Isotropy of quadratic forms over function fields of curves over p-adic fields

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Introduction

Let k be a field of characteristic not equal to 2. We recall the notion of the u-invariant $u(k)$ of k:

$$u(k) = \sup\{\text{dimension of } (q) \mid q \text{ an anisotropic quadratic form over } k\}$$

It is a longstanding question whether the finiteness of $u(k)$ implies the finiteness of $u(k(t))$. This was open even in the case k is a p-adic field. Recently, Hoffmann and Van Geel ([HV], 3.7) showed that if k is a non-dyadic p-adic field, X an irreducible curve over k and $k(X)$ its function field, then $u(k(X))$ is finite, more precisely $u(k(X)) \leq 22$. They used a theorem of Saltman ([S], 3.4, cf. [HV], 2.5), bounding the index of central simple algebras over such fields by the square of the exponent. In this paper, we follow the techniques of Saltman to prove that the u-invariant of $k(X)$ is bounded by 10. We remark that conjecturally $u(k(X)) = 8$. Recall that if F is a finite field, $k = F((t))$ and X is an irreducible curve over k, then $u(k(X)) = 8$.

The main step of the proof is to kill any element in $H^3(k(X), \mathbb{Z}/2)$ in a quadratic extension of $k(X)$ (3.8). This is done by killing the ramification of any element of $H^3(k(X), \mathbb{Z}/2)$ on a regular proper model $\mathcal{X}$ of a quadratic extension L of $k(X)$ and using a theorem of Kato ([K], 5.2) that the unramified cohomology group $H^3_{\mathrm{nr}}(L/\mathcal{X}, \mathbb{Z}/2) = 0$. This shows that every element α in $H^3(k(X), \mathbb{Z}/2)$ is of the form $(f) \cup \beta$, with $(f) \in H^1(k(X), \mathbb{Z}/2) = k(X)^*/k(X)^{*2}$ and $\beta \in H^2(k(X), \mathbb{Z}/2)$. In view of a theorem of Saltman (cf. 2.2), β and hence α, is a sum of two symbols. A subtler choice of a biquadratic extension (2.1) which splits $\beta \in H^2(k(X), \mathbb{Z}/2)$ leads to the fact that every element in $H^3(k(X), \mathbb{Z}/2)$ is a symbol $(f) \cup (g) \cup (h)$. In fact we also prove (3.9) that given $\alpha_i \in H^3(k(X), \mathbb{Z}/2)$, $1 \leq i \leq n$, there exist $f, g, h_i \in k(X)^*$ such that $\alpha_i = (f) \cup (g) \cup (h_i)$. This is a local two-dimensional analogue of a result of Tate for number fields ([T], 5.2).

Using methods of Hoffmann and Van Geel ([HV]) and the fact that every element in $H^3(k(X), \mathbb{Z}/2)$ is a symbol, one can deduce that $u(k(X)) \leq 12$ (4.2). One shows further that given $\alpha \in H^3(k(X), \mathbb{Z}/2)$, a suitable choice of a quadratic extension $L = k(X)(\sqrt{f})$ which splits α can be made so that f is a value of a given binary quadratic form (4.4). This leads to $u(k(X)) \leq 10$ (4.5).

Let k be a p-adic field and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration (cf. [CTSk]) and $CH_0(X/C)$ the kernel of the induced homomorphism $\pi_* : CH_0(X) \to CH_0(C)$, where CH_0 denotes the group of zero-cycles modulo rational equivalence. In ([CTSk]), Colliot-Thélène and Skorobogatov posed the question whether $CH_0(X/C)$ is zero if $\dim(X) \geq 4$. In ([HV], 4.2), Hoffmann and Van Geel showed that if k is non-dyadic and $\dim X \geq 6$ then $CH_0(X/C) = 0$. They further proved that if every element in $H^3(k(X), \mathbb{Z}/2)$ is a symbol and $\dim(X) \geq 4$, then $CH_0(X/C) = 0$ ([HV], 4.4). Thus, as a consequence of our result, it follows that if $\dim(X) \geq 4$, then $CH_0(X/C) = 0$ (5.2), answering the above question of Colliot-Thélène and Skorobogatov in the affirmative.

In ([Se], §8.3), Serre raised the question whether for a p-adic field k, every element in $H^3(k(t), \mathbb{Z}/2)$ is a symbol. Under the assumption that this is true, he has the following explicit description of the set of isomorphism classes of Cayley algebras over $k(t)$ as the set

$$C(P) = \{f : P \to \mathbb{Z}/2 \mid \mathrm{Supp}(f) \text{ finite and } \sum_{x \in P} f(x) = 0\},$$

where P denotes the set of closed points of $\mathbb{P}^1_k$. Using a result of Kato ([K]), we give a description (6.3), following Serre's method, of the set of isomorphism classes of Cayley algebras over $k(X)$, where X is a smooth, irreducible curve over a non-dyadic p-adic field, which reduces to that of Serre in the case $X = \mathbb{P}^1_k$.

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1. Some Preliminaries

We recall (cf. [Sc]) some basic definitions and facts about quadratic forms and (cf. [CT]) various facts on Galois cohomology and unramified cohomology. Let F be a field of characteristic not equal to 2. By a *quadratic form* over F we mean a pair (V, q), where V is a finite dimensional vector space, $q : V \to F$ is a map such that $q(\lambda v) = \lambda^2 q(v)$, for $\lambda \in F$, $v \in V$ and the map $b_q : V \times V \to F$ given by $b_q(v, w) = q(v + w) - q(v) - q(w)$ is a non-singular bilinear form. We shall abbreviate (V, q) by q. Let q be a quadratic form over k. The *rank* of q, denoted by $rk(q)$, is defined as the dimension of V over F. We say that a quadratic form q over F is *isotropic* if there exists $v \in V$, $v \neq 0$, such that $q(v) = 0$; otherwise q is called *anisotropic*. The u-invariant of F, denoted by $u(F)$, is defined as

$$u(F) = \sup\{rk(q) \mid q \text{ an anisotropic quadratic form over } F\}.$$

Let q be a quadratic form over F. Since $\mathrm{char}(F) \neq 2$, q is isometric to a diagonal form $< a_1, \cdots, a_n >$, for some $a_i \in F^*$. A quadratic form is isotropic if and only if $q \simeq < 1, -1 > \perp q'$ for some quadratic form over F. A quadratic form is said to be *hyperbolic* if $q \simeq < 1, -1 > \perp \cdots \perp < 1, -1 >$. Let $W(F)$ be the *Witt group* of quadratic forms over F. Note that every element in $W(F)$ is represented by an anisotropic quadratic form over F. A quadratic form q represents 0 in $W(F)$ if and only if q is hyperbolic. Tensor product of quadratic forms makes $W(F)$ into a ring. Let $I(F)$ be the ideal of $W(F)$ consisting of even rank forms. For $n \geq 1$, let $I^n(F)$ denote the n^{th} power of $I(F)$. The abelian group $I^n(F)$ is generated by quadratic forms of the type $< 1, a_1 > \otimes \cdots \otimes < 1, a_n >$, with $a_i \in F^*$. A quadratic form of the type $< 1, a_1 > \otimes \cdots \otimes < 1, a_n >$ is called an *n-fold Pfister* form. Let $P_n(F)$ denote the set of n-fold Pfister forms over F.

The rank induces an isomorphism $rk : W(F)/I(F) \simeq \mathbb{Z}/2$. For a quadratic form over F, let $d(q)$ be the discriminant of q and $c(q)$ the Clifford invariant of q. Then the discriminant induces an isomorphism $d : I(F)/I^2(F) \to F^*/F^{*2}$. A celebrated theorem of Merkurjev ([M]) asserts that c induces an isomorphism

$$\frac{I^2(F)}{I^3(F)} \xrightarrow{\sim} H^2(F, \mathbb{Z}/2),$$

where for any $n \geq 0$, $H^n(F, \mathbb{Z}/2)$ denotes the n^{th} Galois cohomology group $H^n(Gal(F_s/F), \mathbb{Z}/2)$, F_s denoting the separable closure of F. For $a \in F^*$, let (a) denote the class in $H^1(F, \mathbb{Z}/2) = F^*/F^{*2}$. For $a_1, \cdots, a_n \in F^*$, let $(a_1) \cdot \cdots \cdot (a_n)$ denote the element $(a_1) \cup \cdots \cup (a_n) \in H^n(F, \mathbb{Z}/2)$. Let $n \geq 1$. For $a_1, \cdots, a_n \in F^*$, let

$$e_n : P_n(F) \to H^n(F, \mathbb{Z}/2)$$

be defined by $e_n(< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n >) = (a_1) \cdot \cdots \cdot (a_n) \in H^n(F, \mathbb{Z}/2)$. Then e_1 is the discriminant and e_2 is the Clifford invariant. Suppose that the 2-cohomological dimension $cd_2(F)$ of F is at most 3. Then by a theorem of Arason, Elman and Jacob ([AEJ], Corollary 4 and Theorem 2) $I^4(F) = 0$ and

$$e_3 : I^3(F) \to H^3(F, \mathbb{Z}/2)$$

is an isomorphism.

Let R be a discrete valuation ring, F its quotient field and κ its residue field. Assume that the characteristic of κ is not equal to 2. For $q \geq 1$, let

$$\partial_R : H^q(F, \mathbb{Z}/2) \to H^{q-1}(\kappa, \mathbb{Z}/2)$$

be the residue homomorphism defined with respect to R. If P is the maximal ideal of R, then sometimes we denote ∂_R by ∂_P. For u_i units in R, $1 \leq i \leq q-1$ and π a parameter in R, we have $\partial_R((u_1) \cdots (u_{q-1}) \cdot (\pi)) = (\overline{u}_1) \cdots (\overline{u}_{q-1})$, where bar denotes the image in κ.

Let $\mathcal{X}$ be a regular integral scheme of dimension n and F its function field. For $i \geq 0$, let $\mathcal{X}^i$ denote the set of points of $\mathcal{X}$ of codimension i. For any $x \in \mathcal{X}$, let $\kappa(x)$ denote the residue field at x. Assume that the characteristic of $\kappa(x)$ is *not equal* to 2, for any $x \in \mathcal{X}$. For $x \in \mathcal{X}^1$, let $\mathcal{O}_{\mathcal{X},x}$ denote the discrete valuation ring at x and $\partial_x : H^q(F, \mathbb{Z}/2) \to H^{q-1}(\kappa(x), \mathbb{Z}/2)$ the residue homomorphism defined with respect to $\mathcal{O}_{\mathcal{X},x}$. Let

$$H^q_{\text{nr}}(F/\mathcal{X}, \mathbb{Z}/2) = \ker(H^q(F, \mathbb{Z}/2) \xrightarrow{\partial = (\partial_x)} \bigoplus_{x \in \mathcal{X}^1} H^{q-1}(\kappa(x), \mathbb{Z}/2)).$$

An element $\alpha \in H^q(F, \mathbb{Z}/2)$ is called *unramified* at a point $x \in \mathcal{X}^1$, if $\partial_x(\alpha) = 0$; otherwise it is called *ramified* at x. We say that $\alpha \in H^q(F, \mathbb{Z}/2)$ is *unramified on* $\mathcal{X}$ if it is unramified at all points of $\mathcal{X}^1$, i.e., $\alpha \in H^q_{\mathrm{nr}}(F/\mathcal{X}, \mathbb{Z}/2)$. We define the ramification divisor

$$ram_{\mathcal{X}}\alpha = \sum_{\partial_x(\alpha)\neq 0} x.$$

For $f \in F^*$, we denote $\mathrm{Supp}_{\mathcal{X}}(div_{\mathcal{X}}(f))$ by $\mathrm{Supp}_{\mathcal{X}}(f)$

Let k be a p-adic field, $p \neq 2$. Let X be a smooth, projective, integral curve over k and $K = k(X)$ the function field of X. Let $\mathcal{O}_k$ be the ring of integers of k. For $\alpha_i \in H^q(K, \mathbb{Z}/2)$ and $f_j \in K^*$, $1 \leq i \leq n$, $1 \leq j \leq m$, by a result of Lipman on the resolution of singularities (cf. [S], Proof of 2.1), there exists a regular, projective model $\mathcal{X}$ of X over $\mathcal{O}_k$ and two regular curves C and E on $\mathcal{X}$ with only normal crossings (i.e., for every $x \in C \cap E$, the maximal ideal of the local ring $\mathcal{O}_{\mathcal{X},x}$ is generated by local equations of C and E at x), such that

$$\cup_{1\leq i\leq n}\mathrm{Supp}(ram_{\mathcal{X}}(\alpha_i)) \cup \cup_{1\leq j\leq m}\mathrm{Supp}_{\mathcal{X}}(f_j) \subset \mathrm{Supp}(C+E).$$

We use this result throughout this paper without further reference.

Let F be a field of characteristic not equal to 2 and L a field extension of F. For any $\alpha \in H^q(F, \mathbb{Z}/2)$, the image of α in $H^q(L, \mathbb{Z}/2)$ under the restriction map is denoted by α_L. Let $\mathcal{X}$ be a scheme and $x \in \mathcal{X}$. Let $\mathcal{O}_{\mathcal{X},x}$ be the local ring at x. For any $f \in \mathcal{O}_{\mathcal{X},x}$, the image of f in $\kappa(x)$ is denoted by $f(x)$. For any ring A, let A^* denote the group of units in A. Let $A \subset B$ be local rings with maximal ideals m_A and m_B respectively. We say that B *dominates* A if $m_A \subset m_B$. In the rest of the paper, we assume that 2 is *invertible* in all the rings concerned.

2. Cohomology in degree 2

Let k be a non-dyadic p-adic field and $\mathcal{O}_k$ the ring of integers in k. Let X be a smooth, projective, irreducible curve over k and $K = k(X)$ the function field of X over k.

Proposition 2.1 Let k, X and K be as above. Let $\alpha_i \in H^2(K, \mathbb{Z}/2)$, $1 \le i \le n$. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ such that

$$\sum_{i=1}^{n} ram_{\mathcal{X}}(\alpha_i) \subset C + E,$$

where C and E are regular curves on $\mathcal{X}$ having only normal crossings. Suppose there exists $f \in K^*$ such that

$$div_{\mathcal{X}}(f) = C + E + F,$$

where F is a divisor on $\mathcal{X}$ whose support does not contain any point of $C \cap E$ and no component of C or E is contained in F. Let T be the finite set of closed points consisting of $C \cap E$, $C \cap F$, $E \cap F$. Let B be the semi-local ring at T. Let $h \in B$, $h \neq 0$, be such that $\mathrm{Supp}_{\mathrm{Spec}(B)}(h) \subset \mathrm{Supp}(C + E)$ and h is square free in B. Suppose $x \in C \cap E$ is a closed point. Let π_x and δ_x be local equations at x for C and E respectively. We write $h = \pi_x^{\epsilon_1}\delta_x^{\epsilon_2} w_x$ and $f = \pi_x \delta_x w'_x$, where w_x, w'_x are units at x and $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Suppose there exists an element $h_1 \in B^*$ such that for every $x \in T$,

i) if $h(x) \neq 0$, then $(hh_1)(x)$ is not a square in $\kappa(x)$.

ii) if $h(x) = 0$ and either $x \in C \cap F$ or $x \in E \cap F$, then h_1 is a unit at x.

iii) if $h(x) = 0$ and $x \in C \cap E$, then $(w_x w'_x h_1)(x)$ is not a square in $\kappa(x)$.

Then the image of α_i in $H^2(K(\sqrt{f}, \sqrt{hh_1}), \mu_2)$ is zero, for $1 \le i \le n$.

Proof. Let $L = K(\sqrt{f}, \sqrt{hh_1})$ and S be a discrete valuation ring, containing $\mathcal{O}_k$, with quotient field L. Since $\mathcal{X}$ is projective over $\mathcal{O}_k$, there exists a point $x \in \mathcal{X}$ of codimension 1 or 2 such that S dominates the local ring $A = \mathcal{O}_{\mathcal{X},x}$. We show that, for $1 \le i \le n$, $(\alpha_i)_L$ is unramified at S. Fix i, $1 \le i \le n$ and let $\alpha = \alpha_i$.

Suppose that $x \notin C \cup E$. Then α is unramified on A and hence unramified over S ([S], 1.4). Assume that $x \in C \cup E$.

Suppose that $\dim(A) = 1$. Then f is a parameter at x and hence S is ramified over A. Therefore α is unramified on S.

Suppose that $\dim(A) = 2$. Let m_S be the maximal ideal of S and ν_S denote the valuation of S. We show that $\partial_S(\alpha_L) = 0$.

Suppose that $x \in C \setminus (E \cup F)$ (resp. $x \in E \setminus (C \cup F)$). Then f is a local equation for C (resp. E) at x and α can be ramified only at (f) in A. By ([S], 1.2), we have $\alpha = \alpha' + (u)\cdot(f)$, where α' is unramified on A and $u \in A^*$. Since $(u)\cdot(f)_L = (u)\cdot(1) = 0$, $\alpha_L = \alpha'_L$ is unramified at S.

Suppose that $x \in C \cap F$. Then $x \notin E$ and hence, by ([S], 1.2), $\alpha = \alpha' + (u)\cdot(\pi_x)$, where α' is unramified on A, $u \in A^*$. Suppose further that $h(x) \neq 0$. Then $(hh_1)(x)$ is not a square in $\kappa(x)$. We have $\partial_S((u)\cdot(\pi_x)) = \overline{u}^{\nu_S(\pi_x)}$, bar denoting the image modulo m_S. Since $(hh_1(x))$ is not a square in the finite field $\kappa(x)$, $u(x)$ is a square in $\kappa(x)(\sqrt{hh_1(x)})$. Since $\kappa(x)(\sqrt{(hh_1)(x)}) \subset S/m_S$, $\overline{u}$ is a square in S/m_S and hence $(u)\cdot(\pi_x)$ is unramified on S. Suppose that $h(x) = 0$. Since $h_1(x)$ is a unit at x and $\mathrm{Supp}_{\mathrm{Spec}(B)} h \subset \mathrm{Supp}(C + E)$, hh_1 is a local equation for C at x and $\alpha = \alpha' + (u)\cdot(hh_1)$. Since $(u)\cdot(hh_1)_L = (u)\cdot(1)_L = 0$, α is unramified at S. Similarly, one proves that α is unramified at S, if $x \in E \cap F$.

Suppose that $x \in C \cap E$. Let π_x and δ_x be local equations for C and E at x given in the statement of the proposition. Then we have $f = \pi_x \delta_x w'_x$ with $w'_x \in A^*$. We have ([S], 1.2) $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(u)\cdot(\pi_x)$, $(v)\cdot(\delta_x)$ and $(\pi_x)\cdot(\delta_x)$, $u, v \in A^*$. For $u \in A^*$, we have

$$(u)\cdot(\delta_x)_L = (u)\cdot(\delta_x f)_L = (u)\cdot(\pi_x w'_x)_L \qquad (*),$$

$$(u)\cdot(\pi_x)_L = (u)\cdot(\pi_x f)_L = (u)\cdot(\delta_x w'_x)_L \qquad (**),$$

$$(\pi_x)\cdot(\delta_x)_L = (\pi_x f)\cdot(\delta_x)_L = (\delta_x w'_x)\cdot(\delta_x)_L = (-w'_x)\cdot(\delta_x)_L \qquad (***).$$

Suppose further that $h(x) \neq 0$. Then $hh_1(x)$ is not a square in $\kappa(x)$. As before $(v)\cdot(\pi_x)_L$ and $(v)\cdot(\delta_x)_L$ are unramified at S for any $v \in A^*$. Therefore α_L is unramified at S. Suppose that $h(x) = 0$. Then either $h = \pi_x w_x$ or $h = \delta_x w_x$ or $h = \pi_x \delta_x w_x$, where $w_x \in A^*$. If $h = \pi_x w_x$ or $\delta_x w_x$, then, by $(*), (**), (***)$ it follows that $\partial_S(\alpha'') = 0$ and hence α is unramified at S. Suppose $h = \pi_x \delta_x w_x$. Since $\sqrt{f}, \sqrt{hh_1} \in L^*$, $\sqrt{w'_x w_x h_1} \in L^*$. Since $(w'_x w_x h_1)(x)$ is not a square in $\kappa(x)$, once again using $(***)$ and arguing as above it follows that α'' and hence α is unramified at S.

Let k' be the field of constants in L. Let X' be the smooth, projective, irreducible curve over k' with L as its function field. Let $\mathcal{X}'$ be a regular, projective model of X' over $\mathcal{O}_{k'}$. For every $x' \in \mathcal{X}$ of codimension 1, $\mathcal{O}_{\mathcal{X}',x'}$ dominates $\mathcal{O}_{\mathcal{X},x}$, where $x \in \mathcal{X}$ is a point of codimension 1 or 2. We have α_L

unramified at x' for every $x' \in \mathcal{X}'^1$. Since the Brauer group of $\mathcal{X}'$ is trivial (cf. [L], Theorem 4 or [Gr], 2.15 and 3.1), it follows that $\alpha_L = 0$. This completes the proof of the proposition. □

Corollary 2.2 ([S], 3.4) Let D be a central division algebra over K of exponent 2 in the Brauer group of K. Then the degree of D is at most 4. In particular, every element in $H^2(K, \mathbb{Z}/2)$ is a sum of two symbols.

Proof. Let $\alpha \in H^2(K, \mathbb{Z}/2)$ denote the class of D. Let $\mathcal{X}$, C and E be as in (2.1) defined with respect to α. By a semi-local argument, due to Colliot-Thélène (cf. [HV], Lemma 2.4), we choose $f \in K^*$ such that

$$div_{\mathcal{X}}(f) = C + E + F,$$

where F is a divisor on $\mathcal{X}$ whose support does not contain any point of $C \cap E$ and any component of C or E. Let T and B be as in (2.1). Let $h \in B^*$ be such that for every $x \in T$, $h(x)$ is not a square in $\kappa(x)$ and $h_1 = 1$. Then h and h_1 satisfy the hypotheses of (2.1). Therefore by (2.1), the image of α in $H^2(K(\sqrt{f}, \sqrt{h}), \mathbb{Z}/2)$ is zero. Hence $D \otimes K(\sqrt{f}, \sqrt{h})$ is a split algebra. In particular, the degree of D is at most 4 and D is a tensor product of two quaternion algebras ([A]). Hence α is a sum of two symbols. □

3. Cohomology in degree 3

Let k be a non-dyadic p-adic field and $\mathcal{O}_k$ the ring of integers in k. Let X be a smooth, projective, irreducible curve over k and $K = k(X)$ the function field of X over k.

Let $\alpha \in H^3(K, \mathbb{Z}/2)$. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ such that

$$ram_{\mathcal{X}}(\alpha) \subset C + E,$$

where C and E are regular curves on $\mathcal{X}$ having only normal crossings. Let $T = C \cap E$ and B the semi-local ring at T. Since $\mathcal{X}$ is regular, B is a regular. semi-local and hence unique factorisation domain.

□

Lemma 3.1 Let F be a finite field of characteristic not equal to 2 and Y a smooth, projective curve over F. Let $\beta \in H^2(F(Y), \mathbb{Z}/2)$ and $P_1, \cdots, P_n$ be the closed points where β is ramified. Let $f \in F(Y)^*$ be such that at each P_i either f has odd valuation or f is a unit at P_i and $f(P_i)$ is not a square in $\kappa(P_i)$. Then $\beta \otimes F(Y)(\sqrt{f}) = 0$.

Proof. By class field theory, it is enough to prove that $\beta \otimes F(Y)(\sqrt{f})$ is unramified at each discrete valuation ring of $F(Y)(\sqrt{f})$. Let S be a discrete valuation ring with $F(Y)(\sqrt{f})$ as its quotient field. Let R be the discrete valuation ring of $F(Y)$ such that $R \subset S$. If β is unramified at R, then β is unramified at S. Suppose that β is ramified at R and $R = \mathcal{O}_{Y,P_i}$ for some i. If f has odd valuation at P_i, then S over R is ramified and hence β is unramified at S. If f has even valuation at P_i, then by the choice of f, f is a unit at P_i and not a square in $\kappa(P_i)$. Therefore the residue field $\overline{S}$ of S is a quadratic extension of the residue field $\kappa(P_i)$ at R. Since S over R is unramified, $\partial_S(\alpha_L) = \partial_R(\alpha) \otimes_{\kappa(P_i)} \overline{S}$ (cf. [S]. 1.3). Since $\overline{S}$ is a quadratic extension of $\kappa(P_i)$ and $\kappa(P_i)$ is a finite field, every element of $\kappa(P_i)$ is a square in $\overline{S}$. Therefore β is unramified at S. $\square$.

Lemma 3.2 Let R be a discrete valuation ring, K its quotient field and κ its residue field. Let δ be a parameter in R and $u \in R^*$. If $(u)\cdot(\delta)$ is unramified at R, then $(u)\cdot(\delta) = (u)\cdot(u')$ for some $u' \in R^*$.

Proof. Suppose that $(u)\cdot(\delta)$ is unramified at R. Since $\partial_R((u)\cdot(\delta)) = (\overline{u})$, where bar denotes the image in κ, $\overline{u}$ is a square in κ. Therefore the quadratic form $< 1, -\overline{u} >$ is isotropic over κ. Let $a, b \in R$ such that $\overline{a}^2 - \overline{b}^2\overline{u} = 0$ and at least one of $\overline{a}$, $\overline{b}$ is non-zero. We write $a^2 - b^2u = v\delta^r$ for some $r \geq 1$ and $v \in R^*$. Suppose that $r \geq 2$. We have $(a+\delta)^2 - b^2u = v\delta^r + \delta^2 + 2a\delta = \delta(v\delta^{r-1} + \delta + a)$. Since $r \geq 2$, if a is a unit in R, by replacing a by $a + \delta$ we assume that $r = 1$. Similarly, if b is a unit in R, then by replacing b by $b+\delta$, we assume that $r = 1$. Therefore we have $< 1, -u > \simeq v\delta < 1, -u >$. We have $< 1, -u, -\delta, u\delta > \simeq v\delta < 1, -u > \perp < -\delta, u\delta > = v\delta < 1, -u, -v, vu >$ and $v\delta$ is a value of $< 1, -u, -\delta, u\delta >$. Hence $< 1, -u, -\delta, u\delta > \simeq v\delta < 1, -u, -\delta, u\delta > \simeq < 1, -u, -v, uv >$ and $(u)\cdot(\delta) = (u)\cdot(v)$. $\square$

Proposition 3.3 Let A be a regular local ring of dimension 2, K its quotient field and κ its residue field. Let π be a prime element of A and $\kappa(\pi)$ the residue field at π. Assume that every element of $H^2(\kappa(\pi), \mathbb{Z}/2)$ is represented by a symbol $(a)\cdot(b)$ for some $a, b \in \kappa(\pi)^*$. Let $\alpha \in H^3(K, \mathbb{Z}/2)$.

i) Suppose α is ramified only at π among the prime elements of A. Assume that π is a regular parameter in A, i.e., $A/(\pi)$ is regular. Then

$$\alpha = \alpha' + (u)\cdot(v)\cdot(\pi)$$

for some $\alpha' \in H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2)$ and $u, v \in A^*$.

ii) Suppose α is ramified only at π and δ among the prime elements of A. Further assume that π and δ generate the maximal ideal m of A. Then

$$\alpha = \alpha_1 + \alpha_2,$$

where $\alpha_1 \in H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2)$ and α_2 is a sum of symbols of the type

$$(u)\cdot(v)\cdot(\pi), \quad (u)\cdot(v)\cdot(\delta), \quad (u)\cdot(\delta)\cdot(\pi),$$

u, v running over the units of A.

Proof. Let α and π be as in i). Since π is a regular parameter of A, there exists a prime element δ in A such that the maximal ideal m of A is generated by π and δ. We have a complex ([K], Prop. 1.7)

$$H^3(K, \mathbb{Z}/2) \xrightarrow{\partial} \bigoplus_{x \in \mathrm{Spec}(A)^1} H^2(\kappa(x), \mathbb{Z}/2) \xrightarrow{\partial} H^1(\kappa, \mathbb{Z}/2).$$

By the assumption on $\kappa(\pi)$, there exist $a, b \in A$ such that $\partial_\pi(\alpha) = (\overline{a})\cdot(\overline{b})$, where for any element $c \in A$, $\overline{c}$ denotes the image of c in $A/(\pi)$. Since m is generated by π and δ, $A/(\pi)$ is a discrete valuation ring with $\overline{\delta}$ as a parameter. Without loss of generality we assume that $\partial_\pi(\alpha)$ is equal to either $(\overline{u})\cdot(\overline{v})$ or $(\overline{u})\cdot(\overline{v\delta})$ for some $u, v \in A^*$. Suppose $\partial_\pi(\alpha) = (\overline{u})\cdot(\overline{v\delta})$. Since α has residue only at π, $\partial\partial(\alpha) = \partial((\overline{u})\cdot(\overline{v\delta}))$ is the square class of the image of u in κ^*. Since $\partial\partial = 0$, u is a square modulo m. Thus $(\overline{u})\cdot(\overline{v\delta})$ over $\kappa(\pi)$ is unramified at $\overline{\delta}$ and by (3.2) $(\overline{u})\cdot(\overline{v\delta}) = (\overline{u})\cdot(\overline{v'})$ for some $v' \in A^*$. Let $\alpha' = \alpha - (u)\cdot(v')\cdot(\pi)$. Since $\partial_\pi(\alpha) = \partial_{(\pi)}((\overline{u})\cdot(\overline{v'})\cdot(\pi))$ and

$\partial_{\pi'}((u)\cdot(v)\cdot(\pi)) = \partial_{\pi'}(\alpha) = 0$ for any prime element π' of A not equal to π, we have $\partial(\alpha') = 0$. Hence $\alpha' \in H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2)$ and $\alpha = \alpha' + (u)\cdot(v)\cdot(\pi)$.

Now let α, π and δ be as in *ii*). Since every element in $H^2(\kappa(\pi), \mathbb{Z}/2)$ is represented by a symbol, one finds $u, v \in A^*$, such that $\partial_\pi(\alpha)$ is equal to $(\overline{u})\cdot(\overline{v})$ or $(\overline{u})\cdot(\overline{v\delta})$. Set $\alpha_1 = \alpha - (u)\cdot(v)\cdot(\pi)$ if $\partial_\pi(\alpha) = (\overline{u})\cdot(\overline{v})$ and $\alpha_1 = \alpha - (u)\cdot(v\delta)\cdot(\pi)$ if $\partial_\pi(\alpha) = (\overline{u})\cdot(\overline{v\delta})$. Since α is ramified only at π and δ, α_1 is unramified except possibly at δ. Now we can apply i) to describe α_1. This completes the proof of the proposition. □

Remark 3.4 Suppose that in the above proposition K is a function field in one variable over a non-dyadic local field. Then for every prime $\pi \in A$, the residue field $\kappa(\pi)$ at π is either a local field or a function field in one variable over a finite field. Therefore every element in $H^2(\kappa(\pi), \mathbb{Z}/2)$ is represented by a symbol. Thus A satisfies the hypothesis of (3.3).

Lemma 3.5 Let k, K and $\mathcal{X}$ be as above. Let x be a closed point of $\mathcal{X}$ and $A = \mathcal{O}_{\mathcal{X},x}$. Let S be a discrete valuation ring which dominates A. Then every symbol of the type $(u)\cdot(v)\cdot(\pi)$, with $u, v \in A^*$ and $\pi \in K^*$, is unramified at S.

Proof. Let $u, v \in A^*$. We have $\partial_S((u)\cdot(v)\cdot(\pi)) = (\overline{u}, \overline{v})^{\nu_S(\pi)}$, bar denoting the image in the residue field of S and ν_S denoting the valuation of S. Since $u, v \in A^*$ and $\kappa(x)$ is a finite field, it follows that $(\overline{u}, \overline{v}) = 0$. Hence $(u)\cdot(v)\cdot(\pi)$ is unramified at S. □

Lemma 3.6 Let k, K, $\alpha \in H^3(K, \mathbb{Z}/2)$, $\mathcal{X}$, C and E be as above. Let L be an extension of K and S a discrete valuation ring with quotient field L. Suppose that there exists $x \in C \cap E$ a closed point of $\mathcal{X}$ such that S dominates $\mathcal{O}_{\mathcal{X},x}$. Suppose one of the following conditions holds.

i) The residue field of S contains a quadratic extension of $\kappa(x)$.

ii) there exist local equations π_x, δ_x for C and E respectively at x such that either π_x or δ_x or $\pi_x\delta_x$ is of the form $w\theta^2$, $w \in S^*$ whose image in the residue field of S has its square class coming from $\kappa(x)^*$ and $\theta \in S$.

Then α_L is unramified at S.

Proof. Let $A = \mathcal{O}_{\mathcal{X},x}$. By (3.3), $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of the symbols of the type $(u)\cdot(v)\cdot(\pi_x)$, $(u)\cdot(v)\cdot(\delta_x)$, $(u)\cdot(\pi_x)\cdot(\delta_x)$, with $u, v \in A^*$. Let ν_S denote the discrete valuation of S, ∂_S denote the residue homomorphism at S and m_S denote the maximal ideal of S. For $u, v \in A^*$, by (3.5), $(u)\cdot(v)\cdot(\pi_x)$, $(u)\cdot(v)\cdot(\delta_x)$ are unramified at S

Suppose that the residue field of S contains a quadratic extension of $\kappa(x)$. We have

$$\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = (\overline{u}) \cup \partial_S((\pi_x)\cdot(\delta_x)).$$

Since the unique quadratic extension of $\kappa(x)$ is contained in the residue field of S, $\overline{u}$ is a square in the residue field of S. Therefore $\partial_S(\alpha_L) = 0$.

Suppose that $\pi_x = w\theta^2$ for some $w \in S^*$ such that $\overline{w} = \lambda\lambda_1^2$ with $\lambda \in \kappa(x)^*$, and $\theta \in S$. Then, we have $((u)\cdot(\pi_x)\cdot(\delta_x))_L = ((u)\cdot(w)\cdot(\delta_x))_L$. We have $\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = ((\overline{u})\cdot(\overline{w}))^{\nu_S(\delta_x)} = ((\overline{u})\cdot(\lambda))^{\nu_S(\delta_x)}$. Since $\overline{u}, \lambda \in \kappa(x)^*$, as above, it follows that $(\overline{u})\cdot(\lambda) = 0$. Similarly, one can prove that if $\delta_x = w\theta^2$, with w, θ as above, then $\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = 0$. Suppose that $\pi_x\delta_x = w\theta^2$, with w, θ as above. Since $(u)\cdot(\pi_x)\cdot(\delta_x) = (u)\cdot(-\pi_x\delta_x)\cdot(\delta_x)$, we have $((u)\cdot(\pi_x)\cdot(\delta_x))_L = ((u)\cdot(-w)\cdot(\pi_x))_L$ and $\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)_L) = ((\overline{u})\cdot(-\overline{w}))^{\nu_S(\pi_x)} = 0$. Therefore α is unramified at S.

□

Lemma 3.7 Let k and K be as in (3.6). Let A be a regular local ring of dimension 2 with K as its quotient field. and S a discrete valuation ring containing A. Then the map $H^3(K, \mathbb{Z}/2) \to H^3(L, \mathbb{Z}/2)$ restricts to a map

$$H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2) \to H^3_{\mathrm{nr}}(L/\mathrm{Spec}(S), \mathbb{Z}/2).$$

Proof. The lemma follows from the absolute purity theorem of Gabber for two dimensional regular local rings. We give a proof here for the sake of completeness.

Let $W(A)$ denote the Witt group of A. Since A is a two-dimensional regular local ring, one has the following exact sequence ([O], [CTS])

$$0 \to W(A) \to W(K) \to \bigoplus_{x \in \mathrm{Spec}(A)^1} W(\kappa(x)).$$

For $n \geq 0$, let $I_n(A) := I^n(K) \cap W(A)$. Since $\mathrm{cd}(K) \leq 3$ and $\mathrm{cd}(\kappa(x)) \leq 2$, in view of ([AEJ], Theorem 2), the homomorphisms $e_n : I^n(F) \to H^n(F, \mathbb{Z}/2)$

exist and are surjective with kernel $I^{n+1}(F)$, for $F = K$ or $\kappa(x)$. Since the following diagram is commutative (cf. [P]),

$$\begin{array}{ccc} I^3(K) & \stackrel{\partial}{\to} & \bigoplus_{x\in \mathrm{Spec}(A)^1} I^2(\kappa(x)) \\ \downarrow e_3 & & \downarrow e_2 \\ H^3(K,\mathbb{Z}/2) & \stackrel{\partial}{\to} & \bigoplus_{x\in \mathrm{Spec}(A)^1} H^2(\kappa(x),\mathbb{Z}/2) \end{array}$$

with e_3 and e_2 isomorphisms, e_3 induces an isomorphism

$$e_3 : I_3(A) \to H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2).$$

Let $\alpha \in H^3_{\mathrm{nr}}(K/\mathrm{Spec}(A), \mathbb{Z}/2)$ and $q \in I_3(A)$ with $e_3(q) = \alpha$. Then $q_L \in I_3(S)$ and α_L is precisely the image of $e_3(q_L)$ in $H^3(L, \mathbb{Z}/2)$. Since the following diagram commutes

$$\begin{array}{ccc} I^3(L) & \stackrel{\partial_S}{\to} & I^2(S/m_S) \\ \downarrow e_3 & & \downarrow e_2 \\ H^3(L,\mathbb{Z}/2) & \stackrel{\partial_S}{\to} & H^2(S/m_S,\mathbb{Z}/2) \end{array}$$

we have $\partial_S(e_3(q_L)) = \partial_S(\alpha_L)$ so that $\partial_S(e_3(q_L)) = 0$.

Thus $\alpha_L \in H^3_{\mathrm{nr}}(L/\mathrm{Spec}(S), \mathbb{Z}/2)$.

□

Theorem 3.8 Let k be a non-dyadic p-adic field, X a smooth, projective, irreducible curve over k. Let $K = k(X)$ and $\alpha_i \in H^3(K, \mathbb{Z}/2)$, $1 \le i \le n$. Then there exists $f \in K^*$ such that $\alpha_i \otimes K(\sqrt{f}) = 0$ for $1 \le i \le n$.

Proof. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ with

$$\cup_{i=1}^n \mathrm{Supp}(ram_{\mathcal{X}}\alpha_i) \subset \mathrm{Supp}(C + E),$$

where C and E regular curves on $\mathcal{X}$ with only normal crossings. Let $f \in K^*$ be such that

$$div_{\mathcal{X}}(f) = C + E + F,$$

where F is a divisor on $\mathcal{X}$ whose support does not contain any point of $C \cap E$ and any component of C or E. Let $L = K(\sqrt{f})$. Let k' be the field of constants in L. Let X' be the smooth, projective, irreducible curve over k' with function field L. Let $\mathcal{X}'$ be a regular, projective model for X' over $\mathcal{O}_{k'}$. Fix i, $1 \leq i \leq n$ and let $\alpha = \alpha_i$. We show that $\alpha_L \in H^3_{\rm nr}(L/\mathcal{X}', \mathbb{Z}/2)$. Let $y \in \mathcal{X}'$ be a point of codimension 1 and $S = \mathcal{O}_{\mathcal{X}',y}$ be the discrete valuation ring at y. Since $\mathcal{X}$ is proper over $\mathcal{O}_k$, there exists a point $x \in \mathcal{X}$ of codimension 1 or 2, such that the S dominates the local ring $A = \mathcal{O}_{\mathcal{X},x}$.

Suppose $\dim(A) = 1$. Then A is a discrete valuation ring. If x corresponds to a component of C or E, then f is a parameter at x and S over A is ramified. Hence, α_L is unramified at S. Suppose that x does not correspond to a component of C or E. Since $ram_{\mathcal{X}}(\alpha) \subset C + E$, α is unramified at R and hence α_L is unramified at S.

Suppose $\dim A = 2$. Suppose first that x does not belong to $\mathrm{Supp}(C) \cup \mathrm{Supp}(E)$. Then α is unramified on A and hence unramified at S (3.7). Suppose $x \in \mathrm{Supp}(C) \setminus \mathrm{Supp}(E)$ or $x \in \mathrm{Supp}(E) \setminus \mathrm{Supp}(C)$, then by (3.3 and 3.5), α is unramified on A and hence by (3.7), α_L is unramified at S. Suppose that $x \in \mathrm{Supp}(C) \cap \mathrm{Supp}(E)$. Let π_x and δ_x be local equations for C and E at x respectively. Then we have $f = \pi_x\delta_x w$ for some $w \in A^*$. Since f is a square in S, it follows from (3.6) that α_L is unramified at S. Therefore $\alpha_L \in H^3_{\rm nr}(L/\mathcal{X}', \mathbb{Z}/2)$. Since $H^3_{\rm nr}(L/\mathcal{X}', \mathbb{Z}/2) = 0$ ([K], 5.2), we have $\alpha_L = 0$.

□

Theorem 3.9 Let k be a non-dyadic p-adic field and K a function field in one variable over k. Let $\alpha_i \in H^3(K, \mathbb{Z}/2)$, $1 \leq i \leq n$. Then there exist $f, g, h_i \in K^*$ such that $\alpha_i = (f)\cdot(g)\cdot(h_i)$, with $h_i \in K^*$. In particular, every element in $H^3(K, \mathbb{Z}/2)$ is a symbol.

Proof. By (3.8), there exists $h \in K^*$ such that $\alpha_i \otimes K(\sqrt{h}) = 0$, for $1 \leq i \leq n$. Therefore there exist ([Ar], 4.6) $\beta_i \in H^2(K, \mathbb{Z}/2)$ such that $\alpha_i = (h) \cup \beta_i$, for $1 \leq i \leq n$. Let X be a smooth, projective, irreducible curve over k with $k(X) = K$. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ such that

$$\cup_{i=1}^n \mathrm{Supp}(ram_{\mathcal{X}}(\beta_i)) \cup \mathrm{Supp}(h) \subset \mathrm{Supp}(C + E)$$

where C and E as above. Let $f \in K^*$ be such that

$$div_{\mathcal{X}}(f) = C + E + F,$$

where F is a divisor on $\mathcal{X}$ whose support does not contain any point of $C \cap E$ and any component of C or E. Let T be the finite set of closed points of $\mathcal{X}$ consisting of $C \cap E$, $C \cap F$ and $E \cap F$. Let B be the semi local ring at T. Since $\mathcal{X}$ is regular, B is a regular ring. For $x \in C \cap E$, let π_x and δ_x be local equations at x for C and E respectively. Let $x \in C \cap E$. Then $h = \pi_x^{\epsilon_1}\delta_x^{\epsilon_2}w_x$ and $f = \pi_x\delta_x w'_x$, where $w_x, w'_x \in B$ are units at x and $\epsilon_1, \epsilon_2 \in \{0,1\}$. Choose $w \in B^*$ such that w is a unit at one closed point of each component of C and E and $-w(x)w_x(x)w'_x(x)$ is not a square in $\kappa(x)$. Replacing f by wf, we assume that $-w_x w'_x(x)$ is not a square in $\kappa(x)$ for all $x \in C \cap E$ and $div_{\mathcal{X}}(f) = C + E + F$, with C, E and F as above. We claim that there exist $a_i \in K^*$ such that $\alpha_i = (h, f, a_i)$, $1 \leq i \leq n$. Since B is a unique factorisation domain with K as its quotient field, without loss of generality, we assume that $h \in B$ is square free with $\mathrm{Supp}_{\mathrm{Spec}(B)}(h) \subset \mathrm{Supp}(C + E)$. For $x \in T$,

i) if $h(x) \neq 0$, let $a_x, b_x \in \kappa(x)$ be such that $h(x)(h(x)a_x^2 - b_x^2)$ is not a square.

ii) if $h(x) = 0$ and either $x \in C \cap F$ or $x \in \cap E \cap F$, let $a_x = 0$ and $b_x = 1$ in $\kappa(x)$.

iii) if $h(x) = 0$ and $x \in C \cap E$, let $a_x = 0$ and $b_x = 1$ in $\kappa(x)$.

Let $a, b \in B$ be such that, $a(x) = a_x$ and $b(x) = b_x$ for all $x \in T$. Let $h_1 = ha^2 - b^2$. Then it is easy to see that f, h, h_1 satisfies the conditions in (2.1). Therefore, by (2.1), $\beta_i \otimes K(\sqrt{f}, \sqrt{hh_1}) = 0$, for $1 \leq i \leq n$. Hence there exist $a_i, b_i \in K^*$ such that $\beta_i = (f)\cdot(a_i) + (hh_1)\cdot(b_i)$, for $1 \leq i \leq n$ (cf. [HV], 3.1). Since $hh_1 = (ha)^2 - hb^2$, hh_1 is norm from $K(\sqrt{h})$ and hence $(h)\cdot(hh_1) = 0$. For $1 \leq i \leq n$, we have

$$\begin{aligned}\alpha_i &= h \cup \beta_i \\ &= (h)\cdot(f)\cdot(a_i) + (h)\cdot(hh_1)\cdot(b_i) \\ &= (h)\cdot(f)\cdot(a_i).\end{aligned}$$

This completes the proof of the theorem. □

4. u-invariant

Let k be a non-dyadic p-adic field and $\mathcal{O}_k$ the ring of integers in k. Let K be a function field in one variable over k.

Theorem 4.1 Let K be as above. Then every anisotropic quadratic form over K representing an element of $I^3(K)$ is a 3-fold Pfister form.

Proof. Let q be an anisotropic quadratic form over K representing an element of $I^3(K)$. Let $\alpha = e_3(q)$. Then by (3.9), $\alpha = (f)\cdot(g)\cdot(h)$. Since $e_3 : I^3(K) \to H^3(K, \mathbb{Z}/2)$ is an isomorphism ([AEJ], Theorem 2), $q =< 1, -f >< 1, -g >< 1, -h >$ in $I^3(K)$. Since q is anisotropic, $q \simeq< 1, -f >< 1, -g >< 1, -h >$. $\square$

Corollary 4.2 Let K be as above. Then every quadratic form over K of rank at least 13 is isotropic.

Proof. Let q be a quadratic form over q of rank 13. By the theorem of Saltman (cf. 2.2), $c(q)$ is a biquaternion algebra over K. Let q_0 be a quadratic form over K such that $rk(q_0) = 5$, $d(q + q_0) = 1$ and $c(q + q_0) = 0$ (cf. [HV], 3.2). Then $q + q_0 \in I^3(K)$ ([M]). By (4.1), we have $q + q_0 =< 1, f >< 1, g >< 1, h >$ for some $f, g, h \in K^*$. Since $rk(q) = 13$, $q \simeq< 1, f >< 1, g >< 1, h >\perp -q_0$. Since $I^4(K) = 0$, every element in $I^3(K)$ represents every element of K^*. In particular $< 1, f >< 1, g >< 1, h >$ represents a value of q_0. Therefore q is isotropic. $\square$

To prove that every quadratic form over K of rank at least 11 is isotropic, we need a subtler choice of a quadratic extension which splits the given element in $H^3(K, \mathbb{Z}/2)$.

Let k be a non-dyadic p-adic field and X a smooth, projective, integral curve over k. Let $K = k(X)$. Let $\alpha \in H^3(K, \mathbb{Z}/2)$. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ such that

$$ram_{\mathcal{X}}\alpha \subset C + E,$$

where C and E are regular curves on $\mathcal{X}$ such that C and E have only normal crossings. Let $T = C \cap E$ and B the semi-local ring at T. Since $\mathcal{X}$ is regular, B is a regular semi-local ring and hence it is a unique factorisation domain.

Lemma 4.3 With the notation as above, let L be a quadratic extension of K. Let S be a discrete valuation ring with L as its quotient field. Assume that $S \cap K = B_{(\pi)}$, where π is a prime element in B giving a local equation for a component C_1 of C. If $C_1 \cap E \neq \emptyset$, let $C_1 \cap E = \{x_1, \cdots, x_r\}$ and δ_{x_i} a local equation of E at x_i, $1 \leq i \leq r$. Suppose that either $C_1 \cap E = \emptyset$ or $L = K(\sqrt{f})$ with $f \in B$ satisfying *one* of the following conditions:

i) f is a parameter in $B_{(\pi)}$,

ii) f is a unit in $B_{(\pi)}$ such that either $v_{\overline{\delta}_{x_i}}(\overline{f}) = 1$ or $f(x_i)$ is not a square in $\kappa(x_i)$, $1 \leq i \leq r$, bar denoting the image modulo (π) and $v_{\overline{\delta}_{x_i}}$ denoting the discrete valuation of $B/(\pi)$ at $\overline{\delta}_{x_i}$.

Then α_L is unramified at S.

Proof. Let $A = B_{(\pi)}$. Then the residue field $\kappa(\pi)$ of A is the quotient field of $B/(\pi)$. Suppose that $C_1 \cap E = \emptyset$. Since $ram_{\mathcal{X}}\alpha \subset C + E$ and C_1 is regular curve on $\mathcal{X}$, it follows from the complex ([K], 1.7)

$$H^3(K, \mathbb{Z}/2) \xrightarrow{\partial} \bigoplus_{\eta \in \mathcal{X}^1} H^2(\kappa(\eta), \mathbb{Z}/2) \xrightarrow{\partial} \bigoplus_{y \in \mathcal{X}^2} H^1(\kappa(y), \mathbb{Z}/2)$$

that $\partial_{(\pi)}(\alpha)$ is possibly ramified only at the discrete valuations of $\kappa(\pi)$ corresponding to $C_1 \cap E$. Since $C_1 \cap E = \emptyset$, it follows that $\partial_{(\pi)}(\alpha)$ is unramified at every discrete valuation ring of $\kappa(\pi)$. Since $\kappa(\pi)$ is either a global field or a local field, by class field theory, we have $\partial_{(\pi)}(\alpha) = 0$ and hence α_L is unramified at S.

Suppose that $C_1 \cap E \neq \emptyset$. Suppose that f is a parameter in A. Then S over R is ramified and hence α_S is unramified at S. Assume that f is a unit in A. Suppose that f is as in ii). Since $v_{\overline{\delta}_{x_i}}(\overline{f}) = 1$ or $f(x_i)$ is not a square in $\kappa(x_i)$, for $1 \leq i \leq r$, it follows that $\overline{f}$ is not a square in $B/(\pi)$. Since C and E have only normal crossings, $B/(\pi)$ is a regular semi local ring and is integrally closed. Hence $\overline{f}$ is not a square in the residue field $\kappa(\pi)$ of A. Since $H^3(K, \mathbb{Z}/2)$ is generated by symbols, using the explicit description of the residue map on the symbols, one easily sees that if S over A is unramified, then $\partial_S(\alpha_L) = \partial_A(\alpha) \otimes \kappa(\pi)(\sqrt{\overline{f}})$. Suppose that $\kappa(\pi)$ is a p-adic field. Since the residue field of S is the quadratic extension $\kappa(\pi)(\sqrt{\overline{f}})$, it follows that $\partial_A(\alpha) \in H^2(\kappa(\pi), \mathbb{Z}/2)$ is split over $\kappa(\pi)(\sqrt{\overline{f}})$. Since f is a unit in A, S over R is unramified and hence $\partial_S(\alpha_L) = \partial_A(\alpha) \otimes \kappa(\pi)(\sqrt{\overline{f}}) = 0$ and α_L is unramified at S. Suppose that $\kappa(\pi)$ is a function field in one variable over a finite field. As above it follows that $\partial_A(\alpha)$ is possibly ramified only at the discrete valuation rings of $\kappa(\pi)$ given by the prime elements $\overline{\delta}_{x_i}$ in $B/(\pi)$, $1 \leq i \leq r$. By the assumption on $\overline{f}$, in view of (3.1), $\partial_A(\alpha) \otimes \kappa(\pi)(\sqrt{\overline{f}}) = 0$ and the lemma follows. □

Proposition 4.4 Let k, K be as above. Let $\alpha \in H^3(K, \mathbb{Z}/2)$ and $a, b \in K^*$. Then there exists an $f \in K^*$ which is a value of the quadratic form $< a, b >$ such that $\alpha \otimes K(\sqrt{f}) = 0$.

Proof. Let $\mathcal{X}$ be a regular, projective model of X over $\mathcal{O}_k$ such that

$$\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \cup \mathrm{Supp}(ram_{\mathcal{X}}\alpha) \subset \mathrm{Supp}(C + E),$$

where C and E are regular curves on $\mathcal{X}$ with only normal crossings. Let $T = C \cap E$. Let B be the semi-local ring at T. For $x \in T$, let $\pi_x, \delta_x \in B$ be local equations for C and E at x, respectively. Since B is a unique factorisation domain with quotient field K, without loss of generality we assume that a, b are square free in B and $\mathrm{Supp}_{\mathrm{Spec}(B)}(ab) \subset \mathrm{Supp}(C + E)$. Let $c \in B$ be the greatest common divisor of a and b, so that $a = ca'$, $b = cb'$, with $a', b' \in B$. Since a and b are square free, c, a', b' are pairwise coprime. For $x \in T$, choose $u_x, v_x \in \kappa(x)$ as follows:

i) Suppose $c(x) = 0$. Let m_x denote the maximal ideal of B at x. Since c, a', b' are pairwise coprime and the only prime elements of B_{m_x} which divide $ca'b'$ are π_x, δ_x, at least one of a' and b' is coprime to π_x and δ_x, and hence is a unit at x. Thus $a'(x) \neq 0$ or $b'(x) \neq 0$. Let $u_x, v_x \in \kappa(x)$ be such that $a'(x)u_x^2 + b'(x)v_x^2 \neq 0$.

ii) Suppose that $c(x) \neq 0$ and $a'b'(x) = 0$. Let $u_x = v_x = 1$.

iii) Suppose that $c(x)a'(x)b'(x) \neq 0$. Since $\kappa(x)$ is a finite field of characteristic not equal to 2, every element of $\kappa(x)$ is represented by the quadratic form $< a'(x), b'(x) >$. Let $u_x, v_x \in \kappa(x)$ be such that $c(x)a'(x)b'(x)(a'(x)u_x^2 + b'(x)v_x^2) \notin \kappa(x)^{*2}$.

Let $u, v \in B$ be such that $u(x) = u_x$ and $v(x) = v_x$ for all $x \in T$. Let $f = ca'b'(a'u^2 + b'v^2)$. Clearly f is a value of $c < a', b' >=< a, b >$. We now show that $\alpha \otimes K(\sqrt{f}) = 0$. Let $L = K(\sqrt{f})$ and k' be the field of constants in L. Let X' be a smooth, projective, irreducible curve over k' with $k'(X') = L$. Let $\mathcal{X}'$ be a regular proper model of X' over $\mathcal{O}_{k'}$ and $y \in \mathcal{X}'$ be a point of codimension one. Let $S = \mathcal{O}_{\mathcal{X}',y}$ be the discrete valuation ring at y. As in the proof of (3.9), it is enough to show that α_L is unramified at S. Since $\mathcal{X}$ is projective over $\mathcal{O}_k$, there exists a point $z \in \mathcal{X}$ of codimension 1 or 2, such that S dominates the local ring $A = \mathcal{O}_{\mathcal{X},z}$.

Suppose $\dim(A) = 1$. Then A is a discrete valuation ring. Suppose that z does not correspond to a component of C or E. Then α is unramified at A and hence α_L is unramified at S. Assume that z corresponds to a component of C or E. Let z correspond to a component C_1 of C.

Suppose that $C_1 \cap E = \emptyset$. Then by (4.3), α_L is unramified at S.

Suppose that $C_1 \cap E \neq \emptyset$. Let π be a prime element of B corresponding to the component C_1. Since c, a', b' are pairwise coprime in B, it follows that at most one of c, a', b' is divisible by π.

Suppose π divides c. Then by i), $a'u^2 + b'v^2$ is a unit at every point of $C_1 \cap E$. Since A is a localisation of $\mathcal{O}_{\mathcal{X},x}$, for $x \in C_1 \cap E$, $a'u^2 + b'v^2$ is a unit in A. Further, since π divides c, both a' and b' are units in A. Therefore f is a parameter in A and hence by (4.3), α_L is unramified at S.

Suppose π does not divides c and divides a' or b'. Let $x \in C_1 \cap E$. If $c(x) = 0$, then by i), $a'u^2 + b'v^2$ is a unit at x and hence it is a unit in A. If $c(x) \neq 0$, then by ii), u and v are units at x and hence units in A. Since only one of the a', b' is divisible by π, $a'u^2 + b'v^2$ is a unit in A. Therefore, as above f is a parameter in A and α_L is unramified at S.

Suppose that π does not divides $ca'b'$. Let $x \in C_1 \cap E$. If $c(x) = 0$, then by i), $a'u^2 + b'v^2$ is a unit at x and hence a unit in A. Suppose that $c(x) \neq 0$. Since π does not divides $a'b'$, the only prime elements of B_{m_x} which divide $a'b'$ being π and δ_x, either $a'(x) \neq 0$ or $b'(x) \neq 0$. Therefore if $a'b'(x) = 0$, then by ii), $a'u^2 + b'v^2$ is a unit at x and if $a'b'(x) \neq 0$, then by iii), $a'u^2 + b'v^2$ is a unit at x. Therefore $a'u^2 + b'v^2$ is a unit in A and hence $v_{\overline{\delta}_x}(\overline{f}) = v_{\overline{\delta}_x}(\overline{ca'b'})$, which is equal to 0 or 1. Further if $v_{\overline{\delta}_x}(\overline{f}) = 0$, by iii), $f(x)$ is not a square in $\kappa(x)$. Therefore, by (4.3), α_L is unramified at S.

Suppose $\dim(A) = 2$. Then z is closed point of $\mathcal{X}$. If $z \notin C \cup E$, then α is unramified on A and hence unramified at S (3.7). Assume that $z \in C \cup E$. If $z \notin C \cap E$, then by (3.3 and 3.5), α_L is unramified at S. Further assume that $z \in C \cap E$. Then $A = B_{m_z}$, where m_z is the maximal ideal of B at z.

Suppose that $c(z) = 0$. Then, as in i), at least one of a', b' is a unit at z and by the choice of u, v, $a'u^2 + b'v^2$ is a unit at z. Since the only prime elements of A which divide $ca'b'$ are π_z, δ_z, it follows that $f = ca'b'(a'u^2+b'v^2)$ is of the form $w\pi_z$ or $w\delta_z$ or $w\pi_z\delta_z$, with $w \in A^*$. Since $f \in L^{*2}$, by (3.6), α_L is unramified at S.

Suppose that $c(z) \neq 0$ and either $a'(z) = 0$ or $b'(z) = 0$. If $a'(z) \neq 0$ or $b'(z) \neq 0$, then as above, one shows that either π_z or δ_z or $\pi_z\delta_z$ is as in (3.6, ii) and hence, by (3.6), α_L is unramified at S. Suppose that $a'(z) = b'(z) = 0$. Since the only prime elements of A which divide a', b' are π_z, δ_z, a', b' are coprime and a', b' are not units at z, we have $a' = w\pi_z$ and $b' = w'\delta_z$ or $a' = w\delta_z$ and $b' = w'\pi_z$ for some $w, w' \in A^*$. Consider the case where $a' = w\pi_z$ and $b' = w'\delta_z$, with $w, w' \in A^*$ (the other case being similar). Let ν_S denote the valuation at S. Since S dominates A, we have $\nu_S(a') \geq 1$ and $\nu_S(b') \geq 1$. We assume without loss of generality that $\nu_S(a') \leq \nu_S(b')$. Then, $b'/a' \in S$ and

$$f = cb'(u^2 + \left(\frac{b'}{a'}\right) v^2)(a')^2.$$

Suppose that $\nu_S(a') < \nu_S(b')$. Then $u^2 + \frac{b'}{a'}v^2 \in S^*$. Since $b' = w'\delta_z$, $w' \in A^*$, $c \in A^*$ and $f \in L^{*2}$, it follows that δ_z is as in (3.6, ii) and α_L is unramified at S. Suppose that $\nu_S(a') = \nu_S(b')$. If $u^2 + \frac{b'}{a'}v^2 \in S^*$, then $\nu_S(f) = \nu_S(b') + 2\nu_S(a') = 3\nu_S(b')$. Since $\nu_S(f)$ is even, it follows that $\nu_S(a') = \nu_S(b')$ is even. In particular $\nu_S(\pi_z) = \nu_S(\delta_z)$ is even. By (3.3, iii), we have $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(\mu)\cdot(\mu')\cdot(\pi_z)$, $(\mu)\cdot(\mu')\cdot(\delta_z)$, $(\mu)\cdot(\pi_z)\cdot(\delta_z)$, with μ, μ' running over A^*. Since π_z and δ_z have even valuations at S, clearly α''_L is unramified at S. By (3.7), α'_L, and hence α_L, is unramified at S. Assume that $u^2 + \frac{b'}{a'}v^2$ is not a unit in S. Let $n = \nu_S(a') = \nu_S(b')$. Let θ be a parameter in S and write $a' = w_1\theta^n$, $b' = w_2\theta^n$, with $w_1, w_2 \in S^*$. By (ii), $u, v \in A^*$. Since $u^2 + \frac{b'}{a'}v^2$ is not a unit in S, we have

$$\frac{\overline{u}^2}{\overline{v}^2} = -\overline{\left(\frac{b'}{a'}\right)} = \frac{\overline{w_2}}{\overline{w_1}}.$$

By (3.7, iii), we have $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(\mu)\cdot(\mu')\cdot(\pi_z)$, $(\mu)\cdot(\mu')\cdot(\delta_z)$, $(\mu)\cdot(\pi_z)\cdot(\delta_z)$, with μ, μ' running over A^*. As in the proof of (3.6), $(\mu)\cdot(\mu')\cdot(\pi_z)$ and $(\mu)\cdot(\mu')\cdot(\delta_z)$

are unramified at S. Since $a' = w\pi_z = w_1\theta^n$, $b' = w'\delta_z = w_2\theta^n$, we have

$$(\mu)\cdot(\pi_z)\cdot(\delta_z) = (\mu)\cdot(ww_1\theta^n)\cdot(w'w_2\theta^n).$$

If n is even, then clearly $(\mu)\cdot(\pi_z)\cdot(\delta_z)$ is unramified at S. Assume that n is odd. Then, we have

$$(\mu)\cdot(\pi_z)\cdot(\delta_z) = (\mu)\cdot(ww_1\theta)\cdot(w'w_2\theta) = (\mu)\cdot(ww_1\theta)\cdot(-ww_1w'w_2)$$

and

$$\partial_S((\mu)\cdot(\pi_z)\cdot(\delta_z))_L) = (\overline{\mu})\cdot(-\overline{ww_1w'w_2}).$$

Since $\overline{w_2}/\overline{w_1}$ is a square in the residue field of S, we have $\partial_L((\mu)\cdot(\pi_z)\cdot(\delta_z)) = (\overline{\mu})\cdot(-\overline{ww'})$. Since $\mu, w, w' \in A^*$ and $\kappa(z)$ is a finite field, it follows that $(\overline{\mu})\cdot(-\overline{ww'}) = 0$. Hence α_L is unramified at S.

Suppose that $c(z)a'(z)b'(z) \neq 0$. Then by the choice of u, v it follows that $f(z) \notin \kappa(z)^{*2}$. Since f is a square in S, it follows from (3.6) that α_L is unramified at S. This completes the proof of the proposition. □

Theorem 4.5 Let k be a non-dyadic p-adic field and K be a function field in one variable over k. Then every quadratic form over K of rank at least 11 is isotropic.

Proof. Let q be a quadratic form over K of rank 11. Then by the theorem of Saltman (cf. 2.2) $c(q)$ is a biquaternion algebra. Let q_0 be a quadratic form over K with $rk(q_0) = 5$, $d(q+q_0) = 1$ and $c(q+q_0) = 0$. Then $q+q_0 \in I^3(K)$. Therefore, by (4.1), there exists a 3-fold Pfister form q_1 over K such that $q = q_1 - q_0$. Since q is isotropic if and only if λq is isotropic for any $\lambda \in K^*$, we assume that $q_0 =< 1, a, b, c, d >$ for some $a, b, c, d \in K^*$. Let $\alpha = e_3(q_1)$. Then by (4.4), there exists $f \in K^*$ which is a value of $< -a, -b >$ such that $\alpha_1 \otimes K(\sqrt{f}) = 0$. Since e_3 is an isomorphism, $q_1 \otimes K(\sqrt{f})$ is hyperbolic. Therefore there exists $g, h \in K^*$ such that $q_1 =< 1, -f >< 1, g >< 1, h >$. Since $-f$ is a value of $< a, b >$, there exists $f' \in K^*$ such that $< a, b > \simeq < -f, f' >$. We have

$$\begin{aligned} q &= q_1 - q_0 \\ &= < 1, -f >< 1, g >< 1, h > - < 1, -f, f', c, d > \\ &= < 1, -f >< g, h, gh > - < f', c, d > . \end{aligned}$$

Since $rk(q) = 11$ and the rank of $< 1, f >< g, h, gh > - < f', c, d >$ is 9, it follows that q is isotropic over K. □

Theorem 4.6 Let K be as above and q a quadratic form over K of rank at least 9. Suppose that $c(q)$ is of index at most 2. Then q is isotropic.

Proof. By (4.5), if the rank of q is at least 11, then q is isotropic. Assume that rank of q is 9 or 10. Since $c(q)$ is of index at most 2, there exist $a, b \in K^*$ such that $c(q) = (-a, -b)$. Suppose that the rank of q is 9. By scaling, we can assume that $d(q) = 1$. Let $q_0 =< a, b, ab >$. Then $d(q - q_0) = 1$ and $c(q - q_0) = 0$. Therefore $q = q_0 + q_1$ for some $q_1 \in I^3(K)$. As in the proof of (4.5), there exists $f \in K^*$ which is a value of $< a, b >$ and $q_1 \otimes K(\sqrt{-f})$ is hyperbolic. Therefore we have $< a, b >=< f, f' >$ and $q_1 =< 1, f >< 1, g >< 1, h >$ for some $f', g, h \in K^*$. Since $q = q_0 + q_1$ and $I^4(K) = 0$, we have

$$\begin{aligned} (-ab)q &= (-ab)q_0 + q_1 \\ &= < -b, -a, -1 > + q_1 \\ &= < -f, -f', -1 > + < 1, f > + < 1, f >< g, h, gh > \\ &= < -f' > + < 1, f >< g, h, gh > . \end{aligned}$$

Therefore q is isotropic. Suppose that the rank of q is 10. Let $q' = q \perp < 1 >$. Then $c(q) = c(q')$. Since the rank of q' is 11, it is isotropic by (4.5). Write $q' =< 1, -1 > \perp q''$. Then the rank of q'' is 9 and $c(q'') = c(q)$. Therefore q'' is isotropic. Since $q = q'' \perp < -1 >$, q is isotropic. □

5. Zero-cycles on quadric fibrations

Let k be a p-adic field and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration over C (cf. [CTSk]). For a variety Y, let $CH_0(Y)$ denote the Chow group of zero-cycles on Y. Let $\pi_* : CH_0(X) \to CH_0(C)$ be the induced homomorphism and $CH_0(X/C) = \ker(\pi_*)$. If $\dim(X) = 2$, then it was proved in ([G]) that the group $CH_0(X/C)$ is finite. In ([CTSk]), Colliot-Thélène and Skorobogatov proved that if $\dim(X) = 3$, then $CH_0(X/C)$ is finite and raised the following question:
If dim$(X) \geq 4$, is the group $CH_0(X/C)$ zero or at least finite?
In ([PS], 4.8), it was shown that the group $CH_0(X/C)$ is finite, answering the latter part of the above question. Recently Hoffmann and Van Geel ([HV], 4.2) proved that if k is non-dyadic and $\dim(X) \geq 6$, then $CH_0(X/C) = 0$.

Using results proved in §4, we show that $CH_0(X/C) = 0$ if $\dim(X) \geq 4$ and k is a non-dyadic p-adic field.

We recall the identification of $CH_0(X/C)$ with a certain subquotient of $k(C)^*$ given in ([CTSk]). Let k be a field of characteristic not equal to 2 and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration of relative dimension at least 1. Let q be a quadratic form over $k(C)$ defining the generic fibre of π. Let $N_q(k(C))$ be the subgroup of $k(C)^*$ generated by elements of the type ab with $a, b \in k(C)^*$, which are values of q over $k(C)$. Let $k(C)^*_{\text{dn}}$ be the subgroup of $k(C)^*$ consisting of functions, which, at each closed point P of C, can be written as a product of a unit at P and an element of $N_q(k(C))$. We recall the following result from ([CTSk]).

Proposition 5.1 There is an isomorphism $CH_0(X/C) \xrightarrow{\sim} k(C)^*_{\text{dn}}/k^*N_q(k(C))$.

Theorem 5.2 Let k be a non-dyadic p-adic field and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration. If $\dim(X) \geq 4$, then $CH_0(X/C) = 0$.

Proof. Let q be a quadratic form over $k(C)$ defining the generic fibre of π. Since $\dim(X) \geq 4$, rank of q is at least 5. If q is isotropic, then every element in $k(C)^*$ is represented by q over $k(C)$ and hence $N_q(k(C)) = k(C)^*$. Assume that q is anisotropic over $k(C)$. Let $f \in k(C)^*$. Since $< 1, -f > \otimes q \otimes k(C)(\sqrt{f})$ is hyperbolic, $c(q \otimes < 1, -f >) \otimes k(C)(\sqrt{f})$ is zero and hence the index of $c(q \otimes < 1, -f >)$ is at most 2. Therefore by (4.6), $< 1, -f > \otimes q$ is isotropic. That is, there exist v, w in the underlying vector space of q, with at least one of them non-zero such that $q(v) - fq(w) = 0$. Since q is anisotropic $q(v)q(w) \neq 0$. Therefore $f = q(v)q(w)^{-1} \in N_q(k(C))$ and hence $N_q(k(C)) = k(C)^*$. By (5.1), it follows that $CH_0(X/C) = 0$. □

6. Cayley algebras

Let K be a field of characteristic not equal to 2. Let G be a split simple algebraic group of type G_2 defined over K. We recall from ([Se], §8.3) the following description of the set of isomorphism classes of Cayley algebras over K.

Theorem 6.1 There are canonical bijections between the following:

i) $H^1(K,G)$,

ii) $H^3_{\text{dec}}(K) = \{\alpha \in H^3)K, \mathbb{Z}/2), \alpha = a \cup b \cup c, a, b, c \in K^*\}$,

iii) The set of isomorphism classes of K-forms of G,

iv) The set of isomorphism classes of Cayley algebras over K,

v) The set of isomorphism classes of 3-fold Pfister forms.

Let k be a p-adic field. Let P be the set of closed points of $\mathbb{P}^1_{k\,k}$ and

$$C(P) = \{f : P \to \mathbb{Z}/2 \mid \text{supp}(f) \text{ finite and } \sum_{x \in P} f(x) = 0\}.$$

The exact sequence

$$0 \to H^3(k(t), \mathbb{Z}/2) \to \oplus_{x\in P} H^2(k(x), \mathbb{Z}/2) \to H^2(k, \mathbb{Z}/2) \to 0$$

identifies $H^3(k(t), \mathbb{Z}/2)$ with $C(P)$, noting that $H^2(k(x), \mathbb{Z}/2) = \mathbb{Z}/2$ for every $x \in P$ and the map $\oplus_{x\in P} H^2(k(x), \mathbb{Z}/2) \to H^2(k, \mathbb{Z}/2)$ is the addition. In ([Se], §8.3), Serre raises the question whether $H^1(k(t), G)$ is in bijection with $C(P)$. This is equivalent to the question whether $H^3_{\text{dec}}(k(t)) = H^3(k(t), \mathbb{Z}/2)$. In view of (3.9), this is indeed true if k is non-dyadic.

Let k be a p-adic field, $p \neq 2$ and X a smooth, projective, integral curve over k. Using a result of Kato ([K]) and following Serre, we give a description of $H^1(k(X), G)$ as follows. Let $\mathcal{X}$ be a regular, proper model of X over $\mathcal{O}_k$. Let $Y = \mathcal{X} \times_{\text{Spec}(\mathcal{O}_k)} \text{Spec}(\mathbf{F}_q)$ be the special fibre, where $\mathbf{F}_q$ is the residue field of k. Let $\pi : \tilde{Y} \to Y$ be the normalisation of Y. Let Y_{sing} denote the set of singular points of Y and $Q = \pi^{-1}(Y_{sing})$. Let $\tilde{Y} = \cup_1^r \tilde{Y}_i$, $\tilde{Y}_i$ denoting the irreducible components of $\tilde{Y}$. Let

$$C(Q) = \{f : Q \to \mathbb{Z}/2 \mid supp(f) \text{ finite}, \sum_{x\in\tilde{Y}_i} f(x) = 0, 1 \leq i \leq r\}.$$

For $y \in Y^1$, let

$$\partial_i^y : H^2(k(\tilde{Y}_i), \mathbb{Z}/2) \to H^1(k(y), \mathbb{Z}/2)$$

be the residue homomorphism defined by $\partial_i^y = 0$ if $\pi^{-1}(y) \cap \tilde{Y}_i = \emptyset$ and otherwise

$$\partial_i^y = \sum_{\tilde{y} \in \pi^{-1}(y) \cap \tilde{Y}_i} \partial_i^{\tilde{y}},$$

where $\partial_i^{\tilde{y}}$ denotes the residue map at $\tilde{y}$. By a result of Kato ([K], 5.2), we have an isomorphism

$$H^3_{\rm nr}(\kappa(X)/X, \mathbb{Z}/2) \xrightarrow{\sim} \ker(\oplus_i H^2(k((\tilde{Y})_i), \mathbb{Z}/2) \stackrel{\partial=(\partial_i^y)}{\rightarrow} \oplus_{y \in Y^1} H^1(k(y), \mathbb{Z}/2)).$$

Lemma 6.2 We have an isomorphism

$$\ker(\oplus_i H^2(k((\tilde{Y})_i), \mathbb{Z}/2) \stackrel{\partial=(\partial_i^y)}{\rightarrow} \oplus_{y \in Y^1} H^1(k(y), \mathbb{Z}/2)) \simeq C(Q)$$

Proof. Let $(\alpha_i) \in \oplus_i H^2(k(\tilde{Y}_i), \mathbb{Z}/2)$ be such that $\partial((\alpha_i)) = 0$. Then for a closed point $\tilde{y} \in \tilde{Y}_i \setminus Q$, $\partial_{\tilde{y}}(\alpha_i) = 0$. For $\tilde{y} \in Q \cap \tilde{Y}_i$, let $f(\tilde{y}) = \partial_{\tilde{y}}(\alpha_i) \in H^1(\kappa(\tilde{y}), \mathbb{Z}/2) = \mathbb{Z}/2$. Then, by class field theory for function fields in one variable over finite fields, it follows that $f \in C(Q)$. Conversely, let $f \in C(Q)$. Then by class field theory, there exist $\alpha_i \in H^2(k(\tilde{y}_i), \mathbb{Z}/2)$ such that for $\tilde{y} \in Q \cap \tilde{Y}_i$, $\partial_{\tilde{y}}(\alpha_i) = f(\tilde{y})$ and if $\tilde{y} \in \cup \tilde{Y}_i \setminus Q$, then $\partial_{\tilde{y}}(\alpha_i) = 0$ for all i. Since $f \in C(Q)$, $\partial(\alpha_i) = 0$. This proves the lemma. □

Let P be the set of closed points of X. Let

$$C(P) = \{f : P \to \mathbb{Z}/2 \mid supp(f) \text{ finite and } \sum_{x \in P} f(x) = 0\}.$$

We have an exact sequence ([K], 5.2)

$$0 \to H^3_{\rm nr}(k(X)/X, \mathbb{Z}/2) \to H^3(k(X), \mathbb{Z}/2) \to \oplus_{x \in P} H^2(k(x), \mathbb{Z}/2) \to \mathbb{Z}/2 \to 0.$$

This sequence induces an exact sequence

$$0 \to H^3_{\rm nr}(k(X)/X, \mathbb{Z}/2) \to H^3(k(X), \mathbb{Z}/2) \to C(P) \to 0.$$

By (6.2), we have $H^3_{\rm nr}(k(X)/X, \mathbb{Z}/2) \simeq C(Q)$. In view of (3.9), we have $H^3_{\rm dec}(k(X), \mathbb{Z}/2) = H^3(k(X), \mathbb{Z}/2)$ and we have the following the following

Theorem 6.3 Let k be a non-dyadic p-adic field and X a smooth, projective, irreducible curve over k. The bijection $H^1(k(X), G) \simeq H^3_{dec}(k(X), \mathbb{Z}/2) = H^3(k(X), \mathbb{Z}/2)$ makes $H^1(k(X), G)$ a $\mathbb{Z}/2$-vector space which fits into an exact sequence

$$0 \to C(Q) \to H^1(k(X), G) \to C(P) \to 0.$$

References

[A] Albert, A.A.: Normal division algebras of degree four over an algebraic field, Trans. Amer. Math. Soc. 34, 363-372 (1931).

[Ar] Arason, J.K.: Cohomologische Invarianten quadratischer Formen, J. Algebra 36, 448-491 (1975).

[AEJ] Arason, J.K.: Elman, R. and Jacob, B.: Fields of cohomological 2-dimension three, Math. Ann. 274, 649-657 (1986).

[CT] Colliot-Thélène, J.-L.: Birational invariants, purity, and the Gresten conjecture, Proceedings of Symposia in Pure Math. 55, Part 1, 1-64.

[CTS] Colliot-Thélène, J.-L. et Sansuc, J.-J.: Fibrés quadratiques et composantes connexes réelles, Math. Ann. 244 (1979), 105-134.

[CTSk] J.-L. Colliot-Thélène et A.N. Skorobogatov: Groupe de Chow des zérocycles sur les fibrés en quadriques, K-Theory, 7, 477-500 (1993).

[G] M. Gros: 0-cycles de degré zéro sur les surface fibrées en coniques, J. für die reine und angew. Math. 373, 166-184 (1987).

[Gr] A. Grothendieck: Le groupe de Brauer III, Dix exposés sur la cohomologie des schémas, Amsterdam, North-Holland, 88-188 (1968).

[HV] Hoffmann, D.W. and Van Geel, J.: Zeroes and norm groups of quadratic forms over function fields in one variable over a local non-dyadic field, preprint (1997).

[K] Kato, K.: A Hasse principle for two-dimensional global fields, J. reine Angew. Math. 366, 142-181 (1986).

[L] Lichtenbaum, S.: Duality theorems for curves over p-adic fields, Invent. Math. 7, 120-136 (1969).

[M] Merkurjev, A.S.: On the norm residue symbol of degree 2, Dokl. Akad. Nauk. SSSR 261, 542-547 (1981).

[O] Ojanguren, M.: Quadratic forms over regular rings, J. Indian Math. Soc. 44, 109-116 (1980).

[P] Parimala, R.: Witt groups of affine three folds, Duke Math. J. 57, 947-954 (1989).

[PS] Parimala, R. and Suresh, V.: Zero-cycles on quadric fibrations: finiteness theorems and the cycle map, Invent. Math. 122, 83-117 (1995).

[S] Saltman, D.J.: Division Algebras over p-adic curves, J. Ramanujan Math. Soc. 12, no. 1, 25-47 (1997).

[Sc] Scharlau, W.: Quadratic and Hermitian Forms, Grundlehren der Math. Wiss., Vol. 270, Berlin, Heidelberg, New York 1985.

[Se] Serre, J.P.: Cohomologie Galoisienne : Progrès et Problèmes, Séminaire Bourbaki, 783 (1993-94).

[T] Tate, J.: Relations between K_2 and Galois cohomology, Invent. Math. 36, 257-274 (1976).

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hep-th/9705074

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The gluon/charm content of the $\eta\prime$ meson and instantons
hep-ph/9706316; Phys.Rev. D57 (1998) 2001-2004

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On chain graph models for description of conditional independence structures

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On special pieces in the unipotent variety

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A unified theory of density models and auto-encoders
DERA report DERA/CIS/CIS5/651/FUN/STIT/5-4 31 October 1997

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Distribution of Eigenvalues in non-Hermitian Anderson models
Phys. Rev. Lett. 80 (1998) No.13, 2897-2900

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