Non-local properties of multi-particle density matrices

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Abstract

As far as entanglement is concerned, two density matrices of n particles are completely equivalent if one can be transformed into the other by local unitary transformations. We present two methods of finding whether or not two generic density matrices of arbitrary numbers of spin-1/2 particles are equivalent. Both methods describe density matrices in terms of a finite number of invariant parameters.

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Non-locality is a fundamental characteristic of quantum mechanics. Its importance lies not only in philosophical considerations of the nature of quantum theory, but also in applications where it has emerged recently that non-locality is the key ingredient in quantum computation [1] and communication [2] and plays an important role in cryptography [3,4]. It has become clear that entanglement is a resource which may be manipulated (for example by concentration [5], dilution or purification [6–8]) and transformed from one form into another.

From the point of view of non-locality, two states are completely equivalent if one can be transformed into the other by means of local unitary transformations. How to determine whether or not two states are locally equivalent is still an open question. In this letter we solve this problem for systems containing arbitrary numbers of spin-1/2 particles in generic mixed states.

As discussed in [9], the space of pure states of n spin-1/2 particles is $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$; however not all the 2^n complex parameters have non-local significance: the group of local transformations, $U(2)^n$ acts on the space of states and two states which may be reached from each other by local actions are equivalent as far as their non-local properties are concerned. Each equivalence class of locally equivalent density matrices is an *orbit* of this group. For many purposes, only parameters describing non-local properties are significant; an example is that any good measure of entanglement must be invariant under local transformations [6,10–13], and thus it should be a function of non-local parameters only (here and henceforth we will refer to parameters which are invariant under local transformations as invariants); invariants are also relevant in discussions of Bell inequalities [14,15] and teleportation etc [16]. A key question is to identify the invariants.

In this paper we will focus on density matrices and show that for $n \geq 2$, of the $2^{2n} - 1$ real parameters describing density matrices of n spin-1/2 particles $2^{2n} - 3n - 1$ are invariant under local transformations, $U(2)^n$. This generalises to an arbitrary set of n particles as $\prod_r d_r^2 - \sum_r d_r^2 + n - 1$ where d_r is the dimension of the state space of the rth particle. For n spin-$\frac{1}{2}$ particles we also show how to characterise generic orbits, both by giving an explicit parametrisation of the orbits and by finding a finite set of polynomial invariants which separate the orbits. Thus given two density matrices we can compute explicitly whether they are on the same orbit or not. Other authors have also discussed the use of invariants in discussing entanglement [17,18] and applied invariant theory to quantum codes [19].

In order to calculate the number of functionally independent invariants it will be convenient to find the dimension of the orbit of a generic density matrix under the group of local transformations. The dimension of the orbit is the number of parameters describing the location of a density matrix on the orbit. The total number of parameters ($2^{2n} - 1$ real parameters) describing the space of density matrices minus the number of parameters describing a generic orbit (the dimension of the orbit) gives the number of parameters describing the location of the orbit in the space of orbits, i.e. the number of parameters describing the non-local properties of the density matrices.

To fix notation, it will be convenient to consider the case of a one-particle density matrix first. The space of pure states of a single spin-1/2 particle is $\mathbb{C}^2$ and thus a density matrix is a 2×2 complex matrix which is hermitian, positive and with trace one, and may therefore be described by three real parameters. A particularly convenient representation of such a matrix is

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$$\rho = \frac{1}{2}1_2 + \alpha_i\sigma_i, \tag{1}$$

where α_i, $i = 1, 2, 3$ are real and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

We note that

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}. \tag{3}$$

Under a local transformation by a unitary matrix U, ρ is transformed as

$$\rho \mapsto U\rho U^\dagger. \tag{4}$$

Every unitary matrix may be written as a product of an element of $SU(2)$ and a phase transformation, represented by a unitary matrix $e^{i\phi}1_2$. This latter element clearly leaves any density matrix invariant under the transformation (4) so that when considering the action (4) we may restrict attention to elements of $SU(2)$. In order to find the number of invariants it will be more convenient to find the dimension of a generic orbit under the action of $SU(2)$. To do so one may work infinitesimally. Thus, associated to the action of the Lie algebra of the group of local transformations acting on the space of density matrices there is a vector field: if we take an element T of a basis for the Lie algebra, the action of the group element $k = \exp i\epsilon T \in K$ on an element ρ induces an action on functions from ρ to $\mathbb{C}$; and the vector field, X_T, associated to the Lie algebra element T is found by differentiating:

$$X_T f(v) \stackrel{\text{def}}{=} \frac{\partial}{\partial\epsilon} f(e^{i\epsilon T}\rho)|_{\epsilon=0} = \frac{\partial}{\partial\epsilon} f(\rho + \delta\rho)|_{\epsilon=0}. \tag{5}$$

The linear span of vector fields at the point ρ associated with the whole Lie algebra forms the tangent space to the orbit at the point ρ and so the number of linearly independent vector fields at this point gives the dimension of the orbit.

A general element of the Lie algebra in the spin-1/2 representation is given by

$$T = \eta_i\sigma_i \tag{6}$$

and its action on the density matrix is to give an infinitesimal transformation

$$\delta\rho = i[T, \rho] \tag{7}$$

where $[,]$ is the matrix commutator.

We may therefore calculate the three vector fields X_1, X_2 and X_3 associated to the Lie algebra elements σ_1, σ_2 and σ_3 as

$$X_1 = \alpha_2\frac{\partial}{\partial\alpha_3} - \alpha_3\frac{\partial}{\partial\alpha_2}, \quad X_2 = \alpha_3\frac{\partial}{\partial\alpha_1} - \alpha_1\frac{\partial}{\partial\alpha_3}, \quad X_3 = \alpha_1\frac{\partial}{\partial\alpha_2} - \alpha_2\frac{\partial}{\partial\alpha_1}. \tag{8}$$

We note that at generic values of $\alpha_1, \alpha_2, \alpha_3$ only two of these vector fields are linearly independent since

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0. \tag{9}$$

Thus the dimension of the generic orbit is two and therefore of the three parameters describing a generic density matrix, two are non-invariant leaving only one invariant parameter, as one expects since only the single independent eigenvalue of ρ is invariant under local transformations.

We note that the effect of the transformations (4) is to act on the vector α by rotation by an orthogonal matrix, i.e. an element of $SO(3)$ - this follows from the fact that $\alpha_i \sigma_i$ is the representative of a Lie algebra element and the conjugation action (4) is the adjoint action of the group on its Lie algebra. We may thus find a way of exhibiting the invariant under local transformations:

$$I = \alpha_i \alpha_j \delta_{ij} = |\alpha|^2 \tag{10}$$

where we have used the fact that $SO(3)$ has an invariant tensor δ^{ij}. We note that this invariant may also be expressed as

$$I = \text{Tr}(\rho^2) - \frac{1}{2}. \tag{11}$$

We now turn to the case of two-particle density matrices. Such a density matrix has 15 real parameters, and the maximum dimension that a generic orbit could have is 6 (corresponding to two copies of $SU(2)$) if all the vector fields corresponding to a basis of the Lie algebra were independent. We will show that the vector fields do indeed span 6 dimensions, and thus that there are 9 non-local parameters.

We may write a density matrix as

$$\rho = \frac{1}{4} 1_2 \otimes 1_2 + \alpha_i \sigma_i \otimes 1_2 + \beta_i 1_2 \otimes \sigma_i + R_{ij} \sigma_i \otimes \sigma_j. \tag{12}$$

The action of a Lie algebra element of the subgroup $SU(2)$ acting on the first component of the tensor product is

$$\delta^{(1)} \rho = i[\eta_k \sigma_k \otimes 1_2, \rho]$$
$$= \alpha_k \eta_m \epsilon_{mki} \sigma_i \otimes 1_2 + R_{kj} \eta_m \epsilon_{mki} \sigma_i \otimes \sigma_j, \tag{13}$$

and that corresponding to a Lie algebra element of the subgroup $SU(2)$ acting on the second component of the tensor product,

$$\delta^{(2)} \rho = i[\eta_k 1_2 \otimes \sigma_k, \rho]$$
$$= \beta_k \eta_m \epsilon_{mki} 1_2 \otimes \sigma_i + R_{ik} \eta_m \epsilon_{mkj} \sigma_i \otimes \sigma_j. \tag{14}$$

The vector fields corresponding to the six basis elements $\sigma_i \otimes 1_2, 1_2 \otimes \sigma_i$ are

$$X_k = \epsilon_{kim} \left(\alpha_i \frac{\partial}{\partial \alpha_m} + R_{ij} \frac{\partial}{\partial R_{mj}} \right),$$
$$Y_k = \epsilon_{kim} \left(\beta_i \frac{\partial}{\partial \beta_m} + R_{ji} \frac{\partial}{\partial R_{jm}} \right) \tag{15}$$

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Consider the set X_i first: one can see that these three are linearly independent at generic points by considering the coefficients of $\partial/\partial\alpha_i$, since a linear relation would have to be of the form $\alpha_k X_k = 0$, but one can see that this relation will not hold for non-zero α's by looking at the coefficients of the partial derivatives with respect to R_{ij}. Similarly by considering the coefficients of the partial derivatives with respect to $\beta_1, \beta_2, \beta_3$, one sees that Y_1, Y_2, Y_3 are linearly independent. Finally, we note that the coefficients of the partial derivatives with respect to $\beta_1, \beta_2, \beta_3$ are zero for X_1, X_2, X_3 and the coefficients of the partial derivatives with respect to $\alpha_1, \alpha_2, \alpha_3$ are zero for Y_1, Y_2, Y_3 so that there can be no linear relation at all between the six vector fields $X_1, X_2, X_3, Y_1, Y_2, Y_3$. Thus the dimension of the orbit of a generic density matrix is 6 and thus the number of non-local parameters, $15 - 6 = 9$.

In general, we can consider a system of n particles with individual state spaces of dimensions $d_1, \ldots, d_n$. The density matrix is a hermitian $D \times D$ matrix with trace 1, where $D = d_1 d_2 \ldots d_n$, and therefore requires $D^2 - 1$ real parameters which can be taken to be the coefficients $\alpha^{(1)}, \ldots, \alpha^{(n)}, \ldots, R$ in an expansion

$$\rho = \frac{1}{D} 1_{d_1} \otimes \cdots 1_{d_n} + \sum_{r=1}^{n} \alpha_{i_r}^{(r)} 1 \otimes \cdots \otimes T_{i_r}^{(r)} \otimes \cdots \otimes 1 + \cdots + R_{i_1 \ldots i_n} T_{i_1}^{(1)} \otimes \cdots \otimes T_{i_n}^{(n)} \quad (16)$$

where $T_{i_r}^{(r)}$ $(i_r = 1, \ldots, d_r^2 - 1)$ are a basis set of traceless hermitian $d_r \times d_r$ matrices (generators of $SU(d_r)$). The action of an infinitesimal generator of $SU(d_r)$ acting on the rth factor of the tensor product is

$$\delta^{(r)} \rho = c_{ijk}^{(r)} \eta_i (\alpha_j^{(r)} 1 \otimes \cdots \otimes T_k^{(r)} \otimes \cdots \otimes 1 + \cdots) \qquad (i, j, k = 1, \ldots, d_r^2 - 1) \qquad (17)$$

where $c_{ijk}^{(r)}$ are the structure constants of $SU(d_r)$. Thus the infinitesimal action of local transformations is given by a set of vector fields

$$X_i^{(r)} = c_{ijk}^{(r)} \left(\alpha_j^{(r)} \frac{\partial}{\partial \alpha_k^{(r)}} + \cdots + R_{i_1 \ldots j \ldots i_n} \frac{\partial}{\partial R_{i_1 \ldots k \ldots i_n}} \right). \qquad (18)$$

Similar considerations to those used above for the case of two spin-$\frac{1}{2}$ particles show that these vector fields are generically all independent. Thus the generic orbit has dimension $d_1^2 + \cdots d_n^2 - n$. Since the space of density matrices has dimension $d_1^2 \ldots d_n^2$, there are a total of

$$\prod_r d_r^2 - \sum_r d_r^2 + n - 1 \qquad (19)$$

non-local invariants.

Let us now return to the case of $n \geq 2$ spin-1/2 particles and explicitly identify a set of invariant parameters which characterise generic orbits. To be explicit, consider the case of three spin-1/2 particles with density matrix which may be written as

$$\begin{aligned}
\rho = \frac{1}{8} 1_2 \otimes 1_2 \otimes 1_2 &+ \alpha_i \sigma_i \otimes 1_2 \otimes 1_2 + \beta_i 1_2 \otimes \sigma_i \otimes 1_2 + \gamma_i 1_2 \otimes 1_2 \otimes \sigma_i \\
&+ R_{ij} \sigma_i \otimes \sigma_j \otimes 1_2 + S_{ij} \sigma_i \otimes 1_2 \otimes \sigma_j + T_{ij} 1_2 \otimes \sigma_i \otimes \sigma_j \\
&+ Q_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k.
\end{aligned} \qquad (20)$$

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The action by a local unitary transformation on the first component in the tensor product induces the following transformations on the components of ρ

$$\alpha_i \mapsto L_{ij}\alpha_j; \quad R_{ij} \mapsto L_{ik}R_{kj}; \quad S_{ij} \mapsto L_{ik}S_{kj}; \quad Q_{ijk} \mapsto L_{im}Q_{mjk} \tag{21}$$

where L_{ij} is an orthogonal matrix, and the other components of ρ do not change. Similarly actions by a local transformations on the second and third components of the tensor product induce

$$\beta_i \mapsto M_{ij}\beta_j; \quad R_{ij} \mapsto M_{jk}R_{ik}; \quad T_{ij} \mapsto M_{ik}T_{kj}; \quad Q_{ijk} \mapsto M_{jm}Q_{imk} \tag{22}$$

and

$$\gamma_i \mapsto N_{ij}\gamma_j; \quad S_{ij} \mapsto N_{jk}S_{ik}; \quad T_{ij} \mapsto N_{jk}T_{ik}; \quad Q_{ijk} \mapsto N_{km}Q_{ijm} \tag{23}$$

respectively, where M_{ij} and N_{ij} are orthogonal matrices independent of L.

We max fix a canonical point on a generic orbit as follows: firstly let us define

$$X_{ii'} = Q_{ijk}Q_{i'jk}; \quad Y_{jj'} = Q_{ijk}Q_{ij'k}; \quad Z_{kk'} = Q_{ijk}Q_{ijk'}, \tag{24}$$

and perform unitary transformations on particles 1, 2 and 3 so as to move to a point on the orbit in which X, Y and Z are diagonal; generically the diagonal entries are distinct and we can arrange them in decreasing order (X, Y and Z are hermitian, positive matrices). The only remaining transformations which leave X, Y and Z in these forms are local unitary transformations which induce orthogonal transformations in which L_{ij}, M_{ij} and N_{ij} are one of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

We may specify a canonical point on the generic orbit uniquely by specifying that all the components of α have the same sign, and similarly for β and γ. This method works as long as X, Y and Z have distinct eigenvalues and the components of α, β and γ are not zero at the canonical point on the orbit. The parameters which describe the generic orbits are the components of $\alpha, \beta, \gamma, R, S, T$ and Q at the canonical point on the orbit. We note that the number of parameters describing the canonical point are the $2^6 - 1 = 63$ components of $\alpha, \beta, \gamma, R, S, T$ and Q minus the $3 \times 3 = 9$ constraints that the non-diagonal elements of X, Y and Z are zero; thus the number of non-local parameters is 54 as given by the general formula.

We note that the fact that the canonical point, as constructed, is unique means that all points on the same orbit will have the same canonical representative: conversely, if two density matrices ρ_1 and ρ_2 have the same canonical form, then

$$U_1\rho_1 U_1^\dagger = \rho_{\text{canonical}} = U_2\rho_2 U_2^\dagger \tag{26}$$

for some U_1 and U_2, so that

$$\rho_2 = (U_2^\dagger U_1)\rho_1(U_2^\dagger U_1)^\dagger \tag{27}$$

and thus ρ_1 and ρ_2 are on the same orbit.

We now describe a *finite* set of polynomial invariants which separate generic orbits by finding a set which allows one to calculate the components of $\alpha, \beta, \gamma, R, S, T$ and Q at this canonical point. The complete infinite set of polynomial invariants is found by contracting the indices of $\alpha, \beta, \gamma, R, S, T$ and Q with the invariant tensors δ_{ij} and ϵ_{ijk}. However we may find a finite set of invariants which separates generic orbits. Firstly we note that $\mathrm{tr}(X)$, $\mathrm{tr}(X^2)$ and $\mathrm{tr}(X^3)$ determine the diagonal elements λ_1^2, λ_2^2 and λ_3^2 of X, and similarly for Y and Z. Now consider the three invariants $A_{2n} = \alpha^T X^{n-1}\alpha$, $n = 1, 2, 3$. We may write these three invariants in the following way:

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix} \begin{pmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{pmatrix} = \begin{pmatrix} A_2 \\ A_4 \\ A_6 \end{pmatrix}, \tag{28}$$

where a_1, a_2 and a_3 are the components of α at the canonical point on the orbit. The Vandermonde matrix

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix} \tag{29}$$

has determinant $(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)$, and we may solve for a_1^2, a_2^2 and a_3^2 as long as $\det \Lambda$ is non-zero. Also if the invariant

$$A_9 = \epsilon_{ijk}\alpha_i (X\alpha)_j (X^2\alpha)_k = \alpha.(X\alpha) \wedge (X^2\alpha) = a_1 a_2 a_3 \det \Lambda \tag{30}$$

is non-zero, then we may determine the sign of the components of α; recall that, by definition, all the components of α have the same sign at the canonical point. The analogous expressions B_9, C_9 determine the values of β and γ at the canonical point. The values of the components of R at the canonical point may be calculated from the following nine invariants:

$$I_{r,s} = (X^{r-1}\alpha)_i (Y^{s-1}\beta)_j R_{ij}. \tag{31}$$

These nine equations may be put together into a matrix form

$$I = ((\Lambda F) \otimes (MG)) R, \tag{32}$$

where I and R are column vectors with nine components and the matrices Λ, F, M and G are

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 \end{pmatrix}; \quad F = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}; \quad M = \begin{pmatrix} 1 & 1 & 1 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 \\ \mu_1^4 & \mu_2^4 & \mu_3^4 \end{pmatrix}; \quad G = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix},$$

$$\tag{33}$$

where μ_1^2, μ_2^2 and μ_3^2 are the diagonal elements of Y. We note that $\det(\Lambda F) = A_9$ and $\det(MG) = B_9$, so since we are assuming that these are non-zero we may invert the matrix equation to find the components R_{ij}. The components of S and T may be found in a similar way. Finally we may use the 27 invariants

$$I_{r,s,t} = (X^{r-1}\alpha)_i(Y^{s-1}\beta)_j(Z^{t-1}\gamma)_k Q_{ijk}. \tag{34}$$

to find the components of Q at the canonical point on the orbit in terms of the $I_{r,s,t}$ (there will, of course, be some relations between these components due the constraints that X, Y and Z are diagonal).

Thus, by showing that the following set of polynomial invariants is sufficient to calculate the components of a generic density matrix at the canonical point we have demonstrated that they characterise generic orbits:

$$\text{tr}X^r, \quad \text{tr}Y^r, \quad \text{tr}Z^r$$
$$\alpha^{\mathrm{T}}X^{r-1}\alpha, \quad \beta^{\mathrm{T}}Y^{r-1}\beta, \quad \gamma^{\mathrm{T}}Z^{r-1}\gamma;$$
$$\alpha.(X\alpha) \wedge (X^2\alpha), \quad \beta.(Y\beta) \wedge (Y^2\beta), \quad \gamma.(Z\gamma) \wedge (Z^2\gamma)$$
$$(X^{r-1}\alpha)_i(Y^{s-1}\beta)_j R_{ij}, \quad (Y^{r-1}\beta)_i(Z^{s-1}\gamma)_j T_{ij}, \quad (X^{s-1}\alpha)_i(Z^{r-1}\gamma)_j S_{ij};$$
$$(X^{r-1}\alpha)_i(Y^{s-1}\beta)_j(Z^{t-1}\gamma)_k Q_{ijk}; \tag{35}$$

the indices r, s, t range over the values $1, 2, 3$.

If two density matrices have different values of any of these invariants they are not on the same orbit; if they have same value of all of these invariants, and if A_9, B_9 and C_9 are non-zero, then the density matrices are locally equivalent.

We note that the number of independent components of a generic density matrix at the canonical point is equal to the number of functionally independent parameters calculated at the beginning of this letter. However, the number of polynomial invariants needed to characterise the generic orbit is greater than this; this is related to the fact that the ring of invariants is non-polynomial, i.e. that the geometry of the space of orbits is non-trivial.

The procedure given above can be used for all $n \geq 2$: use the tensors of highest rank and rank one in the expression for ρ to fix a canonical point on the orbit; the polynomials which separate the generic orbits are the analogues of those used in the case $n = 3$.

In the case of $n = 2$ this method can be used but there is some redundancy in the description we have given: the matrices $X_{ii'} = R_{ij}R_{i'j}$ and $Y_{jj'} = R_{ij}R_{ij'}$ (using the notation of (12)) have the same eigenvalues and the matrix R_{ij} is diagonal at the canonical point. In this case there are nine functionally independent invariants which specify the squares of the non-zero components of α, β and R at the canonical point on a generic orbit: $\text{tr}X^n$, $\alpha^T X^{m-1}\alpha$ and $\beta^T Y^{p-1}\beta$, where n, m, p take the values $1, 2, 3$. Additional invariants are needed to specify the signs of the non-zero components. The five invariants $\alpha.(X\alpha) \wedge (X^2\alpha)$, $\beta.(Y\beta) \wedge (Y^2\beta)$ and $\alpha X^{r-1}R\beta$, $r = 1, 2, 3$, are sufficient to determine these signs for generic orbits and hence separate these orbits. In fact, using slightly different arguments, one can show that, in this case, one can reduce the number of polynomial invariants to ten, namely $\text{tr}X$, $\text{tr}X^2$, $\det R$, $\alpha^T X^{r-1}\alpha$, $\alpha^T X^{r-1}R\beta$, $r = 1, 2, 3$ and A_9, which are subject to a single relation expressing A_9^2 as a function of the other invariants.

The general idea of investigating canonical points on orbits in the way we have described is also appropriate for higher spins, but the situation is somewhat more complicated. Consider the example of two particles of spin one in which case the unitary group under which ρ transforms is $SU(3)$. ρ may be written as

$$\rho = \frac{1}{64}1_8 \otimes 1_8 + \alpha_i T_i \otimes 1_8 + \beta_i 1_8 \otimes T_i + R_{ij}T_i \otimes T_j \tag{36}$$

where T_i, $i = 1...8$ are representatives of a basis for the Lie algebra of $SU(3)$ in the adjoint representation and 1_8 is the 8×8 identity matrix. However, the adjoint representation of $SU(3)$ is equivalent not to $SO(8)$ but to an eight dimensional subgroup of it; this means that we cannot transform ρ so that RR^T and $R^T R$ are diagonal so the canonical form is rather more complicated than in the case of spin-1/2 particles.

In summary we have shown how to calculate the number of functionally independent parameters needed to determine whether or not two density matrices are locally equivalent. We have also shown how to characterise the generic classes of locally equivalent density matrices of n spin-1/2 particles by two methods: (a) by finding an explicit set of non-polynomial invariants (the components of the density matrices at the canonical points on the orbits) and (b) by finding an explicit finite set of polynomial invariants. These methods work for generic density matrices; in a future publication we intend to give a systematic method for characterising classes of locally equivalent non-generic density matrices. In particular this will give a basis for the ring of invariants. We note that the canonical point on certain types of non-generic orbit has non-trivial stability group; this is a signature that density matrices on this orbit have special types of entanglement [9].

Acknowledgments

REFERENCES

[1] R. Jozsa, in *Geometric Issues in the Foundations of Science*, eds. S. Huggett, L. Mason, K. P. Tod, S. T. Tsou and N. M. J. Woodhouse (Oxford University Press, 1997).

[2] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett **70**, 1895 (1993).

[3] A. Ekert, Phys. Rev. Lett **67**, 661 (1991).

[4] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett **77**, 2818 (1996).

[5] C.H. Bennett, H.J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. **A 70**, 2046 (1996).

[6] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin and W.K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[7] N. Gisin, Phys. Lett. **A 210**, 151 (1996).

[8] M. Horodecki, P. Horodecki and R. Horodecki quant-ph/9801069.

[9] N. Linden and S. Popescu, quant-ph/9711017 to be published in Forschritte der Physik.

[10] A. Shimony, in *The Dilemma of Einstein, Podolsy and Rosen 60 Years Later* (Annals of the Israel Physical Society, **12**), A. Mann and M. Revzen, eds, Adam Hilger, to appear.

[11] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[12] S. Popescu and D. Rohrlich, Phys. Rev. **A 56**, 3219 (1997).

[13] M. Horodecki and R. Horodecki, quant-ph/9705003.

[14] M. Horodecki, P. Horodecki and R. Horodecki, Physics Letters **A 200** (1995), 340.

[15] P. Horodecki and R. Horodecki, Physics Letters **A 210** (1996), 227.

[16] M. Horodecki, P. Horodecki and R. Horodecki, Physics Letters **A 222** (1996), 21.

[17] J. Schlienz and G. Mahler, Physics Letters **A 224** (1996) 39.

[18] M. Grassl, M. Rötteler and T. Beth, quant-ph/9712040.

[19] E. Rains, quant-ph/9704042

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Orientifold limit of F-theory vacua
hep-th/9702165; Phys.Rev. D55 (1997) 7345-7349

NI97014-RAG   **A Kondratiev and VI Trofimov**
Vertex stabilizers of finite symmetric graphs and a strong version of the Sims conjecture

NI97015-NQF   **JP Gauntlett, GW Gibbons, G Papadopoulos et al**
Hyper-Kähler manifolds and multiply intersecting branes
hep-th/9702202; Nucl.Phys. B500 (1997) 133-162

NI97016-NQF   **O Bärwald, RW Gebert and H Nicolai**
On the imaginary simple roots of the Borcherds algebra $\mathfrak{g}_{II_{9,1}}$
hep-th/9705144; Nuclear Physics B510 [PM] (1998) 721-738

NI97017-NQF   **A Sen and S Sethi**
The mirror transform of type I vacua in six dimensions
hep-th/9703157; Nucl.Phys. B499 (1997) 45-54

NI97018-NQF   **I Halperin and A Zhitnitsky**
Polarized intrinsic charm as a possible solution to the proton spin problem
hep-ph/9706251

NI97019-NQF   **N Dorey, VV Khoze and MP Mattis**
Instantons, three-dimensional gauge theory and the Atiyah-Hitchin manifold
hep-th/9703228; Phys.Lett. B408 (1997) 213-221

NI97020-NQF   **N Dorey, VV Khoze, MP Mattis et al**
Multi-instantons, three-dimensional guage theory, and the Gauss-Bonnet-Chern theorem
hep-th/9704197; Nucl.Phys. B502 (1997) 94-106

NI97021-NQF   **JM Figueroa-O'Farrill, C Köhl and B Spence**
Supersymmmetry and the cohomology of (hyper) Kähler manifolds
hep-th/9705161; Nucl.Phys. B503 (1997) 614-626

NI97022-NQF   **JP Gauntlett**
Duality and supersymmetric monopoles
hep-th/9705025; Nucl.Phys.Proc.Suppl. 61A (1998) 137-148

NI97023-NQF   **JP Gauntlett**
Intersecting branes
hep-th/9705011

NI97024-RAG   **J Rickard**
Triangulated categories in the modular representation theory of finite groups

NI97025-NQF   **P van Baal**
Intermediate volumes and the role of instantons

NI97026-NQF   **CJ Houghton, NS Manton and PM Sutcliffe**
Rational maps, monopoles and skyrmions
hep-th/9705151; Nuclear Physics B510 [PM] (1998) 507-537

NI97027-NQF   **PS Howe, E Sezgin and PC West**
Aspects of superembeddings
hep-th/9705903

NI97028-NQF   **CM Hull**
Gravitational duality, branes and charges
hep-th/9705162; Nucl.Phys. B509 (1998) 216-251

NI97029-NQF   **M Abou Zeid and CM Hull**
Instrinsic geometry of D-branes
hep-th/9704021; Phys.Lett. B404 (1997) 264-270

NI97030-NQF   **CP Bachas, MR Douglas and MB Green**
Anomalous creation of branes
hep-th/9705074

NI97031-RAG   **M Broué, G Malle and J Michel**
Complex reflection groups, braid groups, Hecke algebras

NI97032-NQF   **D Zwanziger**
Renormalization in the Coulomb gauge and order parameter for confinement in QCD

NI97033-NQF   **E Shuryak and A Zhitnisky**
The gluon/charm content of the $\eta\prime$ meson and instantons
hep-ph/9706316; Phys.Rev. D57 (1998) 2001-2004

NI97035-RAG   **K Magaard and G Malle**
Irreducibility of alternating and symmetric squares

NI97036-STA   **N Linden and S Popescu**
On multi-particle entanglement

NI97037-NNM   **M Studeny and RR Bouckaert**
On chain graph models for description of conditional independence structures

NI97038-RAG   **M Geck and G Malle**
On special pieces in the unipotent variety

NI97039-NNM   **SP Luttrell**
A unified theory of density models and auto-encoders
DERA report DERA/CIS/CIS5/651/FUN/STIT/5-4 31 October 1997

NI97040-NNM   **CKI Williams and D Barber**
Bayesian Classification with Gaussian Processes

NI97041-NNM   **TS Richardson**
Chain graphs and symmetric associations

NI97042-NNM   **A Roverato and J Whittaker**
An importance sampler for graphical Gaussian model inference

NI97043-DQC   **MR Haggerty, JB Delos, N Spellmeyer et al**
Extracting classical trajectories from atomic spectra

NI97044-DQC   **S Zelditch**
Large level spacings for quantum maps in genus zero

NI97045-DQC   **U Smilansky**
Semiclassical quantization of maps and spectral correlations

NI97046-DQC   **IY Goldscheid and BA Khoruzhenko**
Distribution of Eigenvalues in non-Hermitian Anderson models
Phys. Rev. Lett. 80 (1998) No.13, 2897-2900

NI97047-DQC   **G Casati, G Maspero and DL Shepelyansky**
Quantum fractal Eigenstates

NI98001-STA   **N Linden and S Popescu**
Non-local properties of multi-particle density matrices