SEMICLASSICAL QUANTIZATION OF MAPS AND SPECTRAL CORRELATIONS

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Abstract

The semiclassical theory of the spectral statistics of quantized maps is reviewed. The results are compared with the predictions of random matrix theory for the circular ensembles. Systematic deviations are expected, and they vanish only in the limit where the underlying classical dynamics evolves to the completely mixed state (equilibrium) infinitely fast.

I. INTRODUCTION

Maps appear naturally in the analysis of classical chaotic systems: The Poincaré map, which maps the Poincaré section onto itself, is a powerful tool which was designed to simplify the description of complex multidimensional flows, without loosing their essential features. Its most important property is that the phase space measure of the section is preserved, thus maintaining the Hamiltonian character of the original system. Maps with similar properties arise also when the Hamiltonian is a periodic function of time. The stroboscopic description – where the dynamics is recorded and analyzed only at times which are integer multiples of the period – is carried out in terms of an area preserving classical map. We shall confine our attention to measure preserving maps which act on a compact phase space in two dimensions. These are the simplest, yet non trivial maps, which correspond to the simplest flows that display classical chaos.

The quantum mechanical analogues of the area preserving maps introduced above are unitary operators which act on Hilbert spaces of finite dimension, N. They are often referred to as evolution, scattering or Floquet operators, depending on the physical context where they are used. Their eigenvalues consist of N points on the unit circle (eigenphases). The eigenphases statistics was shown to depend on the nature of the underlying classical dynamics. The quantum analogues of classically integrable maps display Poissonian statistics. This was found numerically in many systems, and for a few particular cases, there exist some rigorous results [1], [2]. Numerical studies of various classically chaotic systems suggest that the eigenphases statistics conform quite accurately with the results of Random Matrix Theory (RMT) for Dyson's "circular ensembles" [3–6]. This universal behavior which is directly linked with the underlying classical dynamics is one of the central issues in quantum chaology, and is the subject of the present paper.

There are three reasons why the study of quantum maps gained considerable interest during the past years.

- The quantum energies (either exact or semiclassical) of hamiltonian systems, can be derived from secular equations which involve the eigenvalues of certain unitary matrices, which, in the classical limit, correspond to Poincaré maps on a properly defined section. Notable examples are Bogomolny's semiclassical T matrix method, [7] and the scattering method developed by the present author and his coworkers [8–10]. In the semiclassical limit, the statistics of the *energy levels* can be expressed in terms of the statistics of the *eigenphases* of the corresponding unitary operators. This correspondence enables one to take advantage of the structural simplicity of unitary matrices and their spectrum.

- Even though the original development and applications of Random Matrix Theory were focussed on the Gaussian ensembles of *hermitian* matrices, it was soon realized by Dyson [11] that the circular ensembles of *unitary* matrices are much more convenient and natural to study. The spectra are confined to the unit circle and not to the entire real line as for the hermitian operators. Moreover, the measures which define the ensembles are well understood, and appear naturally in the classical theory of continuous groups. [12].

- Recently, important progress was achieved by Zirnbauer and his coworkers [13], who extended the field theoretical methods of Efetov to problems involving unitary operators and the corresponding circular ensembles.

At present, there exist two distinct theoretical approaches to the study of spectral statistics of unitary operators whose classical analogues are chaotic: The first is the field theoretical approach which is introduced and explained in M. Zirnbauer's contribution to this volume. The second is the semiclassical approximation, which will be the topic of this article. The material will be presented in the following way. The next section will introduce the statistical measures which we shall study, and summarize the predictions of RMT for these quantities. Preparing the background for the semiclassical approximation, a few topics from the theory of classical maps will be reviewed. Then, the semiclassical expression of the quantum evolution operator will be discussed, and the semiclassical trace formula appropriate to the study of unitary operators will be presented. In order to study spectral statistics, the "semiclassical ensemble" will be introduced and will be used to obtain expressions for the statistical measures which can be directly compared with the RMT predictions. In particular, two-points correlation function will be discussed using two complementary approaches. The first is an adaptation of a formalism introduced recently by Bogomolny and Keating [14], and the other, which emphasizes the rôle of periodic orbits correlations, follows the ideas developed by Argaman *et. al.* [15] and Cohen *et. al.* [17]. The summary section is dedicated to a critical discussion of the present status of the semiclassical approach, and it includes a list of open problems.

II. SPECTRAL STATISTICS - DEFINITIONS AND RESULTS FROM RMT

Consider an arbitrary $N \times N$ unitary matrix U with a spectrum $\left\{e^{i\theta_l}\right\}_{l=1,\dots N}$. The spectral density can be written as

$$d_U(\theta) \equiv \sum_{l=1}^{N} \delta(\theta - \theta_l) = \frac{N}{2\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(e^{-in\theta} \mathrm{tr} U^n + e^{in\theta} \mathrm{tr} U^{-n}\right) \tag{1}$$

The corresponding counting (staircase) function is

$$N_U(\theta) = \frac{N\theta - \Theta}{2\pi} + \frac{i}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{-in\theta} \mathrm{tr} U^n - e^{in\theta} \mathrm{tr} U^{-n}\right) , \tag{2}$$

where $\det(-U) \equiv e^{i\Theta}$, and we set $N(0) = 0$. From now on, we shall use the notation $t_n = \mathrm{tr} U^n$. We write the characteristic polynom as

$$p_U(z) \equiv \det(I - zU) = \sum_{n=0}^{N} a_n z^n \tag{3}$$

and its roots are $z_l = e^{-i\theta_l}$. An important consequence of the unitarity of U is the *inversive symmetry* of the coefficients a_n,

$$a_n = e^{i\Theta} a^*_{N-n} \ . \tag{4}$$

Starting from

$$\det(I - xU) = \sum_{n=0}^{N} a_n x^n = \exp\left(-\sum_{k=1}^{\infty} \frac{x^k}{k} \mathrm{tr} U^k\right) \ , \tag{5}$$

one can derive Newton's identities which relate the traces $t_n = \mathrm{tr} U^n$ and the coefficients of the secular polynom

$$a_n = -\frac{1}{n}\left(t_n + \sum_{k=1}^{n-1} a_k t_{n-k}\right) \ . \tag{6}$$

Since $a_n = 0$ for $n > N$, the t_n for all $n > N$ depend linearly on the lower $n \leq N$ traces.

The two-point spectral statistics of interest here are defined as follows - Consider the oscillatory part of the number counting function:

$$\tilde{N}_U(\theta) \equiv N_U(\theta) - \frac{N\theta - \Theta}{2\pi} \tag{7}$$

The number-number correlation function is

$$\begin{aligned}\mathcal{N}(x,y) &\equiv \int_0^{2\pi} \frac{\mathrm{d}\omega}{2\pi} \left\langle \tilde{N}_U(\omega + x)\tilde{N}_U(\omega + y)\right\rangle \\ &= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{<|t_n|^2>}{n^2} \cos(n(x-y))\end{aligned} \tag{8}$$

where we denote by $< \cdot >$ the average over an ensemble of Unitary matrices which can be e.g., one of Dyson's circular ensembles or any other ensemble such as the semiclassical ensemble which will be introduced in the sequel. Due to the averaging over the variable ω, $\mathcal{N}(x,y)$ depends on its arguments only through the difference $\eta = x - y$. The more frequently used density-density correlation function can be derived from the number-number correlators by

$$R_2(\eta) \equiv \left(\frac{2\pi}{N}\right)^2 \int_0^{2\pi} \frac{\mathrm{d}\omega}{2\pi} \left\langle \tilde{d}_U(\omega + \eta/2)\tilde{d}_U(\omega - \eta/2)\right\rangle = -\left(\frac{2\pi}{N}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\mathcal{N}(\eta) \tag{9}$$

Another statistical measure which we shall study is the auto-correlation function of the characteristic polynomial [6,18]

$$\begin{aligned}C(\eta) &\equiv e^{i\eta\frac{N}{2}} \int_0^{2\pi} \frac{\mathrm{d}\omega}{2\pi} \left\langle p_U(e^{-i(\omega+\eta/2)})p_U^*(e^{-i(\omega-\eta/2)})\right\rangle \\ &= \sum_{n=0}^{N} < |a_n|^2 > \cos((n - N/2)\eta)\end{aligned} \tag{10}$$

where the phase factor $e^{i\eta\frac{N}{2}}$ is introduced to keep the correlation function real. The inversive symmetry (4) implies that the Fourier components of $C(\eta)$ are symmetric about $N/2$. This statistical measure contains in it correlation between more than just pairs of eigenvalues, as can be easily shown by writing the a_n in terms of the eigenvalues. Hence, $C(\eta)$ tests aspects of the eigenvalues distribution which are not accessible by the study of the two-points functions defined previously.

There are a few important identities which will be used throughout this work, and which provide alternative expressions for the spectral functions which were introduced above. Using Cauchy's theorem one can calculate the number of eigenphases in the interval $[0,\theta]$ on the unit circle

$$N(\theta) = \frac{1}{2\pi i}\left[\log p_U(e^{i\theta+\epsilon}) - \log p_U(e^{i\theta-\epsilon})\right]_{\epsilon\downarrow 0} - \frac{1}{2\pi i}\left[\log p_U(e^{\epsilon}) - \log p_U(e^{-\epsilon})\right]_{\epsilon\downarrow 0} \qquad (11)$$

The unitarity of U implies the "functional equation"

$$p_U(z) = z^N e^{-i\Theta}\left(p_U(\frac{1}{z^*})\right)^* . \qquad (12)$$

Substituting in (11) one can get an expression for the ϵ smoothed stair-case function

$$N_\epsilon(\theta) = \frac{\theta N - \Theta}{2\pi} + \frac{1}{2\pi i}\log\frac{\sum_{n=0}^{N} a_n^* e^{-in\theta}e^{-n\epsilon}}{\sum_{n=0}^{N} a_n e^{in\theta}e^{-n\epsilon}} \ , \qquad (13)$$

and $\lim_{\epsilon\downarrow 0} N_\epsilon(\theta) = N(\theta)$. The argument of the log in (13) does not have zeros or poles as long as $\epsilon > 0$. Hence, the phase

$$\phi_\epsilon(\theta) = \frac{1}{i}\log\frac{\sum_{n=0}^{N} a_n^* e^{-in\theta}e^{-n\epsilon}}{\sum_{n=0}^{N} a_n e^{in\theta}e^{-n\epsilon}} \qquad (14)$$

is a well defined function for real θ and it assumes values in $[0,2\pi]$. The limit $\epsilon \downarrow 0$ is not always possible to follow. Thus, we are sometimes forced to proceed with a small but finite value of ϵ, for which the number function $N_\epsilon(\theta)$ is smooth, and retains the stair-case character in an approximate way [14]. This is realized by observing that $N_\epsilon(\theta)$ for ϵ which is smaller than the mean spacing $2\pi/N$ maintains some of the staircase characteristics, namely, it assumes approximately constant integer values which are interrupted by steeply increasing sections, located in the neighborhood of the eigenphases θ_l. One can obtain an approximate spectrum as the solutions $\theta_l(\epsilon)$ of the equation

$$N_\epsilon(\theta_l(\epsilon)) = l - \frac{1}{2} \quad \text{for} \quad l = 1\cdots N \ . \qquad (15)$$

It is tacitly assumed that $N_\epsilon(\theta)$ is a monotonic increasing function, because otherwise the above equation might have more than N solutions. The corresponding sharpened spectral density is

$$d^{\sharp}\epsilon(\theta) = \sum_{l=1}^{N} \delta\left(N_\epsilon(\theta) - (l - \frac{1}{2})\right) \frac{\mathrm{d}N_\epsilon(\theta)}{\mathrm{d}\theta} \tag{16}$$

Using the definitions (13,14), one gets

$$N_\epsilon^\sharp(\theta) = N_\epsilon(\theta) + \frac{1}{2\pi}\sum_{m\neq 0} \frac{(-1)^m}{m} e^{im(\phi_\epsilon(\theta)+\theta N-\Theta)} \tag{17}$$

$N_\epsilon^\sharp(\theta)$ is a staircase function, which approximates the true counting function better and better as $\epsilon \downarrow 0$. It is hoped that even for finite but sufficiently small $\epsilon \approx \mathcal{O}(N^{-1})$, the spectral statistics calculated for $N_\epsilon^\sharp(\theta)$ imitates the exact spectral statistics. This equation will be the starting point for the semiclassical theory for the two-point correlations $\mathcal{N}(\eta)$.

To end this section, we shall quote the expressions provided by Random Matrix Theory for $< |t_n|^2 >$ and $< |a_n|^2 >$, for Dyson's circular ensembles COE and CUE, and for the circular Poisson ensemble [19].

COE :

$$< |t_n|^2 > = \begin{cases} 2n - n\sum_{m=1}^{n} \frac{1}{m+\frac{N-1}{2}} & \text{for } n < N \\ 2N - n\sum_{m=1}^{N} \frac{1}{m+n-\frac{N+1}{2}} & \text{for } n \geq N \end{cases}$$

$$< |a_n|^2 > = 1 + \frac{n(N-n)}{N+1} \tag{18}$$

CUE :

$$< |t_n|^2 > = \begin{cases} n & \text{for } n < N \\ N & \text{for } n \geq N \end{cases}$$

$$< |a_n|^2 > = 1 \tag{19}$$

POISSON :

$$< |t_n|^2 > = N \qquad \text{for all } n$$

$$< |a_n|^2 > = \begin{pmatrix} N \\ n \end{pmatrix} \tag{20}$$

Substituting these expressions in (8),(9) or (10), one can get the RMT expression for $\mathcal{N}(\eta)$, $R_2(\eta)$ or $C(\eta)$. As was mentioned previously, the numerical studies of the spectral statistics of quantum maps show that chaotic maps which obey time reversal symmetry, adhere to the predictions of the COE ensemble, those which violate time reversal reproduce the predictions of the CUE ensemble, while for integrable maps, the statistical measures are given by the predictions of the Poisson ensemble. The main purpose of this paper is to show how these observations can be explained by the semiclassical theory of quantized maps, and to identify the circumstances where one would expect systematic deviations from the predictions of RMT.

III. SEMICLASSICAL QUANTIZATION OF MAPS

A. Classical maps

The quantum unitary operator U which we consider, is assumed to be the analogue of an area preserving map $\mathcal{F}$ acting on a finite phase space domain $\mathcal{M}$ with area $|\mathcal{M}|$. (For the present discussion we shall confine our attention to maps with the twist property. The semiclassical treatment can be extended to the general case). The phase space coordinates are denoted by $\gamma = (q,p)$ and γ is mapped to $\gamma' = \mathcal{F}(\gamma)$. The map can be defined in terms of a generating function (action) $\Phi(q,q')$

$$p = -\frac{\partial \Phi(q,q')}{\partial q} \quad ; \quad p' = \frac{\partial \Phi(q,q')}{\partial q'}. \tag{21}$$

The explicit mapping function $\gamma' = \mathcal{F}(\gamma)$ is obtained by solving the implicit relations (21). The twist condition ensures that the implicit equations (21) have a unique solution. As an example, consider an integrable map, and denote by (I,ϕ) the action-angle variables, where I is the invariant momentum under the action of the map. The domain of the mapping is $I \in [I_{min}, I_{max}], \phi \in [0, 2\pi]$. In this representation, the generating function must take the form $\Phi(\phi,\phi') = \Phi(\phi' - \phi)$. The explicit map is

$$I' = I \quad ; \quad \Delta\phi = \phi' - \phi = f(I) \tag{22}$$

Here, $f(I)$ (the angular velocity) is the inverse of the generating relation $I = \frac{\mathrm{d}\Phi(\Delta\phi)}{\mathrm{d}\Delta\phi}$, which gives $\Delta\phi$ in terms of I. The twist condition is fulfilled if $\frac{\mathrm{d}^2\Phi(\Delta\phi)}{\mathrm{d}\Delta\phi^2} \neq 0$.

A classical trajectory is obtained by applying the map to an arbitrary initial point in $\mathcal{M}$, and the corresponding action is accumulated along the trajectory.

Classical dynamics can be also viewed as the evolution of phase space densities in time. For maps, the evolution is mediated by the Perron-Frobenius operator,

$$W(\gamma,\gamma') \equiv \delta(\gamma' - \mathcal{F}(\gamma)), \tag{23}$$

and a phase space density $\rho(\gamma)$ evolves under the map as

$$\rho'(\gamma') = \int_{\mathcal{M}} \mathrm{d}\gamma \rho(\gamma) W(\gamma,\gamma') \tag{24}$$

The Ruelle ζ function of the map $\mathcal{F}$ is defined as

$$\zeta_{cl}(z) \equiv (\det(I - zW))^{-1} = \exp\left[-\log\det(I - zW)\right] = \exp\left[\sum_{n=1}^{\infty} \frac{w_n}{n} z^n\right], \tag{25}$$

where, for hyperbolic maps,

$$w_n = \mathrm{tr}W^n = \sum_{p\in\mathcal{P}_n} \frac{n_p g_p}{|\det(I - T_p^r)|}. \tag{26}$$

This sum extends over the set $\mathcal{P}_n$ of all the primitive periodic orbits of $\mathcal{F}$, with periods n_p which are divisors of n, so that $n = n_p r$. Orbits which are related by a discrete symmetry are counted once, and their multiplicity is denoted by g_p. T_p is the linearization of the map $\mathcal{F}^{n_p}$ about the p periodic orbit, and its eigenvalues are λ_p and λ_p^{-1}. We shall use the convention that $|\lambda_p| > 1$. The parameters w_n can be interpreted as the probability to perform n-periodic motion. The ergodicity of the map dynamics implies that w_n approaches 1 as $n \to \infty$. Thus, the infinite sum in (25) converges absolutely for $|z| < 1$. One can substitute (26) in (25) and perform the summation over repetitions, to get the well known expression

$$\zeta_{cl}(z) = \left[\prod_{m=1}^{\infty} \prod_{p\in\mathcal{P}} \left(1 - \left(\frac{z^{n_p}}{\lambda_p^m}\right)\right)^{g_p m}\right]^{-1} \tag{27}$$

The summation extends over the set of primitive periodic orbits $p \in \mathcal{P}$. $\zeta_{cl}(z)$ is analytic in the open unit circle, with a pole at $z = 1$. The rate of mixing which is induced by the map is determined by the gap between the main pole and the next one. In the limit of infinitely fast mixing,

$$\zeta_{cl}^{(m)}(z) \approx \frac{1}{1-z} \,. \tag{28}$$

This is as much classical mechanics as we need, and the semiclassical theory follows in the next subsection.

B. The semiclassical approximation for trU^n

The semiclassical expression for the matrix elements of U in the q representation is [21], [23].

$$< q|U|q' >= \left(\frac{1}{2\pi\hbar i}\right)^{\frac{1}{2}} \left[\frac{\partial^2\Phi(q,q')}{\partial q \partial q'}\right]^{\frac{1}{2}} e^{i\Phi(q,q')/\hbar} \tag{29}$$

In the semiclassical limit, N, the dimension of the Hilbert space where U acts, is the integer part of $\frac{|\mathcal{M}|}{2\pi\hbar}$.

We have seen above, that the main building blocks of the spectral statistics are the traces $t_n = \mathrm{tr}U^n$. The semiclassical approximation for t_n involves the periodic manifolds of the classical map. For hyperbolic maps [22], [23],

$$t_n \equiv \mathrm{tr}U^n \approx \sum_{p\in\mathcal{P}_n} \frac{g_p n_p e^{ir(\Phi_p/\hbar - \nu_p \frac{\pi}{2})}}{|\det(I - T_p^r)|^{\frac{1}{2}}} \tag{30}$$

The semiclassical approximation for t_n involves the same periodic orbits as the classical trace trW^n (26), namely, $\mathcal{P}_n$ is the set of all primitive periodic orbits of $\mathcal{F}$, with periods n_p which are divisors of n, so that $n = n_p r$. T_p is the monodromy matrix, and orbits which are related by a discrete symmetry and multiplicity g_p are counted once. Each periodic orbit contribution is endowed with a phase which is the sum of the action along the periodic orbit is

$$\Phi_p = \sum_{j=1}^{n_p} \Phi(q_j, q_{j+1}) \quad (\text{with} \qquad q_{n_p+1} = q_1), \tag{31}$$

and the Maslov contribution $-\frac{\nu_p \pi}{2}$.

For integrable maps, we use the phase space variables (I, ϕ) where I is the classical invariant. In the quantum picture, I is quantized to integer multiples of $\hbar$ so that $I_j = j\hbar$ and $1 \leq j \leq N$. The matrix U is diagonal in the j representation. The semiclassical approximation for the eigenphases can be carried out directly,

$$\theta_j(\hbar) \approx \frac{1}{\hbar}\left(\Phi(f(I_j)) - I_j f(I_j)\right) \ . \tag{32}$$

where $f(I)$ is the angular frequency (22). One can use the above equation to calculate $\mathrm{tr}U^n$,

$$t_n \equiv \mathrm{tr}U^n \approx \sum_{m=1}^{n}\left[\frac{2\pi}{n\hbar f'(I_{n,m})}\right]^{1/2} e^{i\left[n\Phi(\Delta\phi=2\pi\frac{m}{n})/\hbar-(n+\frac{1}{2})\frac{\pi}{2}\right]} \tag{33}$$

Where the summation is carried over the periodic manifolds of period n and winding number m. They occur at values of I for which the angular frequency is rational $\frac{f(I_{n,m})}{2\pi} = \frac{m}{n}$.

The expressions for t_n in the classically integrable and classically chaotic cases are the necessary input for the computation of the spectral measures which were introduced in the previous section.

IV. A SEMICLASSICAL THEORY FOR THE SPECTRAL STATISTICS

Before we can proceed any further, we must clarify an essential point. In contrast with the RMT, the semiclassical theory deals with a *single* system, and not with an ensemble of systems. However, averaging is mandatory in order to get a meaningful theory since the quantities we calculate fluctuate and are not self averaging. We generate the "semiclassical ensemble" by considering the inverse Planck constant $\beta = \hbar^{-1}$ as a parameter, and different realizations of the ensemble are distinguished by the value of the parameter β. We restrict β to the interval $|\beta - \beta_0| < \Delta/2$ and $\Delta = \frac{2\pi}{|\mathcal{M}|}$. This way, the matrices in the ensemble have the same dimension $N(\beta_0)$. The mean value β_0 is assumed to be sufficiently large to justify the use of the semiclassical approximation. That is, for typical orbits $\beta_0|\Phi_t - \Phi_{t'}| >> 1$. The interval Δ is taken to be small on the scale of β_0, but sufficiently large so that

$\Delta|\Phi_t - \Phi_{t'}| > 2\pi$. In this way, the phases (mod2π) of the semiclassical expressions can be considered random. The averaging over the "semiclassical ensemble" is effected by

$$< A >_\hbar \equiv \frac{1}{\Delta} \int_{\beta_0-\Delta/2}^{\beta_0+\Delta/2} \mathrm{d}\beta A(\beta). \tag{34}$$

With this definition of the ensemble average, we get for both classically integrable and chaotic maps,

$$< t_n >_\hbar = 0. \tag{35}$$

The variance for the classically chaotic case reads,

$$< |t_n|^2 >_\hbar \approx \sum_{p\in P_n} \frac{g_p^2 n_p^2}{|\det(I - T_p^r)|} =< g_p n_p >_{cl(n)} w_n \ , \tag{36}$$

where we made use of the fact that averaging with respect to the classical phase space measure can be approximated by a periodic point average. If the repetitions of primitive orbits are neglected, we can write $< g_p n_p >_{cl(n)} w_n = gnw_n$, which defines the mean multiplicity, and it is further assumed for simplicity that the mean multiplicity is independent of n.

For integrable maps we get

$$< |t_n|^2 >_\hbar \approx \frac{2\pi(I_{max} - I_{min})}{2\pi\hbar} = N \tag{37}$$

Thus, the β averaging provides the well know diagonal (random phase) approximation. The diagonal approximation is not valid uniformly in n, and it leads to completely wrong results for $n > N$. This issue will be explained and discussed further in the chapter on action correlations. We shall use the diagonal approximation in the restricted range $n < N(\beta_0) = \frac{\mathcal{M}\beta_0}{2\pi}$ where it can be justified. For integrable maps, the variances of t_n are independent of n. The result $< |t_n|^2 >_\hbar \approx N$ for integrable systems implies that the spectral two-point correlation function in this case is Poisson [20]. Let us consider higher moments of the t_n distribution, such as e.,g., $< (t_n)^k (t_m^*)^l >_\hbar$. If n and m are relatively prime, the actions Φ_t which contribute to t_n and t_m are sufficiently different. Thus, all the terms in the product $(t_n)^k(t_m^*)^l$ are oscillatory and will yield a vanishing result upon averaging. If n and m have a common divisor, j, choose $k = m/j, l = n/j$, and all the amplitudes which involve repetitions of the primitive orbits of length j will contribute non oscillatory terms to the correlator. However, for hyperbolic maps, the number of periodic orbits which involve repetitions is exponentially smaller than the total number of periodic orbits, and the statistical independence of the variables t_n and t_m is ensured. If we check all other correlators using the approximation that repetitions can be neglected, we find that for $n < N/2$, t_n are Gaussian random variables. The corresponding approximation is more difficult to justify for integrable maps because the proliferation of periodic manifolds is only algebraic.

In summary, the approximation in which **repetitions of periodic orbits are neglected**, implies that we may replace the β averaging by averaging over the ensemble of the t_n variables, with $n \leq N/2$, whose distribution is defined as follows:

I . The semiclassical ensemble of $\{t_n\}$, $(n < N/2)$ *is an ensemble of independent random Gaussian variables.*

II . For classical chaotic systems, the variances $< |t_n|^2 >_\hbar \approx g n w_n$*, where* w_n *are the traces of the classical evolution operator (26), and* g *is the mean multiplicity.*

III. For classical integrable systems, the variances are $< |t_n|^2 >_\hbar \approx N$

This set of rules will be now used to represent the averaging with respect to the "semiclassical ensemble". In particular, they will be used to calculate the semiclassical expressions for the spectral statistics of interest here. The resulting expressions are very similar to results obtained by other groups using completely different methods. This gives some confidence in the validity of the approximation in the present context.

Finally, it should be noted that in RMT, the traces t_n are indeed random Gaussian variables [19] in the limit where n is fixed and $N \to \infty$. This property, which is shared by the statistical and the semiclassical ensembles, has far reaching consequences, and it constitutes a strong link between RMT and quantum chaos.

1. A. *The auto-correlation function of the characteristic polynomial*

The Fourier coefficients of the auto-correlation function $< |a_n|^2 >_\hbar$ will be derived from the generating function

$$\begin{aligned} G_\hbar(x,y) &= \left\langle \det(I - xU)\det(I - yU^\dagger)\right\rangle_\hbar \\ &= \left\langle \exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}(x^k t_k + y^k t_k^*)\right)\right\rangle_\hbar \\ &= \exp\left(\sum_{k=1}^{\infty}(xy)^k \frac{< |t_k|^2 >_\hbar}{k^2}\right) \end{aligned} \tag{38}$$

so that

$$< a_n a_m^* >_\hbar = \delta_{n,m}\,\frac{1}{n!}\frac{\partial^n}{\partial^n v} Z_\hbar(v)\Big|_{v=0} \tag{39}$$

with

$$Z_\hbar(v) = \exp\left(\sum_{k=1}^{\infty} < |t_k|^2 >_\hbar \frac{v^k}{k^2}\right) \tag{40}$$

The $< |a_n|^2 >_\hbar$ can be obtained from the recursion relations

$$< |a_l|^2 >_\hbar = \frac{1}{l}\sum_{k=1}^{l} < |a_{l-k}|^2 >_\hbar \frac{< |t_k|^2 >_\hbar}{k} \tag{41}$$

It is important to remember that for the purpose of calculating the autocorrelation function (10), it suffices to obtain the $< |a_n|^2 >_\hbar$ for $n \le N/2$. The rest are provided by the inversive relation $< |a_n|^2 >_\hbar = < |a_{N-n}|^2 >_\hbar$. Note that the iteration procedure (41) is constructed in such a manner, that the only input necessary for the calculation of the $N/2$ lower $< |a_n|^2 >_\hbar$ consists only of the lowest $N/2$ values of $< |t_n|^2 >_\hbar$, for which the working hypothesis which underlies the present derivation can be justified.

The semiclassical expressions for $< |t_k|^2 >_\hbar$, depend on the nature of the corresponding classical dynamics (36),(37). For chaotic classical dynamics we have

$$< |t_n|^2 >_\hbar \approx g n w_n \tag{42}$$

From which it follows that (40) can be written as

$$
\begin{aligned}
Z_\hbar(v) &= \exp\left(\sum_{k=1}^{\infty} < |t_k|^2 >_\hbar \frac{v^k}{k^2}\right) \\
&\approx \exp\left(g \sum_{k=1}^{\infty} w_k \frac{v^k}{k}\right) \\
&= (\zeta_{cl}(v))^g
\end{aligned}
\tag{43}
$$

Where $\zeta_{cl}(z)$ is the Ruelle ζ function for the classical mapping $\mathcal{F}$ defined previously in (25). This can be derived by substituting (42) in (41) and the recursion relations take the form

$$
< |a_m|^2 >_\hbar = \frac{g}{m} \sum_{k=1}^{m} < |a_{m-k}|^2 >_\hbar w_k \tag{44}
$$

which should be solved with the initial condition $< |a_0|^2 >_\hbar = 1$.

Let us consider systems which are strongly mixing and for which all transients die out on a short time scale. In such cases, $\zeta_{cl}(v) = \frac{1}{1-v}$ and $Z_\hbar(v) = \frac{1}{(1-v)^g}$. Direct substitution gives the following results:

Chaotic systems which *violate* time reversal symmetry (TRS) have $g = 1$, and

$$
< |a_m|^2 >_\hbar = 1 \ . \tag{45}
$$

Thus, the semiclassical result coincides with the prediction of RMT for the CUE, (see (19)), $< |a_m|^2 >_{CUE} = 1$.

Chaotic systems which *respect* TRS, have $g = 2$ and

$$
< |a_l|^2 >_\hbar \ \approx \ \begin{cases} 1 + l & \text{for} \ \ 1 < l \le N/2 \\ 1 + N - l & \text{for} \ \ N > l \ge N/2 \end{cases} \tag{46}
$$

This expression does not reproduce the RMT result for the COE case (18) However, for large N, where the semiclassical approximation is justified, the semiclassical result agrees with the exact expression in a domain of l values of size $\sqrt{N}$ in the vicinity of the end points of the l interval, $l = 0$ and $l = N$. The deterioration of the quality of the agreement between the semiclassical and the RMT expressions when TRS is imposed is typical, and its explanation remains one of the open problems in quantum chaos [20].

So far, we discussed systems for which all transients die out on a fast time scale, which was imposed by setting $w_k = \mathrm{tr}W^k = 1$ for all k. This is possible only when one eigenvalue of U is 1 and all the rest vanish. In generic systems, the spectrum is not degenerate in this extreme way. Rather, beside the eigenvalue 1 which corresponds to the invariant phase - space measure, the spectrum is in the interval $[0, 1]$, and it accumulates at 0. The rate of decay of transients is determined by the magnitude of the eigenvalues of U which are less then 1. To get the leading correction due to the non vanishing eigenvalues of U, one can expand the recursion relation (44) to first order in $\mu = w_1 - 1$. One obtains in this way recursion relations for the correction to $< |a_m|^2 >_\hbar$. They are particularly simple for the cases with $g = 1, 2$ and the corrected coefficients are

$$\begin{aligned} < |a_l|^2 >_\hbar \ &= 1 + \mu \qquad\qquad \text{for} \quad g = 1 \\ &= 1 + l + 2\mu l \quad \text{for} \quad g = 2 \end{aligned} \tag{47}$$

The symmetry $< |a_l|^2 >_\hbar = < |a_{N-l}|^2 >_\hbar$ should be implemented for $l > N/2$. Recently, the Essen group studied numerically the variances of the coefficients of the characteristic polynomial for the quantum kicked top [19]. They checked systems with and without TRS, and their numerical results show systematic deviations from the RMT predictions which are consistent with the expressions (47). The numerical results for the case without TRS are particularly convincing. Other numerical tests which involve $S(k)$ matrices which appear in the theory of quantized graphs [24], where recently carried out. For these systems, the strong mixing limit is not justified, and indeed, the resulting $< |a_l|^2 >_\hbar$ deviate appreciably from the predictions of RMT. However, the semiclassical theory which uses the classical Ruelle ζ as input, reproduces the main features of the numerical data.

For integrable maps we have (37)

$$< |t_n|^2 >_\hbar \approx N \tag{48}$$

The resulting recursion relations for the coefficients $< |a_m|^2 >_\hbar$ are

$$< |a_m|^2 >_\hbar = \frac{N}{m} \sum_{k=1}^{m} \frac{< |a_{m-k}|^2 >_\hbar}{k} \tag{49}$$

We were not able to find a closed form for the solution of this equation. However, to leading order $< |a_m|^2 >_\hbar \approx \frac{N^m}{m!}$ which coincides with the leading term of the result for the Poisson ensemble (20).

2. B. The spectral two point correlation functions.

Encouraged by the results of the previous chapter, we shall try to use the same strategy to compute the two-point correlation functions. The essential point in this approach is to express all quantities of interest in terms of quantities which involve the short periodic orbits. This will be achieved here by following the ideas presented in the first chapter, namely, one generates a *synthetic* point spectrum which might not coincide with the true spectrum in detail, but is expected to reproduce its statistical measures when $\epsilon \approx 1/N$. In this spirit, we calculate the number-number correlation function of the "sharpened" number function (17)

$$N_\epsilon^\sharp(\theta) = N_\epsilon(\theta) + \frac{1}{2\pi} \sum_{m\neq 0} \frac{(-1)^m}{m} e^{im(\phi_\epsilon(\theta)+\theta N-\Theta)} \tag{50}$$

where (14),

$$\phi_\epsilon(\theta) = \frac{1}{i} \log \frac{\sum_{n=0}^N a_n^* e^{-in\theta} e^{-n\epsilon}}{\sum_{n=0}^N a_n e^{in\theta} e^{-n\epsilon}} \ . \tag{51}$$

The spectral statistics will be performed for the semiclassical ensemble defined previously. It will be shown bellow, that setting $\epsilon = c/N$ amounts to the introduction of a smooth cut-off which suppresses the dependence of the present theory on t_n with $n > N/2c$.

The oscillatory part of $N_\epsilon^\sharp(\theta)$ comes from the oscillatory part of the first term,

$$\tilde{N}_\epsilon(\theta) = N_\epsilon(\theta) - \frac{\theta N - \Theta}{2\pi} \ , \tag{52}$$

and from the infinite sum

$$\tilde{N}_{\epsilon,m\neq 0}(\theta) = \frac{1}{2\pi} \sum_{m\neq 0} \frac{(-1)^m}{m} e^{im(\phi_\epsilon(\theta)+\theta N-\Theta)} \ . \tag{53}$$

$\tilde{N}_{\epsilon,m\neq 0}(\theta)$ involves much higher frequencies then those involved in $\tilde{N}_\epsilon(\theta)$, and therefore there are no cross correlations. Accordingly, we shall write

$$\mathcal{N}^{\hbar}(\eta) = \mathcal{N}^{diag}(\eta) + \mathcal{N}^{off}(\eta) \tag{54}$$

where $\mathcal{N}^{diag}$ and $\mathcal{N}^{off}$ stand for the contributions from correlations in $\tilde{N}_\epsilon(\theta)$ and from $\tilde{N}_{\epsilon,m\neq 0}(\theta)$, respectively. Starting with $\mathcal{N}^{diag}(\eta)$, we write

$$\begin{aligned}\tilde{N}_\epsilon(\theta) &= \frac{1}{2\pi i}\left[\log\sum_{n=1}^{N} a_n^* e^{(-i\theta-\epsilon)n} - \log\sum_{n=1}^{N} a_n e^{(i\theta-\epsilon)n}\right] \\ &= \frac{1}{2\pi i}\sum_{l=1}^{\infty}\frac{e^{-\epsilon l}}{l}\left(t_l^* e^{-i\theta l} - t_l e^{i\theta l}\right) ,\end{aligned} \tag{55}$$

and get,

$$\begin{aligned}\mathcal{N}^{diag}(\eta) &= \int_0^{2\pi}\frac{\mathrm{d}\omega}{2\pi}\left\langle \tilde{N}_\epsilon(\omega+\eta/2)\tilde{N}_\epsilon(\omega-\eta/2)\right\rangle_\hbar \\ &= \frac{g}{(2\pi)^2}\left(\sum_{l=1}^{\infty}\frac{w_l}{l}(e^{(i\eta-2\epsilon)l} + e^{(-i\eta-2\epsilon)l})\right) = \frac{1}{2\pi^2}\Re e\log\left(\zeta_{cl}(e^{-2\epsilon+i\eta})\right)^g .\end{aligned} \tag{56}$$

Above we used the approximate relation $<|t_l|^2>_\hbar = glw_l$. Had we chosen $c=1$, we would have obtained a damping factor $e^{-\frac{2}{N}l}$ which suppresses the $l > N/2$ terms in (??), in a way which justifies *a posteriori* the use of the semiclassical ensemble as defined above. Unfortunately, one has to choose $c = 1/2$ in order that the off - diagonal contribution agrees with the CUE result, rendering the damping less effective. This will be shown in the sequel.

To compute $\mathcal{N}^{off}(\eta)$, we have to evaluate integrals of the type

$$\int_0^{2\pi}\frac{\mathrm{d}\omega}{2\pi}\left\langle e^{im(\phi_\epsilon(\omega+x)+(\omega+x)N-\Theta)}e^{im'(\phi_\epsilon(\omega+y)+(\omega+y)N-\Theta)}\right\rangle_\hbar . \tag{57}$$

For this purpose we write

$$\begin{aligned}\left\langle e^{im\phi_\epsilon(\omega+x)}e^{im'\phi_\epsilon(\omega+y)}\right\rangle_\hbar &= \left\langle\left(\frac{\exp[-\sum_{l=1}^{\infty}\frac{t_l^*}{l}(\psi^*)^l]}{\exp[-\sum_{l=1}^{\infty}\frac{t_l}{l}(\psi)^l]}\right)^m\left(\frac{\exp[-\sum_{l=1}^{\infty}\frac{t_l^*}{l}(\chi^*)^l]}{\exp[-\sum_{l=1}^{\infty}\frac{t_l}{l}(\chi)^l]}\right)^{m'}\right\rangle_\hbar \\ &= \exp\left[-\sum_{l=1}^{\infty}\frac{<|t_l|^2>_\hbar}{l^2}\left|m\psi^l + m'\chi^l\right|^2\right]\end{aligned} \tag{58}$$

where, $\psi = e^{i(\omega+x)-\epsilon}$ and $\chi = e^{i(\omega+y)-\epsilon}$. The last line in (58) is independent of ω, and therefore, when the ω integration is performed in (57), only terms with $m = -m'$ contribute. Recalling the approximate relation $< |t_l|^2 >_{\hbar} = lgw_l$ and the definition of the Ruelle ζ function,

$$\begin{aligned}\mathcal{N}^{\hbar}(\eta) &= \mathcal{N}^{diag}(\eta) + \mathcal{N}^{off}(\eta) \\ &= \frac{1}{2\pi^2}\Re e \log\left(\zeta_{cl}(e^{-2\epsilon+i\eta})\right)^g \\ &+ \frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{\cos(Nm\eta)}{m^2}\left(\frac{\zeta_{cl}(e^{-2\epsilon+i\eta})\zeta_{cl}(e^{-2\epsilon-i\eta})}{\zeta_{cl}{}^2(e^{-2\epsilon})}\right)^{gm^2} \end{aligned} \tag{59}$$

The two-point correlation function is obtained by taking the second derivative of (59). Expressing it in terms of the "unfolded" phase difference $s = \eta\frac{N}{2\pi}$, and neglecting corrections which are of order $1/N$ we get

$$\begin{aligned}R_2^{(\hbar)}(s) &= R^{diag}(s) + R^{off}(s) \\ &= -\frac{1}{2\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\Re e \log\left(\zeta_{cl}(e^{-2\epsilon+i\frac{2\pi}{N}s})\right)^g\right] \\ &+ 2\left[\sum_{m=1}^{\infty}\cos(2\pi m s)\left(\frac{\zeta_{cl}(e^{-2\epsilon+i\frac{2\pi}{N}s})\zeta_{cl}(e^{-2\epsilon-i\frac{2\pi}{N}s})}{\zeta_{cl}{}^2(e^{-2\epsilon})}\right)^{gm^2}\right]\end{aligned} \tag{60}$$

This is the central result of the present chapter,expressing the two point statistics in terms of the Ruelle ζ function of the classical map. Analogueous expressions were previously derived for the Gaussian ensembles using field theoretical methods [25], [26], and using the semiclassical trace formula [14]. The partitioning of R_2 to its "diagonal" and "off-diagonal" parts conforms with the notations used in [14]. M. Zirnbauer presents a field theoretical derivation for maps in his contribution to this volume. To bring our result into a form which is closer to the expressions derived in previous studies, we note that for $\pi|s| > 1$, the m sum in $R^{off}(s)$ converges very rapidly, so that the $m = 1$ term is sufficient. This is not true in the vicinity of $s = 0$, since there, the entire m sum is necessary to reproduce the $\delta(s)$ singularity.

We shall now test to what extent (60) reproduces the RMT limit when we approach the strong mixing limit, for which $\zeta_{cl}(z) = \zeta_{cl}^{(m)}(z) = (1-z)^{-1}$ and assume that TRS is violated

$(g = 1)$ (see (28)). This should be compared with the CUE result in the semiclassical limit $N \to \infty$. We shall set $\epsilon = \frac{c}{N}$ and the yet unknown constant c will be chosen such that the best agreement with the CUE expression is achieved. For the diagonal term we get

$$R^{diag}(s) \approx \frac{1}{2} \; \frac{c^2 - (\pi s)^2}{(c^2 + (\pi s)^2)^2} \; \xrightarrow[\pi s >> 1]{} \; - \frac{1}{2(\pi s)^2}. \tag{61}$$

For $\pi s >> c$ this coincides with the CUE expression. $R^{diag}(s)$ can also be written as

$$R^{diag}(s) = \frac{1}{N} \sum_{n=1}^{\infty} \frac{n}{N} e^{-2c\frac{n}{N}} \left(e^{i2\pi s \frac{n}{N}} + e^{-i2\pi s \frac{n}{N}} \right). \tag{62}$$

Thus, the Fourier coefficients of $R^{diag}(s)$ almost coincide with the CUE coefficient for $n < \frac{N}{2c}$, and for $n > \frac{N}{2c}$ the Fourier coefficients approach 0 exponentially.

As was indicated above, the calculation of the "off diagonal" term requires different approximations depending on whether $|\pi s|$ is larger or smaller than 1. For $|\pi s| > 1$ one can truncate the series at the $m = 1$ term. The $m = 1$ contribution is

$$R^{off}_{m=1}(s) = 2 \cos 2\pi s \, \frac{c^2}{c^2 + (\pi s)^2} \tag{63}$$

To get agreement with the CUE expression for $\pi s >> 1$, one must choose the regularization constant to be $c = 1/2$. This is disappointing, because it implies that the high n terms are less effectively damped, and this is not consistent with the assumptions which underlie the semiclassical ensemble.

In the domain $\pi s \le 1$ one can approximate $(1 + (\frac{\pi s}{c})^2)^{-1} \approx e^{-(\frac{\pi s}{c})^2}$, taking $c = 1/2$ and using the Poisson summation formula to re-sum the m series we obtain

$$R^{off}(s) = -1 + \frac{1}{\sqrt{4\pi s^2}} \sum_{m=-\infty}^{\infty} e^{-(\frac{s-m}{2s})^2} \tag{64}$$

Since for $t \to 0$

$$\frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} e^{-(\frac{x-m}{2t})^2} \quad \to \quad \delta(x) \tag{65}$$

We can identify $x = s$ and $t = s^2$ and get

$$R^{off}(s) \quad \approx -1 + \delta(s) \quad \text{for} \quad |\pi s| << 1. \tag{66}$$

which shows that the expected $\delta(s)$ singularity is reproduced.

The reconstruction of the main features of the CUE expression manifests a few of the difficulties encountered in this approach. They stem from the regularization procedure, which requires a specific choice of the regularization parameter to get the right answer. The smoothing parameter $\epsilon = \frac{1}{2N}$ which should be used, is not large enough to damp the contribution of the $n > N/2$. This pushes the "semiclassical ensemble" beyond its strict domain of validity. The same difficulty arises also in the work of Keating and Bogomolny [14], where the "cut-off" time must be taken as the Heisenberg time, and not its half.

The purpose of the approach presented above was to circumvent the need to use the semiclassical approximation for the higher traces t_n for $n > N$. It is very successful for the calculation of the autocorrelations of the characteristic polynom. However, it meets with the difficulties mentioned above, when unitarity (pure point spectrum on the unit circle) is to be restored within the semiclassical approximation. In the next chapter we shall present another approach which does not suffer from these problems, but which introduces a new concept which still requires much more study - periodic orbits correlations.

V. PERIODIC ORBITS CORRELATIONS

In the previous chapter we derived the semiclassical theory of spectral fluctuation using semiclassical information on the periodic orbits with period $n \leq N$. In other words, we based the theory on the evolution of the classical system during times which are shorter than the relevant Heisenberg time. However, this by itself is not sufficient to reproduce the most distinctive property of the quantum spectrum - the fact that it consists of N points on the unit circle. This feature was incorporated by generating the "synthetic" spectrum (15) out of the ϵ smoothed spectrum, which served the desired end at the cost of obtaining a theory which depends critically on the smoothing parameter.

In the present chapter we shall take a completely different route. We shall use the semiclassical expression (30) or (33) for $t_n = \mathrm{tr}U^n$ for classically chaotic or integrable systems, respectively. Averaging $|t_n|^2$ over the domain of β values (34), and *retaining the off diagonal terms* we shall be able to express the two point quantum correlations in terms of correlations in the spectrum of periodic orbits of the classical system. In a way, we do not solve the problem which we have set to solve, but defer the problem to another unknown function which is the classical correlation function. However, recent detailed numerical studies actually confirmed the existence of such classical correlations. [15], [16], [17]. Once the classical correlations are studied and theoretically confirmed, the present formalism will provide the desired semiclassical theory of spectral statistics.

Periodic orbit correlations appear also in the discussion of integrable systems (where one should refer to periodic tori). To be able to treat the two types of systems in a uniform way, we shall introduce a short-hand notation and rewrite (30) and (33) as

$$t_n(\beta) \equiv \sum_{l=1}^{N(\beta)} e^{in\theta_l(\beta)} \approx \sum_{p\in\mathcal{P}_n} A_p e^{i\beta\Phi_p} \ . \tag{67}$$

The summation over p goes over the set of unstable n - periodic orbits (primitive and repetitions) for chaotic dynamics, and over the set of n - periodic tori for integrable dynamics. The complex coefficients A_p are given in (30), (33) and their phase is determined by the Maslov index. The dependence of the quantum spectrum on the value of $\hbar = \beta^{-1}$ is explicitly indicated. Also, $N(\beta)$ is the integer part of $\left[\frac{|\mathcal{M}|\beta}{2\pi}\right]$ and $|\mathcal{M}|$ is the classical phase space area.

We define a classical density

$$d_{cl}(x;n) \equiv \sum_{p\in\mathcal{P}_n} A_p \delta(x-\Phi_p) \ , \tag{68}$$

where x has the dimension of action. Recalling the quantum density of eigenphases on the unit circle

$$d_{qm}(\omega;\beta) \equiv \sum_{l=1}^{N} \delta(\omega-\theta_l(\beta)) \tag{69}$$

we can write (67) as

$$\int_0^{2\pi} \mathrm{d}\omega e^{i\omega n} d_{qm}(\omega;\beta) \approx \int_{\Phi_{min}(n)}^{\Phi_{max}(n)} \mathrm{d}x e^{i\beta x} d_{cl}(x;n) \qquad (70)$$

In this way, the trace formulae (30) or (33) are interpreted as a relationship between a strictly quantum density and a strictly classical density. The quantum density involves N points on the unit circle, which have equal (positive) weights. $d_{qm}(\omega;\beta)$ is a function (distribution) in ω and it depends *parametrically* on β. The classical density corresponds to unstable n - periodic orbits (n - periodic tori) whose number increases exponentially (algebraically) with n, and which are weighted by *complex* coefficients which are different from each other in generic systems. $d_{cl}(x;n)$ is a function (distribution) in x and it depends *parametrically* on n. The actions of n-periodic orbits (tori) are bounded from bellow and from above so that $\Phi_{min}(n) \le x \le \Phi_{max}(n)$.

Defining the oscillatory part of the classical density,

$$\tilde{d}_{cl}(x;n) \equiv d_{cl}(x;n) - \frac{1}{\Phi_{max}(n) - \Phi_{min}(n)} \sum_{p\in\mathcal{P}_n} A_p \ , \qquad (71)$$

we can construct the *classical* two point correlation function

$$\rho_{cl}(\xi;n) \equiv \frac{\sin\frac{\xi\pi}{|\mathcal{M}|}}{\frac{\xi\pi}{|\mathcal{M}|}} \int_{\Phi_{min}(n)}^{\Phi_{max}(n)} \mathrm{d}x \ \ \tilde{d}_{cl}(x+\xi/2;n)\tilde{d}_{cl}(x-\xi/2;n) \qquad (72)$$

The factor in front of the integral comes from the β averaging, and it limits the range of action correlations to $|\xi| < |\mathcal{M}|$. As we shall see above, the correlations of interest here occur on much smaller action differences.

In the present context, it is convenient to normalize the *quantum* two point correlation function (9) in a different way

$$\rho_{qm}(\eta;N) \equiv \int_0^{2\pi} \frac{\mathrm{d}\omega}{2\pi} < \tilde{d}_{qm}(\omega+\eta/2;\beta)\tilde{d}_{qm}(\omega-\eta/2;\beta) >_{\hbar} = \left(\frac{N}{2\pi}\right)^2 R_2(\eta). \qquad (73)$$

Comparing (72) and (73) we get

$$\int_0^{2\pi} \mathrm{d}\eta e^{-i\eta n} \rho_{qm}(\eta;N) = \frac{1}{2\pi} \int_{\Phi_{min}(n)}^{\Phi_{max}(n)} \mathrm{d}\xi e^{i(N+1/2)\Delta\xi} \rho_{cl}(\xi;n) \equiv \rho(n;N) \qquad (74)$$

The function $\rho(n;N)$ expresses the duality between the quantum and classical spectra in the clearest way, because it generates the two-point correlations of the two spectra by taking Fourier transforms with respect to either variable:

$$\rho_{qm}(\eta; N) = \frac{1}{2\pi} \sum_n e^{i\eta n} \rho(n; N) \tag{75}$$

and

$$\rho_{cl}(\xi; n) = \frac{2\pi}{|\mathcal{M}|} \sum_N e^{-i\frac{\xi\pi}{|\mathcal{M}|}(2N+1)} \rho(n; N). \tag{76}$$

It is easy to show that

$$\rho(-n; N) = \rho(n; -N) = \rho(-n; -N) = \rho(n; N) \ , \tag{77}$$

which completes the definition of $\rho(n; N)$. $\rho(n; N)$ can be considered as a function of n for a fixed N, and then it stands for the *quantum* spectral form factor for the system which is quantized with a Planck constant $\hbar = \frac{|\mathcal{M}|}{2\pi N}$. If n is kept fixed then $\rho(n; N)$ is the *classical* form factor for the *classical* spectrum of n-periodic orbits (tori).

Since both the classical and the quantum spectral densities consist of isolated δ functions, one can extract their diagonal parts,

$$\rho_{qm}(\eta; N) = \frac{N}{2\pi} \left[\delta(\eta) - p_{qm}(\eta; N)\right] \quad \text{with} \ \int p_{qm}(\eta; N) \mathrm{d}\eta = 1 \ . \tag{78}$$

Similarily,

$$\rho_{cl}(\xi; n) = \left[\sum_{p \in \mathcal{P}_n} A_p^2\right] \left[\delta(\xi) - p_{cl}(\xi; n)\right] \quad \text{with} \ \int p_{cl}(\xi; n) \mathrm{d}\xi = 1 \ . \tag{79}$$

The normalization of the functions p_{qm} and p_{cl} follows from the fact that the correlators ρ_{qm} and ρ_{cl} were constructed from the oscillatory parts of the corresponding densities.

The implementation of these relations to systems which are integrable classically will be discussed first, as a transparent example of the duality idea. The quantum spectrun in this case is known to be Poisson, hence $\rho_{qm}^{Integ}(\eta; N) = \frac{N}{2\pi}\delta(\eta)$, which implies $\rho^{Integ}(n; N) = \frac{N}{2\pi}$. For classically integrable systems, we know that $\sum_{p \in \mathcal{P}_n} A_p^2 = N$ (37). When this is substituted in (79) and in (74), one obtains $\rho^{Integ}(n; N) = \frac{N}{2\pi}$ if the classical correlation $p_{cl}^{Integ}(\xi; n) = 0$. That is, the classical - quantum spectral duality implies that both spectra have to be Poisson! This result can be also substantiated on different grounds. The quantum eigenphases (in the semiclassical approximation) are obtained by quantizing the action variables to integer multiples of $\hbar$, and with $I_j = j\hbar$,

$$\theta_j(\hbar) \approx \frac{1}{\hbar}\left(\Phi(f(I_j)) - I_j f(I_j)\right) \ . \tag{80}$$

$f(I)$ is the angular frequency (22). The reason why this series of phases is Poissonian is because the correlations are lost when the phases are considered mod 2π. This happens in the semiclassical limit, and when e.g., $\Phi(f(I)) \approx I^\mu$, with $\mu \geq 2$. The same argument applies also for the spectrum of actions of n- periodic tori

$$\Phi_m(n) = n\Phi(2\pi\frac{m}{n}) \quad ; \quad 1 \leq m \leq n \ . \tag{81}$$

These actions are related to the eigenphases by a Legendre transformation, and will have a Poisson distribution on the unit circle provided n is large enough.

For system which are chaotic in the classical limit, we can write $gnw_n = \sum_{p\in\mathcal{P}_n} A_p^2$ so that

$$\rho_{cl}(n;N) = gnw_n\left[\delta(\xi) - p_{cl}(\xi;n)\right] . \tag{82}$$

To obtain the *universal* behavior of $p_{cl}(\xi;n)$ we use the strong mixing limit $w_n \approx 1$. We also assume that the quantum two points correlation function converges as $N \to \infty$ to a well defined limit when it is expressed in terms of the unfolded variable $s = \eta\frac{N}{2\pi}$. This is a strong requirement which leads automatically to a scaling form for $\rho(n;N)$

$$\rho_{universal}(n;N) = \frac{N}{2\pi}q(n/N) \tag{83}$$

and $q(x) \to 1$ for $x \gg 1$. With this information and using (76), $p_{cl}(\xi;n)$ can be extracted. Replacing the N summation by an integral,

$$p_{cl}(\xi;n) = \frac{1}{g}\frac{2}{(|\mathcal{M}|/n)}\int_0^\infty \cos\left(2\pi s\frac{\xi}{(|\mathcal{M}|/n)}\right)\left[g - sq(\frac{1}{s})\right] \mathrm{d}s = \frac{1}{g}\frac{1}{\lambda(n)}Q\left(\frac{\xi}{\lambda(n)}\right) \tag{84}$$

This is the main result of the present chapter. It expresses the *classical* correlation function in terms of the function Q which is an appropriate Fourier transform of the quantum function q. The action correlation length $\lambda(n) = \frac{|\mathcal{M}|}{n}$ can be interpreted as the mean phase space area per point on the periodic orbit. It has the dimension of action, and it exceeds by far the mean spacing between actions in the classical spectrum, which are exponentially small in n. This is consistent with the results of numerical studies which show that the action spectrum is Poisson when studied on the scale of a few mean level spacings. For systems which violate time reversal symmetry, $g = 1$ and RMT provides $q(x) = \min(x, 1)$. Then,

$$Q_{CUE}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \tag{85}$$

which coincides with the RMT expression for the quantum two point correlation function. The expression for Q_{COE} was derived in [15].

Numerical tests of the Baker map [15,?] and the deformed cat map [15] confirm the existence of classical correlations in these systems, and reproduce the scaling behavior. However, the numerical correlation functions deviate from the predicted universal functions by up to a factor 2. In [17] the classical quantum duality was investigated for the Sinai billiard in two and three dimensions. For these systems, the classical origin of the action correlations was investigated, and a possible mechanism was identified and was shown to be consistent with the numerical data. This is, however, a mechanism which is particular to these billiards. The origin of the correlations in generic hyperbolic systems remains an enigma.

VI. DISCUSSION AND CONCLUSIONS

In the previous chapters we described a few methods to substantiate the empirical finding that the spectral correlations of unitary matrices which represent quantum evolution of classical maps reflect the underlying classical dynamics. In particular, we have shown under what conditions, the spectral correlation functions approach the predictions of random matrix theory.

It should have been made clear from the outset, that typical evolution operators which pertain to actual physical systems can rarely reproduce the full joint probability distribution of the N eigenphases, as given by RMT. (a non trivial exception for $N = 2$ is discussed in [27]). The reason for this observation is that short time evolution usually does not reflect details on the forces and constraints which characterize the system, and which dominate the long time and non trivial evolution. This feature is mediated in our formalism by the appearance of the Ruelle ζ function in the semiclassical expressions of the two spectral measures which we derived. Only if the actual ζ_{cl} is replaced by the function $\zeta_{cl}^{(m)}$ which reflects the long time - and hence the strong mixing - properties of the system, do we recover the RMT results. In other words, the only limit in which the RMT can be reproduced to any desired accuracy is the semiclassical limit ($N \rightarrow \infty$) where short time effects can be neglected.

Once we agree that the universal behavior can be reproduced only in the semiclassical limit, we can now ask what are the main attributes which characterize the approach to this limit. The short time evolution in bound systems which approach complete mixing limit via phase space diffusion were the first systems which were shown to reveal systematic deviations from the RMT results [28]. The classical evolution operator on the relevant time scale can be approximated by the diffusion evolution operator, which gives, for the 1-d diffusion discussed in [28], $w_n \approx 1/\sqrt{nD}$ for $n < L^2/D$ where D is the diffusion constant and L is the length of the system. Only for $n > L^2/D$ one recovers the strong mixing limit $w_n = 1$, and hence the RMT attributes. In most systems, however, one cannot identify a systematic approach to the strong mixing limit, and the approach to the RMT limit is affected by the idiosyncratic behavior of short periodic orbits. A few examples will clarify this point. Consider the billiard bounce maps. This map has no fixed points in the interior of the phase space. (The phase space boundaries form a continuous set of fixed points which do not appear in the semiclassical treatment). Hence, for any billiard map, $t_1 = 0$, which is at variance with the RMT result $< |t_1|^2 > \approx g$. Billiards with parallel boundary sections support families of bouncing ball orbits. They and their repetitions affect all the

$< |t_n|^2 >$, and should be subtracted away if any degree of agreement with RMT is to be expected.

The appearance of "freak" short periodic orbits affect not only the spectral measures which are related to short time dynamics, (such as e.g., the lowest $< |t_n|^2 >$), but also the other statistical measures. Even if we do not consider their repetitions, they appear in other indirect ways. The quantities which are most affected by the short periodic orbits are the coefficients of the characteristic polynom. This can be easily understood from the Newton identities through which they are constructed (6). t_k appears in the relations which define all a_n with $n \geq k$, and any possible anomaly which is due to a k - periodic orbit will be noticed. This is why the auto-correlation of the characteristic polynom is a statistical measure which is most sensitive to the deviation from universality. Another venue through which short periodic orbits become noticeable, is also connected with the Newton identities, and the self inversive symmetry (4)which is imposed by unitarity. Consider

$$a_k = -\frac{1}{k}\left(t_k + \sum_{l=1}^{k-1} a_l t_{k-l}\right) \tag{86}$$

and

$$a_{N-k} = -\frac{1}{N-k}\left(t_{N-k} + \sum_{l=1}^{N-k-1} a_l t_{N-k-l}\right) \tag{87}$$

Since $a_k = e^{i\Theta} a^*_{N-k}$, we must have

$$\frac{1}{N-k} t_{N-k} = \frac{1}{k} e^{i\Theta} t^*_k \ + \ \cdots \tag{88}$$

Which shows how the inversive symmetry may also affect the 2-point spectral statistics.

The universal classical correlation function is also expected to be modified due to the deviations from the strong mixing limit at short times. This can be clearly seen in derivation of the scaling properties of $\rho_{cl}(\xi; n)$ which could be derived only when the w_n in (82) were set to their strong mixing limit $w_n = 1$.

Finally let me point out a few problems which remain open, and deserve in my opinion further work. The classical correlations which were discussed above are far from being understood. These correlations are at the heart of any theory which attempts to explain a phenomenon which depends crucially on quantum interference. Eckhardt [30] have shown that periodic orbit correlations are necessary to explain the distribution of the dwell times in chaotic scattering. Doron Cohen [29] used the concept of classical correlations to provide a semiclassical explanation of the scaling theory for Anderson localization. However, to achieve this goal, the classical correlation function was not derived from first principles, but was supported by physical arguments and intuition.

Spectral correlations were discussed in this paper from the semiclassical point of view. The results presented in the previous chapter but one are formally similar to the results obtained using the field theoretical approach. The semiclassical theory is based on one important assumption, namely, that repetitions of short periodic orbits can be neglected. Can this assumption be reformulated in a way which will agree with the approximations which underlie the field theoretical approach? The comparison of the two methods calls for further study to elucidate this point.

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