Arithmetic of diagonal quartic surfaces, II

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1. *Introduction.* In this paper we shall be concerned with the solubility or insolubility in $\mathbf{Q}$ of equations of the form

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0 \qquad (1)$$

where the a_i are nonzero elements of $\mathbf{Z}$ and $a_0a_1a_2a_3$ is a square. When we refer to a point of (1), we shall normally assume that the X_i are integers with highest common factor 1. We shall always assume that (1) is everywhere locally soluble, since otherwise insolubility is trivial; as well as $2, \infty$ and the primes which divide $a_0a_1a_2a_3$ it is only necessary to check local solubility at 5. To avoid certain special cases, we shall from time to time also assume that none of the $-a_ia_j$ is a square. Though our main results do not follow from those of [3], this paper can be regarded as an illustration in down-to-earth terms of the general theory expounded there.

The reason why we confine ourselves to the special case when $a_0a_1a_2a_3$ is a square is because it is more tractable than the general one, for the following reason. Let $V \subset \mathbf{P}^3$ be the surface defined by (1) and let W be the quadric

$$a_0Y_0^2 + a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 = 0. \qquad (2)$$

Setting $Y_i = X_i^2$ for each i gives a map $V \to W$; so W is everywhere locally soluble since V is, and hence W is soluble in $\mathbf{Q}$ by Hasse's theorem. If $a_0a_1a_2a_3$ is a square, this implies that each of the two families of lines on W is parametrised over $\mathbf{Q}$ by $\mathbf{P}^1$. The pull-back of a general line of either family to V is a curve of genus 1 defined over $\mathbf{Q}(\mathbf{P}^1)$; and it turns out that these curves are 2-coverings of elliptic curves all of whose 2-division points are also defined over $\mathbf{Q}(\mathbf{P}^1)$. As in [3], our methods really address the question which fibres contain rational points. We cannot simply apply the main theorem of [3], because one of its hypotheses is not satisfied, but we can apply the same ideas; the result of doing so is Theorem 2 in §3.

The conclusion of Theorem 2 is conditional on two major unproved conjectures: the finiteness of the Tate-Shafarevich group of an elliptic curve, and Schinzel's Hypothesis. The former is used to prove Lemma 4. The way in which we shall use the latter is given in Lemma 1 below; for a discussion of Schinzel's Hypothesis and the (straightforward) proof of the Lemma, see [6], §2.

To state Lemma 1, we need some definitions. Let $\mathcal{S}$ be a finite set of places of $\mathbf{Q}$ which includes ∞, and let $\mathcal{Y}$ be the set of ordered pairs of coprime non-zero integers $\eta \times \zeta$. For each v in $\mathcal{S}$ let U_v be any open subset of $\mathbf{Q}_v^* \times \mathbf{Q}_v^*$ in the usual topology such that U_v contains $c^2 y \times c^2 z$ for all c in $\mathbf{Q}_v^*$ whenever it contains $y \times z$. We shall define the $\mathcal{S}$-topology on $\mathcal{Y}$ to be the topology for which the $\prod U_v$ form a base of open sets. The reason for choosing this somewhat exotic topology is that it makes the functions L introduced in §3 continuous, whereas they would not be continuous functions of η/ζ in the standard topology. If $\mathcal{S}_1 \supset \mathcal{S}$ then the $\mathcal{S}$-topology on $\mathcal{Y}$ is coarser than the $\mathcal{S}_1$-topology. It should be emphasized that although the curves (7) with which we work depend only on y/z, the continuity statements which follow (17) would not be true if we used a topology which only depends on η/ζ.

Lemma 1 *Let $Q_1(y,z), \ldots, Q_n(y,z)$ be homogeneous irreducible non-constant polynomials in $\mathbf{Z}[y,z]$ and let $\mathcal{S}_0$ be a finite set of primes containing ∞ and all finite $p \leq \sum \deg Q_\nu$. Assume Schinzel's Hypothesis. Then given any non-empty set U in $\mathcal{Y}$, open in the topology induced by $\mathcal{S}_0$, we can find $\eta \times \zeta$ in U and primes p_ν outside $\mathcal{S}_0$ such that each $Q_\nu(\eta,\zeta)$ is a uniformizing parameter at p_ν and a unit at all other primes outside $\mathcal{S}_0$.* □

A computer search reveals that there are exactly four surfaces (1) with $a_0 a_1 a_2 a_3$ square and all $|a_i| < 16$ which are everywhere locally soluble but which have no integer solutions with $|X_i| \leq 300$. The corresponding sets (a_0, a_1, a_2, a_3) are

$$\left.\begin{array}{ll} (4,9,-8,-8), & (2,9,-6,-12), \\ (2,8,-13,-13), & (7,8,-9,-14). \end{array}\right\} \tag{3}$$

We discuss the first, second and fourth of these surfaces in §6 and the third in §8. We shall show that the first two are insoluble in $\mathbf{Q}$ and that the obstruction to solubility in each case is Brauer-Manin. We use the machinery of this paper to exhibit a rational point on the fourth surface — something which is probably beyond the scope of a straightforward search. A similar process yields the point $(995, 1227, 71, 1115)$ on the third surface (3). This surface is much more susceptible to generalization, and in §8 we discuss the surface

$$X_0^4 + 4X_1^4 = d(X_2^4 + X_3^4) \tag{4}$$

where without loss of generality we can assume that d is fourth-power free and not divisible by 4. Local solubility now requires that $d > 0$, that $d \equiv 1, 2, 5$

or 10 mod 16 and that d is not divisible by any prime $p \equiv 3 \bmod 4$. The third surface (3) corresponds to $d = 26$.

Theorem 1 *Suppose that d is such that* (4) *is everywhere locally soluble. It is still insoluble in $\mathbf{Q}$ if $d \equiv 2 \bmod 16$, no prime $p \equiv 5 \bmod 8$ divides d to an odd power, and $r \equiv 3$ or $5 \bmod 8$ where $d = r^2 + s^2$. Assuming that the Tate-Shafarevich groups of elliptic curves are finite and that Schinzel's Hypothesis holds, it is soluble in all other cases except perhaps when $d \equiv 1 \bmod 16$ and d is not divisible to an odd power by any prime $p \equiv 5 \bmod 8$. In this last case, write $d = \xi_2^2 - 2\xi_3^2$; then* (4) *has no solutions with X_2 odd, $2 \| X_3$ if $|\xi_2| \equiv 1$ or $3 \bmod 8$, and no solutions with X_2 odd, $4 | X_3$ if $|\xi_2| \equiv 5$ or $7 \bmod 8$.*

All but the last sentence of this theorem corresponds to the assertions of Theorem 2 restricted to the special case (4). For the last sentence (which does not use any unproved hypothesis) we have to carry out a second descent, using the ideas of Cassels[1]; it asserts a failure of weak approximation, but one which appears to us not to arise from a Brauer-Manin obstruction. We can transform this last sentence in a number of ways, of which the following seems the most interesting:

Corollary *Suppose that*

$$X_0^4 + 4X_1^4 = W_0^2 - 2W_1^2 \tag{5}$$

for integers X_0, X_1, W_0, W_1 such that no prime $p \equiv 7 \bmod 8$ divides both W_0 and W_1. Then $|W_0| \not\equiv 5$ or $7 \bmod 8$.

Because of the automorphism

$$(X_0, X_1, W_0, W_1) \mapsto (2X_1, X_0, 2W_0, 2W_1)$$

it is enough to study solutions of (5) with X_0 odd and therefore W_0 odd and X_1, W_1 even. Now the conclusion of the Corollary becomes $|W_0| \equiv 1$ or $3 \bmod 8$. We can regard (5) as a pencil of conics, with X_0/X_1 as parameter; but the conclusion is not a Brauer-Manin obstruction to weak approximation, and the requirement that W_0, W_1 should have no common prime factor congruent to $7 \bmod 8$ is essential. We are indebted to Kevin Buzzard for showing us that the result, in the stronger form $|X_0| \equiv 1$ or $3 \bmod 16$, can be proved by classical class field theory.

2. *The associated pencils.* As was explained in the introduction, the assumption that $a_0a_1a_2a_3$ is a square implies that (2) can be written (non-uniquely) in the form

$$A(Y)D(Y) = B(Y)C(Y) \tag{6}$$

where

$$A(Y) = \alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3$$

and so on. Of the two pencils of curves of genus 1 which are the liftings of the families of lines on (2), one is given in an obvious notation by

$$yA(X^2) = zB(X^2), \quad yC(X^2) = zD(X^2) \tag{7}$$

and the other by

$$yA(X^2) = zC(X^2), \quad yB(X^2) = zD(X^2). \tag{8}$$

We shall often denote the curve (7) or (8) by Γ^0. For the time being, we work with (7). Eliminating each of the variables in turn, we obtain four equations of the same shape, only two of which are linearly independent. Each equation has the form

$$d_{i\ell}X_i^2 + d_{j\ell}X_j^2 + d_{k\ell}X_k^2 = 0 \tag{9}$$

where i, j, k, ℓ is any permutation of 0,1,2,3 and d_{ij} is the value of the determinant formed by columns i and j of the matrix

$$\begin{pmatrix} \alpha_0 y - \beta_0 z & \alpha_1 y - \beta_1 z & \alpha_2 y - \beta_2 z & \alpha_3 y - \beta_3 z \\ \gamma_0 y - \delta_0 z & \gamma_1 y - \delta_1 z & \gamma_2 y - \delta_2 z & \gamma_3 y - \delta_3 z \end{pmatrix}.$$

Provided we interchange (7) and (8) if necessary, the effect on the d_{ij} of altering the formulation (6) is simply to make a linear transformation on y, z and to multiply all the d_{ij} by the same constant. By means of bilinear transformations on y, z, we can arrange that all the d_{ij} have nonzero leading coefficients. The Jacobian of the curve (7) has the form

$$Y^2 = (X - c_1)(X - c_2)(X - c_3) \tag{10}$$

where

$$c_1 - c_2 = d_{03}d_{21}, \; c_2 - c_3 = d_{01}d_{32}, \; c_3 - c_1 = d_{02}d_{13}. \tag{11}$$

A map from the curve (7) to its Jacobian (10) is given by

$$Y = d_{12}d_{23}d_{31}X_1X_2X_3/X_0^3, \quad X - c_i = d_{ij}d_{ki}X_i^2/X_0^2$$

where i, j, k is any permutation of $1, 2, 3$. A useful identity implicit in (11) is that

$$d_{01}d_{23} + d_{02}d_{31} + d_{03}d_{12} = 0. \tag{12}$$

Up to this point, the formulae hold for any K3 surface which can be written in the form (6). For the special K3 surfaces of the form (1), we have the unexpected result that each $d_{k\ell}$ is a constant multiple of d_{ij}, where i, j, k, ℓ is any permutation of $0, 1, 2, 3$. (It is because of this that we cannot simply apply the main theorem of [3].) The simplest way to prove this property is to make the decomposition (6) explicit. It follows from the solubility of (2) and the fact that $a_0a_1a_2a_3$ is square that $-a_1$ is represented by $a_2Y_2^2 + a_3Y_3^2$ over $\mathbf{Q}$. In other words, there exist integers r_1, r_2, r_3 and θ such that

$$a_1r_1^2 + a_2r_2^2 + a_3r_3^2 = 0, \quad \theta^2 = a_0a_1a_2a_3. \tag{13}$$

In (6) to (8) we can now take

$$\begin{aligned} A &= a_3r_1^{-1}(\theta r_2X_0^2/a_1a_3 + r_3X_1^2 - r_1X_3^2), \\ B &= a_2r_1^{-1}(\theta r_3X_0^2/a_1a_2 - r_2X_1^2 - r_1X_2^2), \\ C &= r_1^{-1}(\theta r_3X_0^2/a_1a_2 - r_2X_1^2 + r_1X_2^2), \\ D &= -r_1^{-1}(\theta r_2X_0^2/a_1a_3 + r_3X_1^2 + r_1X_3^2); \end{aligned}$$

and the d_{ij} are given by

$$\begin{aligned} d_{23} &= a_3y^2 + a_2z^2, & d_{01} &= (\theta/a_2a_3)d_{23}, \\ d_{31} &= r_1^{-1}(a_3r_2y^2 - 2a_3r_3yz - a_2r_2z^2), & d_{02} &= (\theta/a_3a_1)d_{31}, \\ d_{12} &= r_1^{-1}(a_3r_3y^2 + 2a_2r_2yz - a_2r_3z^2), & d_{03} &= (\theta/a_1a_2)d_{12}. \end{aligned}$$

Changing the values of the r_i can be compensated by a linear transformation on y, z; changing the sign of θ gives the pencil (8) instead of (7). In the numerical examples later in the paper, we shall allow ourselves to adopt somewhat different notations for the sake of integrality. In general the pencils (7) and (8) are not isomorphic. For suppose for example that a_0 and a_1 are positive. Then the forms d_{01} and d_{23}, which are both definite, have the same sign for one pencil but opposite signs for the other. However, there is a natural one-one correspondence between the two $\mathbf{P}^1$ associated with the two fibrations. The Jacobians of fibres at corresponding points are twists of each other by -1, and corresponding triples m^0, as defined in (23) below, are equal.

It follows that the three d_{0i} are coprime in pairs as elements of $\mathbf{Q}[y, z]$ and that, after suitable scaling if necessary, the discriminant of d_{ij} is equal to $-a_i a_j$; this in particular tells us that d_{ij} has no repeated linear factor and that it is a product of two linear factors over $\mathbf{Q}$ if and only if $-a_i a_j$ is in $\mathbf{Q}^{*2}$. We have already noted that if i, j, k is a cyclic permutation of $1, 2, 3$ then

$$d_{0i}/d_{jk} = -a_0 a_i/\theta = -\theta/a_j a_k. \tag{14}$$

Different ratios are connected by equations like

$$d_{ij} d_{ik}/d_{j\ell} d_{k\ell} = -a_i/a_\ell. \tag{15}$$

With the same scaling as before, the resultant of d_{ij} and d_{ik} is $-4a_i^2 a_j a_k$; this is relevent to the remark following (18). The pencil (7) has six singular fibres, given by the roots of $d_{01}d_{02}d_{03} = 0$; and for each pair i, j the roots of $d_{0i} = 0$ harmonically separate the roots of $d_{0j} = 0$. It follows easily from (9) that each singular fibre consists of four lines, which form a skew quadrilateral. Thus each of the 48 lines on (1) is part of a singular fibre of either (7) or (8). We possess a dictionary which gives the Néron-Severi group of each surface (1); some of this information can be found in Table 2 of [4]. In particular, the rank of the Néron-Severi group is at least 2 whenever $a_0a_1a_2a_3$ is a square; it exceeds 2 if and only if there is a relation of the form $a_j = 4a_i$ or $a_j = -a_i$ or $a_i a_j = a_k a_\ell$ up to fourth powers. Because the Néron-Severi group of (1) over $\mathbf{C}$ is spanned by the classes of the 48 lines, it is not hard to verify that the Mordell-Weil group of (10) over $\mathbf{C}(y, z)$ is generated by the three 4-division points, a typical one of which is given by

$$X - c_1 = d_{12}d_{13}\sqrt{-a_0/a_1}, \quad Y = d_{12}d_{13}\theta^{1/2}\{d_{12}\sqrt{-a_0/a_2} + d_{13}\sqrt{a_0/a_3}\}/a_1.$$

Here the signs are so chosen that

$$\sqrt{-a_0/a_1}.\sqrt{-a_0/a_2}.\sqrt{a_0/a_3} = -a_0^2/\theta.$$

¿From this result it is straightforward to read off the Mordell-Weil group of (10) for any intermediate field.

3. *The main theorem.* We must now introduce some notation. Let $f(y, z)$, $g(y, z)$ be homogeneous polynomials in $\mathbf{Z}[y, z]$ such that fg has no non-constant squared factor, and let $\mathcal{S}$ be a finite set of places of $\mathbf{Q}$ containing 2, ∞ and all the odd primes which divide the discriminant of fg. Let $L = L(\mathcal{S}; f, g)$ be the function

$$\lambda \times \mu \mapsto \prod{}' (f(\lambda, \mu), g(\lambda, \mu))_p$$

defined for all pairs of coprime integers λ, μ such that $f(\lambda,\mu)g(\lambda,\mu)$ is nonzero; here the outer bracket is the Hilbert symbol, defined by

$$(a,b)_v = \begin{cases} 1 \text{ if } ax^2+by^2=1 \text{ is soluble in } \mathbf{Q}_v, \\ -1 \text{ otherwise} \end{cases} \tag{16}$$

and the product is taken over all p not in $\mathcal{S}$ which divide $g(\lambda,\mu)$. By the condition imposed on $\mathcal{S}$, this implies that p does not divide $f(\lambda,\mu)$. Clearly L is multiplicative in each of f and g, and satisfies $L(f-gh,g)=L(f,g)$ and

$$L(\mathcal{S};f,g)L(\mathcal{S};g,f) = \prod_{v \text{ in } \mathcal{S}} (f(\lambda,\mu), g(\lambda,\mu))_v; \tag{17}$$

for any f,g these facts are enough to enable one to express $L(\mathcal{S};f,g)$ as a multiple of $L(\mathcal{S};y,z)^{\deg f.\deg g}$. Hence provided that at least one of f,g has even degree, $L(\lambda,\mu)$ is continuous in the topology induced by $\mathcal{S}$ and there is an explicit description of L which makes the continuity evident. If the degrees of f and g are both odd, then the same holds for $L(f,g;\lambda,\mu)L(y,z;\lambda,\mu)$. However, in the construction used to prove Lemma 1 we can always ensure that $L(y,z;\eta,\zeta)=1$; for the proofs of all these assertions, see [6], Lemma 3 and [6], §6. If $\mathcal{S}$ is replaced by $\mathcal{S}_1 \supset \mathcal{S}$ the effect on L corresponds to deleting the factors associated with the p in $\mathcal{S}_1 \backslash \mathcal{S}$.

In what follows we shall repeatedly have to evaluate expressions of the form $L(\mathcal{B};f_1,f_2;\eta,\zeta)$ where $f_i=\alpha_i y^2+\beta_i yz+\gamma_i z^2$ for $i=1,2$ and $\mathcal{B}$ contains $2,\infty$ and the odd primes which divide

$$R = (\alpha_1\gamma_2-\alpha_2\gamma_1)^2 - \beta_1\beta_2(\alpha_1\gamma_2+\alpha_2\gamma_1) + \alpha_1\gamma_1\beta_2^2 + \alpha_2\gamma_2\beta_1^2,$$

the resultant of f_1 and f_2. Suppose that $\eta\times\zeta$ and $\rho\times\sigma$ are in $\mathcal{Y}$. Using the algorithm in [6], we find

$$\begin{aligned} L(\mathcal{B};f_1,f_2;\eta,\zeta)\,L(\mathcal{B};f_1,f_2;\rho,\sigma) = \textstyle\prod_{v \text{ in } \mathcal{B}}\{&(g/(\sigma\eta-\rho\zeta),R)_v \\ (f_2(\rho,\sigma),f_2(\eta,\zeta))_v\,(g,f_2(\eta,\zeta))_v\,&(-g,f_2(\rho,\sigma))_v\} \end{aligned} \tag{18}$$

where

$$g = f_1(\eta,\zeta)f_2(\rho,\sigma) - f_1(\rho,\sigma)f_2(\eta,\zeta).$$

In various previous papers the *non-classical* part of $L(\eta,\zeta)$ has been defined as the part which depends continuously on η and ζ, but not on η/ζ alone. In this case we see that the non-classical part is just $\prod(g/(\sigma\eta-\rho\zeta),R)_v$.

Now let U_ℓ be the surface fibred by the pencil of conics (9). Because of the projection maps $V \to U_\ell$ and the basic hypothesis that V is everywhere locally soluble, each U_ℓ is everywhere locally soluble; and a necessary condition for V to be soluble in $\mathbf{Q}$ is that each U_ℓ should be so. If we fix y, z then a similar result holds for the curve (7) and the curves given by (9). But there is a computable obstruction to weak approximation on U_ℓ, which is in fact equivalent to the Brauer-Manin obstruction but which for our purpose is more conveniently put in the following form. Let $\mathcal{B}_0$, the set of genuinely bad places for (1), consist of 2, ∞ and the odd primes which divide $a_0a_1a_2a_3$. Let $\mathcal{B}_1$, which may depend on the decomposition (6), be a large enough fixed finite set of places; we shall require $\mathcal{B}_1$ to contain $\mathcal{B}_0$ and the primes which divide the discriminants of all the expressions like $d_{01}d_{02}d_{03}$. For $i = 1, 2, 3$ let $c_{0i\tau}(y, z)$ run through the nonconstant irreducible factors of d_{0i} in $\mathbf{Z}[y, z]$; let $\xi = \xi_{0i\tau}$ satisfy $c_{0i\tau}(\xi, 1) = 0$ and write

$$L_{0i\tau} = \mathbf{Q}(\xi) = \mathbf{Q}(\sqrt{-a_0a_i}). \tag{19}$$

Taking for example $i = 3$, there are two such $c_{03\tau}$ if $-a_1a_2$ is a square and one otherwise. Recall that $c_{03\tau}$ is also an irreducible factor of d_{12}; thus for (9) to be soluble in $\mathbf{Q}$ with $y = \lambda, z = \mu$ and λ, μ coprime integers, it is necessary that λ, μ should satisfy

$$(-d_{13}d_{23}, c_{03\tau})_p = (-d_{02}d_{32}, c_{03\tau})_p = 1 \tag{20}$$

for each prime p not in $\mathcal{B}_1$ which divides the value of $c_{03\tau}$ to an odd power. If p divides the value of $c_{03\tau}$ to a positive even power then it does not divide the value of $-d_{13}d_{23}$ or $-d_{02}d_{32}$; so (20) holds automatically. In the notation above, local solubility everywhere therefore requires all the conditions like

$$L(\mathcal{B}_1; -d_{13}d_{23}, c_{03\tau}; \lambda, \mu) = L(\mathcal{B}_1; -d_{02}d_{32}, c_{03\tau}; \lambda, \mu) = 1. \tag{21}$$

We denote by $\mathcal{B}$ the union of the $\mathcal{B}_1$ corresponding to the various functions like $L(\mathcal{B}_1; -d_{13}d_{23}, c_{03\tau})$ together with 3 or 5 if either of these divides the value of $d_{01}d_{02}d_{03}$ for all pairs λ, μ. The reason for including 3 or 5 is to be able to apply Lemma 1; the third surface (3) is one for which we do have to adjoin 5. For the solubility of (1) it is therefore necessary that the set $\mathcal{X} \subset \mathcal{Y}$ of pairs λ, μ which satisfy all the conditions like (21), and for which (7) is locally soluble at all places in $\mathcal{B}$, should not be empty; and in looking for solutions by means of the fibring (7) we can confine ourselves to pairs

y, z in $\mathcal{X}$. We actually go further; we choose a pair $\lambda_0 \times \mu_0$ in $\mathcal{X}$ and confine ourselves to pairs $\lambda \times \mu$ such that $\lambda \times \mu$ is in $\mathcal{X}$ and close to $\lambda_0 \times \mu_0$ in the topology induced by $\mathcal{B}$.

Now let G^* be the set of triples $m = (m_1, m_2, m_3)$ where the $m_i(y, z)$ are homogeneous squarefree polynomials in $\mathbf{Z}[y, z]$ subject to the conditions

(i) $m_1 m_2 m_3$ is a square in $\mathbf{Z}[y, z]$;

(ii) m_1 divides $d_{02} d_{03}$ in $\mathbf{Q}[y, z]$ and so on;

(iii) no m_i is divisible in $\mathbf{Z}[y, z]$ by any prime outside $\mathcal{B}$.

Let G be the subset of G^* which satisfies the additional condition

(iv) all the m_i have even degree.

Both G^* and G have a natural structure of abelian groups, the law of composition being to multiply triples elementwise and then remove squared factors. To each element of G we can associate a 2-covering of (10) given by

$$m_i Z_i^2 = X - c_i \quad \text{for} \quad i = 1, 2, 3; \tag{22}$$

the particular 2-covering given by the four equations (9) corresponds to

$$m_1^0 = -d_{21} d_{31}, \; m_2^0 = -d_{12} d_{32}, \; m_3^0 = -d_{13} d_{23}, \tag{23}$$

where we have written $Z_i^2 = X_i^2 / X_0^2$. By (15), the 2-covering corresponding to the 2-division point $(c_1, 0)$, for example, is given by

$$m_1 = -a_0 a_1, \; m_2 = d_{03} d_{21}, \; m_3 = d_{02} d_{31}, \tag{24}$$

again after removal of squared factors. For any triple m in G^*, let $\delta_{03\tau}(m)$ denote the class of $m_3(\xi_{03\tau}, 1)$ in

$$L_{03\tau}^* / < L_{03\tau}^{*2}, m_3^0(\xi_{03\tau}, 1), d_{02}/d_{31}, d_{01}/d_{32} >;$$

here one of the last two expressions is redundant in view of (15), but both have been included to preserve the symmetry. We define $\delta_{01\tau}(m)$ and $\delta_{02\tau}(m)$ similarly. The following condition is the analogue for our problem of the condition introduced in [5] and refined in [3]:

Condition D. *The kernel of $\oplus \delta_{0i\tau}$ acting on G^* lies in the group generated by m^0 and the image of the Mordell-Weil group of* (10) *over* $\mathbf{Q}(y, z)$.

Using the results of [5], §3 or of Lemma 5 below, it is easy to check that the triples m in the kernel of $\oplus\delta_{0i\mathcal{T}}$ are precisely those with the following property: for any $\eta \times \zeta$ in $\mathcal{Y}$ and any p not in $\mathcal{B}$, if the 2-covering associated with m^0 is soluble in $\mathbf{Q}_p$ so is the 2-covering associated with m. At the end of §5 we show how to replace Condition D by a weaker but computationally less convenient one.

Lemma 2 *Suppose that no $-a_i a_j$ is a square. Then for any m in the kernel of $\oplus\delta_{oi\mathcal{T}}$ either m or mm^0 is independent of y and z; and the kernel has order 8 if and only if $a_0a_1a_2a_3$ is not a fourth power and no a_ia_j is a square. In this case Condition D holds.*

Proof. We have $G = G^*$, and for m in G the possible values of m_3, for example, are generated by

$$-1, 2, d_{01}, d_{02} \text{ and the odd primes in } \mathcal{B}.$$

Let ξ_{03} satisfy $d_{03}(\xi_{03}, 1) = 0$; in the notation of §2 we have

$$\xi_{03} = (-a_2r_2 \pm r_1\sqrt{-a_1a_2})/a_3r_3.$$

Thus $d_{02}(\xi_{03}, 1) = \pm d_{01}(\xi_{03}, 1)\sqrt{-a_2/a_1}$ and $\operatorname{norm} d_{01}(\xi_{03}, 1)$ is $-a_0a_2$ times a square. But for m to be in the kernel of δ_{03} it is necessary that

$$\operatorname{norm} m_3(\xi_{03}, 1) \text{ is in } < \mathbf{Q}^{*2}, \operatorname{norm} m_3^0(\xi_{03}, 1) >,$$

and it follows that m_3 must contain both or neither of d_{01} and d_{02} as factors. Applying a similar argument to m_1 and m_2, we deduce that if m is to be in the kernel of $\oplus\delta_{0i}$ then either m or mm^0 is independent of y/z. It is enough to consider the former case; now as elements of $\mathbf{Q}^*/\mathbf{Q}^{*2}$,

$$\begin{aligned} m_1 &\text{ is in } \{1, -a_2a_3, \theta a_0a_3, -\theta a_0a_2\}, \\ m_2 &\text{ is in } \{1, -a_1a_3, \theta a_0a_1, -\theta a_0a_3\}, \\ m_3 &\text{ is in } \{1, -a_1a_2, \theta a_0a_2, -\theta a_0a_1\}. \end{aligned}$$

In general the condition that $m_1m_2m_3$ is a square allows us only four choices, which correspond to the origin and the three 2-division points. Additional possibilities happen only when one of

$$\theta, \theta a_0a_1, \theta a_0a_2, \theta a_0a_3, a_1a_2, a_2a_3, a_3a_1$$

is in $\pm\mathbf{Q}^{*2}$. But if $\theta a_0 a_i$ is in $\pm\mathbf{Q}^{*2}$ then all we obtain is a new way of describing triples m which are already known to be in the kernel; so these cases can be ignored. The others give the exceptions listed. $\square$

If for example a_2a_3 is a square then $(1, -a_1a_2, -a_1a_2)$ is an additional element of the kernel. Again, if θ is in $-\mathbf{Q}^{*2}$ then (a_1a_3, a_1a_2, a_2a_3) is in the kernel, whereas if θ is in $\mathbf{Q}^{*2}$ then (a_1a_2, a_2a_3, a_3a_1) is in the kernel. If some a_ia_j is a square, then in general the rank of the Néron-Severi group remains 2 but the recipe of Lemma 11 below can be used to generate elements of order 2 in the Brauer-Manin group additional to those listed in §7; one such element is that called F_{ij} in §6. But if $a_0a_1a_2a_3$ is a fourth power then in general the rank of the Néron-Severi group remains 2 and there is no obvious implication for the Brauer-Manin group.

We can now state the main result of this paper:

Theorem 2 *Assume that the Tate-Shafarevich groups of elliptic curves are finite and that Schinzel's Hypothesis holds. Let*

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0 \tag{25}$$

be a nonsingular surface defined over $\mathbf{Q}$*, everywhere locally soluble and such that* $a_0a_1a_2a_3$ *is square. If* $\mathcal{X}$ *is not empty and Condition D holds, then* (25) *contains rational points.*

We do in fact obtain a stronger conclusion: that if U is any non-empty open subset of $\mathcal{X}$ then we can find $\lambda \times \mu$ in $\mathcal{X}$ such that (7) is soluble for $y = \lambda, z = \mu$. This statement is analogous to weak approximation, but we are unable to obtain any conclusion about weak approximation in the strict sense because we have no result for weak approximation analogous to Lemma 4 below.

4. *The symmetry property.* For the reader's convenience we describe here the symmetry property which is an essential tool for the methods used in [3] and in the proof of Theorem 2 of this paper. For a fuller account, with proofs, see [3], §1.2 or [6], §5, where the results are stated and proved for an arbitrary number field.

An elliptic curve all of whose 2-division points are rational can be written in the form

$$E : Y^2 = (X - c_1)(X - c_2)(X - c_3) \tag{26}$$

where without loss of generality we can assume that the c_i are integers. We shall denote by $\mathcal{S} = \mathcal{S}(E)$ a fixed finite set of places of $\mathbf{Q}$ which contains 2,

∞ and the odd primes of bad reduction for E. To any triple (m_1, m_2, m_3) of elements of $\mathbf{Q}^*$ with $m_1 m_2 m_3 = 1$ we associate the 2-covering given by

$$m_i Z_i^2 = X - c_i \quad \text{for} \quad i = 1, 2, 3$$

and $Y = Z_1 Z_2 Z_3$; in this way we obtain an isomorphism between $(\mathbf{Q}^*/\mathbf{Q}^{*2})^2$ and the $\mathbf{F}_2$-vector space of all 2-coverings of E. The 2-coverings soluble in $\mathbf{Q}_p$ for every p outside $\mathcal{S}$ can be identified with the elements of $(\mathfrak{o}_\mathcal{S}^*/\mathfrak{o}_\mathcal{S}^{*2})^2$, where $\mathfrak{o}_\mathcal{S}^*$ consists of the elements of $\mathbf{Q}$ which are units outside $\mathcal{S}$. Write

$$X_\mathcal{S} = \mathfrak{o}_\mathcal{S}^*/\mathfrak{o}_\mathcal{S}^{*2}, \quad Y_v = k_v^*/k_v^{*2}, \quad Y_\mathcal{S} = \oplus_\mathcal{S} Y_v;$$

then as $\mathbf{F}_2$-vector spaces $X_\mathcal{S}$ has dimension n and $Y_\mathcal{S}$ has dimension $2n$, where n is the order of $\mathcal{S}$. Moreover $X_\mathcal{S} \to Y_\mathcal{S}$ is injective.

Now write $V_v = Y_v \times Y_v$ and $V_\mathcal{S} = \oplus_\mathcal{S} V_v = Y_\mathcal{S} \times Y_\mathcal{S}$; and let $U_\mathcal{S}$ be the image of $X_\mathcal{S} \times X_\mathcal{S}$ in $V_\mathcal{S}$. Thus $\dim U_\mathcal{S} = \frac{1}{2} \dim V_\mathcal{S}$. We can define a non-degenerate alternating bilinear form e_v on V_v by

$$e_v((a,b),(c,d)) = (a,d)_v (b,c)_v \tag{27}$$

where the factors on the right are the Hilbert symbols. This induces a non-degenerate alternating bilinear form $e_\mathcal{S} = \prod_\mathcal{S} e_v$ on $V_\mathcal{S}$. By the Hilbert product formula $U_\mathcal{S}$ is isotropic in $V_\mathcal{S}$, and comparison of dimensions shows that it is maximal isotropic.

Let T_v be the image of $(\mathfrak{o}_v^*/\mathfrak{o}_v^{*2})^2$ in V_v, where $\mathfrak{o}_v$ is the ring of integers of $\mathbf{Q}_v$, and let W_v be the image of $E(\mathbf{Q}_v)$ in V_v under the Kummer map

$$\partial : P = (X, Y) \mapsto (X - c_1, X - c_2) \tag{28}$$

It is known that W_v is a maximal isotropic subspace of V_v for the alternating form e_v, and $W_v = T_v$ if v is not in $\mathcal{S}$. (An explicit description of the W_v can be found in [5], §3; this is useful for computational purposes.) A 2-covering of E is soluble in $\mathbf{Q}_v$ for v in $\mathcal{S}$ if and only if the corresponding point $m_1 \times m_2$ of $(\mathbf{Q}^*/\mathbf{Q}^{*2})^2$ is in W_v. We have already remarked that a 2-covering of E is soluble in $\mathbf{Q}_v$ for all v not in $\mathcal{S}$ if and only if the corresponding point of $(\mathbf{Q}^*/\mathbf{Q}^{*2})^2$ is in $U_\mathcal{S}$. Thus if we write $W_\mathcal{S}$ for the subset $\oplus_\mathcal{S} W_v$ of $V_\mathcal{S}$, the 2-Selmer group of E can be identified with $U_\mathcal{S} \cap W_\mathcal{S}$. Thus it is both the left and the right kernel of the bilinear map $U_\mathcal{S} \times W_\mathcal{S} \to \{\pm 1\}$ induced by $e_\mathcal{S}$.

It can be shown that there exist maximal isotropic subspaces $K_v \subset V_v$ for all v in $\mathcal{S}$ such that $V_\mathcal{S} = U_\mathcal{S} \oplus K_\mathcal{S}$ where $K_\mathcal{S} = \oplus_\mathcal{S} K_v$; and if v is not 2 or ∞ we can take $K_v = T_v$. Let $t_S : V_S \to U_S$ be the projection along K_S and write

$$U'_S = U_S \cap (W_S + K_S), \quad W'_S = W_S/(W_S \cap K_S) = \oplus_S W'_v$$

where $W'_v = W_v/(W_v \cap K_v)$. The map t_S induces an isomorphism

$$\tau_S : W'_S \to U'_S;$$

and the bilinear function e_S induces a bilinear function

$$e'_S : U'_S \times W'_S \to \{\pm 1\}.$$

The symmetry property which we need is as follows:

Lemma 3 *The bilinear functions $U'_S \times U'_S \to \{\pm 1\}$ and $W'_S \times W'_S \to \{\pm 1\}$ defined respectively by*

$$u'_1 \times u'_2 \mapsto e'_S(u'_1, \tau_S^{-1}(u'_2)) \quad \textit{and} \quad w'_1 \times w'_2 \mapsto e'_S(\tau_S w'_1, w'_2)$$

are symmetric and their kernels are isomorphic to the 2-Selmer group of E.

5. *Proof of Theorem* . At the beginning of §2 we showed how to fibre V by 2-coverings of elliptic curves E whose equations have the form (26). To prove Theorem 2 we have to show that we can choose a fibre which contains rational points; clearly y and z must be in $\mathbf{Q}$ and the fibre must be everywhere locally soluble, so that it must lie in the 2-Selmer group of E. The question which elements of the 2-Selmer group contain points defined over $\mathbf{Q}$, or equivalently come from points of $E(\mathbf{Q})$, has been vexatious since the time of Fermat and remains unsolved; but the following partial answer will meet our needs.

Lemma 4 *Suppose that the Tate-Shafarevich group of E is finite and the image of the Mordell-Weil group of E in the 2-Selmer group of E is at least half the latter. Then every element of the 2-Selmer group is represented by curves containing points defined over $\mathbf{Q}$.*

Proof. The elements of the 2-Selmer group which lie in the image of the Mordell-Weil group certainly contain points defined over $\mathbf{Q}$, so the 2-torsion of the Tate-Shafarevich group of E has order at most 2. But if the order of the Tate-Shafarevich group is finite, then it follows from the existence of the

Cassels alternating form that the order of the 2-torsion is a square. Hence it must be 1, whence every element of the 2-Selmer group contains a point defined over $\mathbf{Q}$. $\square$

Our strategy for proving Theorem 2 will be as follows. For $i = 1, 2, 3$ and each irreducible nonconstant factor $c_{0i\tau}$ of d_{0i} we shall construct a finite set $\mathcal{G}_{0i\tau}$ of primes p, these sets being disjoint from $\mathcal{B}$ and from each other, such that there exist integers η_p, ζ_p with $p \| c_{0i\tau}(\eta_p, \zeta_p)$; when $c_{0i\tau}$ is linear we further require that η_p, ζ_p are not both divisible by p. It will be convenient to require also that no p in $\mathcal{G}_{0i\tau}$ divides the leading coefficient of $c_{0i\tau}$, so that each ζ_p is prime to p. (Recall that we have already arranged for all these leading coefficients to be nonzero.) Choose neighbourhoods $\mathcal{N}_p$ of $\eta_p \times \zeta_p$ in $\mathbf{Q}_p \times \mathbf{Q}_p$ so small that $p \| c_{0i\tau}(\eta, \zeta)$ and $\eta - \eta_p, \zeta - \zeta_p$ are both divisible by p for each $\eta \times \zeta$ in $\mathcal{N}_p$; since p is not in $\mathcal{B}$, for $\eta \times \zeta$ in $\mathcal{N}_p$ no other $c_{0j\sigma}(\eta, \zeta)$ is divisible by p. The sets $\mathcal{G}_{0i\tau}$ will be chosen to have the additional properties (i) and (iv) below. Let $\mathcal{S}_0$ be the union of $\mathcal{B}$ and the $\mathcal{G}_{0i\tau}$, and choose $\eta \times \zeta$ in $\mathcal{X}$ according to the recipe in Lemma 1 for the $c_{0i\tau}(y, z)$, and with the additional property that $L(\mathcal{B}; y, z; \eta, \zeta) = 1$. Denote by $p_{0i\tau}$ the additional prime which divides $c_{0i\tau}(\eta, \zeta)$; then each $c_{0i\tau}(\eta, \zeta)$ is a unit outside the set $\mathcal{S}$ obtained by adjoining to $\mathcal{S}_0$ all the $p_{0i\tau}$, and it is divisible to precisely the first power by $p_{0i\tau}$ and by each of the primes in $\mathcal{G}_{0i\tau}$. We shall take the set $\mathcal{S}$ just defined to be the appropriate $\mathcal{S}$ for the notation of §4.

We now apply the ideas above to the curve Γ^0 given by (7) or (8) with $y = \eta, z = \zeta$ and to its Jacobian E, which we already know has the form (26). We need to ensure that

(i) the curve Γ^0 is soluble in $\mathbf{Q}_p$ for every p in $\cup \mathcal{G}_{0i\tau}$;

by requiring that $\eta \times \zeta$ is in $\mathcal{X}$ we have already ensured the corresponding result for every v in $\mathcal{B}$. From this we shall need to deduce

(ii) the curve Γ^0 is soluble in $\mathbf{Q}_p$ for each $p = p_{0i\tau}$.

Thus (i) and (ii) prove that the class of Γ^0 is in the 2-Selmer group of E. The choice of the $\mathcal{G}_{0i\tau}$ does not determine the $p_{0i\tau}$; but this is unimportant because we can prove

(iii) the bilinear form $e_S^* : W_S' \times W_S' \to \{\pm 1\}$ defined in Lemma 3 is effectively independent of the choice of $\eta \times \zeta$ and hence of the $p_{0i\tau}$. In particular, the order of the 2-Selmer group only depends on the $\mathcal{G}_{0i\tau}$.

The other condition which we shall impose on the choice of the $\mathcal{G}_{0i\tau}$, which is only meaningful once we have proved (iii), is that

(iv) the kernel of e_S^* is generated by the images of m^0 and of the elements of the Mordell-Weil group of E over $\mathbf{Q}(y, z)$.

It is in the proof of (iv) that we use Condition D. Once we have (iv), it follows from Lemma 4 that Γ^0 has rational solutions for $y = \eta, z = \zeta$ and therefore (25) has rational solutions. The Lemmas which follow correspond to these four claims. It is however convenient to state Lemma 5 in a form applicable to any 2-covering of E and to give the application to Γ^0 as a Corollary. We again employ the notation of (19).

Lemma 5 *Suppose that p is a prime not in $\mathcal{B}_1$. Let i, j, k be any permutation of $1, 2, 3$. There exists $\eta \times \zeta$ in $\mathcal{Y}$ with $p | c_{0i\tau}(\eta, \zeta)$ if and only if p splits in $L_{0i\tau}$. If so, the 2-covering corresponding to m is soluble in $\mathbf{Q}_p$ if and only if one of the two following conditions holds:*

(i) *$p | m_j$ and both m_i and d_{0j}/d_{ik} are in $\mathbf{Q}_p^{*2}$.*

(ii) *$p \not| m_j$ and at least one of m_i and $(d_{0j}/d_{ik})m_i$ is in $\mathbf{Q}_p^{*2}$.*

Proof. Recall that by convention m_j is square-free. The Lemma follows immediately from the results for Case I in [6], §3. It is also straightforward to prove it directly. □

Corollary *Suppose that p is a prime not in $\mathcal{B}_1$. There exists $\eta \times \zeta$ in $\mathcal{Y}$ with $p \| c_{0i\tau}(\eta, \zeta)$ if and only if p splits in $L_{0i\tau}$. If so, let $\mathfrak{p} = (p, \eta - \xi\zeta)$ where $c_{0i\tau}(\xi, 1) = 0$; thus $\mathfrak{p}$ is a first degree prime factor of p in $L_{0i\tau}$. Let i, j, k be any permutation of $1, 2, 3$; then Γ^0 is soluble in $\mathbf{Q}_p$ if and only if $\mathfrak{p}$ splits completely in*

$$L_{0i\tau}(\sqrt{m_i^0(\xi, 1)}, \sqrt{d_{0j}/d_{ik}}). \tag{29}$$

Proof. The first two assertions are standard, and since $p \| m_j^0(\eta, \zeta)$ it follows from the Lemma that Γ^0 is soluble in $\mathbf{Q}_p$ if and only if $m_i^0(\eta, \zeta)$ and d_{0j}/d_{ik} are in $\mathbf{Q}_p^{*2}$. But we can embed ξ in $\mathbf{Z}_p$ with $p | (\xi\zeta - \eta)$, and from this the Corollary follows. The apparent asymmetry in the criteria in both the Lemma and the Corollary disappears when one takes account of $L_{0i\tau} = \mathbf{Q}(\sqrt{-a_0 a_i})$ and (15). □

To ensure (i), we need only require that each p in each $\mathcal{G}_{0i\tau}$ satisfies the conditions of the Corollary. Henceforth we assume that this is so.

Lemma 6 *If Γ^0 is locally soluble at p for each p in $\mathcal{G}_{0i\tau}$ then Γ^0 is also locally soluble at $p_{0i\tau}$.*

Proof. To fix ideas, we set $i = 3$. The condition

$$L(\mathcal{B}; -d_{13}d_{23}, c_{03\tau}; \eta, \zeta) = 1$$

is equivalent to the statement that the number of p in $\mathcal{G}_{03\tau} \cup \{p_{03\tau}\}$ at which the equation (9) with $\ell = 3$ is locally insoluble is even. But by hypothesis this equation is locally soluble at each p in $\mathcal{G}_{03\tau}$; so it is locally soluble at $p_{03\tau}$. The same argument works for each equation (9), so that $m_3^0 = -d_{13}d_{23}$ and $-d_{02}d_{32}$ are $p_{03\tau}$-adic squares. Hence the same is true of d_{02}/d_{31}. Now the Lemma follows from Lemma 5(i). □

For the next stage of the argument we shall need to construct partitions of U_S' and W_S' which are compatible and which separate the effects of $\mathcal{B}$, the primes in $\cup\mathcal{G}_{0i\tau}$ and the primes $p_{0i\tau}$. For each p in $\mathcal{S} \setminus \mathcal{B}$ denote by P_p the generator of the one-dimensional space W_p'; here P_p is the class of $(1, p)$, $(p, 1)$ or (p, p) according as $i = 1, 2$ or 3. Let $Z_1' \subset W_S'$ be the $\mathbf{F}_2$-vector space generated by the P_p with p in $\cup\mathcal{G}_{0i\tau}$. Write $Q_{0i\tau} = \sum P_p$ where the sum is taken over all p in $\mathcal{G}_{0i\tau} \cup \{p_{0i\tau}\}$, and let $Z_2' \subset W_S'$ be the $\mathbf{F}_2$-vector space generated by the $Q_{0i\tau}$. Then

$$W_S' = W_{\mathcal{B}}' \oplus Z_1' \oplus Z_2', \quad U_S' = U_{\mathcal{B}}' \oplus \tau_S Z_1' \oplus \tau_S Z_2'$$

and τ_S induces the isomorphism $\tau_{\mathcal{B}} : W_{\mathcal{B}}' \to U_{\mathcal{B}}'$. Moreover $\tau_S Q_{0i\tau}$ is of the form

$$(\epsilon_{0i\tau}', \epsilon_{0i\tau}'' c_{0i\tau}), \ (\epsilon_{0i\tau}' c_{0i\tau}, \epsilon_{0i\tau}'') \text{ or } (\epsilon_{0i\tau}' c_{0i\tau}, \epsilon_{0i\tau}'' c_{0i\tau})$$

according as $i = 1, 2$ or 3; here the arguments of $c_{0i\tau}$ are η, ζ and $\epsilon_{0i\tau}', \epsilon_{0i\tau}''$ are in $\mathfrak{o}_{\mathcal{B}}^*$. A similar but stronger statement holds for the $\tau_S P_p$. By continuity the $\epsilon_{0i\tau}', \epsilon_{0i\tau}''$ do not depend on η, ζ.

If G^* is as in §3, so that each element of G^* defines a 2-covering, then there are two subgroups of G^* (depending on the choice of the $\mathcal{N}_p$ but not on $\eta \times \zeta$) each of which has some claim to be described as a geometric 2-Selmer group. The first and larger is

$$(\mathrm{Im}(G^* \to U_S)) \cap (W_{\mathcal{B}} \oplus (\oplus_{\mathcal{S}\setminus\mathcal{B}} V_p)); \tag{30}$$

it consists of those 2-coverings induced by elements of G^* which are locally soluble for each v in $\mathcal{B}$. There is a natural injection of (30) into $U_{\mathcal{B}}' \oplus \tau_S Z_2'$, and the second and smaller candidate for being called a geometric 2-Selmer group consists of those elements of the image which are orthogonal to Z_2' under e_S'; when the $\mathcal{G}_{0i\tau}$ are empty these are the elements of (30) which are also locally soluble at each potential $p_{0i\tau}$. It is natural to identify each of these groups with its inverse image in G^*; each of them contains the elements corresponding to the 2-division points and to m^0. The smaller of the two groups will appear again as G_2 in Condition E below.

Lemma 7 *The values taken by the bilinear form* $e_S^* : W_S' \times W_S' \to \{\pm 1\}$ *do not depend on the choice of* η, ζ *or on the* $p_{0i\tau}$.

Proof. Here it is of course implicit that if we change the values of η, ζ, and thereby also the values of the $p_{0i\tau}$, we make use of the natural isomorphism between the old $\mathcal{S}$ and the new one. We need to exhibit rules for evaluating e_S^* which do not depend on η, ζ or on the $p_{0i\tau}$; and for later purposes we shall need to know how they depend on the p in $\cup\mathcal{G}_{0i\tau}$. In $(W_B' \oplus Z_2') \times W_B'$ the value of e_S^* is a product of Hilbert symbols at places of $\mathcal{B}$, whose arguments are either fixed elements of $\mathbf{Z}$ or of the form $f_{0i\tau}(\eta, \zeta)$; by continuity such a value is constant. At a point of $Z_2' \times Q_{0i\tau}$ the value is a product of expressions $L(\mathcal{B}; c_{0j\sigma}, c_{0i\tau}; \eta, \zeta)$, where $c_{0j\sigma}$ is distinct from $c_{0i\tau}$, and expressions $L(\mathcal{B}; \epsilon, c_{0i\tau}; \eta, \zeta)$ where ϵ is in $\mathfrak{o}_B^*$, and we noted in §2 that these are constant even if $c_{0j\sigma}$ and $c_{0i\tau}$ both have odd degrees. Now suppose that p is in $\mathcal{G}_{0i\tau}$ for some i and τ, and let β be an element of $W_B' \oplus Z_2'$; then $\tau_S\beta$ corresponds to a triple (m_1, m_2, m_3) in G^*. By considering each value of i separately, and remembering that $m_1 m_2 m_3$ is a square, we obtain

$$e_S^*(\beta, P_p) = (m_i(\eta, \zeta), p)_p. \tag{31}$$

If we define $\mathfrak{p}$ as in Lemma 5, then arguments similar to those in the proof of that Lemma show that the value of (31) is 1 if and only if $\mathfrak{p}$ splits in $L_{0i\tau}(\sqrt{m_i(\xi, 1)})$. In particular, this criterion does not depend on the values of η, ζ. Finally, the values of e_S^* on $Z_1' \times Z_1'$ are clearly independent of the choices of η, ζ; it is not difficult to give explicit formulae for these values, but we shall not need them. □

It remains to show that, provided Condition D holds, we can choose the $\mathcal{G}_{0i\tau}$ so as to satisfy (iv). Here we shall be motivated by the following result from linear algebra.

Lemma 8 *Let* Ψ *be a symmetric bilinear form on a vector space* $H \oplus H'$ *over a field* F. *Let* $H_0 \subset H$ *be the kernel of the restriction of* Ψ *to a bilinear* *form on H. Suppose that the pairing between H_0 and H' given by Ψ is non-degenerate (that is, defines an isomorphism between H' and the dual H_0^* of H_0). Then Ψ is non-degenerate.*

Proof. Let $H_1 \subset H$ be the left kernel of the restriction of Ψ to $H \times H'$; thus $H = H_0 \oplus H_1$ and H_1 is orthogonal to $H_0 \oplus H'$. The restriction of Ψ to H_1 is non-degenerate; and the same is true of the restriction of Ψ to $H_0 \oplus H'$ because Ψ identifies H' with H_0^* and the restriction of Ψ to H_0 is zero. □

Lemma 9 *We can choose the $\mathcal{G}_{0i\tau}$ so that the conditions of Lemma 5 hold for every p in $\cup\mathcal{G}_{0i\tau}$ and the kernel of e_S^* is generated by the images of m^0 and of the Mordell-Weil group of E over $\mathbf{Q}(y, z)$.*

Proof. Recall that $U_S \cap W_S$ has been identified with the 2-Selmer group. Using the notation immediately before the statement of Condition D in §3, denote by G_0 the subgroup of G generated by m^0 and the image of the Mordell-Weil group of E over $\mathbf{Q}(y, z)$, and by U_0' its image under the map $G_0 \to U_S \cap W_S \to U_S'$; clearly $U_0' \subset U_B' \oplus \tau_S Z_2'$. In the notation of Lemma 8 we shall take $H = (W_B' \oplus Z_2')/\tau_S^{-1} U_0'$ and $H' = Z_1'$; and Ψ will be the bilinear form induced by e_S^*. Thus H_0 is the image of the smaller geometric 2-Selmer group. We choose the primes p in $\cup\mathcal{G}_{0i\tau}$ one by one. Suppose that we have already chosen $p_1, \ldots, p_r$. If the left kernel of the restriction of Ψ to

$$H_0 \times (W_{p_1}' \oplus \ldots \oplus W_{p_r}') \qquad (32)$$

is trivial, then we are done. If not, we choose a non-trivial element β of the left kernel and lift it back via τ_S to an element m of G^*. This is possible, because the image of the injection $G^* \to U_S$ obtained by evaluating m_1, m_2 at $\eta \times \zeta$ is precisely $U_B \oplus \tau_S Z_2'$, so we obtain an isomorphism $U_B \oplus \tau_S Z_2' \to G^*$. By Condition D, we can choose i, τ so that m is not in the kernel of $\delta_{0i\tau}$. To fix ideas, let $i = 3$; then the field $L_{0i\tau}(\sqrt{m_i(\xi, 1)})$ is not contained in the field (29), and we can choose a $\mathfrak{p}_{r+1}$ which splits in the second of these fields but not in the first. The underlying p_{r+1} satisfies the constraints on $\mathcal{G}_{0i\tau}$ imposed after the proof of Lemma 5; but $\Psi(h_0^*, P_{p_{r+1}}) = -1$ as in the proof of Lemma 7. Thus every time we increase r by 1, we halve the order of the left kernel of the restriction of Ψ to (32). Let $H' = Z_1'$ be the direct sum of the W_p' obtained in this way; then the conditions of Lemma 8 are satisfied and Lemma 9 follows. □

It will be seen that we are far from having used the full force of Condition D; indeed we have only used it for those elements of G^* which are not in G_0

but lie in the pull-back of the kernel of the restriction of e_S^* to $W_B' \oplus Z_2'$. Let G_1 be the image of

$$W_B' \oplus Z_2' \sim U_B' \oplus \tau_S Z_2' \hookrightarrow U_B \oplus \tau_S Z_2' \sim G^*$$

where the second isomorphism is that exhibited in the proof of Lemma 9. There is a symmetric bilinear form on G_1 induced by e_S^*; an important part of the proof of Lemma 7 is to show that this form is independent of the $\mathcal{G}_{0i\tau}$ and the choice of η, ζ within a small enough open set, and can be evaluated directly. Let G_2, which can be identified with the smaller geometric 2-Selmer group, be its kernel; then Condition D in the statement and proof of Theorem 2 can be replaced by

Condition E. *The kernel of* $\oplus\delta_{0i\tau}$ *acting on* G_2 *lies in the group generated by* m^0 *and the image of the Mordell-Weil group of* E *over* $\mathbf{Q}(y, z)$.

In view of the remark after the statement of Condition D, the kernel of $\oplus\delta_{oi\tau}$ consists of those triples m such that, for any $\eta \times \zeta$ in a small enough open set, the 2-covering induced by m lies in the 2-Selmer group of (10) whenever that induced by m^0 does. Note however that G_2 does depend on $\eta \times \zeta$, although it is locally constant. A particularly favourable case is when G_2 itself is generated by m^0 and the image of the Mordell-Weil group, so that this condition is trivial; in this case, provided we assume Schinzel's Hypothesis and the finiteness of the Tate-Shafarevich group, a necessary and sufficient condition for the existence of $\lambda \times \mu$ arbitrarily close to $\lambda_0 \times \mu_0$ at which (7) is soluble for $y = \lambda, z = \mu$ is that $\lambda_0 \times \mu_0$ is in $\mathcal{X}$.

Theorem 2 was proved by carrying out a first descent on elements of the pencil (10) so chosen that we have good control over the prime factors of the $c_{ij}(\lambda, \mu)$. Cassels [1] has shown how to perform a second descent for elliptic curves (10) — or more precisely how to determine which elements of the 2-Selmer group survive the second descent. For this purpose he defines a skew-symmetric bilinear form $< ., . >$ on the 2-Selmer group, whose kernel consists of those elements which survive the second descent. Assume that the c_i in (10) are in $\mathbf{Z}$ and let S consist of $2, \infty$ and the odd primes of bad reduction for (10). Let $m^\sharp$ and $m^\flat$ be two triples which generate elements of the 2-Selmer group; thus we can assume that the components of $m^\sharp$ and $m^\flat$ are square-free integers all of whose prime factors lie in S. Now let i, j, k be any cyclic permutation of 1,2,3 and denote by $C_i^\sharp$ the conic

$$m_j^\sharp Z_j^2 - m_k^\sharp Z_k^2 = (c_k - c_j) Z_0^2.$$

$C_i^\sharp$ is an image of the 2-covering associated with $m^\sharp$ and is soluble everywhere locally and therefore globally; let $P_i^\sharp$ be a rational point on $C_i^\sharp$ and let L_i be a linear form in the Z_j such that $L_i^\sharp = 0$ is the tangent to $C_i^\sharp$ at $P_i^\sharp$. For any v in $\mathcal{S}$ let $Q_v^\sharp$ be a v-adic point on the affine 2-covering induced by $m^\sharp$, which is given by (22); we fix a coordinate representation of $Q_v^\sharp$ and, by abuse of language, denote by $L_i^\sharp(Q_v^\sharp)$ the elementof $\mathbf{Q}_v$ obtained by substituting that coordinate representation into $L_i^\sharp$. Then Cassels defines

$$<m^\sharp, m^\flat> = \prod_v \prod_i (L_i^\sharp(Q_v^\sharp), m_i^\flat)_v \tag{33}$$

where the product is taken over all v. However, we can require each $L_i^\sharp$ to have coefficients integral outside $\mathcal{S}$ and with no common factor outside $\mathcal{S}$, and if p is not in $\mathcal{S}$ we can then choose $Q_p^\sharp$ and its representation so that each $L_i^\sharp(Q_p^\sharp)$ is a p-adic unit and therefore the corresponding Hilbert symbols are trivial. It is certainly sometimes possible to implement this algorithm while treating y and z as parameters; see for example §8.2. But we have not been able to do this systematically.

6. *Evaluation of the obstructions.* For any particular surface of the kind we are considering, the application of the ideas of §§3-5 begins in the same way. By writing the surface in the form (6), we fibre it by a suitable pencil of curves (7). For each place in $\mathcal{B}_0$ we identify the local conditions on y/z for (7) to be locally soluble. The simplest way to do this is by first identifying the relevent W_v in the sense of §4 and then seeing which elements of W_v are possible, given the expressions for the m_i^0 as polynomials in y, z. Alternatively, we can fall back on the calculations in [5]. Denote the resulting set of $y \times z$ by $\mathcal{X}_0$. We shall confine ourselves to surfaces for which none of the $-a_i a_j$ is a square, so that for each i, j there is just one $c_{ij\tau}$ which we shall denote by c_{ij}. To identify $\mathcal{X}$, it is enough to evaluate the functions

$$F_{ij}(\lambda, \mu) = L(\mathcal{B}_1; -d_{kj}d_{\ell j}, c_{ij}; \lambda, \mu)$$

on $\mathcal{X}_0$, where i, j, k, ℓ is any permutation of $0, 1, 2, 3$. In view of (18), the non-classical part of F_{ij} has the form $\prod_{v \text{ in } \mathcal{B}_1} (\phi_{ij}, a_i a_j)_v$ where ϕ_{ij} is a linear form in y, z. Thus, as noted after the proof of Lemma 2, the condition $F_{01} = 1$ is classical if and only if $a_0 a_1$ is a square. Because $c_{ij} = c_{k\ell}$, certain combinations of the F_{ij} are in general easier to evaluate than the individual

F_{ij}; for example, if i, j, k is a cyclic permutation of $1, 2, 3$ then

$$F_{0i}F_{jk} = L(\mathcal{B}_1; d_{ij}d_{0k}, c_{0i}) = L(\mathcal{B}_1; -a_0a_k/\theta, c_{0i}) = \prod_{v \text{ in } \mathcal{B}_1} (-a_0a_k/\theta, c_{0i})_v \tag{34}$$

where the second equality follows from (14) and the last one from the Hilbert product formula. Similarly

$$F_{ij}F_{ik}F_{i\ell} = \prod_{v \text{ in } \mathcal{B}_1} \{(d_{jk}, d_{k\ell})_v(d_{k\ell}, d_{\ell j})_v(d_{\ell j}, d_{jk})_v\}. \tag{35}$$

We shall show that each condition $F_{0i}F_{jk} = 1$ and $F_{ij}F_{ik}F_{i\ell} = 1$ is a Brauer-Manin condition, as is $F_{ij} = 1$ when a_ia_j is a square.

Lemma 10 *$F_{ij} = F_{ji}$ for each pair i, j and for all $\lambda \times \mu$ in $\mathcal{Y}$.*

Proof. By (15) we have

$$F_{ij}F_{ji} = L(\mathcal{B}_1; d_{ik}d_{j\ell}d_{i\ell}d_{jk}, c_{ij}) = \prod(-a_ia_j, c_{ij})_p \tag{36}$$

where the product is taken over all primes p outside $\mathcal{B}_1$ which divide $c_{ij}(\lambda, \mu)$. Such p must split in the splitting field of $c_{ij}(y, z)$, which is $\mathbf{Q}(\sqrt{-a_ia_j})$; so $-a_ia_j$ is in $\mathbf{Q}_p^{*2}$ and each factor on the right of (36) is equal to 1. □

Examination of special cases suggests that on $\mathcal{X}_0$ all the $F_{ij}F_{k\ell}$ are equal, as are all the $F_{ij}F_{ik}F_{i\ell}$. These two statements are equivalent. In Lemma 12 below we shall prove the stronger statement that the elements of $\mathrm{Br}(V)/\mathrm{Br}(\mathbf{Q})$ corresponding to any of the $F_{ij}F_{k\ell}$ coming from either pencil are equal. At points at which all the $F_{ij}F_{k\ell} = 1$, it is almost inevitable that the $F_{ij}F_{ik}F_{i\ell}$ coming from the two pencils should take the same value, but at points at which $F_{ij}F_{k\ell} = -1$ there appears to be no relation between the values of the $F_{ij}F_{ik}F_{i\ell}$ for the two pencils.

Ideally we would like to be able to write down all the elements of order 2 in the Brauer group of the surface V given by (1), or, which is equivalent, all quaternion Azumaya algebras on V. But since we do not know how to do this, we fall back on the less ambitious objective of listing those elements which it is easy to write down. For this purpose we have two obvious tools. The first is the following general geometric lemma, which we shall use only in the special case when $k = \mathbf{Q}$ and V is given by (1).

Lemma 11 *Let k be an algebraic number field and V any complete nonsingular surface defined over k having points in every completion of k. Let $K = k(a^{1/2})$ be a quadratic extension of k. For any nonzero function f in $k(V)$ let $\mathcal{A}$ be the quaternion algebra whose norm form is*

$$Z_0^2 - aZ_1^2 - fZ_2^2 + afZ_3^2. \tag{37}$$

Then $\mathcal{A}$ is Azumaya if and only if $(f) = \mathfrak{d}' + \mathfrak{d}''$ where $\mathfrak{d}'$ is a divisor on V defined over K and $\mathfrak{d}''$ is its conjugate over k. Moreover, the class of $\mathcal{A}$ is in $\mathrm{Br}(k)$ *if and only if $\mathfrak{d}'$ can be chosen to be principal.* □

In practice, one starts by choosing $\mathfrak{d}'$ so that $\mathfrak{d}' + \mathfrak{d}''$ is principal. This determines f up to multiplication by a constant, and for calculating the Brauer-Manin obstruction the constant is unimportant because it corresponds to varying the algebra whose norm form is (37) by a constant algebra. Note also that if we replace $\mathfrak{d}'$ by a divisor $\mathfrak{d}'_1$ linearly equivalent to it, so that $\mathfrak{d}'_1 - \mathfrak{d}'$ is the divisor of $g_2 + a^{1/2}g_3$ where g_2 and g_3 are in $k(V)$, then $f_1 = (g_2^2 - ag_3^2)f$ defines the same quaternion algebra and has divisor $\mathfrak{d}'_1 + \mathfrak{d}''_1$. Thus it is actually only the class of $\mathfrak{d}'$ which matters.

To apply the Lemma, we need to know the Néron-Severi group of V over $\mathbf{Q}$ and over quadratic extensions. In what follows, we consider the applications of Lemma 11 when $a_0a_1a_2a_3$ is a square; the case when this condition does not hold is considered in §9. If there are no other relevent relations between the a_i, as is the case in particular for the first, second and fourth surfaces (3), then we are in case 23 of Table 2 of [4]. If we denote by $\mathfrak{a}^\sharp$ or $\mathfrak{a}^\flat$ the divisor of a typical curve of the family (7) or (8) respectively, and by π the divisor of a plane section of V, then $\mathfrak{a}^\sharp + \mathfrak{a}^\flat \sim 2\pi$ and in case 23 the classes of $\mathfrak{a}^\sharp$ and π form a base for the Néron-Severi group. The circumstances in which the Néron-Severi group has more than two generators are cases 1-22 of that table; the third surface (3) is in case 21, the relevent additional condition being $a_1 = 4a_0$. For convenience, we shall continue to assume that none of the $-a_0a_i$ are squares, which as we have already seen is equivalent to saying that the d_{0i} are irreducible over $\mathbf{Q}$. Now take $\theta^2 = a_0a_1a_2a_3$ as in (13) and assume that $a = a_0a_1/\theta$ is not a square in $\mathbf{Q}$; then the Néron-Severi group of V over $K = \mathbf{Q}(a^{1/2})$ is strictly greater than it is over $\mathbf{Q}$, an extra element being the class of the divisor $\mathfrak{c}'_2$ which is the union of the four lines

$$X_0 = c^{1/4}X_2, \quad X_3 = a^{1/2}c^{1/4}X_1 \tag{38}$$

where $c = -a_2/a_0$. If the Néron-Severi group over $\mathbf{Q}$ has only two generators, quadratic extensions like $K/\mathbf{Q}$ are the only ones which give rise to an increase in the Néron-Severi group. The four lines (38) and their conjugates over $\mathbf{Q}$ (which are obtained by inserting a minus sign into the second equation (38)) belong to singular fibres of (7); changing the sign of θ would give us components of singular fibres of (8). Thus $\mathfrak{c}_2'$ consists of a pair of non-intersecting lines from each of two of the degenerate fibres (7).

Denote the divisor $A = B = 0$ by $\mathfrak{a}_{AB}$ and so on. In the language of Lemma 11 we can now take $\mathfrak{d}' = \mathfrak{c}_2' - \mathfrak{a}_{BD}$, which gives $f = c_{02}(D, C)/D^2$. Call the resulting algebra $\mathcal{A}_0$; then $\mathcal{A}_0$ has good reduction outside $\mathcal{B}_1$, so that the Brauer-Manin obstruction reduces to

$$\sum_{v \text{ in } \mathcal{B}_1} \mathrm{inv}_v(\mathcal{A}_0) = 0.$$

Replacing sums by products transforms the left hand side here into $F_{02}F_{13}$ by (34); so we can interpret each condition $F_{ij}F_{k\ell} = 1$ as a Brauer-Manin obstruction. Permuting the subscripts apparently gives us six norm forms (37), and we obtain another six by changing the sign of θ; but actually they all come from the same algebra. For let $\mathfrak{c}_3'$ be the union of the four lines

$$X_0 = d^{1/4}X_3, \quad X_2 = a^{1/2}d^{1/4}X_1 \tag{39}$$

where $d = -a_3/a_0$. Each of the lines (39) meets each of the lines (38); so $\mathfrak{c}_2' + \mathfrak{c}_3'$ is the intersection of V with a quadric, the latter being given by $X_2X_3 = a^{1/2}X_0X_1$. It follows easily that we can replace $\mathfrak{d}'$ by $\mathfrak{d}_1' = \mathfrak{a}_{CD} - \mathfrak{c}_3'$. If we again use $\flat$ to denote objects associated with the fibration (8) then we can take $f_1 = y^{\flat 2}/c_{03}^\flat(y^\flat, z^\flat)$. Hence $F_{02}F_{13} = F_{03}^\flat F_{12}^\flat$. Using symmetry, it follows, as claimed above, that at any adelic point of V the values of all the $F_{ij}F_{k\ell}$ coming from either pencil are equal.

The second tool depends on the concept of the vertical part of the Brauer group of a fibred surface $\pi : V \to \mathbf{P}^1$ defined over k. This was introduced in [3]; it is defined as

$$\mathrm{Br}_{\mathrm{vert}}(V) = \mathrm{Br}(V) \cap \pi^*(\mathrm{Br}(k(\mathbf{P}^1))).$$

These are the algebras which are constant on each irreducible fibre. For a given fibration, $\mathrm{Br}_{\mathrm{vert}}$ is computable, and it is killed by 2 when V is given by (1). For the time being, we suppose merely that V is fibred by curves of genus 1 and the singular fibres have no multiple components, and we take t to be the parameter on $\mathbf{P}^1$. Without loss of generality, we can assume that the fibre at $t = \infty$ is non-singular. Let $t_1, \ldots, t_r$ be a maximal set of elements of $\bar{\mathbf{Q}}$ such that no two t_i are conjugate over $\mathbf{Q}$ and the fibre at $t = t_i$ is reducible; let the W_{ij} denote the absolutely irreducible components of the fibre at $t = t_i$ and let $K_{ij} \supset \mathbf{Q}(t_i)$ be the least field of definition of W_{ij}. In the notation of [3] any element of order 2 in $\mathrm{Br}_{\mathrm{vert}}(V)/\mathrm{Br}(\mathbf{Q})$ can be written as

$$\sum\nolimits_i n_i \mathrm{Cores}_{\mathbf{Q}(t_i)/\mathbf{Q}}(\alpha_i, t - t_i) \tag{40}$$

where each n_i is 0 or 1 and α_i is in $\mathbf{Q}(t_i)^*/\mathbf{Q}(t_i)^{*2}$. We can always force $n_i = 1$ by taking $\alpha_i = 1$ if necessary. For (40) to have good reduction at $t = t_i$ and its conjugates, for each i with $n_i \neq 0$ there must be elements ζ_{0i}, ζ_{1i} of $\mathbf{Q}(t_i)(V)$ which do not vanish identically on any W_{ij} but are such that $\zeta_{0i}^2 - \alpha_i \zeta_{1i}^2$ vanishes identically on the fibre at $t = t_i$. If each W_{ij} contains a point defined over K_{ij}, this is equivalent to requiring α_i to be a square in each K_{ij}. For (40) to be an Azumaya algebra, it must also have good reduction on the fibre at infinity; the condition for this is that

$$\prod\nolimits_i \mathrm{Norm}_{\mathbf{Q}(t_i)/\mathbf{Q}}(\alpha_i^{n_i}) \text{ is in } \mathbf{Q}^{*2}. \tag{41}$$

Now let f_i, g_i be homogeneous polynomials in $\mathbf{Q}[y, z]$ with f_i irreducible and $\deg g_i$ even, such that

$$f_i(t_i, 1) = 0, \quad \alpha_i = g_i(t_i, 1).$$

If $\mathcal{B}$ is the set of bad primes, in a sense which we leave the reader to supply, then it follows from Lemma 7.2.1 of [2] that the Brauer-Manin condition associated with the Azumaya algebra (40) is

$$\prod\nolimits_i (L(\mathcal{B}; f_i, g_i))^{n_i} = 1. \tag{42}$$

This is the machinery which, as we shall see below, identifies the obstructions $F_{ij}F_{k\ell} = 1$ and $F_{ij}F_{ik}F_{i\ell} = 1$ with Brauer-Manin obstructions.

Lemma 12 *Suppose that V is given by* (1) *with $a_0a_1a_2a_3$ square and that none of the $-a_ia_j$ is a square. Then the group* $\mathrm{Br}_{\mathrm{vert}}(V)/\mathrm{Br}(\mathbf{Q})$ *for either pencil* (7) *or* (8) *is non-cyclic of order* 4. *The intersection of the two groups corresponding to* (7) *and* (8) *has order* 2 *and its nontrivial element is the class of the $\mathcal{A}_0$ defined above.*

Proof. Suppose initially that the fibration $V \to \mathbf{P}^1$ is that associated with the pencil (7). In the notation of §2 we can now take $t = y/z$; and we number the t_i so that $d_{0i}(t_i, 1) = 0$. For fixed i there are four W_{ij}, each of which is a straight line; and the four fields K_{ij} are all equal and are biquadratic extensions of $\mathbf{Q}(t_i)$. Explicitly, if we write $u_i = (-a_0a_i)^{-1/2}$ then

$$\begin{aligned} t_1 &= u_1, & K_{1j} &= \mathbf{Q}(u_1, \sqrt{a_1a_3\theta}, \sqrt{a_1u_1}); \\ t_2 &= (r - a_2u_2)/a_1s, & K_{2j} &= \mathbf{Q}(u_2, \sqrt{a_2a_1\theta}, \sqrt{a_2u_2}); \\ t_3 &= (-a_1s + \theta u_3)/a_0a_1r, & K_{3j} &= \mathbf{Q}(u_3, \sqrt{a_3a_2\theta}, \sqrt{a_3u_3}). \end{aligned}$$

Thus for example α_1 in (40) must have the value $1, a_1a_3\theta, a_1u_1$ or $a_3u_1\theta$. It only remains to satisfy the condition (41). This is satisfied by any one of the three elements

$$\mathrm{Cores}_{\mathbf{Q}(t_i)/\mathbf{Q}}(a_ia_k\theta, t - t_i) \tag{43}$$

where i, j, k is a cyclic permutation of $1, 2, 3$. By (42) and (34), the corresponding Brauer-Manin condition is $F_{0i}F_{jk} = 1$. In looking for the remaining elements of a base for $\mathrm{Br}_{\mathrm{vert}}$ we can restrict each α_i to take the value 1 or a_iu_i, with not all α_i being 1. Now we always obtain the additional possibility

$$\sum_i \mathrm{Cores}_{\mathbf{Q}(t_i)/\mathbf{Q}}(a_iu_i, t - t_i); \tag{44}$$

and if for example a_0a_1 is a square we also have

$$\mathrm{Cores}_{\mathbf{Q}(t_1)/\mathbf{Q}}(a_1u_1, t - t_1) \tag{45}$$

But the value of $-d_{02}d_{12}$ at $(t_1, 1)$ is $4(a_0ru_1 + s)^2a_1u_1$ and therefore the Brauer-Manin condition associated with (45) is $F_{23} = 1$. Similarly the condition associated with (44) is $F_{23}F_{31}F_{12} = 1$. Thus the Brauer-Manin conditions arising from the elements of $\mathrm{Br}_{\mathrm{vert}}$ are just the obstructions already obtained at the beginning of this section.

We have already seen that $\mathcal{A}_0$ belongs to $\mathrm{Br}_{\mathrm{vert}}$, and by comparing the induced obstructions we see that $\mathcal{A}_0$ must be equal to (43) with $i = 2$. But permutation of subscripts and replacing the pencil (7) by the pencil (8) are both operations which leave the class of $\mathcal{A}_0$ unchanged; so the three algebras (43) and the corresponding algebras associated with (8) all lie in the same class in $\mathrm{Br}(V)/\mathrm{Br}(\mathbf{Q})$. It only remains to show that in general the algebra corresponding to (44) for (8) is not in the $\mathrm{Br}_{\mathrm{vert}}$ associated with (7). But if it were, this would still be true if a_0, a_1, a_2 and θ were independent transcendentals over $\mathbf{Q}$ and we worked over $K = \mathbf{Q}(a_0, a_1, a_2, \theta)$ instead of over $\mathbf{Q}$. If this algebra was in the class of the algebra corresponding to (44) for (7), then for any a_0, a_1, a_2, θ in $\mathbf{Q}^*$ the values of the $F_{12}F_{23}F_{31}$ associated with the pencils (7) and (8) would be equal, and we shall see below that this is untrue for (49). If instead the classes of the algebras corresponding to (44) for (7) and (8) are different then they must differ by the class of $\mathcal{A}_0$, and for any a_0, a_1, a_2, θ in $\mathbf{Q}^*$ the values of $F_{12}F_{23}F_{31}$ for one pencil and of $F_{01}F_{02}F_{03}$ for the other would be equal. This fails for (52). □

It may appear surprising, particularly in view of the terminology, that there can be a non-constant algebra which belongs to $\mathrm{Br}_{\mathrm{vert}}$ for more than one fibration. Indeed, the fact that the two systems of fibres cross creates a large subset of V on which $\mathcal{A}_0$ is constant; but this subset need not be the whole of V. For example, on the surface (52) the 3-adic points of a fibre of either system lie either all in $3|X_0$ or all in $3|X_3$.

7. *The special surfaces* (3). In this section, we apply the methodology outlined at the beginning of the previous section to the first, second and fourth of the surfaces (3). For each of these surfaces, we are able to choose the pencil (7) so that $\mathcal{B}_1 = \mathcal{B}_0$. Because of its special features, we postpone consideration of the third surface (3) to §8.

The first of the surfaces (3) is

$$4X_0^4 + 9X_1^4 = 8X_2^4 + 8X_3^4 \tag{46}$$

and we have $\mathcal{B}_0 = \{2, 3, \infty\}$. It is easy to see that all primitive solutions must satisfy $4|X_0$ and $2\|X_1$, with X_2, X_3 odd. Similarly, none of X_0, X_2 and X_3 can be divisible by 3. We can take the formulation (6) to be

$$\begin{aligned}4(X_0^2 + X_2^2 + X_3^2)(X_0^2 - X_2^2 - X_3^2)\\ = -(3X_1^2 - 2X_2^2 + 2X_3^2)(3X_1^2 + 2X_2^2 - 2X_3^2);\end{aligned}$$

thus the pencil (7) can be taken in the form

$$\left.\begin{aligned}2y(X_0^2 + X_2^2 + X_3^2) + z(3X_1^2 - 2X_2^2 + 2X_3^2) = 0,\\ y(3X_1^2 + 2X_2^2 - 2X_3^2) - 2z(X_0^2 - X_2^2 - X_3^2) = 0.\end{aligned}\right\} \tag{47}$$

Now the d_{ij} are given by

$$\begin{aligned}d_{01} &= 6(y^2 + z^2), & d_{23} &= -8(y^2 + z^2),\\ d_{02} &= 4(y^2 + 2yz - z^2), & d_{31} &= 6(y^2 + 2yz - z^2),\\ d_{03} &= -4(y^2 - 2yz - z^2), & d_{12} &= -6(y^2 - 2yz - z^2).\end{aligned}$$

In $\mathbf{R}$ we have $c_2 > c_1 > c_3$, so (47) is soluble in $\mathbf{R}$ if and only if $m_3^0 > 0$ in the notation of (23); this is equivalent to

$$y^2 + 2yz - z^2 = (y+z)^2 - 2z^2 < 0. \tag{48}$$

For solubility in $\mathbf{Q}_2$ the statements above for the X_i imply that $y \equiv z \bmod 4$, whence W_2 is generated by the classes of the three triples like (24), together with the class of (5,7,3). But m_1^0 is in $\mathbf{Q}_2^{*2}$, so that for (47) to be soluble in $\mathbf{Q}_2$ is equivalent to m_2^0 being in $\mathbf{Q}_2^{*2}$, and therefore to $4 \| (y-z)$. Similarly W_3 is generated by the classes of the three triples like (24). But m_2^0 and m_3^0 are both divisible by odd powers of 3, so m_1^0 is in $-\mathbf{Q}_3^{*2}$ and m_2^0 is in $3\mathbf{Q}_3^{*2}$ and hence $3|z$. Now all the $F_{ij}F_{k\ell} = 1$ on $\mathcal{X}_0$, but all the $F_{ij}F_{ik}F_{i\ell} = -1$ there; so (46) is insoluble. We could have used the pencil (8) instead of (7); but the effect of this would simply be to interchange the roles of X_2 and X_3, and the subsequent argument would be essentially the same.

The second surface (3) is

$$2X_0^4 + 9X_1^4 = 6X_2^4 + 12X_3^4, \tag{49}$$

for which again $\mathcal{B}_0 = \{2, 3, \infty\}$. All primitive solutions must have X_0, X_2, X_3 odd and $2\|X_1$; and $3|X_0$ but X_2, X_3 are prime to 3. We can take the formulation (6) to be

$$\begin{aligned} &2(X_0^2 - X_2^2 - 2X_3^2)(X_0^2 + X_2^2 + 2X_3^2) \\ &\quad = -(3X_1^2 - 2X_2^2 + 2X_3^2)(3X_1^2 + 2X_2^2 - 2X_3^2); \end{aligned}$$

thus the pencil (7) can be taken in the form

$$\left.\begin{aligned} 2y(X_0^2 - X_2^2 - 2X_3^2) + z(3X_1^2 - 2X_2^2 + 2X_3^2) = 0, \\ y(3X_1^2 + 2X_2^2 - 2X_3^2) - z(X_0^2 + X_2^2 + 2X_3^2) = 0. \end{aligned}\right\} \tag{50}$$

Now the d_{ij} are given by

$$\begin{aligned} d_{01} &= 3(2y^2 + z^2), & d_{23} &= 6(2y^2 + z^2), \\ d_{02} &= 2(2y^2 - 2yz - z^2), & d_{31} &= -6(2y^2 - 2yz - z^2), \\ d_{03} &= -2(2y^2 + 4yz - z^2), & d_{12} &= 3(2y^2 + 4yz - z^2). \end{aligned}$$

Arguments similar to those which we used for (47) show that (50) is soluble in $\mathbf{R}$ if and only if $m_2^0 > 0$, which is to say

$$2y^2 + 4yz - z^2 = 2(y+z)^2 - 3z^2 > 0.$$

For solubility in $\mathbf{Q}_2$ the statements above for the X_i imply that $y+z \equiv 0 \bmod 4$; and a full analysis shows that this condition is sufficient as well as necessary. The analysis of solubility in $\mathbf{Q}_3$ is more tedious. W_3 is again generated by the classes of the three triples like (24); this is the same as saying that m_1 is in $\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$ and m_3 is in $\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$. But

$$m_1^0 = -18\{3y^2 - (y+z)^2\}\{2(y+z)^2 - 3z^2\}, \tag{51}$$

so $3 \not| (y+z)$ and consideration of m_3^0 shows that

$$2y^2 + z^2 \text{ is in } \mathbf{Q}_3^{*2} \text{ or } 6\mathbf{Q}_3^{*2};$$

this incidentally implies that $3 \not| z$. Now all the $F_{ij}F_{k\ell} = -1$ on $\mathcal{X}_0$, and all the $F_{ij}F_{ik}F_{i\ell} = -1$ there; so (49) is insoluble.

We could instead have used the pencil (8), which it is convenient to write in the form

$$\left.\begin{aligned} -2y(X_0^2 + X_2^2 + 2X_3^2) + z(3X_1^2 - 2X_2^2 + 2X_3^2) = 0,\\ y(3X_1^2 + 2X_2^2 - 2X_3^2) + z(X_0^2 - X_2^2 - 2X_3^2) = 0. \end{aligned}\right\}$$

Now the d_{ij} are given by

$$\begin{aligned} d_{01} &= -3(2y^2 + z^2), & d_{23} &= 6(2y^2 + z^2),\\ d_{02} &= -2(2y^2 - 2yz - z^2), & d_{31} &= -6(2y^2 - 2yz - z^2),\\ d_{03} &= 2(2y^2 + 4yz - z^2), & d_{12} &= 3(2y^2 + 4yz - z^2), \end{aligned}$$

the difference from the values derived from (50) being that the signs of the d_{0i} have been reversed. Solubility in $\mathbf{R}$ now requires

$$2y^2 - 2yz - z^2 = 3y^2 - (y+z)^2 < 0$$

and solubility in $\mathbf{Q}_2$ requires $2y + z \equiv 0 \bmod 8$. The condition that m should be in W_3 is that m_1 should be in $\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$ and m_2 should be in $\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$; since (51) still holds, we again have $3 \not| (y+z)$, but this time consideration of m_2^0 shows that

$$2y^2 + z^2 \text{ is in } \mathbf{Q}_3^{*2} \text{ or } 3\mathbf{Q}_3^{*2}$$

whence again $3 \not| z$. Again all the $F_{ij}F_{k\ell} = -1$ on $\mathcal{X}_0$, but this time all the $F_{ij}F_{ik}F_{i\ell} = 1$.

The fourth surface (3) is

$$7X_0^4 + 8X_1^4 = 9X_2^4 + 14X_3^4 \tag{52}$$

for which $\mathcal{B}_0 = \{2, 3, 7, \infty\}$. All primitive solutions have X_0, X_2, X_3 odd and X_1 even; and either $3|X_0$, $9|(X_1^2 + 2X_3^2)$ or $3|X_3$, $9|(X_0^2 + 2X_1^2)$. We can take the formulation (6) to be

$$\begin{aligned}(7X_0^2 + 4X_1^2 + 9X_2^2)&(7X_0^2 + 4X_1^2 - 9X_2^2)\\ &= -14(X_0^2 - 2X_1^2 + 3X_3^2)(X_0^2 - 2X_1^2 - 3X_3^2);\end{aligned}$$

thus the pencil (7) can be taken in the form

$$\left.\begin{aligned} y(7X_0^2 + 4X_1^2 + 9X_2^2) + 7z(X_0^2 - 2X_1^2 + 3X_3^2) = 0,\\ 2y(X_0^2 - 2X_1^2 - 3X_3^2) - z(7X_0^2 + 4X_1^2 - 9X_2^2) = 0.\end{aligned}\right\} \qquad (53)$$

Now the d_{ij} are given by

$$\begin{aligned} d_{01} &= -18(2y^2 + 7z^2), & d_{23} &= -27(2y^2 + 7z^2),\\ d_{02} &= -9(2y^2 - 14yz - 7z^2), & d_{31} &= 12(2y^2 - 14yz - 7z^2),\\ d_{03} &= -21(2y^2 + 4yz - 7z^2), & d_{12} &= 18(2y^2 + 4yz - 7z^2).\end{aligned}$$

In $\mathbf{R}$ we have $c_3 > c_1 > c_2$, so (53) is soluble in $\mathbf{R}$ if and only if $m_2^0 > 0$ in the notation of (23); this is equivalent to

$$2y^2 + 4yz - 7z^2 = 2(y + z)^2 - 9z^2 < 0.$$

For solubility in $\mathbf{Q}_2$ the statements above for the X_i imply that $2\|z$. Hence $2^5\|m_1^0$, $2^3\|m_2^0$, $2^4\|m_3^0$ and comparison with W_2 implies that m_1^0 is in $2\mathbf{Q}_2^{*2}$ and m_2^0 in $10\mathbf{Q}_2^{*2}$ or m_1^0 is in $6\mathbf{Q}_2^{*2}$ and m_2^0 in $2\mathbf{Q}_2^{*2}$. Consideration of d_{12} and d_{23} shows that only the first case is possible, and that $2y \equiv z \bmod 8$. Again, the statements above for the X_i imply that if $3|X_0$ then $9|(z - 2y)$, whereas if $3|X_3$ then $9|(y + z)$; and m in W_3 is equivalent to m_1 in $\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$ and m_2 in $\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$. Now if $9|(z - 2y)$ then $2y^2 + 4yz - 7z^2$ is in $\mathbf{Q}_3^{*2}$ and $2y^2 + 7z^2$ in $3\mathbf{Q}_3^{*2}$, and hence $2y^2 - 14yz - 7z^2$ must be in $2\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$; but if $9|(y + z)$ then $2y^2 + 4yz - 7z^2$ is in $2\mathbf{Q}_3^{*2}$ and $2y^2 - 14yz - 7z^2$ in $\mathbf{Q}_3^{*2}$, and hence $2y^2 + 7z^2$ must be in $\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$. Finally, m in W_7 is equivalent to m_3^0 in $\mathbf{Q}_7^{*2}$ and hence to $7|y$. Now all the $F_{ij}F_{k\ell}$ are equal to 1 if $9|(z - 2y)$ and to -1 if $9|(y + z)$; and in either case all the $F_{ij}F_{ik}F_{i\ell} = 1$. Thus we have a failure of weak approximation, in that (52) has no primitive solutions with $3|X_3$. On the other hand, Theorem 2 shows that it does have primitive solutions with $3|X_0$. To find one, we can proceed as follows. On the part of $\mathcal{X}_0$ on which $9|(z - 2y)$ we have

$$F_{01} = (-2, y)_2(7, z)_7, \quad F_{03} = (-2, y)_2(-1, z)_\infty. \qquad (54)$$

Thus $F_{01} = F_{03} = 1$ is an additional condition on $\mathcal{X}$. It follows from Lemma 2 that Condition D holds. The larger geometric 2-Selmer group has order 2^5; its quotient by the subgroup generated by m^0 and the triples like (24) consists of the classes generated by $(1,1,1)$, $(1,-7d_{01},-7d_{01})$, $(-42d_{02},1,-42d_{02})$ and $(42d_{03},42d_{03},1)$. The simplest way to proceed is to require that each d_{0i} is (up to sign) the product of a prime p_{0i} and powers of primes from $\mathcal{B}_1$, and that none of these last three triples lies in G_2. For $(1,-7d_{01},-7d_{01})$ not to lie in G_2, for example, is equivalent to $-7d_{01}$ being a quadratic non-residue $\bmod p_{02}$ or $\bmod p_{03}$. The six conditions arising in this way are equivalent on $\mathcal{X}$ and each of them reduces to $(2,y)_2 = 1$. Combining this with (54) and reversing the signs of y, z if necessary, we find that we need

$$y \equiv 1 \bmod 8, \ z \text{ in } \mathbf{Q}_7^{*2}, \ z > 0.$$

An easy search program yields pairs (y,z) which satisfy all these conditions; there are just three such pairs with $|y|$ and $|z|$ less than 500, and they are

$$(-7,58), \ (161,394) \text{ and } (-175,442).$$

In each of these cases, standard conjectures imply that (53) is soluble in Z, that the Mordell-Weil group of its Jacobian has rank 1 and that a non-trivial point on the Jacobian is given by Heegner's recipe. This recipe is constructive, but the calculations involved are excessively lengthy. However, the Jacobian of the curve derived from the first of these pairs can be handled by one of the standard packages for elliptic curves. In this way we obtain the solution

$$(5145, 18832, 11843, 15623)$$

of (52).

We could instead have used the pencil (8), which it is convenient to write in the form

$$\left.\begin{array}{l} y(7X_0^2 + 4X_1^2 + 9X_2^2) + 7z(X_0^2 - 2X_1^2 - 3X_3^2) = 0, \\ 2y(X_0^2 - 2X_1^2 + 3X_3^2) - z(7X_0^2 + 4X_1^2 - 9X_2^2) = 0. \end{array}\right\}$$

Now the d_{ij} are given by

$$\begin{array}{ll} d_{01} = -18(2y^2 + 7z^2), & d_{23} = 27(2y^2 + 7z^2), \\ d_{02} = -9(2y^2 - 14yz - 7z^2), & d_{31} = -12(2y^2 - 14yz - 7z^2), \\ d_{03} = 21(2y^2 + 4yz - 7z^2), & d_{12} = 18(2y^2 + 4yz - 7z^2), \end{array}$$

the difference from the values derived from (53) being that the signs of the d_{i3} have been reversed. Solubility in $\mathbf{R}$ now requires

$$2y^2 - 14yz - 7z^2 = 9y^2 - 7(y+z)^2 < 0$$

and solubility in $\mathbf{Q}_2$ requires $4\|z$. This time, if $3|X_0$ then $9|(y-2z)$ whereas if $3|X_3$ then $9|(y+z)$; and m in W_3 is equivalent to m_1 in $\mathbf{Q}_3^{*2}$ or $6\mathbf{Q}_3^{*2}$ and m_2 in $\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$. If $9|(y-2z)$ then $2y^2+4yz-7z^2$ is in $\mathbf{Q}_3^{*2}$ and $2y^2+7z^2$ in $6\mathbf{Q}_3^{*2}$, and hence $2y^2-14yz-7z^2$ is in $2\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$; but if $9|(y+z)$ then $2y^2+4yz-7z^2$ is in $2\mathbf{Q}_3^{*2}$ and $2y^2-14yz-7z^2$ in $\mathbf{Q}_3^{*2}$, and hence $2y^2+7z^2$ is in $\mathbf{Q}_3^{*2}$ or $3\mathbf{Q}_3^{*2}$. Finally, direct calculation shows that solubility in $\mathbf{Q}_7$ requires $7|y$ or $7|(y+2z)$. By considering W_7 we see that if $7|y$ then m_3^0 is in $\mathbf{Q}_7^{*2}$, whence m_1^0 is divisible by an even power of 7 and so $2y^2+4yz-7z^2$ is divisible by an odd power of 7; but if $7|(y+2z)$ then m_3^0 is in $-\mathbf{Q}_7^{*2}$, whence m_1^0 is divisible by an odd power of 7 and so is $2y^2+4yz-7z^2$. Now all the $F_{ij}F_{k\ell}$ are equal to 1 if $9|(y-2z)$ and to -1 if $9|(y+z)$; and all the $F_{ij}F_{ik}F_{i\ell}=1$.

8. *The case* $a_1 = 4a_0$. This is one of the simplest cases in which the Néron-Severi group has rank greater than 2, and we treat it in greater depth. For the reader's convenience, a brief introduction is followed by three subsections. The first of these applies the ideas of §§3-5 to prove all but the last sentence of Theorem 1. The second proves the last sentence of Theorem 1, and its Corollary, using the second descent algorithm of Cassels as described at the end of §5. The third is primarily concerned with the geometry.

The most general surface (1) with $a_0a_1a_2a_3$ square and $a_1 = 4a_0$ can be written in the form

$$X_0^4 + 4X_1^4 = du^2X_2^4 + dw^2X_3^4 \tag{55}$$

where d is fourth-power-free and u, w are square-free, positive and coprime. In accordance with our standard requirement that no $-a_ia_j$ is a square, we shall assume that d is not a square. We shall also assume that neither duw nor uw is in $2\mathbf{Q}^{*2}$, since otherwise the rank of the Néron-Severi group of (55) would exceed 3. The set $\mathcal{B}_0$ consists of $2, \infty$ and the odd primes which divide duw. Write $d = r^2+s^2$ where r, s are not divisible by any odd prime factor of d; then we can take the formulation (6) to be

$$\begin{aligned}(X_0^2 + urX_2^2 + wsX_3^2)&(X_0^2 - urX_2^2 - wsX_3^2)\\ &= -(2X_1^2 + usX_2^2 - wrX_3^2)(2X_1^2 - usX_2^2 + wrX_3^2).\end{aligned}$$

The pencil (7) now takes the form

$$\left.\begin{array}{l} y(X_0^2 + urX_2^2 + wsX_3^2) + z(2X_1^2 + usX_2^2 - wrX_3^2) = 0, \\ y(2X_1^2 - usX_2^2 + wrX_3^2) - z(X_0^2 - urX_2^2 - wsX_3^2) = 0. \end{array}\right\} \tag{56}$$

As in §6, we shall write π for the class of a plane section of (55), and $\mathfrak{a}^\sharp$ for the class of a curve (56); and we shall write $\mathfrak{b}_1 = \pi - \mathfrak{a}^\sharp$. The pencil (8) can be obtained from (56) by interchanging r and s, u and w, and X_2 and X_3, and the class of any curve in this pencil is $\pi + \mathfrak{b}_1$. The d_{ij} for (56) are given by

$$\begin{array}{ll} d_{01} = 2(y^2 + z^2), & d_{23} = duw(y^2 + z^2), \\ d_{02} = -u(sy^2 - 2ryz - sz^2), & d_{31} = 2w(sy^2 - 2ryz - sz^2), \\ d_{03} = w(ry^2 + 2syz - rz^2), & d_{12} = -2u(ry^2 + 2syz - rz^2). \end{array}$$

For convenience we shall write

$$c_{01} = y^2 + z^2, \ c_{02} = sy^2 - 2ryz - sz^2, \ c_{03} = ry^2 + 2syz - rz^2.$$

Here $\mathcal{B}_1$ is the same as $\mathcal{B}_0$. To obtain $\mathcal{B}$ we have to adjoin $p = 3$ if $3|rs$ and $p = 5$ if $5|rs$; for in each of these cases p will divide the value of one of the c_{0i} whatever the values of r, s.

We note the curious fact that $F_{23} = 1$ identically; for the identity

$$c_{02}c_{03} = 4yz(sy - rz)^2 + (y^2 + z^2)(rsy^2 - 2dyz + rsz^2)$$

implies

$$F_{23} = L(-d_{03}d_{13}, c_{01}) = L(2yz, y^2 + z^2) = 1.$$

Thus the additional Brauer-Manin condition associated with a_0a_1 being a square is now trivial.

To avoid an intolerable profusion of cases, we shall henceforth assume that $u = w = 1$.

8.1 *The first descent.* After a change of variables if necessary, we can assume that $4 \nmid d$. Local solubility requires that $d > 0$, that $d \equiv 1, 2, 5$ or $10 \bmod 16$ and that d is not divisible by any prime $p \equiv 3 \bmod 4$. In $\mathbf{R}$ we have $c_3 > c_1 > c_2$, so (56) is soluble in $\mathbf{R}$ if and only if $m_2^0 > 0$ in the notation of (23); this is equivalent to

$$c_{03} = ry^2 + 2syz - rz^2 < 0. \tag{57}$$

The conditions for solubility in $\mathbf{Q}_2$ are obtained in the same way as in the examples in §7, though one has to consider separately the possible classes for d in $\mathbf{Q}^*/\mathbf{Q}^{*4}$. The results are given in the following table, in which d is an element of $\mathbf{Q}^*/\mathbf{Q}^{*4}$ and the m_i^0 are elements of $\mathbf{Q}^*/\mathbf{Q}^{*2}$:

d	rules for r, s	m_1^0	m_2^0	m_3^0
1	$4\|r, 8\|(s-1)$	1, 2, 7, 14	2	1, 2, 7, 14
		5, 7, 10, 14	1, 2, 3, 6	14
5	$2\|\|r, 8\|(s-1)$	2, 14	10	3, 5
		10, 14	5, 7	6
2, 18	$4\|(r-1), 8\|(r-s)$	1	1	1
10, 26	$8\|(r-1), 8\|(s-5)$	5	7	3

Here the first and third rows correspond to X_2 odd and $4|(y+z)$, the second and fourth to X_3 odd and $2|y$, the fifth to $4|(y+z)$ and the sixth to $4|(y-z)$.

Now let $p \equiv 1 \bmod 4$ be an odd prime which divides d, and define ν by $p^\nu \| d$. If X_0, X_1 are not divisible by p then it follows from (55) that

$$(rX_0^2 - 2sX_1^2)(rX_0^2 + 2sX_1^2) \equiv 0 \bmod p^\nu;$$

and (56) implies that

$$y(rX_0^2 - 2sX_1^2 + dX_2^2) + z(sX_0^2 + 2rX_1^2 - dX_3^2) = 0.$$

If $p^\nu|(rX_0^2+2sX_1^2)$ then also $p^\nu|(sX_0^2-2rX_1^2)$ and so $p^\nu|(ry+sz)$. If conversely $p|(ry+sz)$ then it follows from the first equation (56) that $p|(sX_0^2 - 2rX_1^2)$; so $p^\nu|(rX_0^2 - 2sX_1^2)$ implies $p \nmid (ry+sz)$ and therefore $p^\nu|(ry-sz)$. If X_0, X_1 are divisible by p, which can only happen if $p \equiv 1 \bmod 8$, then similarly $p^{4-\nu}|(rX_2^2 - sX_3^2)$ implies $p^\mu|(ry+sz)$ where $\mu = \min(2,\nu)$; on the other hand, $p^{4-\nu}|(rX_2^2 + sX_3^2)$ implies $p^\lambda \nmid (ry+sz)$ where $\lambda = \max(1, \nu-1)$. But

$$rsm_1^0 = -4\{dy^2 - (ry+sz)^2\}\{(ry+sz)^2 - dz^2\}$$

and we have just shown that the two terms inside either curly bracket cannot be divisible by the same power of p. Since also $2rs \equiv (r+s)^2 \bmod p$, it follows that m_1^0 is in $2\mathbf{Q}_p^{*2}$. If $p \equiv 5 \bmod 8$ then W_p is generated by the three triples like (24), and it follows that one of m_2^0 and m_3^0 is in $d\mathbf{Q}_p^{*2}$ and the other is in $2d\mathbf{Q}_p^{*2}$; but if $p \equiv 1 \bmod 8$ it can be shown that solubility in $\mathbf{Q}_p$ yields no further information about m_2^0 and m_3^0.

The obstructions imposed by $F_{01}F_{23} = 1$ and $F_{01}F_{02}F_{03} = 1$ can be calculated by means of (34) and (35). In the first condition, the contribution from $v = \infty$ is 1, while the contribution from $v = 2$ is 1 if $8|(y \pm z)$ or $4|y$ and -1 otherwise. If p is an odd prime which divides d, then the contribution from $v = p$ is 1 if $p \equiv 1 \bmod 8$ or if $p \equiv 5 \bmod 8$ and $(y^2 + z^2)$ is divisible by an even power of p, and -1 otherwise. It can now be verified that the only case in which $F_{01}F_{23} = -1$ for all allowable y, z is when $d \equiv 2 \bmod 16$, no prime $p \equiv 5 \bmod 8$ divides d to an odd power and $r \equiv 5 \bmod 8$. The smallest such example is when $d = 34$. There is in general more than one way of writing d as $r^2 + s^2$; but let $d = 2\prod q_\nu$ where the q_ν are prime powers coprime to each other and $q_\nu = r_\nu^2 + s_\nu^2$ with r_ν, s_ν prime to q_ν. Then $r \equiv 5 \bmod 8$ if and only if the number of q_ν with $r_\nu + s_\nu \equiv \pm 5 \bmod 8$ is odd. As to the second condition, the contribution to (35) from $v = \infty$ is 1 and that from $v = 2$ is 1 if $d \equiv 1$ or $2 \bmod 16$ and -1 otherwise. The contribution from $v = p$ with p odd is -1 if $p \equiv 5 \bmod 8$ and $p^\nu \| d$ with ν odd; and it is 1 in all other cases. Hence $F_{01}F_{02}F_{03} = 1$ always. Using (18) we can obtain, with a slight abuse of notation

$$F_{02} = L(-2dr, s)\prod\nolimits_{v \text{ in } \mathcal{B}}\{(ry + sz, d)_v\,(y, -1)_v\,(-2y(ry + sz), sc_{02})_v\}.$$

It follows that (in the topology induced by $\mathcal{B}$) it does not depend continuously on y/z alone. Hence it cannot provide a decisive obstruction to solubility. As for Condition D or E, an argument like that in the proof of Lemma 2 shows that if d is not in $\pm\mathbf{Q}^{*2}$ or $\pm 2\mathbf{Q}^{*2}$ then the kernel of $\oplus\delta_{0i\tau}$ has just one extra generator, which can be taken to be $(2, 2, 1)$. The corresponding 2-covering is insoluble at some place in $\mathcal{B}$ if and only if d is even or d is divisible to an odd power by some prime $p \equiv 5 \bmod 8$; in this case Condition E holds, though Condition D does not. This proves all but the last sentence of Theorem 1.

8.2 *The second descent.* If the 2-covering corresponding to (2,2,1) is soluble at every place of $\mathcal{B}$, then we can draw no conclusion from a first descent; however, we can use the ideas of Cassels [1], as outlined at the end of §5, to perform a second descent. Here we have $d \equiv 1 \bmod 16$ and d is not divisible to an odd power by any $p \equiv 5 \bmod 8$; and we wish to prove, under some additional conditions, that Γ is insoluble.

It follows from

$$X_0^4 + 4X_1^4 = (X_0^2 + 2X_0X_1 + 2X_1^2)(X_0^2 - 2X_0X_1 + 2X_1^2),$$

where each factor on the right must be congruent to 1 mod 8 because it divides $X_2^4 + X_3^4$, that $4|X_1$. It is clearly enough to look for solutions with X_3 even. Now elementary algebra shows that

$$8|(y+z), \quad c_{01} \equiv 2 \bmod 16, \quad c_{03} \equiv -2 \bmod 16;$$

moreover if $2\|X_3$ then $4\|r$ and $c_{02} \equiv -8 \bmod 64$, whereas if $4|X_3$ then $8|r$ and $2^5|c_{02}$.

Let $\eta \times \zeta$ be any element of $\mathcal{Y}$ satisfying the conditions of the Corollary to Lemma 5. If we take $m^\sharp$ to be (2,2,1), we can choose the $P_i^\sharp$ as follows:

$P_1^\sharp$ has $Z_2 = \xi_2(\eta^2+\zeta^2), Z_3 = 2\xi_3(\eta^2+\zeta^2)$ where $\xi_2^2 - 2\xi_3^2 = d$,
$P_2^\sharp$ has $Z_3 = 0, Z_1 = s\eta^2 - 2r\eta\zeta - s\zeta^2$,
$P_3^\sharp$ has $Z_1 = 0, Z_2 = r\eta^2 + 2s\eta\zeta - r\zeta^2$.

The eventual value of $< m^\sharp, m^\flat >$ will not depend on the choice of ξ_2, ξ_3. The corresponding $L_i^\sharp$ can be taken to be

$$L_1^\sharp = 2\xi_2 Z_2 - 2\xi_3 Z_3 - 2d(\eta^2+\zeta^2),$$
$$L_2^\sharp = Z_1 - (s\eta^2 - 2r\eta\zeta - s\zeta^2),$$
$$L_3^\sharp = Z_2 - (r\eta^2 + 2s\eta\zeta - r\zeta^2).$$

We now take $m^\flat = m^0$ and evaluate $< m^\sharp, m^\flat >$ by means of (33). For the rest of this paragraph and the whole of the next one, the arguments of the c_{0i} and m_i^0 will be either y, z or η, ζ according to context. The conditions which we shall impose on the points $Q_v^\sharp$, and the resulting contributions to the product (33), are as follows.

$$Q_2^\sharp = Q_\infty^\sharp = (-c_{02}(\eta,\zeta), d^{1/2}c_{01}(\eta,\zeta), 0),$$

the contribution from $v = \infty$ is $(\xi_2, c_{02})_\infty$ and the contribution from $v = 2$ is

$$(-2(d^{1/2}\xi_2 - d), m_1^0)_2\,(2c_{03}(d^{1/2}c_{01} - c_{03}), m_3^0)_2 \qquad (58)$$

where $d^{1/2}$ can take either value in $\mathbf{Q}_2$. Changing the value of $d^{1/2}$ reverses the sign of each factor in (58); if we take $d^{1/2} \equiv 1 \bmod 8$ then the second factor reduces to $(3, m_3^0)_2$. If $p|d$ and $p \equiv 1 \bmod 8$ then we can take each $p^2 Z_i$ to be a p-adic unit and the contribution to the product is 1. If $p|d$ and $p \equiv 5 \bmod 8$ then $p^2\|d$. Now we can take

$$Z_2 = -\xi_2 c_{01}(\eta,\zeta), \quad Z_3 = 2\xi_3 c_{01}(\eta,\zeta)$$

because $p|\xi_3$ and we can deduce from

$$c_{02}^2 + c_{03}^2 = dc_{01}^2 \tag{59}$$

and the known properties of the m_i^0 that c_{01} is divisible by at least as high a power of p as c_{02} or c_{03}. Hence the contribution to the product from such p is 1.

For odd primes p which do not divide d, we take

$$Z_2 = -\xi_2 c_{01}(\eta,\zeta),\ Z_3 = -2\xi_3 c_{01}(\eta,\zeta) \ \text{ if } \ p|c_{01}(\eta,\zeta),$$

and since by Lemma 5 solubility of Γ in $\mathbf{Q}_p$ implies that either c_{01} is divisible by an even power of p or 2 is in $\mathbf{Q}_p^{*2}$, the contribution to the product (33) is 1. Finally

$$Q_p^\sharp = (-c_{02}(\eta,\zeta), d^{1/2}c_{01}(\eta,\zeta), 0) \ \text{ if } \ p|(c_{02}(\eta,\zeta)c_{03}(\eta,\zeta)).$$

This last choice is allowable because (59) shows that $d^{1/2}$ is in $\mathbf{Q}_p$ for such primes. It will be convenient to take i, j to be $2, 3$ in some order, determined by the condition that $p|c_{0i}(\eta,\zeta)$. It now follows from Lemma 5 that for the 2-covering corresponding to m^0 to be soluble in $\mathbf{Q}_p$ requires that if $c_{0i}(\eta,\zeta)$ is divisible by an odd power of p then 2 is in $\mathbf{Q}_p^{*2}$ if $p|c_{02}$ and -2 is in $\mathbf{Q}_p^{*2}$ if $p|c_{03}$. Now let $\mathcal{S}$ be the union of $\mathcal{B}$ and the odd primes which divide ξ_3; the reason for introducing $\mathcal{S}$ is that, for example, the resultant of $\xi_2 c_{03} + dc_{01}$ and c_{02} is $-8d^2\xi_3^2$ which is a unit outside $\mathcal{S}$. If also p is in $\mathcal{S}\setminus\mathcal{B}$, the contribution to the product (33) from p reduces to $(-2, c_{02})_p$ if $i = 2$ and $p \nmid (2\xi_3 c_{03} + dc_{01})$, and 1 in all other cases. If p is not in $\mathcal{S}$ then the contribution is $(\xi_2 d^{1/2} - d, c_{0i})_p$ where we choose that value of $d^{1/2}$ which is congruent to $-c_{0j}/c_{01} \bmod p$; and this is equal to

$$(-c_{01}(\eta,\zeta)(\xi_2 c_{0j}(\eta,\zeta) + dc_{01}(\eta,\zeta)), c_{0i}(\eta,\zeta))_p.$$

The product of the contributions from the primes not in $\mathcal{S}$ is therefore

$$L(\mathcal{S}; -c_{01}, c_{02}c_{03})L(\mathcal{S}; \xi_2 c_{03} + dc_{01}, c_{02})L(\mathcal{S}; \xi_2 c_{02} + dc_{01}, c_{03}) \tag{60}$$

Now write

$$f_1 = \xi_2 c_{03} + dc_{01},\ f_2 = c_{02},\ \rho = \eta - \zeta,\ \sigma = \eta + \zeta$$

in (18); the factors coming from the primes in $\mathcal{S} \setminus \mathcal{B}$ are trivial, and using the fact that the right hand side of (35) is equal to 1 we obtain

$$L(\mathcal{S}; \xi_2 c_{03} + d c_{01}, c_{02}) L(\mathcal{S}; \xi_2 c_{02} + d c_{01}, c_{03}) \\ = \prod_{v \text{ in } \mathcal{S}} \{(-c_{01}, -2)_v \, (c_{02}, c_{03})_v \, (d c_{01}(\xi_2 c_{01} + c_{02} + c_{03}), c_{02} c_{03})_v\}.$$

If p is in $\mathcal{S} \setminus \mathcal{B}$ the corresponding factor on the right is equal to the product of the factor from $L(\mathcal{S}; -c_{01}, c_{02}c_{03})$ and the additional factor $(-2, c_{02})_p$ when $p | c_{02}$ and $p \nmid (\xi_2 c_{03} + d c_{01})$; so we can transfer these factors to $L(\mathcal{S}; -c_{01}, c_{02}c_{03})$, thereby turning it into $L(\mathcal{B}; -c_{01}, c_{02}c_{03})$. Thus the contribution to (33) from primes outside $\mathcal{B}$ reduces to

$$\prod_{v \text{ in } \mathcal{B}} \{(-c_{01}, -2)_v \, (c_{02}, c_{03})_v \, (-d(\xi_2 c_{01} + c_{02} + c_{03}), c_{02} c_{03})_v\}.$$

Bearing in mind the fact that $p \equiv 5 \bmod 8$ implies $p | \xi_2$, it can be shown that the factor corresponding to an odd prime p is 1. The factor corresponding to $v = \infty$ is $-(-\xi_2, -c_{02})_\infty \, (-1, c_{02})_\infty$, and the product of this with the earlier factor $(\xi_2, c_{02})_\infty$ is $(-1, \xi_2)_\infty$. Similarly we multiply the factor corresponding to $v = 2$ by (58). The resulting expression reduces to $\epsilon(\xi_2, -2)_2$ where $\epsilon = 1$ if $2^5 | c_{02}$ and $\epsilon = -1$ otherwise. Hence there are no solutions with $2 \| X_3$ if $|\xi_2| \equiv 1$ or $3 \bmod 8$, and no solutions with $4 | X_3$ if $|\xi_2| \equiv 5$ or $7 \bmod 8$. This is just what is claimed in the last sentence of Theorem 1.

We can transform this last statement in an interesting way; for the equation (4) with $d = \xi_2^2 - 2\xi_3^2$ can be written

$$X_0^4 + 4X_1^4 = W_0^2 - 2W_1^2$$

where

$$W_0 = \xi_2(X_2^2 + X_3^2) + 2\xi_3 X_2 X_3, \quad W_1 = \xi_3(X_2^2 + X_3^2) + \xi_2 X_2 X_3$$

and the assertion becomes that, under constraints corresponding to the previous ones, there are no solutions with $|W_0| \equiv 5$ or $7 \bmod 8$. All these constraints can be dropped other than the condition that no prime $p \equiv 7 \bmod 8$ divides both W_0 and W_1. This easily translates into the same condition on X_0 and X_1, and this proves the Corollary to Theorem 1.

8.3 *Geometric aspects.* The Néron-Severi group of (55) has an extra generator, the class of which we shall denote by $\mathfrak{a}^\natural - \mathfrak{b}_2$; it is given by the union of the four lines

$$X_0 = (1 - i)X_1, \quad X_2 = \pm(iw/u)^{1/2} X_3$$

and

$$X_0 = (1+i)X_1, \quad X_2 = \pm(-iw/u)^{1/2}X_3.$$

This union consists of a pair of intersecting lines from each of two degenerate fibres of the pencil (56). Subject to the hypotheses already made, any divisor class on (55) defined over $\mathbf{Q}$ has the form $\mathfrak{x} = x_0\pi + x_1\mathfrak{b}_1 + x_2\mathfrak{b}_2$ for some integers x_0, x_1, x_2; it has degree $4x_0$ and self-intersection number

$$(\mathfrak{x}\cdot\mathfrak{x}) = 4(x_0^2 - x_1^2 - x_2^2).$$

In particular, the class $\mathfrak{x}$ is effective and represents a pencil of curves of genus 1 if and only if

$$x_0 > 0, \quad x_0^2 - x_1^2 - x_2^2 = 0, \quad (x_0, x_1, x_2) = 1$$

where the last bracket denotes the highest common factor. In particular, we have four such pencils of degree 4, given by $\pi \pm \mathfrak{b}_1$ and $\pi \pm \mathfrak{b}_2$; the first two of these we have already considered.

The general curve of the pencil $\pi + \mathfrak{b}_2$ has the form

$$\left.\begin{aligned} \hat{y}(X_0^2 - 2X_1^2 + ruX_2^2 + swX_3^2) + \hat{z}(2X_0X_1 - suX_2^2 + rwX_3^2) = 0,\\ \hat{y}(-2X_0X_1 - suX_2^2 + rwX_3^2) + \hat{z}(X_0^2 - 2X_1^2 - ruX_2^2 - swX_3^2) = 0. \end{aligned}\right\} \tag{61}$$

In contrast with what happens for $\pi \pm \mathfrak{b}_1$, the two pencils $\pi \pm \mathfrak{b}_2$ are isomorphic, each being taken into the other by the involution of (55) which changes the sign of X_0. In what follows, we shall for convenience write

$$G = r\hat{y}^2 - 2s\hat{y}\hat{z} - r\hat{z}^2, \quad H = s\hat{y}^2 + 2r\hat{y}\hat{z} - s\hat{z}^2,$$

and note that $G^2 + H^2 = d(\hat{y}^2 + \hat{z}^2)^2$. The Jacobian of (61) is

$$Y^2 = X(X^2 - 2uwGHX + u^2w^2(G^2 + 2H^2)(2G^2 + H^2)) \tag{62}$$

where we have written

$$X = u^2(G^2 + 2H^2)X_2^2/X_3^2, \quad Y = u^2(\hat{y}^2 + \hat{z}^2)(G^2 + 2H^2)X_2(X_0^2 + 2X_1^2)/X_3^3.$$

The second factor on the right in (62) splits in the field $\mathbf{Q}(\sqrt{-2})$, which does not depend on the value of $\hat{y}/\hat{z}$; so it would be possible to investigate this pencil by methods rather like those of this paper, but we see no benefit in doing so. Two of the singular fibres of (61) are given by $\hat{y}^2 + \hat{z}^2 = 0$;

and this holds automatically because $G^2 + H^2$ is a multiple of $\hat{y}^2 + \hat{z}^2$. Similar remarks apply when $i = 3$.

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each of them is a skew quadrilateral, each of whose lines is defined over $\mathbf{Q}(i, \sqrt{2uw})$. Of the eight other singular fibres, four are given by $2G^2+H^2 = 0$ or equivalently

$$\hat{y}(s - r\sqrt{-2}) + \hat{z}(r + s\sqrt{-2} + \sqrt{-d}) = 0 \tag{63}$$

and the other four by $G^2 + 2H^2 = 0$ or equivalently

$$\hat{y}(r - s\sqrt{-2}) = \hat{z}(s + r\sqrt{-2} + \sqrt{-d}) = 0. \tag{64}$$

Each of these eight consists of a pair of conics having two points in common; the least field of definition of each conic is $\mathbf{Q}(\sqrt{-2}, \sqrt[4]{-d})$.

As we did for (10), we can now deduce the Mordell-Weil group of (62) over $\mathbf{C}(\hat{y}, \hat{z})$ from the fact that the Néron-Severi group of (1) over $\mathbf{C}$ is spanned by the classes of the 48 lines. Any line for which X_2/X_0 or X_3/X_0 is constant meets each curve of the pencil (61) once and therefore defines a section. It turns out that the Mordell-Weil group of (62) has rank 4, and its torsion subgroup is generated by the 2-division points. The values of X listed below give 4 independent points:

$$\begin{aligned} X &= GH + \sqrt{-d}(\hat{y}^2 + \hat{z}^2)(G + H), \\ X &= (G + (\hat{y}^2 + \hat{z}^2)\sqrt{-d})(H + (\hat{y}^2 + \hat{z}^2)\sqrt{-d}), \\ X &= i(2G^2 + H^2), \\ X &= i(G^2 + 2H^2). \end{aligned}$$

The first two points in this table are defined over $\mathbf{Q}(\sqrt{-d}, \sqrt{2uw})$ and the last two are defined over $\mathbf{Q}(i, \sqrt{2uw})$.

We can use the methods of §6 to find the group $\mathrm{Br}_{\mathrm{vert}}$ associated with the pencil (61). Write $t = \hat{y}/\hat{z}$, and let t_2, t_3 be the values of t coming from (63), (64) respectively. Then $\mathrm{Br}_{\mathrm{vert}}$ is generated by the two elements

$$\mathrm{Cores}_{\mathbf{Q}(t_i)/\mathbf{Q}}(\sqrt{-d}, t - t_i) \tag{65}$$

for $i = 2, 3$. At $(t_2, 1)$ we have $G = (\hat{y}^2 + \hat{z}^2)\sqrt{-d}$ and there is a similar identity at $(t_3, 1)$; thus in the notation of (42) we can take

$$\begin{aligned} f_2(\hat{y}, \hat{z}) = 2G^2 + H^2, \quad g_2(\hat{y}, \hat{z}) = G(\hat{y}^2 + \hat{z}^2), \\ f_3(\hat{y}, \hat{z}) = G^2 + 2H^2, \quad g_3(\hat{y}, \hat{z}) = H(\hat{y}^2 + \hat{z}^2). \end{aligned}$$

Thus the obstruction associated with (65) for $i = 2$ has the form

$$L(\mathcal{B}; 2G^2 + H^2, \hat{y}^2 + \hat{z}^2) = 1;$$

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