X. The passage to non-homogeneous Toeplitz operators is precisly parallel to that of Fourier integral operators and we omit the details.

Thus, $U_{t,N}$ and $V_{t,N}$ are both Toeplitz Fourier integral operators associated to the graph of ϕ^t. Both have principal symbols equal to the graph 1/2-density. Hence they differ by operators in the same class and of order -1. But the method of quantum maps in [Z] just begins with a Toeplitz Fourier integral operator which is unitary up to order -1 and applies the functional calculus to improve it. Hence we may begin with $V_{t,N}$ and since it is already unitary the improvement doesn't change it. ■

It follows that the PCF of $V_{t,N}$ equals that of $U_{t,N}$. The pair correlation function is therefore given by the formula in §2.1. We now turn to the WKB analysis of the eigenvalues of $\hat{H}_N$.

4.2 WKB in genus zero

We will assume $H : S^2 \to \mathbb{R}$ is a perfect Morse function, with a non-degenerate minimum value equal to zero, and a non-degenerate maximum value equal to A. We then let μ denote the distribution function of $H : M \to [0, A]$, i.e. $\mu(E) = |\{H \leq E\}|$ where $|\cdot|$ denotes the symplectic area. Under our assumptions it is a strictly increasing smooth function on $(0, A)$. We denote its inverse function by ϕ. The following Lemma is of a now standard kind (cf. [CV]).

Lemma 4.2.1 *There exists a sequence of smooth function ϕ_j, with $\phi_o = \phi$ such that:*

$$\lambda_{N,j} = \phi(\frac{j}{N}) + N^{-1}\phi_{-1}(\frac{j}{N}) + N^{-2}\phi_{-2}(\frac{j}{N}) + \dots.$$

Proof: Let $I : S^2 \to [0, 4\pi]$ be defined by $I(z) = \mu(H(z))$. Then I is an action variable for H: as functions on the symplectic S^2, I and H Poisson commute, $\{I, H\} = 0$, and the Hamilton flow of I is 2π- periodic. It is obvious that I generates the algebra $\mathbf{a} = \{f \in C^\infty(S^2) : \{f, H\} = 0\}$ and hence we may write $H = \phi(I)$ where ϕ is the inverse function to μ.

The Toeplitz quantization $\Pi I \Pi$ of I is a positive operator satisfying $e^{2\pi i D[\Pi I \Pi]} = I + K$ where K is a Toeplitz operator of order -1. In a well-known way [CV, loc.cit.], we may add lower order terms to $\Pi I \Pi$ to arrive at a positive operator $\hat{I}$ satisfying $[\hat{I}, D] = [\hat{H}, \hat{I}] = 0$ and $e^{2\pi i D\hat{I}} = I$. Therefore $D\hat{I}$ has only integral eigenvalues. Since $D = N$ on $H^2(N)$, it follows that

$$Sp(\hat{I}|_{H^2(N)}) = \{\frac{j}{N} : j \in \{0, 1, \dots, N\}\}. \tag{13}$$

Moreover, $\hat{I}$ generates the commutant of $\hat{H} = \{A \in \mathcal{T}_\Pi : [A, \hat{H}] = 0\}$, so there exists a polyhomogeneous function Φ such that

$$\hat{H} = \Phi(\hat{I}), \qquad \Phi \sim \phi + \phi_{-1} + \dots. \tag{14}$$

It follows that the eigenvalues of $\hat{H}$ in $H^2(N)$ are given by

$$\lambda_{N,j} = \Phi(\frac{j}{N}) \sim \phi(\frac{j}{N}) + \dots. \tag{15}$$

■

In general, we will call a function $I : S^2 \to \mathbb{R}$ an 'action' variable if the flow $exp t\Xi_I$ of its Hamilton vector field Ξ_f is 2π-periodic. That is, I generates a circular symmetry. Up to symplectic diffeomorphism, the action variable is unique and could be taken to be the generator of rotations around the z-axis. Indeed, any global action variable on S^2 must have precisely two critical points and generate a flow with these as its fixed points. This flow will be diffeomorphic to the rotational flow and will have a global transversal connecting the fixed points. The travel time from this transversal defines an angle variable θ symplectically dual to I and the transformation $\chi(I, \theta) = (cos r, \theta)$ is a global symplectic transformation.

It follows that any perfect Morse Hamiltonian has the form $\phi(I)$ for some smooth function ϕ on $(-1, 1)$. Moreover, non-degeneracy of the critical points forces $d\phi \neq 0$ since

$$d^2H = \phi'(I)d^2I + \phi''(I)dI \otimes dI \tag{16}$$

so that $d^2H = \phi'(I)d^2I \otimes dI$ at critical points.

Thus, $S_t(N, \ell)$ is an oscillatory sum of the form

$$S_t(N, \ell) = \sum_{j=1}^{N} e^{it\ell N\Phi(j,N)} \tag{17}$$

with $\Phi(j, N) = \phi(\frac{j}{N}) + \frac{\phi_{-1}(\frac{j}{N})}{N^2} + \dots$. We observe that (without any further assumptions on ϕ) the terms of order $\frac{\phi_{-3}}{N^3}$ or lower in $\Phi(j, N)$ make no contribution to the pair correlation function. That is, let us put:

$$Z_t(N, \ell) = \sum_{j=1}^{N} e^{it\ell N\Phi_2(\frac{j}{N},N)} \tag{18}$$

with $\Phi_2(\frac{j}{N}, N) := [\phi(\frac{j}{N}) + \frac{\phi_{-1}(\frac{j}{N})}{N} \frac{\phi_{-2}(\frac{j}{N})}{N^2}]$. The following is obvious:

Proposition 4.2.2

$$\frac{1}{N} \sum_{\ell \neq 0} \hat{f}(\frac{2\pi \ell}{N}) \frac{1}{N} |\{|S_\ell(N,t)|^2 - |Z_\ell(N,t)|^2\}| = O(1/N).$$

For the sake of simplicity we will often take the quantum Hamiltonian to have the form $\phi(\hat{I})$ so that the lower order terms are zero. It would be straightforward to extend the results to the general case.

4.3 Proof of Theorem B: pair correlation on average

Our goal is to prove:

Theorem 4.3.1 *Suppose that $|\phi'(x)| \geq C_1$. Then the limit pair correlation function ρ_2^t for $U_{t,N}$ is Poisson* on average*:*

$$\frac{1}{b-a} \int_a^b \rho_2^t(f) dt = f(0) + \hat{f}(0)$$

for any interval $[a,b] \subset \mathbb{R}$ and any $f \in C_o^\infty$.

Proof: It suffices to show that

$$\lim_{N \to \infty} \frac{1}{b-a} \int_a^b \frac{1}{N^2} \sum_{\ell \neq 0} \hat{f}(\frac{2\pi \ell}{N}) |Z_t(N,\ell)|^2 dt = f(0)$$

. To prove this, we use the Hilbert inequality (cf. [Mo, §7.6 (28)])

$$\frac{1}{b-a} \int_a^b |\sum_{j=1}^N e^{2\pi i \mu_j t}|^2 dt = N + O(\sum_{j=1}^N \frac{1}{\delta_j})$$

with

$$\delta_j = \min_{1 \leq j \leq N, j \neq k} |\mu_j - \mu_k|.$$

The two terms correspond respectively to the diagonal and to the off-diagonal in the square, and the O-symbol is an absolute constant (which can be taken to be 3/2). In the case at hand, $\mu_j = N\ell[\phi(\frac{j}{N}) + \frac{\phi_{-1}(\frac{j}{N})}{N} \frac{\phi_{-2}(\frac{j}{N})}{N^2}]$ so that

$$\delta_{N,j} \geq \ell \quad [min_{x \in [a,b]} |\phi'(x)| + O(1/N)].$$

It follows that if $|\phi'(x)| \geq C > 0$ then

$$\frac{1}{b-a} \int_a^b |Z_t(N,\ell)|^2 dt = N + O(\frac{N}{\ell}).$$

Therefore the limit equals

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\ell \neq 0} \hat{f}(\frac{2\pi \ell}{N}) + O(\limsup_{N \to \infty} \frac{1}{N} \sum_{\ell \neq 0} \frac{1}{\ell} \hat{f}(\frac{2\pi \ell}{N})).$$

The first term tends to $\int_{\mathbb{R}} \hat{f}(x) dx = f(0)$. When supp $f \subset [-C, C]$ the second is

$$<< \frac{1}{N} [\sum_{\ell \leq CN} \frac{1}{\ell}] << \frac{log N}{N}$$

and tends to zero. ∎

4.4 Proof of Theorem B (a): Number variance on average

Theorem 4.4.1 *With the same hypotheses as above, we have*

$$\lim_{N\to\infty}\frac{1}{b-a}\int_a^b \Sigma_{2,t}^N dt = L.$$

Proof

We have

$$\frac{1}{b-a}\int_a^b \Sigma_{2,t}^N dt = \frac{2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}sin^2(\frac{\pi\ell L}{N})\frac{1}{b-a}\int_a^b |TrU_{t,N}^{\ell}|^2 dt = \tag{19}$$

$$= N\frac{2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}sin^2(\frac{\pi\ell L}{N}) + \frac{2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}sin^2(\frac{\pi\ell L}{N})\ \{\frac{1}{b-a}\int_a^b \sum_{j\neq k, j,k=1}^{N} e^{2\pi i N\ell(\phi(\frac{j}{N})-\phi(\frac{k}{N})}dt\}. \tag{20}$$

The first term may be rewritten as

$$\frac{1}{N}\frac{2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{N^2}{\ell^2}sin^2(\frac{\pi\ell L}{N}) \to \int_0^{\infty}(\frac{sinLx}{x})^2 dx = L.$$

As above, the second is bounded by

$$NlogN\sum_{\ell=1}^{\infty}\frac{1}{\ell^3}sin^2(\frac{\pi\ell L}{N}) = O(\frac{logN}{N}).$$

■

5 Mean square Poisson statistics for quantum spin evolutions: Proof of Theorem B (b)

We have just seen that averages of the pair correlation function $\rho_{2,t}^N$ of quantized Hamilton flows $U_{t,N}$ converge to $\rho_2^{POISSON}$ as long as H is a perfect Morse function. In particular, the result is true for linear functions $H=\alpha I$ in the action. Since exponential sums with linear phases are far from random (see §4.1), it is evident that the averaging is the agent producing the random number behaviour.

A much stronger test of the Poisson behaviour of $U_{t,N}$ is whether the variance

$$\frac{1}{b-a}\int_a^b |\rho_{2,t}^N(f) - \rho_2^{POISSON}(f)|^2 dt$$

tends to zero as $N\to\infty$. It is easy to see that quantum Hamiltonian flows with linear Hamiltonians do not have this property (§4.1). We therefore turn to quadratic Hamiltonians $H=\alpha I^2+\beta I$. Our next result shows that if average in two parameters, namely (α,β), then their quantum Hamilton flows are Poisson in mean square:

Theorem 5.0.2 *Let $H_{\alpha,\beta}=\alpha I^2+\beta I$ and let $\rho_{2;(t,\alpha,\beta)}^N$ be the pair correlation measure for the quantum map $U_{(t,\alpha,\beta),N}=exp(itN\hat{H}_{(\alpha,\beta;N)})$. Then for any $t\neq 0$, any $T>0$ and any $f\in\mathcal{S}(\mathbb{R})$ with $\hat{f}\in C_o^{\infty}(\mathbb{R})$ we have*

$$\lim_{N\to\infty}\frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T}|\rho_{2;(t,\alpha,\beta)}^N(f) - \rho_2^{POISSON}(f)|^2 d\alpha d\beta = 0.$$

Proof: Removing the $\ell = 0$ and diagonal terms as above, it suffices to show that

$$\lim_{N\to\infty} \frac{1}{(2T)^2} \int_{-T}^{T}\int_{-T}^{T} |\frac{1}{N^2}\sum_{\ell\neq 0} \hat{f}(\frac{\ell}{N})[\sum_{j\neq k, j,k=1}^{N} e(t(\alpha\frac{j^2-k^2}{N} + \beta(j-k)))|^2 d\alpha d\beta = 0. \tag{21}$$

To prove this, we use the Beurling - Selberg function $B_{T,|t|\delta}$. It has the properties

- $B_{T,|t|\delta} \geq \chi_{[-T,T]}$;
- Supp $\hat{B}_{T,|t|\delta} \subset (-|t|\delta, |t|\delta)$.

Here, $\chi_{[-T,T]}$ is the characteristic function of $[-T,T]$. Then the integral on the right side above is bounded above by

$$\int_{\mathbb{R}}\int_{\mathbb{R}} B_{T,|t|\delta}(\alpha)B_{T,|t|\delta}(\beta)|\frac{1}{N^2}\sum_{\ell\neq 0}\hat{f}(\frac{\ell}{N})[\sum_{j\neq k,j,k=1}^{N} e(t(\alpha\frac{j^2-k^2}{N}+\beta(j-k)))|^2 d\alpha d\beta. \tag{22}$$

Squaring and evaluating the Fourier transforms gives

$$\frac{1}{N^4}\sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0}\hat{f}(\frac{\ell_1}{N})\bar{\hat{f}}(\frac{\ell_2}{N})[\sum_{j_1\neq k_1,j_1,k_1=1}^{N}\ \sum_{j_2\neq k_2,j_2,k_2=1}^{N} \hat{B}_{T,|t|\delta}(t(\ell_1\frac{j_1^2-k_1^2}{N}-\ell_2\frac{j_2^2-k_2^2}{N}))\hat{B}_{T,|t|\delta}(t(\ell_1(j_1-k_1)-\ell_2(j_2-k_2))). \tag{23}$$

By the support properties of $B_{T,|t|\delta}$, the latter expression is bounded above by

$$\frac{1}{N^4}\sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0}|\hat{f}(\frac{\ell_1}{N})||\hat{f}(\frac{\ell_2}{N})|I(N,\ell_1,\ell_2) \tag{24}$$

with

$$I(N,\ell_1,\ell_2) = \#\{(j_1,k_1,j_2,k_2)\in[1,N]^4 : j_i\neq k_i : |(\ell_1\frac{j_1^2-k_1^2}{N}-\ell_2\frac{j_2^2-k_2^2}{N})|\leq\delta, |(\ell_1(j_1-k_1)-\ell_2(j_2-k_2)))|\leq\delta\}. \tag{25}$$

Introduce new variables $h_i = j_i - k_i, m_i = j_i + k_i$ so that the conditions read

$$\begin{array}{l} |\ell_1\frac{h_1m_1}{N} - \ell_2\frac{h_2m_2}{N})| \leq \delta \\ |\ell_1 h_1 - \ell_2 h_2| \leq \delta \end{array}.$$

The change of variables is invertible so $I(N,\ell_1,\ell_2)$ is the number of integer solutions (h_1,h_2,m_1,m_2) with $|m_i|\leq 2N, |h_i|\leq N-|m_i|$.

Since $|\ell_1h_1-\ell_2h_2|\in\mathbf{N}$ it can only be $<\delta$ if it vanishes. Therefore the second condition is equivalent to

$$\langle \ell, h\rangle = 0 \quad \Rightarrow h_2 = \frac{\ell_1}{\ell_2}h_1.$$

Here we assume $|h_1|\geq|h_2|$ so that $|h_1| = max\{|h_1|,|h_2|\}\sim|h|$ and we abbreviat $\ell = (\ell_1,\ell_2)$ etc. Substituting in the first condition we get

$$\ell_1h_1(m_1-m_2) = O(N\delta).$$

Now let us count solutions. Since h_2 is determined from (h_1,m_1,m_2) it suffices to count these triples. First, there are $O(N)$ choices of m_1. Then put $m_2 = m_1+M$ so that $h_1M = O(\frac{N}{\ell_1}\delta)$. From $M\leq 2N$ the number of pairs (h_1,M) is bounded above by

$$\sum_{M=1}^{2N}\frac{N}{\ell_1}\frac{1}{M} = O(\frac{N}{\ell_1}logN).$$

Hence $I(N,\ell_1,\ell_2) << \frac{N^2 logN}{\ell_1}$.

It follows that

$$\frac{1}{N^4}\sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0}|\hat{f}(\frac{\ell_1}{N})||\hat{f}(\frac{\ell_2}{N})|I(N,\ell_1,\ell_2) << \frac{logN}{N^2}\sum_{\ell_1\neq 0}\sum_{\ell_2\neq 0}|\hat{f}(\frac{\ell_1}{N})||\hat{f}(\frac{\ell_2}{N})|\frac{1}{\ell_1} \tag{26}$$

As above we use that $\hat{f}$ is compactly supported to get that this is

$$<< \frac{(logN)^2}{N}\int_{\mathbb{R}}|\hat{f}(\xi)|d\xi. \tag{27}$$

Hence the remainder term is $O(\frac{(logN)^2}{N})$. ■

5.1 Proof of Theorem B(b) for general non-degenerate phases

Theorem 5.1.1 *Let $H_{\alpha,\beta} = \alpha\phi + \beta I$ where $|\phi''| > C > 0$ on $[-1,1]$ and let $\rho^N_{2;(t,\alpha,\beta)}$ be the pair correlation measure for the quantum map $U_{(t,\alpha,\beta),N} = exp(itN\hat{H}_{(\alpha,\beta;N)})$. Then for any $t \neq 0$, any $T > 0$ and any $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C^\infty_o(\mathbb{R})$ we have*

$$\frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T}|\rho^N_{2;(t,\alpha,\beta)}(f) - \rho_2^{POISSON}(f)|^2 d\alpha d\beta = 0(\frac{(logN)^2}{N}.$$

Proof The above method leads in this general case to the lattice point problem:

$$\begin{pmatrix} \ell_1 h_1 = \ell_2 h_2 \\ \ell_1(\phi(\frac{j_1}{N}) - \phi(\frac{k_1}{N}) - \ell_2(\phi(\frac{j_2}{N}) - \phi(\frac{k_2}{N}) = O(1/N) \end{pmatrix}$$

By the mean value theorem there exist $\xi_{j_ik_i} \in [j_1, k_1]$ such that $\phi(\frac{j_i}{N}) - \phi(\frac{k_i}{N}) = \frac{1}{N}\phi'(\xi_{j_ik_i}/N)(j_i - k_i)$. As above we then get the system of constraints:

$$\begin{pmatrix} \ell_1 h_1 = \ell_2 h_2 \\ h_1\ell_1(\phi'(\xi_{j_1k_1}/N) - \phi'(\xi_{j_1k_1}/N) = O(1) \end{pmatrix}$$

Then writing $\phi'(\xi_{j_1k_1}/N) - \phi'(\xi_{j_1k_1}/N) = \phi''(\xi_{j_1k_1j_2k_2}/N)(\xi_{j_1k_1} - \xi_{j_2k_2})/N$ we get the system:

$$\begin{pmatrix} \ell_1 h_1 = \ell_2 h_2 \\ h_1\ell_1\phi''(\xi_{j_1k_1j_2k_2}/N)(\xi_{j_1k_1} - \xi_{j_2k_2}) = O(N) \end{pmatrix}$$

By the assumption $|\phi''| > c > 0$ this gives

$$\begin{pmatrix} \ell_1 h_1 = \ell_2 h_2 \\ h_1\ell_1(\xi_{j_1k_1} - \xi_{j_2k_2}) = O(N) \end{pmatrix}$$

Let us put:

$$\Omega^N_{(\ell_1,\ell_2)} := \{(j_1,k_1,j_2,k_2) : h_1\ell_1 = h_2\ell_2, \quad h_1\ell_1(\xi_{j_1k_2} - \xi_{j_2k_2}) \leq N\},$$

and

$$\Omega^N_{(\ell_1,\ell_2,h_1)} := \{(j_1,k_1,j_2,k_2) : j_1 - k_1 = h_1, \;\; h_1\ell_1 = h_2\ell_2, \;\; h_1\ell_1(\xi_{j_1k_2} - \xi_{j_2k_2}) \leq N\}$$

so that

$$\Omega^N_{(\ell_1,\ell_2)} = \sqcup_{h_1}\Omega^N_{(\ell_1,\ell_2,h_1)} \Rightarrow |\Omega^N_{(\ell_1,\ell_2)}| = \sum_{h_1}|\Omega^N_{(\ell_1,\ell_2,h_1)}|.$$

Here, $|\cdot|$ is the number of elements in the set. We would like to estimate $|\Omega^N_{(\ell_1,\ell_2,h_1)}|$. Note that besides the diagonal solutions $j_1 = k_1, j_2 = k_2$ which have been removed there is also a trivial class of solutions of the form $\ell_1 = \ell_2, j_1 = j_2, k_1 = k_2$ which gives a lower bound of $1/N$ to the variance. We now aim to show that the non-trivial solutions only change this by a $logN$ factor.

It will be useful to write:

$$\xi_{jk} = N(\phi')^{-1}(\int_0^1 \phi'(s\frac{j}{N} + (1-s)\frac{k}{N})ds)$$

and

$$\xi_{j_1k_1} - \xi_{j_2k_2} = N\int_0^1 \frac{d}{dt}(\phi')^{-1}\{\int_0^1 \phi'(t(s\frac{j_1}{N} + (1-s)\frac{k_1}{N}) + (1-t)(s\frac{j_2}{N} + (1-s)\frac{k_2}{N})ds\}dt$$

$$= \int_0^1 \{[(\phi')^{-1}]'((\int_0^1 \phi'(t(s_1\frac{j_1}{N} + (1-s_1)\frac{k_1}{N}) + (1-t)(s_1\frac{j_2}{N} + (1-s_1)\frac{k_2}{N})ds_1)$$

$$\times(\int_0^1 \phi''(t(s\frac{j_1}{N} + (1-s)\frac{k_1}{N} + (1-t)s\frac{j_2}{N} + (1-s)\frac{k_2}{N})dt)[sj_1 + (1-s)k_1 + -sj_2 + (1-s)k_2)]\}ds).$$

Performing the ds_1dt we get an expression of the form

$$\int_0^1 K(s;j_1,k_1,j_2,k_2,N)[sj_1 + (1-s)k_1 + -sj_2 + (1-s)k_2)]ds = aj_1 + bk_1 - cj_2 - dk_2$$

where $K(s;j_1,k_1,j_2,k_2,N)$ is the integral of a positive function with $0 < c < K < C$ for some constansts c, C independent of the $j-k-N$ parameters. Therefore there exist positive functions a, b, c, d of (j_1,k_1,j_2,k_2,N) each with values in $[c, C]$ such that

$$|\Omega_{\ell_1,\ell_2,h_1}| << |\{(j_1,k_1,j_2,k_2) : h_1\ell_1 = h_2\ell_2, h_1 = j_1 - k_1, |aj_1 + bk_1 - cj_2 - dk_2| << \frac{N}{h_1\ell_1}\}.$$

We have

$$c(j_1 + k_1) - C(j_2 + k_2) \leq aj_1 + bk_1 - cj_2 - dk_2, \qquad C(j_1 + k_1) - c(j_2 + k_2) \geq aj_1 + bk_1 - cj_2 - dk_2.$$

Hence $|\Omega_{\ell_1,\ell_2,h_1}|$ is bounded above by the number of solutions of

$$c(j_1 + k_1) - C(j_2 + k_2) \leq \frac{N}{h_1\ell_1}, \qquad C(j_1 + k_1) - c(j_2 + k_2) \geq \frac{N}{h_1\ell_1}. \tag{28}$$

These inequalities are essentially the same as in the quadratic case. We regard h_1, h_2, m_1, m_2 with $m_i = j_i+k_i$ as the independent variables. For a given m_1 there are at most $\frac{N}{h_1\ell_1}$ solutions of the inequalities in m_2. Hence

$$|\Omega_{\ell_1,\ell_2,h_1}| << \frac{N^2}{h_1|\ell_1|}$$

and therefore

$$|\Omega_{\ell_1,\ell_2}| << \frac{N^2logN}{|\ell_1|}.$$

We are now in the same position as in the quadratic case (cf. (32)) and it follows as before that the variance is $O(\frac{(logN)^2}{N})$. ■

Corollary 5.1.2 *Let $N_m = [m(logm)^2]$ ($[\cdot]$ = integer part). Then for almost all (α, β) with respect to Lebseque measure and all $t \neq 0$ we have*

$$\lim_{m\to\infty} \rho^{N_m}_{2;(t,\alpha,\beta)} = \rho_2^{POISSON}.$$

Proof By the above,

$$\sum_{m=1}^{\infty} \frac{1}{(2T)^2} \int_{-T}^{T}\int_{-T}^{T} |\rho^{N_m}_{2;(t,\alpha,\beta)}(f) - \rho_2^{POISSON}(f)|^2 d\alpha d\beta < \infty.$$

Since the terms are positive it follows that for almost all (α, β),

$$\sum_{m=1}^{\infty} |\rho^{N_m}_{2;(t,\alpha,\beta)}(f) - \rho_2^{POISSON}(f)|^2 d\alpha d\beta < \infty$$

and for these (α, β) the mth term tends to zero. ∎

Remark In this corollary we have adapted an argument from [Sa.2][R.S], where the pair correlation problem is studied for flat tori and for some homogeneous integrable systems. Their main result was that the relevant pair correlation functions are almost everywhere Poisson. After proving the almost everywhere convergence to Poisson along a slightly sparse subsequence (as in the above Corollary), they show that for $N_m < M < N_{m+1}$,

$$\rho^{N_m}_{2;(t,\alpha,\beta)}(f) - \rho^{M}_{2;(t,\alpha,\beta)}(f) = o(1)$$

as $m \to \infty$. This last step seems to be much more difficult in our problem. The difference is that the spectra in [Sa.2][R] increase with increasing N and the common terms cancel in the difference above. On the other hand, our spectra change rapidly with N and there are no (obvious) common terms to cancel.

6 Appendix: Linear and quadratic cases

In the case of linear and pure quadratic Hamiltonians, the exponential sums discussed above are very classical. We briefly discuss what is known and add a few observations of our own.

First, the pair correlation problem for linear Hamiltonians $H = \alpha I$ has been studied since the fifties (V.Sos and S.Swierczkowski). See [Bl.2][R.S] for discussion and references the literature. The result is that only three level spacings can occur for a given α and the pair correlation function is not even mean square Poisson.

In the case of quadratic Hamiltonians, we get the incomplete Gauss sums:

$$S_t(N;\ell) = \sum_{j=1}^{N} e^{2\pi i t N\ell[(\frac{j}{N})^2 + \alpha\frac{j}{N}]}.$$

In the special case $\alpha = 0$ and $t = 1$ they are classical complete Gauss sums

$$G(\ell, 0; N) \sum_{j=1}^{N} e^{2\pi i \ell \frac{j^2}{N}}.$$

If $(\ell, N) = 1$ then

$$|G(\ell, 0, N)| = \begin{cases} \sqrt{N} & if N \equiv 1 (mod 2) \\ \sqrt{2N} & if N \equiv 0 (mod 4) \\ 0 & if N \equiv 2 (mod 4) \end{cases}$$

In general

$$G(\ell,0,N)=(\ell,N)G(\frac{\ell}{(\ell,N)},0;\frac{N}{(\ell,N)}).$$

Hence the values of

$$I_N=\frac{1}{N^2}\sum_{\ell\neq 0}\hat{f}(\frac{2\pi\ell}{N})|S_t(N,\ell)|^2$$

depend on the residue class of N modulo 4. If $N\equiv 2 \pmod 4$ then $I_N=0$. If N is odd, then

$$I_N=\frac{1}{N}\sum_{\ell\neq 0}(\ell,N)^2\hat{f}(\frac{2\pi\ell}{N})=\frac{1}{N}\sum_{k\in\mathbb{Z}:k\neq 0}k^2\sum_{\ell:(\ell,N)=k}\hat{f}(\frac{2\pi\ell}{N}).$$

When $N=p$, a prime number. Then $(\ell,p)=1$ except for multiples kp with $k\in supp(\hat{f})$. They make a neglible contribution, so

$$I_p=\frac{1}{p}\sum_{\ell\neq 0}\hat{f}(\frac{2\pi\ell}{p})\to\int_{\mathbb{R}}\hat{f}(x)dx=f(0).$$

Thus the prime sequence is Poisson.

In general, if $(\ell,N)=k$ then $k|N$ and $\ell=kq$ with $(q,\frac{N}{k})=1$. Since $\hat{f}(\frac{\ell}{N})=\hat{f}(\frac{q}{\frac{N}{k}})$, we have

$$I_N=\sum_{k:k|N}k\frac{k}{N}\sum_{q=1,(q,\frac{N}{k})=1}^{\frac{N}{k}}\hat{f}(\frac{q}{\frac{N}{k}}).$$

The inner sum $I_{N,k}:=\frac{k}{N}\sum_{q=1,(q,\frac{N}{k})=1}^{\frac{N}{k}}\hat{f}(\frac{q}{\frac{N}{k}})$ has somewhat the form of a Riemann sum with mesh $\frac{k}{N}$ except that $\hat{f}$ is only evalued at points $\frac{q}{\frac{N}{k}}$ of the partition whose numerators q are relatively prime to $\frac{N}{k}$. The number of such q equals $\phi(\frac{N}{k})$ (the Euler ϕ-function), so on average the $\frac{q}{\frac{N}{k}}$ are spaced out $\frac{k}{N}\times\frac{N}{k\phi(\frac{N}{k}}=\frac{1}{\phi(\frac{N}{k})}$. The coefficient $k\frac{k}{N}$ could only resemble the mesh if $k\sim\frac{N}{k\phi(\frac{N}{k})}$ on average. But the average order of $\phi(n)$ is $\frac{3n}{\pi^2}$ (Merten's formula) so $\frac{N}{k\phi(\frac{N}{k})}$ is of smaller order than k in general. Hence, the statistics are not Poisson.

However, if we allow t to vary then we do have an average Poisson behaviour:

Proposition 6.0.3 *For any interval $[-T,T]$, the average PCF of $\Pi expitN\hat{I}^2$ is Poisson, i.e.*

$$\frac{1}{N^2}\sum_{\ell\neq 0}\hat{f}(\frac{\ell}{N})\frac{1}{2T}\int_{-T}^{T}\sum_{j/not=k}e^{i\ell t\frac{j^2-k^2}{N}}dt=o(1).$$

Proof: The integral equals

$$\frac{1}{N}\sum_{\ell\neq 0}\frac{1}{\ell}\hat{f}(\frac{\ell}{N})[\sum_{j/not=k}\frac{sin(\ell T\frac{j^2-k^2}{N})}{j^2-k^2}$$

$$\frac{1}{N^2}\sum_{\ell\neq 0}\hat{f}(\frac{\ell}{N})[\sum_{m=1}^{2N}\sum_{0<|h|\leq N-|m|}\frac{sin(\ell T\frac{hm}{N})}{Nhm\ell}$$

where as above $h=j-k, m=j+k$. Using just that $sinx << 1$ this is

$$<<\frac{1}{N}\sum_{\ell\neq 0}\hat{f}(\frac{\ell}{N})[\sum_{m=1}^{2N}\sum_{0<|h|\leq N-|m|}\frac{1}{hm\ell}$$

$$<< \frac{(logN)^2}{N} \sum_{\ell \neq 0} \hat{f}(\frac{\ell}{N})\frac{1}{\ell} = O(\frac{(logN)^2}{N}).$$

■

We do not know if the variance tends to zero in this case.

References

[B.T] M.V.Berry and M.Tabor, Level clustering in the regular spectrum, Proc.R.Soc.Lond.A356(1977), p.375.

[Bl] P.Bleher, Distribution of energy levels of a quantum free particle on a surface of revolution, Duke Math. J. 74 (1994), 45 - 93.

[Bl.2] P.Bleher, The energy level spacing for two harmonic oscillators with generic ratio of frequencies, J.Stat.Phys. 63 (1991), 261-283.

[B.K.S] P.Bleher, D.Kosygin and Ya.G.Sinai, Distribution of energy levels of a quantum free particle on a Liouville surface and trace formulae, Comm.Math.Phys. 179 (1995), 375-403.

[B.I] E.Bombieri and H.Iwaniec, Some mean value theorems for exponential sums, Annali Scuola Normale Superiore - Pisa (4) **13** (1986), 633-649.

[B.G] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Ann. Math. Studies **99**, Princeton U. Press (1981).

[CV] Y.Colin de Verdiere, Spectre conjoint d'operateurs pseudo-differentiels qui commutent II. Le cas integrable, Math.Zeit. 171 (1980), 51-73.

[D.G] J.J.Duistermaat and V.Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent.Math. 29 (1975), 39-79.

[Ca.Gu.Iz] G.Casati, I.Guarneri, and F.M.Izrailev, Statistical properties of the quasi-energy spectrum of a simple integrable system, Phys.Lett.A 124(4,5) (1987), p.263-66.

[G.K] S.W.Graham and G.Kolesnik, *Van der Corput's Method of Exponential Sums*, London Math.Soc. Lecture Note Series 126, Cambridge U.Press (1991).

[G] V.Guillemin, Toeplitz operators in n-dimensions, Int.Eq.Op.Theory 7 (1984), 145-205.

[G.2] V.Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n - Spaces*, Progress in Math. 122, Birkhauser, Basel (1994).

[G.S] V.Guillemin and S.Sternberg, *Symplectic Techniques in Physics*, Cambridge U.Press, Cambridge (1984).

[G.S.2] V.Guillemin and S.Sternberg, Some problems in integral geometry and some related problems in microlocal analysis, Amer.J.Math.101 (1979), 915-955.

[G.U] V.Guillemin and A.Uribe, Monodromy in the quantum mechanical spherical pendulum, Comm.Math.Phys. 122 (1989), 563-574.

[H.1] M.N.Huxley, *Area, Lattice Points, and Exponential Sums*, London Math.Soc. Monographs, New Series 13, Clarendon Press, New York (1996).

[H.2] M.N.Huxley, The fractional parts of a smooth sequence, Mathematika 35 (1988), 292-296.

[Iz] F.M.Izrailev, Limiting quasi-energy statistics for simple quantum systems, Phys.Rev.Letts. 56 (1986), 541-544.

[Kea] J.Keating, The cat maps: quantum mechanics and classical motion, Nonlinearity 4 (1991), 309-341.

[Lox.1] J.H.Loxton, The graphs of exponential sums, Mathematika 30 (1983), 153-163.

[Lox.2] J.H.Loxton, The distribution of exponential sums, Mathematika 32 (1985), 16-25.

[M] M. Mendes-France, Entropie, dimension et thermodynamique des courbes planes, *Seminar on Number Theory, Paris 1981-2*, Progress.Math.38, Birkhauser, Boston (1983).

[M] H.L.Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Lecture Series 84, AMS Publications (1990).

[P.Z] U.Porod and S.Zelditch, Semi-classical limit of random walks II (Preprint, 1997).

[R.S] Z.Rudnick and P.Sarnak, The pair correlation function of fractional parts of polynomials (preprint, 1997).

[Sa.1] P.Sarnak, Arithmetic quantum chaos, in *Israel Mathematical Conference Proc. Vol. 8* (1995), 183-236.

[Sa.2] P.Sarnak, Values at integers of binary quadratic forms (preprint 1996).

[U.Z] A.Uribe and S.Zelditch, Spectral statistics on Zoll surfaces, Comm.Math.Phys. 154 (1993), 313-346.

[Z] S.Zelditch, Index and dynamics of quantized contact transformations, Annales Inst.Fourier, Grenoble 47 (1997), 305-363.