Distribution of Eigenvalues in Non-Hermitian Anderson Models

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(Received 31 July 1997)

We develop a theory which describes the behavior of eigenvalues of a class of one-dimensional random non-Hermitian operators introduced recently by Hatano and Nelson. We prove that the eigenvalues are distributed along a curve in the complex plane. An equation for the curve is derived, and the density of complex eigenvalues is found in terms of spectral characteristics of a "reference" Hermitian disordered system. The generic properties of the eigenvalue distribution are discussed. [S0031-9007(98)05635-X]

PACS numbers: 72.15.Rn, 05.40.+j

Complex eigenvalues of non-Hermitian random Hamiltonians have recently attracted much interest across several areas of physics [1–5]. However, the actual progress in understanding the statistics of such eigenvalues is mostly limited to the so-called "zero-dimensional" case, i.e., to random matrices with no spatial structure. In this case a number of models can be treated analytically and their basic features are relatively well understood, although on different levels of rigor (see, e.g., [4–6], and references therein). In contrast, little is known about spectra of non-Hermitian Hamiltonians in one or more dimensions. One of the challenging problems here involves an unusual localization-delocalization transition predicted by Hatano and Nelson [1].

Motivated by the studies of statistical mechanics of the magnetic flux lines in superconductors with columnar defects, Hatano and Nelson considered a model described by a random Schrödinger operator with a constant imaginary vector potential. Appealing to a qualitative reasoning they argued that already in dimension one some localized states undergo a delocalization transition when the magnitude of the vector potential increases. The eigenvalues corresponding to the localized states are real and those corresponding to the extended states are complex. The results of numerical calculations presented in [1] support these conclusions. They also show a surprising feature of the eigenvalue distribution in the model: the eigenvalues are attracted to a curve in the complex plane.

The aim of our Letter is to explain this feature. Most of our discussion involves the lattice case which is technically simpler. Following [1], we consider a one-dimensional non-Hermitian Anderson model whose eigenvalue equation reads as follows:

$$-e^{\xi_{k-1}}\psi_{k-1} - e^{\eta_k}\psi_{k+1} + q_k\psi_k = z\psi_k, \qquad 1 \le k \le n, \tag{1}$$

$$\psi_0 = \psi_n, \qquad \psi_1 = \psi_{n+1}. \tag{2}$$

Our basic assumptions about the coefficients in Eq. (1) are as follows: $\{(q_k, \xi_k, \eta_k)\}$ is a stationary ergodic sequence of random three-dimensional vectors such that $\langle \ln(1 + |q_0|)\rangle$, $\langle \xi_0\rangle$, $\langle \eta_0\rangle$ are finite. The angle brackets denote averaging over the disorder. The relevance of this non-Hermitian Anderson model to physics of vortex lines in superconductors is explained in [1].

Potential theory approach.—We start our analysis with a standard transformation which is often used in the theory of differential and difference equations. Let us put $\psi_k = w_k\varphi_k$ in Eq. (1) and choose the weight w_k so that to make the resulting equation symmetric. For instance, if we set

$$w_0 = 1, w_k = e^{\frac{1}{2}\sum_{j=0}^{k-1}(\xi_j - \eta_j)}, \quad \text{if } k \ge 1, \tag{3}$$

this transformation reduces the eigenvalue problem (1),(2) to the following one (where $c_k \equiv \exp[(\xi_k + \eta_k)/2]$):

$$-c_{k-1}\varphi_{k-1} - c_k\varphi_{k+1} + q_k\varphi_k = z\varphi_k, \tag{4}$$

$$\varphi_{n+1} = w_{n+1}^{-1}w_1\varphi_1, \qquad \varphi_n = w_n^{-1}\varphi_0. \tag{5}$$

From now on, we will deal with Eqs. (4)–(5). [Obviously, the eigenvalues of (1),(2) and (4),(5) coincide.]

Let us introduce a "reference" symmetric eigenvalue problem which will be used in our analysis and which is specified by Eq. (4) and by the boundary conditions

$$\varphi_{n+1} = 0, \qquad \varphi_0 = 0. \tag{6}$$

One can rewrite the eigenvalue problems (4),(5) and (4),(6) in the matrix form $\mathcal{H}\varphi = z\varphi$ and $H\varphi = z\varphi$, respectively. H is a *symmetric* tridiagonal $n \times n$ matrix with the $\{q_k\}$ on the main diagonal and the $\{-c_k\}$ on the subdiagonals. *Asymmetric* $\mathcal{H}$ is "almost" tridiagonal: the only nonzero elements of the difference $V = \mathcal{H} - H$ are $V_{1n} = -w_1^{-1}w_ne^{\xi_0}$ and $V_{n1} = -w_1w_n^{-1}e^{\eta_n}$.

Our first approach to the eigenvalue problem (4),(5) is based on the calculation of the electrostatic potential (we assign a unit charge to each eigenvalue z_j of $\mathcal{H}$):

$$F_n(z) = \frac{1}{n}\sum_{j=1}^{n}\ln|z - z_j| = \frac{1}{n}\ln|\det(\mathcal{H} - zI)|,$$

where I is the identity matrix. In the language of two-dimensional electrostatics, the charge (eigenvalue) density ρ is determined by the electrostatic potential via

Poisson's equation: $4\pi\rho = -\Delta F$. We will shortly see that $F(z) \equiv \lim_{n\to\infty} F_n(z)$ can be expressed in terms of the electrostatic potential

$$\Phi(z) = \lim_{n\to\infty} \frac{1}{n} \ln|\det(H - z)| = \int_{-\infty}^{+\infty} \ln|\lambda - z|\, dN(\lambda) \tag{7}$$

of the eigenvalues of H (which all are real). Namely,

$$F(z) = \begin{cases} a, & \text{if } \Phi(z) < a\,, \\ \Phi(z), & \text{if } \Phi(z) \geq a, \end{cases} \tag{8}$$

where $a = \max(\langle\xi_0\rangle, \langle\eta_0\rangle)$. $N(\lambda)$ in Eq. (7) is simply the integrated density of states of an infinite disordered system associated with the symmetric reference Eq. (4).

The idea of using potentials to study eigenvalue distributions is not new and goes back to the 1960s at least, to studies of Töplitz matrices [7]. In the context of random matrices this idea has been used since works [8,9].

Obviously, Φ is harmonic in the complex plane $z = x + iy$ off the support Σ of $dN(\lambda)$. In view of the relationship between F and Φ, the complex eigenvalues of $\mathcal{H}$ are distributed (in the limit $n \to \infty$) on the line $\Phi(z) = a$. The density of their distribution $d\nu/ds$ with respect to the arc-length measure, ds, is equal to $(2\pi)^{-1}$ times the jump in the normal derivative of F across the line [10]. Computing the derivative gives

$$\frac{d\nu}{ds} = \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} \frac{dN(\lambda)}{\lambda - z} \right|. \tag{9}$$

The limit distribution of the real eigenvalues of $\mathcal{H}$ is supported exactly by that part of Σ where $\Phi(x + i0) \geq a$ and coincides there with $dN(\lambda)$.

It is very important for our further analysis that $\Phi(z)$ coincides, up to an additive constant, with the Lyapunov exponent $\gamma(z)$ of Eq. (4). Namely, the well known Thouless formula [11,12] states that $\Phi(z) = \gamma(z) + \frac{1}{2}(\langle\xi_0\rangle + \langle\eta_0\rangle)$. Therefore, $\Phi(z) = a$ is equivalent to

$$\gamma(z) = \frac{1}{2} |\langle\xi_0\rangle - \langle\eta_0\rangle|\,. \tag{10}$$

Our main results, Eqs. (8)–(10), can be illustrated with the following example. Let the q_k be Cauchy distributed, i.e., $\text{Prob}\{q_k \in \Delta\} = \pi^{-1} \int_\Delta dq\, b/(q^2 + b^2)$. Also, let $\xi_k \equiv g$ and $\eta_k \equiv -g$, i.e., $c_k \equiv 1$ in Eq. (4). In this case an explicit algebraic expression for $\gamma(z)$ is known (see, e.g., Ref. [11]):

$$4\cosh\gamma(z) = \sqrt{(x + 2)^2 + (b + |y|)^2} + \sqrt{(x - 2)^2 + (b + |y|)^2}\,. \tag{11}$$

Straightforward computations involving Eqs. (10) and (11) show that the limit spectrum of $\mathcal{H}$ in this case has a complex part only if $K \equiv 2\cosh g > K_{\text{cr}} = \sqrt{4 + b^2}$. The complex part consists of the two arcs

$$y(x) = \pm\left[\sqrt{\frac{(K^2 - 4)(K^2 - x^2)}{K^2}} - b\right], \quad -x_b < x < x_b\,,$$

where the end points $\pm x_b$ are determined by the condition $y(x_b) = 0$. The real eigenvalues $\mathcal{H}$ are distributed in the two intervals complementary to $(-x_b, x_b)$ with a density equal to the density of eigenvalues of Eq. (4) in these intervals. In Fig. 1 we compare these analytical results with data obtained from a numerical experiment.

We omit the technical details of our derivation of Eq. (8) and present only the main idea. One can write $\mathcal{H} - zI = (I + VG)(H - zI)$, where $G = (H - zI)^{-1}$ and $V = \mathcal{H} - H$. Therefore

$$F(z) = \Phi(z) + \lim_{n\to\infty} \frac{1}{n} \ln|d_n(z)|\,, \tag{12}$$

where $d_n(z) = \det(I + VG)$. Expanding this determinant yields

$$d_n(z) = [1 + V_{1n}G_{n1}][1 + V_{n1}G_{1n}] - V_{1n}V_{n1}G_{11}G_{nn}\,,$$

where G_{jk} denotes the (j,k) matrix entry of G. Using $G_{1n} = G_{n1} = \prod_{j=1}^{n-1} c_k / \det(H - zI)$ one obtains that

$$|V_{1n}G_{n1}| = \exp\{n[\langle\xi_0\rangle - \Phi(z) + o(1)]\}, \tag{13}$$

$$|V_{n1}G_{1n}| = \exp\{n[\langle\eta_0\rangle - \Phi(z) + o(1)]\}. \tag{14}$$

Applying Eqs. (13),(14) one calculates the second term on the right-hand side in Eq. (12) and obtains Eq. (8). It should be mentioned here that the complete rigorous derivation of Eq. (8) requires additional efforts (see preprint [13] for details).

Transfer matrix approach.—Now we consider another approach to the eigenvalue problem (4),(5). This approach provides more precise information about the finite-n behavior of the eigenvalues of $\mathcal{H}$ and easily extends to the case of differential equations.

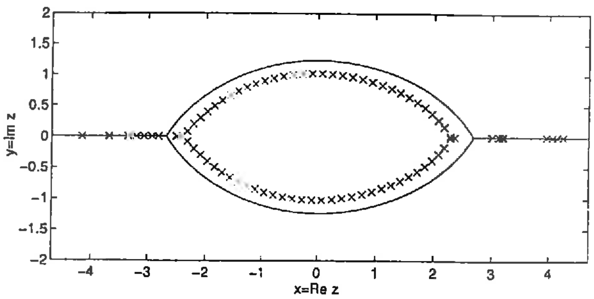

FIG. 1. Spectrum of the non-Hermitian Anderson model with Cauchy distributed q_k. The solid line shows the analytically obtained spectrum of $\mathcal{H}$ for the parameter values $b = 1$ and $K = 3$. This is compared with the numerically obtained eigenvalues of a sample matrix $\mathcal{H}$ of dimension $n = 100$ shown by crosses.

Let us write Eq. (4) in the form $(\varphi_{k+1}, \varphi_k)^T = A_k(\varphi_k, \varphi_{k-1})^T$, where

$$A_k = \frac{1}{c_k}\begin{pmatrix} q_k - z & -c_{k-1} \\ c_k & 0 \end{pmatrix}.$$

Then the solution of Eq. (4) with initial data $(\varphi_1, \varphi_0)^T$ can be written as $(\varphi_{k+1}, \varphi_k)^T = S_k(z)(\varphi_1, \varphi_0)^T$, where $S_k(z) = A_k A_{k-1} \cdots A_1$. Combining this with Eq. (5) one reduces the eigenvalue problem (4),(5) to the following two-dimensional problem:

$$[B_n S_n(z) - w_{n+1}^{-1} I](\varphi_1, \varphi_0)^T = 0, \qquad (15)$$

where $B_n = \mathrm{diag}\{e^{\frac{1}{2}(\eta_0 - \xi_0)}, e^{\frac{1}{2}(\eta_n - \xi_n)}\}$.

It is clear that the analysis of Eq. (15) should rely on the study of the asymptotic behavior of the eigenvalues of the 2×2 matrix $B_n S_n$. In turn, this behavior is controlled by that of the norm of S_n. Put $\omega \equiv \{(q_k, \xi_k, \eta_k)\}$ and define the individual Lyapunov exponent by

$$\gamma(z, \omega) = \lim_{n\to\infty} \frac{1}{n} \ln\|S_n(z)\|. \qquad (16)$$

It is known that for fixed z the limit in Eq. (16) exists for almost all ω and coincides with $\gamma(z) = \langle\gamma(z, \omega)\rangle$.

It is also intuitively clear that the asymptotic behavior of the largest by modulus eigenvalue $\mu_n(z, \omega)$ of $B_n S_n$ [14] should be the same as that of $\|S_n\|$:

$$\lim_{n\to\infty} \frac{1}{n} \ln|\mu_n(z, \omega)| = \gamma(z, \omega). \qquad (17)$$

On the other hand, $n^{-1} \ln w_{n+1} \to (\langle\xi_0\rangle - \langle\eta_0\rangle)/2$. From these facts it is obvious that *all nonreal* solutions of Eq. (15) are asymptotically attracted to the curve (10).

The above argument is heuristic and actually naive. Indeed, the same reasoning would imply that the real eigenvalues of $\mathcal{H}$ have to satisfy, in the limit $n \to \infty$, Eq. (10) which is not true. However, for the complex values of z the above heuristics can be justified. This justification is based on the information about the behavior of $\gamma(z, \omega)$ as a function of z when ω is fixed. This question, in the general context of products of random matrices depending on a parameter, was studied in [15]. It was proved there that with probability 1 the convergence in (16) is uniform in z on compact sets which do not contain any point from Σ. It can be deduced from this fact that $\mu_n(z, \omega)$ exhibits exactly the same behavior: the convergence in (17) is uniform in z in the same domain. And this provides a proof of Eq. (10) by the transfer matrix approach. On the spectrum Σ of H the convergence in Eq. (16) is not uniform [15] and this is why the real eigenvalues of $\mathcal{H}$ do not satisfy Eq. (10).

One can prove a uniform version of the central-limit theorem for $\ln\|S_n(z)\|$ for nonreal z. Using it one can estimate the rate of convergence in Eqs. (15) and (16) and prove that the nonreal eigenvalues of $\mathcal{H}$ for large but finite matrix dimension n belong to a neighborhood of $\mathcal{L}$ of width of the order $1/\sqrt{n}$. This approach also helps to control the behavior of real eigenvalues of $\mathcal{H}$.

From now on we assume that $\{(q_n, \xi_n, \eta_n)\}$ form a sequence of independent identically distributed random vectors. In this case $\overline{\gamma}(x) \equiv \gamma(x + i0)$ is a continuous function of x (see, e.g., Ref. [15]). One can also show that for sufficiently large $|x|$

$$\ln|x| - C_1 \le \overline{\gamma}(x) \le \ln|x| + C_1, \qquad (18)$$

where C_1 depends only on the distribution of (q_0, ξ_0, η_0). Therefore, all solutions of Eq. (10) are confined to a circle of finite radius, i.e., the curve $\mathcal{L}$ which supports the nonreal eigenvalues of $\mathcal{H}$ is bounded. To describe this curve notice that the real part x of every solution of Eq. (10) satisfies the inequality

$$\overline{\gamma}(x) \le \frac{1}{2} |\langle\xi_0\rangle - \langle\eta_0\rangle|. \qquad (19)$$

and, vice versa, for each x satisfying (19) one can find $y(x)$ such that $z = x + iy(x)$ and z^* solve Eq. (10). Because of the continuity of $\overline{\gamma}(x)$ the set of x where (19) holds is a union of disjoint intervals $[a_j, a_j']$ with $a_j < a_j'$. Therefore $\mathcal{L}$ is a union of disconnected contours $\mathcal{L}_j$. Each $\mathcal{L}_j$ consists of two smooth arcs, $y_j(x)$ and $-y_j(x)$, formed by the solutions of Eq. (10) when x is running over $[a_j, a_j']$. It turns out that for every strictly positive ε, the real eigenvalues lie outside the intervals $[a_j + \varepsilon, a_j' - \varepsilon]$ when n is sufficiently large.

We notice that it is easy to construct examples with a prescribed finite number of contours. In general, there is no obvious reason for the number of contours to be finite for an arbitrary distribution of (ξ_0, η_0, q_0). We illustrate our description of $\mathcal{L}$ with Fig. 2.

In general, it is difficult to solve Eq. (10) explicitly. Moreover, we know only one example (mentioned above) when this can be done. However, using Eq. (10) one can describe some generic properties of the eigenvalue distribution of $\mathcal{H}$. Denote $2g \equiv \langle\eta_0\rangle - \langle\xi_0\rangle$. First of all, the limit spectrum of $\mathcal{H}$ is entirely real if $g = 0$. Now consider a special case of the model when $q_k \equiv 0$. In this case

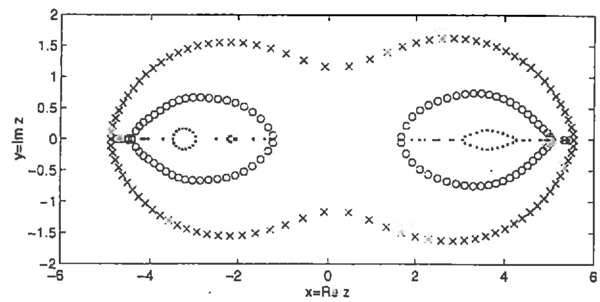

FIG. 2. Numerically obtained spectra of sample matrices $\mathcal{H}$ of dimension $n = 100$. The q_k are drawn from a mixture of the uniform distribution on [3,4] and the normal distribution centered at $x = -3$ with standard deviation 0.01. The off-diagonal elements are constant, $\xi_k \equiv -\eta_k \equiv g$. The parameter g takes three values: $g = 0.75$ (eigenvalues are shown by dots), $g = 1$ (circles), and $g = 1.25$ (crosses).

$\gamma(0) = 0$ (see Ref. [12], p. 347). Hence, if $g \neq 0$ the solutions to the equation $\gamma(z) = |g|$ constitute a nontrivial curve and the limit spectrum of $\mathcal{H}$ has a complex part. This property does not hold if the distribution of q_0 is nondegenerate (i.e., q_0 takes at least two different values with nonzero probabilities). Indeed, in this case $\overline{\gamma}(x) \equiv \gamma(x + i0)$ is strictly positive (see, e.g., Ref. [12]). The continuity of $\overline{\gamma}(x)$ implies that $g_1 \equiv \min_{x\in\Sigma} \overline{\gamma}(x) > 0$. Therefore, $\gamma(x + iy) \geq \overline{\gamma}(x) \geq g_1$ for all $z = x + iy$ and Eq. (10) has no nonreal solutions if the parameter g controlling the degree of non-Hermiticity satisfies $|g| \leq g_1$. If the coefficients in Eq. (4) are bounded, i.e., $c_k^2 + q_k^2 \leq C$ for all k, then Σ is a bounded set and $g_2 \equiv \max_{x\in\Sigma} \overline{\gamma}(x)$ is finite. Therefore, if $|g| \geq g_2$ the inequality (19) is satisfied for every point of Σ. Hence the limit spectrum is purely complex for such parameter values. If either c_0 or q_0 takes arbitrary large values with nonzero probability, then Σ is an unbounded set and, in view of Eq. (18), $g_2 = +\infty$.

In summary, if $|g| \leq g_1$ the limit spectrum of $\mathcal{H}$ is entirely real, if $|g| \geq g_2$ the limit spectrum is entirely complex and if $g_1 < |g| < g_2$ the limit spectrum has real and complex parts. In the latter case the branching points from which the complex branches grow out of the real eigenvalues are determined by $\overline{\gamma}(x) = |g|$.

It should be noticed that the density of the nonreal eigenvalues given by Eq. (9) is analytic everywhere on $\mathcal{L}$ except the (real) end points of the arcs. (If the limit spectrum is entirely complex then this density is analytic everywhere.) The behavior of $d\nu/ds$ near an end point of an arc, a_j say, depends on the regularity properties of $N(\lambda)$ at this point. If the density of states $N'(\lambda)$ of the reference equation is smooth in a neighborhood of $\lambda = a_j$ then $d\nu/ds$ has a finite limit as z approaches a_j along the arc. Also, in this case the tangent to the arc at a_j exists and is not vertical. In other words, if $N'(\lambda)$ is smooth in a neighborhood of a branching point $\lambda = a_j$ the complex branches grow out of this point *linearly*. This may not be the case if $N'(\lambda)$ is not smooth.

Since the original problem (1),(2) is non-Hermitian, its spectrum may depend on the choice of boundary conditions (b.c.). Indeed, all the eigenvalues of Eq. (1) with the Dirichlet b.c., i.e., when $\psi_{n+1} = \psi_0 = 0$, are real. It is remarkable, however, that the boundary conditions of the form $(\psi_{n+1}, \psi_n)^T = B(\psi_1, \psi_0)^T$, where B is a fixed real *nondegenerate* 2×2 matrix, lead to the same Eq. (10) (regardless of the choice of B) in the limit $n \to \infty$. For diagonal B this fact can be readily seen from our derivation of Eq. (8). For nondiagonal B Eq. (10) can be derived using the above mentioned properties of $S_n(z)$.

The derivation of Eq. (8) given above is based only on the existence of $N(\lambda)$ which is ensured by the ergodicity of the coefficients in Eq. (1). Thus one can easily extend our argument to other classes of coefficients. For instance, Eq. (8) holds for periodic q_k, ξ_k, and η_k (in this case the angle brackets denote averaging over the period). The geometry of the limit spectrum of $\mathcal{H}$ in the periodic case follows two patterns. If $\langle\xi\rangle = \langle\eta\rangle$ the limit spectrum is real and coincides with Σ, the spectrum of the symmetric reference equation. If $\langle\xi\rangle \neq \langle\eta\rangle$ the limit spectrum is purely complex and is described by Eq. (10). In either case (real or complex spectrum) the corresponding eigenfunctions are extended.

We conclude the discussion with a remark on the spectrum of the limit operator $\hat{\mathcal{H}}$ defined by the left-hand side of Eq. (1) on $l_2(\mathbf{Z})$. It turns out that for a wide class of distributions of $\{(\xi_k, \eta_k, q_k)\}$ the spectrum of $\hat{\mathcal{H}}$ is a two-dimensional subset of the complex plane and the limit spectrum of $\mathcal{H}$ is embedded into this set. This phenomenon seems to be surprising because $\mathcal{H}\varphi$ converges to $\hat{\mathcal{H}}\varphi$ when $n \to \infty$ for every $\varphi \in l_2(\mathbf{Z})$.

After this work was completed we learned about recent preprints [16] addressing similar problems.

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