Final Version 5 March 1998

Discrete series characters for *GL*(*n*, *q*)

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Introduction

An ordinary, irreducible character χ of the finite general linear group $GL(n,q)$ is said to belong to the "discrete series" if it is not a constituent of the permutation character induced from the radical U of any proper parabolic subgroup $P = L.U$ of $GL(n,q)$. Such a character χ cannot be obtained by "Harish-Chandra induction" from characters of $GL(n',q)$ for $n' < n$, in fact χ cannot be expressed as a linear combination of induced characters from proper parabolic subgroups of $GL(n,q)$.

Three different methods have been used to calculate the discrete series characters for $GL(n,q)$.

(1) In [GL], they are constructed using the "Brauer lifts" of natural modular characters of $GL(n,q)$.

(2) In [L], G. Lusztig constructs a module $D(V)$ which affords a discrete series character for $GL(V) = GL(n,q)$ (V is an n–dimensional vector space over a field of q elements), as an eigenspace of a homology module for a certain simplicial complex made out of affine flags on V.

(3) In their fundamental work [DL], Deligne and Lusztig use the étale cohomology of certain varieties related to a reductive group **G** to construct (generalized) characters of finite subgroups $G = \mathbf{G}^F$ of **G**. Taking $\mathbf{G} = \mathbf{GL_n}$, the discrete series characters of $GL(n,q)$ are (up to a sign) Deligne–Lusztig's $R_{\mathbf{T}}^{\psi}$, where $\mathbf{T}$ is a maximal torus of **G** such that $T = \mathbf{T}^F$ is of order $q^n - 1$, and ψ is a character of T in general position.

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Final Version 5 March 1998

It is the purpose of the present work to present the discrete series characters of $GL(n,q)$ in a rather simpler way, namely as $\mathbf{Z}$–linear combinations of characters induced from linear characters on certain subgroups of $GL(n,q)$. Of course, R. Brauer's theorem ([B], theorem A) shows that it is possible to express any character of any finite group G as $\mathbf{Z}$–linear combination of characters induced from linear characters on "elementary" subgroups of G. But we are able, in our special situation, to achieve our goal much more economically than would be possible by invoking Brauer's general theorem.

Let k be a field of q elements. The discrete series characters χ are determined by certain class-functions $J_n(\psi)$ on $G = GL(n,q)$ described in section 1 (the parameter ψ is equivalent to a character of Deligne–Lusztig's maximal torus T). $J_n(\psi)$ has "degree" $(1-q)(1-q^2)\dots(1-q^{n-1})$, and has primary support , i.e. $J_n(\psi)\ (g) \neq 0$ only if the characteristic polynomial $\det(tI_n - g)$ is a power of an irreducible polynomial in $k[t]$. We describe in section 2 a family $\mathrm{F}(n)$ of subgroups $H_{d,n}(k)$ of G, one for each divisor d of n. For example $H_{1,n}(k)$ is the product of the centre Z of G with the group P of all upper unitriangular matrices in G, while $H_{n,n}(k)$ is a maximal torus T of order $q^n - 1$. Each element g of each $H_{d,n}(k)$ is primary, and between them, the $H_{d,n}(k)$ meet all the primary conjugacy classes of G. In section 3 we define, for each d and for each partition λ of n/d, a character $X_{d,n}(\psi,\lambda)$ of G, which is induced from a linear character of $H_{d,n}(k)$. Our main theorem (theorem 3.2) states that there exists a family of polynomials $r_\lambda(T) \in \mathbf{Z}[T]$, indexed by the set of all partitions λ (of all positive integers), such that for all n, all ψ and all fields k of order q,

$$(3.3)\qquad J_n(\psi) = \sum_{d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) X_{d,n}(\psi,\lambda).$$

Section 4 states without proof some rather technical propositions on the $X_{d,n}(\psi,\lambda)$ and in section 5, the theorem 3.2 is proved on the assumption that these propositions are true. The proofs of the propositions in section 4 require some formulae on the Gelfand–Graev character for $G = GL(n,q)$; these are given in sections 6 and 8 (section 6 is essentially due to Deligne–Lusztig) and may have some interest in their own right. The proofs which were deferred from section 4 are given in sections 7 and 9. An appendix at the end of the paper gives the polynomials $r_\lambda(T)$ for all partitions $\lambda \vdash n \le 5$..

1 Notation. The class function $J_n(\psi)$

n is a positive integer, q is a power of a prime p, and $\bar{k}$ is an algebraically closed field of characteristic p. For each positive integer d, k_d is the unique subfield of $\bar{k}$ of order q^d. Write $k = k_1$.

$M_d = k_d^\times$ and $\hat{M}_d = \mathrm{Hom}(M_d, \mathbf{C}^\times)$ are the multiplicative group of k_d, and the character group of M_d, respectively.

From now on, we denote the group $GL(n,q)$ as $G_n(k)$; similarly $GL(n,q^d) = G_n(k_d)$, etc. For any group G, the set of all conjugacy classes of G is denoted $\mathrm{ccl}\,G$.

t, T are indeterminates over k, $\mathbf{Z}$, respectively.

A sequence of positive integers $\lambda = (\lambda_1,\ldots,\lambda_b)$ is called a *composition* of n if $\lambda_1+\ldots+\lambda_b = n$ (notations $\lambda \models n$ and $|\lambda| = n$). If $\lambda_1 \ge \ldots \ge \lambda_b$, then λ is a *partition* of n (notation $\lambda \vdash n$). We sometimes use the other standard notation $\lambda = 1^{l_1}2^{l_2}\ldots$ for a partition λ, to indicate that λ has l_1 parts equal to 1, l_2 parts equal to 2, etc. Finally if s is a positive integer, $\lambda \cdot s$ will denote the partition $1^{l_1 s}2^{l_2 s}\ldots$ of $n\,s$.[2]

If d and r are positive integers and $X \in GL_d(k)$, then X_r denotes the matrix

1.1
$$X_r = \begin{bmatrix} X & X & 0 & \cdot\cdot & 0 & 0 \\ 0 & X & X & \cdot\cdot & 0 & 0 \\ 0 & 0 & X & \cdot\cdot & 0 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & \cdot\cdot & X & X \\ 0 & 0 & 0 & \cdot\cdot & 0 & X \end{bmatrix} \qquad (r \text{ diagonal blocks } X),$$

which is an element of $G_{dr}(k)$. If $\sigma = (\sigma_1,\ldots,\sigma_b)$ is a partition of a positive integer e, then $X_{(\sigma)}$ denotes the matrix

[2] This should not be confused with the partition $s \cdot \pi = s^{p_1}(2s)^{p_2}\ldots$ defined in [GL], p.435.

1.2 $$X_{(\sigma)} = X_{\sigma_1} \oplus \ldots \oplus X_{\sigma_b}$$

($\oplus$ means "diagonal sum" of matrices); $X_{(\sigma)}$ is an element of $G_{de}(k)$.

Let $\Phi(k)$ be the set of all monic irreducible polynomials $f(t)$ over k, excepting $f(t) = t$. The degree of $f = f(t)$ is denoted $d(f)$.

For any $f \in \Phi(k)$, $C(f)$ denotes a matrix in $GL_{d(f)}(k)$ having characteristic polynomial f. This determines $C(f)$ only up to conjugacy in $G_{d(f)}(k)$, but this will be sufficient for our purposes.

Definition If $d(f) = d$ divides n, and if $\sigma = (\sigma_1, \ldots, \sigma_b)$ is a partition of $e = n/d$, let f^σ be the conjugacy class of $G_n(k)$ which contains the matrix $C(f)_{(\sigma)}$. Every element g of f^σ is primary, and conjugacy classes of form f^σ are called *primary classes*. Every conjugacy class of $G_n(k)$ can be written uniquely as $\oplus_{f \in \Phi(k)} f^{\sigma(f)}$, where the partitions $\sigma(f)$ satisfy $\sum_{f \in \Phi(k)} d(f)|\sigma(f)| = n$. In this work we deal only with primary classes.

Jordan factorization An element $g \in G_n(k)$ is *unipotent* of *type* $\sigma \vdash n$ if it is conjugate in $G_n(k)$ to the matrix $(1)_{(\sigma)}$. Notice that g is unipotent if and only if it has p-power order, i.e. is a p-*element*. An element g is *semisimple* if it has order prime to p, i.e. is a p'-*element*.. A primary element $g \in G_n(k)$ is semisimple if and only if it is conjugate in $G_n(k)$ to an element of the form $C(f)_{(1^{n/d})}$ for some $f \in \Phi(k)$ of degree d dividing n. Each element $g \in G_n(k)$ has a unique factorization $g = g_p g_{p'} = g_{p'} g_p$ as commuting product of a semisimple element $g_{p'}$ and a unipotent element g_p (see [St], p.25). We call $g = g_{p'} g_p$ the *Jordan factorization* of g, and call $g_{p'}$ and g_p the semisimple and unipotent parts, respectively, of g. The semisimple and unipotent parts of g are both powers of g. The Jordan factorization of the element $C(f)_{(\sigma)}$ in the definition above is $C(f)_{(\sigma)} = C(f)_{(1^{n/d})}(I_d)_{(\sigma)}$, notice that the matrix $(I_d)_{(\sigma)}$ is unipotent, because it is conjugate to $(1)_{\sigma \cdot d}$.

Final Version 5 March 1998

Definition of $J_n(\psi)$ Let ψ be any element of $\hat{M}_n$; this will be fixed from now on. If d is a divisor of n, we often identify ψ with the element $\psi|_{M_d}$ of $\hat{M}_d$. Define the class-function $J_n(\psi)$ on $G_n(k)$ as follows. If $c \in \mathrm{ccl}\, G_n(k)$ is not primary, then $J_n(\psi)\{c\} = 0$[3]. If f^σ is the primary class described above, with $d = d(f)$ and σ a partition of $e = n/d$, then

1.3 $$J_n(\psi)\{f^\sigma\} = \psi(f).k(\sigma : q^d),$$

where the symbols $\psi(f)$ and $k(\sigma:T)$ have the following meanings. If $y \in k_d$ is a zero of $f(t)$, so that $f(t) = (t-y)(t-y^q)\ldots(t-y^{q^{d-1}})$, then we define $\psi(f) :=$ $\psi(y) + \psi(y^q) + \ldots + \psi(y^{q^{d-1}})$. If $\sigma = (\sigma_1, \ldots, \sigma_b)$ is any partition, then the polynomial $k(\sigma:T) \in \mathbf{Z}[T]$ is defined to be $(1-T)(1-T^2)\ldots(1-T^{b-1})$ if σ has $b > 1$ parts, and to be 1 if $b = 1$.

$J_n(\psi)$ is a generalized character of $G_n(k)$ for any $\psi \in \hat{M}_n$, and if ψ is *primitive* (or is *in general position*; this means that $\psi, \psi^q, \ldots, \psi^{q^{n-1}}$ are distinct elements of $\hat{M}_n$) then $(-1)^{n-1} J_n(\psi)$ is irreducible [GL, pp.431, 433, 430]. The distinct irreducible characters which you get by taking all primitive $\psi \in \hat{M}_n$ comprise the discrete series for $G_n(k)$. However in the rest of this paper ψ will be an arbitrary element of $\hat{M}_n$.

2 The subgroups $H_{d,n}(k)$ of $G_n(k)$

A class-function F on $G_n(k)$ is said to *have primary support* if $F\{c\} \neq 0$ implies that the class c is primary. A *primary subgroup* H of $G_n(k)$ is one whose elements all lie in primary classes of $G_n(k)$. Clearly $J_n(\psi)$ has primary support, and any character of $G_n(k)$, which is induced from a character of a primary subgroup H, has primary support. In this section we define a family $\mathrm{F}(n)$ of primary subgroups of $G_n(k)$, and we show later that $J_n(\psi)$ can be expressed as a $\mathbf{Z}$-linear combination of characters induced from groups H of $\mathrm{F}(n)$.

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[3]To avoid a confusing forest of parentheses, the value of a class-function F at a class c is given as $F\{c\}$; or sometimes as $F\{g\}$, where g is an element of c.

Let d be a positive integer. The field k_d may be regarded as a k–algebra. It becomes a (simple) left k_d–module by multiplication ($a \in k_d$ acts on $v \in k_d$ to give $a\,v$).Then each k–basis $\{v_1,\ldots,v_d\}$ of k_d provides a k–algebra monomorphism $j_d{:}k_d \to \mathrm{Mat}_d(k)$, which takes $a \in k_d$ to the k–matrix $(a_{i\,j})$ given by the equations $av_j = \sum_i a_{i\,j}v_i$. If we use a different basis of k_d, then j_d is replaced by $\gamma \circ j_d{:}k_d \to \mathrm{Mat}_d(k)$, where γ is conjugation by some element of $G_d(k)$.

Now let e be a positive integer. The map $j_d{:}k_d \to \mathrm{Mat}_d(k)$ induces a group monomorphism $G_e(k_d) \varnothing G_{de}(k)$ which takes $(b_{ij}) \to (j_d(b_{i\,j}))$; we denote this also by j_d .

For any field K, let $Z_e(K)$ and $P_e(K)$ denote, respectively, the centre of $G_e(K)$ and the upper unitriangular subgroup of $G_e(K)$. Let $H_e(K)$ be the group $Z_e(K)P_e(K)$ (this is, of course, the direct product of $Z_e(K)$ and $P_e(K)$).

Now suppose that $d\,|\,n$, and that $e = \frac{n}{d}$. Then we define $Z_{d,n}(k)$, $P_{d,n}(k)$, $H_{d,n}(k)$ and $G_{d\,,n}(k)$ to be the images under the map $j_d : G_e(k_d) \varnothing G_n(k)$ of $Z_e(k_d)$, $P_e(k_d)$, $H_e(k_d)$ and $G_e(k_d)$ respectively.

Examples If $d = 1$, $e = n$ we take the monomorphism $j_1 : G_n(k) \varnothing G_n(k)$ to be the identity map, so that $Z_{1,n}(k) = Z_n(k)$, $P_{1,n}(k) = P_n(k)$ and $H_{1,n}(k) = H_n(k)$. If $d = n$, $e = 1$ then $P_{n,n}(k) = \{1\}$, and $Z_{n,n}(k) = H_{n,n}(k)$ is the image of $j_n : k_n^{\times} \varnothing G_n(k)$, which is a "maximal torus" of $G_n(k)$ (see [M], p.273), and has order $|k_n^{\times}| = q^n - 1$.

Definition Let $\mathrm{F}(n) = \{\, H_{d,n}(k) |\; d$ any positive divisor of $n\,\}$.

The set $\mathrm{F}(n)$ has the following virtues, proved in the lemma below: (i) each member $H_{d,n}(k)$ of $\mathrm{F}(n)$ is a primary subgroup of $G_n(k)$, and (ii) every primary class of $G_n(k)$ meets $H_{d,n}(k)$ for at least one divisor d of n .

2.1 Lemma (i) *Let d be a divisor of n, $e = \frac{n}{d}$ and let $h \in H_{d,n}(k)$. Then the conjugacy class c of $G_n(k)$ which contains h has the form $c = f^{\sigma}$, where*

2.1a *$m = d(f)$ divides d, and*

2.1b *There exists a partition $\pi \vdash e$ such that $\sigma = \pi \cdot \frac{d}{m}$ (for notation, see section 1).*

(ii) *Each primary class f^{σ} of $G_n(k)$ contains an element of $H_{d,n}(k)$, for $d = d(f)$.*

Proof of (i) Each element of $H_e(k_d)$ has the form $x = \zeta I_e.u$, where $\zeta \in k_d^{\times}$ and $u \in P_e(k_d)$. Let $X = (\zeta) \in G_1(k_d)$. Then (using the Jordan normal form) x is conjugate in $G_e(k_d)$ to an element of the form $X_{(\pi)}$, where $\pi = (\pi_1, \ldots, \pi_b)$ is some partition of e (we use here the notation 1.1, 1.2 of section 1). Therefore $h = j_d(x)$ is conjugate in $G_n(k)$ to $j_d(\zeta)_{(\pi)}$.

Now let m be the degree of ζ over k, and let $f \in \Phi(k)$ be the minimal polynomial of ζ over k. Then $m = d(f)$ divides d, and the k–subfield $k(\zeta)$ of k_d which is generated by ζ is isomorphic to k_m, hence is equal to k_m (because this is the only subfield of k_d of order q^m). As left k_m–module, k_d may be written as direct sum of $\frac{d}{m}$ submodules, each isomorphic to k_m. Therefore if we take a k–basis $\{v_1, \ldots, v_d\}$ of k_d adapted to this direct sum decomposition, we can arrange that

2.2 $$j_d(\zeta) = \begin{pmatrix} C(f) & 0 & \ldots & 0 \\ 0 & C(f) & \ldots & 0 \\ .. & .. & .. & .. \\ 0 & 0 & \ldots & C(f) \end{pmatrix} \in G_d(k),$$

where $C(f) \in G_m(k)$ has characteristic polynomial equal to f, and there are $\frac{d}{m}$ diagonal blocks $C(f)$. The reader will now be able to see that $j_d(\zeta)_{(\pi)}$ is conjugate in $G_n(k)$, by an element which permutes suitably the rows and columns of $j_d(\zeta)_{(\pi)}$, to a diagonal sum of d/m copies of the matrix $C(f)_{(\pi)}$. This clearly lies in the conjugacy class f^{σ}, where $\sigma = \pi \cdot \frac{d}{m}$.

This proves part (i) of the lemma.

Proof of (ii) This comes very easily from what has just been proved. If f^{σ} is a primary class of $G_n(k)$, then $d = d(f)$ divides n, and $\sigma \vdash e = \frac{n}{d}$. Let ζ be a zero of f. Then $\zeta \in k_d$ and f is the minimum polynomial of ζ over k. Let $X = (\zeta) \in G_1(k_d)$ and let $x = X_{(\sigma)}$. This is an element of $H_e(k_d)$. The proof of (i), where we now have $m = d$ and $\pi = \sigma$, shows that the class in $G_n(k)$ of $j_d(x)$ is $f^{\sigma \cdot 1} = f^{\sigma}$. Hence this class meets $H_{d,n}(k)$.

3 The characters $X_{d,n}(\psi,\lambda)$ of $G_n(k)$ There is a bijection $\lambda \leftrightarrow J(\lambda)$ between the set of all compositions $\lambda = (n_1,\ldots,n_b)$ of n, and the set of all subsets of the set $I = \{1,\ldots,n-1\}$ (if $n = 1$, take $I = \varnothing$), as follows: if $\lambda = (n)$, then $J(\lambda) = I$, otherwise

$$J(\lambda) = I \setminus \{n_1, n_1 + n_2, \ldots, n_1 + n_2 + \ldots + n_{b-1}\}.$$

Notice that $J((1^n)) = \square$.

For any field K and any $\lambda \models n$ let $\theta_\lambda : P_n(K) \to \mathbf{C}^\times$ denote the linear character of $P_n(K)$ which takes each (upper unitriangular) matrix $(a_{ij}) \in P_n(K)$ to $\omega_1(a_{12})\ldots\omega_{n-1}(a_{n-1,n})$, where $\omega_1,\ldots,\omega_{n-1}$ are elements of the character group $\hat{K}^+ = \mathrm{Hom}(K^+, \mathbf{C}^\times)$ which satisfy the condition

3.1 $\qquad \omega_j \neq 1$ if and only if $j \in J(\lambda)$.

Examples $\theta_{(n)}$ is a *non-degenerate* character of $P_n(K)$, i.e. $\omega_j \neq 1$ for all $j \in I$. The induced character $\mathrm{Ind}_{P_n(K)}^{G_n(K)}(\theta_{(n)})$ is called the *Gelfand–Graev* character of $G_n(K)$ (see section 6, also [DL], p.155 or [Ca], p.254). $\theta_{(1^n)}$ is the trivial (unit) character of $P_n(K)$.

Now let d be a divisor of n, and let $e = \frac{n}{d}$. Recall that $\psi : M_n \to \mathbf{C}^\times$ is a fixed character of $M_n = k_n^\times$. For each $\lambda \models e$ we define a linear character $\psi.\lambda$ of $H_e(k_d) = Z_e(k_d)\, P_e(k_d)$ as follows: if $z = \zeta.I_d \in Z_e(k_d)$ (so that $\zeta \in k_d^\times$), and if $a \in P_e(k_d)$, let $(\psi.\lambda)(za) = \psi(\zeta)\, \theta_\lambda(a)$. Composing $\psi.\lambda$ with the inverse of the map $j_d : H_e(k_d)$ $\varnothing\, H_{d,n}(k)$ we get a linear character of the subgroup $H_{d,n}(k)$ of $G_n(k)$, also denoted $\psi.\lambda$. Finally we make the

Definition $\qquad X_{d,n}(\psi,\lambda) := \mathrm{Ind}_{H_{d,n}(k)}^{G_n(k)}(\psi.\lambda)$.

Remarks (1) $X_{d,n}(\psi,\lambda)$ is independent of the choice of the ω_j, provided that these satisfy 3.1 (see section 6).

(2) If λ and λ' are compositions of the same integer, write $\lambda \approx \lambda'$ to mean that λ' can be obtained from λ by permuting its components. Each $\approx$-class contains exactly one partition. It will turn out that $X_{d,n}(\psi,\lambda) = X_{d,n}(\psi,\lambda')$ if $\lambda' \approx \lambda$ (see section 7). Therefore we may confine ourselves to $X_{d,n}(\psi,\lambda)$ for which $\lambda \vdash n$.

The main result of this paper is that $J_n(\psi)$ is a **Z**-linear combination of these induced characters $X_{d,n}(\psi,\lambda)$ of $G_n(k)$. More precisely, we have the following theorem, whose proof will occupy the rest of this paper.

3.2 Theorem *There exist polynomials* $r_\lambda(T) \in \mathbf{Z}[T]$, *one for each partition* λ, *such that for each positive integer* n, *for each field* k *of finite order* q, *and for each* $\psi \in \hat{M}_n$ = $\mathrm{Hom}(M_n, \mathbf{C}^\times)$ *there holds*

3.3 $$J_n(\psi) = \sum_{d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d)\, X_{d,n}(\psi,\lambda).$$

The polynomials $r_\lambda(T)$ *are determined uniquely by the equations (3.3)*

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4 Some properties of the characters $X_{d,n}(\psi,\lambda)$

Since $X_{d,n}(\psi,\lambda)$ is an induced character from the subgroup $H_{d,n}(k)$ of $G_n(k)$, its value at a class c of $G_n(k)$ is zero unless c contains an element of $H_{d,n}(k)$, i.e. unless c has the form f^σ, where f and σ satisfy conditions 2.1a, 2.1b (see lemma 2.1). This proves statement (i) in the next proposition; the proof of statement (ii) and the definition of the polynomials $x_{\lambda,\pi,l}(T)$ will be deferred to section 7.

Final Version 5 March 1998

4.1 Proposition *For each pair λ, π of partitions for which $|\lambda|=|\pi|$, and for each positive integer l, there exists a polynomial $x_{\lambda,\pi,l}(T) \in \mathbf{Z}[T]$, such that for all positive integers d, n with $d\,|\,n$, for all partitions $\lambda \vdash e = n/d$ and for all $c \in \mathrm{ccl}\, G_n(k)$, there hold*

(i) $X_{d,n}(\psi,\lambda)\{c\} = 0$ *unless $c = f^{\sigma}$, where f, σ satisfy conditions 2.1a and 2.1b, and*

(ii) *If f, σ satisfy conditions 2.1a and 2.1b, so that $m = d(f)$ divides d, and there exists $\pi \vdash e$ such that $\sigma = \pi \cdot d/m$, then $X_{d,n}(\psi,\lambda)\{f^{\sigma}\} = \psi(f).\, x_{\lambda,\pi,d/m}(q^m)$.*

It is sometimes convenient to augment the definition of $x_{\lambda,\pi,l}(T)$ by making the convention: for any partitions λ, σ and any $l \in \mathbf{Q}$, $x_{\lambda,\sigma/l,l}(T)$ is zero unless $l \in \mathbf{Z}$ and there exists π such that $\sigma = \pi \cdot l$, in which case $x_{\lambda,\sigma/l,l}(T) := x_{\lambda,\pi,l}(T)$. Then we have a "short" version of proposition 4.1, namely

4.1a *If $d\,|\,n$, $\lambda \vdash n/d$ and $f^{\sigma} \in \mathrm{ccl}\, G_n(k)$ then $X_{d,n}(\psi,\lambda)\{f^{\sigma}\} = \psi(f).\, x_{\lambda,\sigma/l,l}(q^{d(f)})$, where $l = d/d(f)$.*

In particular we see that $X_{1,n}(\psi,\lambda)\{f^{\sigma}\} = 0$ if $d(f) \neq 1$, while if $d(f) = 1$, i.e. if $f(t) = t - x$ for some $x \in k^{\times}$, then

4.2 $$X_{1,n}(\psi,\lambda)\{(t-x)^{\sigma}\} = \psi(x).\, x_{\lambda,\sigma,1}(q), \textit{ for all } \lambda, \sigma \vdash n \textit{ and all } x \in k^{\times}.$$

We shall see later (section 7) that the polynomials $x_{\lambda,\pi,l}(T)$ are determined in a simple way by the $x_{\lambda,\pi,1}(T)$. If we take $\psi = 1$ in 4.2 we get

4.3 $$X_{1,n}(1,\lambda)\{(t-x)^{\sigma}\} = x_{\lambda,\sigma,1}(q), \textit{ for all } \lambda, \sigma \vdash n \textit{ and all } x \in k^{\times}.$$

For each $\mu \vdash n$ there is an irreducible character I^{μ} of $G_n(k)$, denoted $I_1^0[\mu]$ in [GL], p.437, and first discovered by R. Steinberg (see [S], p. 275. In Steinberg's notation, I^{μ} = $\Gamma(\nu)$, where $\nu = (\mu_n, \mu_{n-1}, \ldots, \mu_1)$). We have the following important relation between the $X_{1,n}(1,\lambda)$ and the I^{μ}. Let • , ® denote the usual scalar product on class-functions on $G_n(k)$ (see 8.3).

4.4 Proposition *Let* $W(\lambda,\mu) = \bullet X_{1,n}(1,\lambda),\ I^{\mu}$®, *for any* $\lambda,\mu \vdash n$. *Then* $W(\lambda,\mu) \in \mathbf{Z}$ *is independent of* k *(i.e. of* q*), and the matrix* $\left(W(\lambda,\mu)\right)_{\lambda,\mu\vdash n}$ *is unimodular.*

It will be useful to record here some information about the character I^{μ}. From its definition as $I^{\mu} = I_1^0[\mu]$, using [GL] , p.441, lemma 8.2 together with [GL], p.423, definition (18), we may verify the first equality in

4.5 $$I^{\mu}\{(t-x)^{\pi}\} = \sum_{\rho\vdash n}\frac{1}{z_\rho}.Q_\rho^{\pi}(q).\chi_\rho^{\mu} = q^{n(\pi)}K_{\mu\pi}(q^{-1}),$$

where the $Q_\rho^{\pi}[T] \in \mathbf{Z}[T]$ are certain polynomials introduced in [GL]—some of whose properties we shall recall in section 8—and χ_ρ^{μ} is standard notation (see e.g. [Le] or [M]) for the value at class ρ of the irreducible character χ^{μ} of the symmetric group $S(n)$. The second equality in 4.5 comes by applying [M], p.248, (7.11), and using the orthogonality relations for the characters of $S(n)$. The polynomials $K_{\mu\sigma}(T) \in \mathbf{Z}[T]$ are defined in [M], p.239; the expression $T^{n(\sigma)}K_{\mu\sigma}(T^{-1})$ is also a polynomial in $\mathbf{Z}[T]$, see [M], p.248.

4.6 Proposition *With* n *given,* $\{X_{d,n}(1,\lambda) | d|n, \lambda \vdash n/d\}$ *is a linearly independent set of class-functions on* $G_n(k)$.

Propositions 4.4 and 4.6 will be proved in section 9.

5 Proof of theorem 3.2

In this section we prove theorem 3.2, on the assumption that the propositions in section 4 are true. It is clear that equation 3.3 holds for $n = 1$ in any case, by taking $r_{(1)}(T) = 1$. For we have $G_1(k) = k^{\times} = M_1$ and $J_1(\psi) = \psi_{|M_1} = X_{1,1}(\psi,(1))$.

We proceed by induction on n. Suppose that $n>1$, and that we have already defined polynomials $r_\lambda(T) \in \mathbf{Z}[T]$ for all $\lambda \vdash \bar{n}$ and all $\bar{n}<n$ in such a way that 3.3 holds, for any appropriate ψ and k, with n replaced by any $\bar{n}<n$. To prove theorem 3.2 for n, we must

show that there exist $r_\lambda(T) \in \mathbf{Z}[T]$ for all $\lambda \vdash n$ so that 3.3 holds using these new $r_\lambda(T)$ (together, of course, with the $r_\lambda(T)$ already defined).

Let s be a divisor of n. Then we define the class-function $R_s(k)$ on $G_n(k)$ by

5.1 $$R_s(k) = J_n(\psi) - \sum_{s|d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) X_{d,n}(\psi, \lambda),$$

where the first sum is over all divisors d of n which are divisible by s. Notice that theorem 3.2 is equivalent to the statement that polynomials $r_\lambda(T)$ exist, such that $R_1(k) = 0$ for all n, ψ and k .

5.2 Lemma *Let $s \neq 1$ be a divisor of n. Then $R_s(k)\{f^\sigma\} = 0$ for all class-functions f^σ of $G_n(k)$ such that $s\,|\,d(f)$.*

Proof Let f^σ be a class of $G_n(k)$ as described, and let $m = d(f)$. By 1.3 and 4.1a, $R_s(k)\{f^\sigma\} = \psi(f)\, U_s(k)$, where

5.3 $$U_s(k) = k(\sigma : q^m) - \sum_{s|d|n} \sum_{\lambda \vdash n/d} r_\lambda(q^d) x_{\lambda, \sigma/(d/m), d/m}(q^m).$$

Notice that s divides all the integers n, m, d appearing in 5.3. Write $\bar{n} = n/s$, $\bar{m} = m/s$, $\bar{d} = d/s$. Take any $\bar{f} \in \Phi(k_s)$ of degree $d(\bar{f}) = \bar{m}$ (for example, we could take $\bar{f}$ to be the minimal polynomial over k_s, of an element $\eta \in k_d$ whose minimum polynomial over k is f), and consider the class $\bar{f}^\sigma$ of $G_{\bar{n}}(k_s)$. The class-function $R_1(k_s)$ on $G_{\bar{n}}(k_s)$ is zero by our induction hypothesis. On the other hand, the analogue of 5.3 gives us, writing $\bar{q} = q^s = |k_s|$,

5.4 $$U_1(k_s) = k(\sigma : \bar{q}^{\bar{m}}) - \sum_{\bar{d}|\bar{n}} \sum_{\lambda \vdash \bar{n}/\bar{d}} r_\lambda(\bar{q}^{\bar{d}}) x_{\lambda, \sigma/(\bar{d}/\bar{m}), \bar{d}/\bar{m}}(\bar{q}^{\bar{m}}).$$

But it is clear that $U_1(k_s) = U_s(k)$. Since $0 = R_1(k_s) = \overline{\psi}(\bar{f}).U_1(k_s)$ holds for any $\overline{\psi} \in \hat{M}_{\overline{n}}$ (including $\overline{\psi} = 1$) we have $0 = U_1(k_s) = U_s(k)$, and so $R_s(k)\{f^\sigma\} = \psi(f)\,U_s(k)$ is zero, which proves the lemma.

Next we define a class-function $B_n(\psi)$ on $G_n(k)$ by

5.5 $$B_n(\psi) = J_n(\psi) - \sum_{d|n,d\neq 1}\ \sum_{\lambda\vdash n/d} r_\lambda(q^d)X_{d,n}(\psi,\lambda).$$

5.6 Lemma *$B_n(\psi)$ is zero on all classes f^σ of $G_n(k)$ for which $d(f) \neq 1$.*

Proof Suppose f^σ is a class on $G_n(k)$ for which $d(f) = s \neq 1$. Then by proposition 4.1(i), $X_{d,n}(\psi,\lambda)\,\{f^\sigma\} = 0$, for all $d\,|\,n$ and $\lambda \vdash n/d$ such that s does not divide d. Therefore $B_n(\psi)\,\{f^\sigma\} = R_s(k)\{f^\sigma\}$, which is zero by lemma 5.2.❮

In order to complete the proof of theorem 3.2, we must construct polynomials $r_\lambda(T)$ $\in \mathbf{Z}[T]$ such that

5.7 $$B_n(\psi) = \sum_{\lambda\vdash n} r_\lambda(q)X_{1,n}(\psi,\lambda)$$

for all $\psi \in \hat{M}_n$. It is enough that 5.7 should hold for $\psi = 1$. For by 1.3 and 4.1, we have $B_n(\psi)\,\{(t-x)^\sigma\} = \psi(x).B_n(1)\,\{(t-1)^\sigma\}$ and $X_{1,n}(\psi,\lambda)\,\{(t-x)^\sigma\} = \psi(x).X_{1,n}(1,\lambda)\,\{(t-1)^\sigma\}$, for all x and σ, and both sides of 5.7 are zero on all classes f^σ of $G_n(k)$ with $d(f) \neq 1$ (see proposition 4.1(i)). Define the class function $B = B_n(1)$. From 1.3 and 4.1a we get, for all $x \in k^\times$ and $\sigma \vdash n$,

5.8 $$B\{(t-x)^\sigma\} = B\{(t-1)^\sigma\} = k(\sigma:q) - \sum_{d|n,d\neq 1}\ \sum_{\lambda\vdash n/d} r_\lambda(q^d)x_{\lambda,\sigma/d,d}(q),$$

Using the notation • , ® for the scalar product on class-functions on $G_n(k)$ (see 8.3) we have by 8.4 the following lemma.

5.9 Lemma *If the class-function F on* $G_n(k)$ *is zero on all classes* f^{σ} *with* $d(f) \neq 1$, *and satisfies* $F\{(t-x)^{\sigma}\} = F\{(t-1)^{\sigma}\}$ *for all* $x \in k^{\times}$ *and* $\sigma \vdash n$, *then*

5.10 $$\bullet F, I^{\mu} \circledR = (q-1) \sum_{\sigma \vdash n} \frac{1}{a_{\sigma}(q)} . F\{(t-1)^{\sigma}\} . I^{\mu}\{(t-1)^{\sigma}\},$$

for all $\mu \vdash n$.

5.11 Corollary *If F is as above, and if* $\bullet F, I^{\mu} \circledR = 0$ *for all* $\mu \vdash n$, *then* $F = 0$.

Proof By [M], p.239 the matrix $\left(q^{n(\sigma)} K_{\mu,\sigma}(q^{-1})\right)_{\mu,\sigma \vdash n}$ is non-singular, hence by 4.5 the matrix $\left(I^{\mu}\{(t-1)^{\sigma}\}\right)_{\mu,\sigma \vdash n}$ is non-singular. So $\bullet F, I^{\mu} \circledR = 0$ for all $\mu \vdash n$ □ $F\{(t-1)^{\sigma}\} = 0$ for all $\sigma \vdash n$ (see 5.10) □ $F = 0$.

Now we define, for each $\lambda \vdash n$,

5.12 $$r_{\lambda}(k) = \sum_{\mu \vdash n} \bullet B, I^{\mu} \circledR V(\mu, \lambda),$$

where $(V(\lambda,\mu))$ is the inverse of the matrix $(W(\lambda,\mu))$ of proposition 4.4.

5.13 Lemma $B = \sum_{\lambda \vdash n} r_{\lambda}(k) X_{1,n}(1,\lambda)$.

Proof Let S denote the right side of the equation above. We check immediately from 4.4 that $\bullet S, I^{\tau} \circledR = \bullet B, I^{\tau} \circledR$ for all $\tau \vdash n$. Hence $S = B$ by corollary 5.11.

We must now show that each coefficient $r_{\lambda}(k)$ defined by 5.12 "belongs to $\mathbf{Z}[q]$" in the sense that there exists a polynomial $r_{\lambda}(T) \in \mathbf{Z}[T]$ such $r_{\lambda}(k) = r_{\lambda}(q)$, for each field k of order q. By 5.12 and 4.4, it will be enough to prove that each $\bullet B, I^{\mu} \circledR$ "belongs to $\mathbf{Z}[q]$" in this sense. We may apply lemma 5.9 to $F = B$. Then 5.10 gives

5.14 $$\bullet B, I^{\mu} \circledR = (q-1) \sum_{\sigma \vdash n} \frac{1}{a_{\sigma}(q)} . B\{(t-1)^{\sigma}\} . I^{\mu}\{(t-1)^{\sigma}\},$$

for all $\mu \vdash n$. But 5.8 shows that $B\{(t-1)^{\sigma}\}$ "belongs to $\mathbf{Z}[q]$", because $k(\sigma : T)$, $r_{\lambda}(T^{d})$ and $x_{\lambda,\sigma/d,d}(T)$ all belong to $\mathbf{Z}[T]$, for all divisors $d \neq 1$ of n and all $\lambda \vdash n/d$. Also

$I^{\mu}\{(t-1)^{\sigma}\} = q^{n(\sigma)}K_{\mu\sigma}(q^{-1})$ "belongs to $\mathbf{Z}[q]$", see the end of section 4. Of course we have in 5.14 denominators $a_{\sigma}(q)$. But the polynomials $a_{\sigma}(T)$ lie in $\mathbf{Z}[T]$ and are monic (see section 8), and we know that $\langle B, I^{\mu}\rangle \in \mathbf{Z}$ for fields k of all p–power orders q, because B is a generalized character of $G_n(k)$ (see 5.5) and I^{μ} is a character of $G_n(k)$. Therefore we deduce that $\langle B, I^{\mu}\rangle$ "belongs to $\mathbf{Z}[q]$" from 5.14 and the following elementary lemma (whose proof we leave to the reader).

5.15 Lemma *Let $\alpha(T)$ and $\beta(T)$ belong to $\mathbf{Z}[T]$, with $\beta(T)$ monic. Let $\kappa(T) = \alpha(T)/\beta(T)$, and suppose that $\kappa(q) \in \mathbf{Z}$ for infinitely many distinct integers q. Then $\kappa(T) \in \mathbf{Z}[T]$.*

We have now proved 5.7, hence that equations 3.3 hold. It remains to prove that the $r_{\lambda}(T)$ are determined uniquely by these equations. But this follows from the case $\psi = 1$ of 3.3, together with proposition 4.6.

Remark We can prove that $\langle B, I^{\mu}\rangle \in \mathbf{Z}$ for all k, without appealing to fact that $J_n(\psi)$ (and in particular $J_n(1)$) is a generalized character. For it is easy to check that $\langle J_n(\psi), I^{\mu}\rangle \in \mathbf{Z}$ by direct calculation, using the definition 1.3 of $J_n(\psi)$. Of course $\langle X_{d,n}(1,\lambda), I^{\mu}\rangle \in \mathbf{Z}$ for all $d\,|\,n$ and $\lambda \vdash n/d$, because the $X_{d,n}(1,\lambda)$ are characters of $G_n(k)$ by definition. We then have $\langle B, I^{\mu}\rangle \in \mathbf{Z}$ as before, from the definition 5.5 of $B_n(\psi)$.

Therefore theorem 3.2 provides a proof (even if rather indirect!) that the functions $J_n(\psi)$ are generalized characters. But, as Robert Steinberg has remarked, this could be proved by a direct application to $J_n(\psi)$ of Brauer's characterization of characters.

6 Gelfand-Graev character for $G_n(k)$, I

The Gelfand–Graev character Γ_n of $G_n(k)$ is by definition the induced character $\mathrm{Ind}_P^G(\theta_{(n)})$, where $G = G_n(k)$, $P = P_n(k)$, and $\theta_{(n)}$ is any non-degenerate linear character of P, see 3.1. (Γ_n is independent of the characters $\omega_j \in \hat{k}^+$, provided these are all $\neq 1$; see

[C], p.254.) Clearly $\Gamma_n\{c\}$ is zero, unless the conjugacy class c meets P, i.e. unless c = $(t-1)^\pi$ for some $\pi \vdash n$. For brevity, we shall henceforth write $F\{\pi\}$ for $F\{(t-1)^\pi\}$, for any class-function F on G.

Deligne and Lusztig have discovered an important property of Γ_n , which they prove for a large class of finite reductive groups ([DL], p. 155, Prop. 10.3). In our case, Deligne–Lusztig's result may be written

6.1 $$\sum_{J\subset I}(-1)^{|J|}.\Gamma_J = \Delta_n,$$

where the class-function Δ_n on G is given by

6.2 $\Delta_n\{c\} = 0$ *unless* $c = (t-1)^{(n)}$*, and* $\Delta_n((t-1)^{(n)}) = \Delta_n((n)) = q^{n-1}(q-1)$.

To define Γ_J, we recall (section 3) that to each subset J of $I = \{1,\ldots,n-1\}$ is associated a composition λ of n, which we shall here denote $\lambda = (n_1,\ldots,n_b)$. To this is associated the parabolic subgroup $P(J)$ of G consisting of all matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1b} \\ 0 & A_{22} & \ldots & A_{2b} \\ \cdot\cdot & \cdot\cdot & \ldots & \cdot\cdot \\ 0 & 0 & \ldots & A_{bb} \end{pmatrix}$$

of $G = G_n(k)$ such that $A_{jj} \in G_{n_j}(k)$ ($j = 1,\ldots,b$). If χ_j is a character on $G_{n_j}(k)$ ($j = 1,\ldots,b$), define the character $\chi_1 \circ \ldots \circ \chi_b$ on G to be $\mathrm{Ind}_{P(J)}^G(\chi)$, where χ is the character $\chi(A) = \chi_1(A_{11})\ldots\chi_b(A_{bb})$ on $P(J)$. We recall from [GL], p. 411 that this "circle-product" $\circ$ is multilinear, associative and commutative. It is easy to see that the character denoted $\mathrm{Ind}_{P(J)^F}^{G^F}(\Gamma_{L(J)})$ in [DL], (10.3.1), is in our notation,

6.3 $\Gamma_J := \Gamma_{n_1} \circ \ldots \circ \Gamma_{n_b}$.

To calculate $\circ$ products like this, we have the formula (see [GL], p.410):

Final Version 5 March 1998

if F_j is a class-function on $G_{n_j}(k)$ ($j = 1,\dots,b$) then for all $\pi \vdash n = n_1+\dots+n_b$

6.4 $$(F_1 \circ \dots \circ F_b)\{\pi\} = \sum g^{\pi}_{\pi_1\dots\pi_b}(q) F_1\{\pi_1\}\dots F_b\{\pi_b\},$$

where the sum is over all rows $\pi_1,\dots,\pi_b$ of partitions of $n_1,\dots,n_b$ respectively, and the integer $g^{\pi}_{\pi_1\dots\pi_b}(q)$ is the value at $T = q$ of Hall's polynomial $g^{\pi}_{\pi_1\dots\pi_b}(T) \in \mathbf{Z}[T]$ (see [M], p.188).

6.5 Lemma *For each $J \subset I = \{1,\dots,n-1\}$ and each $\pi \vdash n$ there exists a polynomial $c_{J,\pi}(T) \in \mathbf{Z}[T]$ such that $\Gamma_J\{\pi\} = (q-1)\, c_{J,\pi}(q)$ for all fields k of order q.*

Proof Deligne and Lusztig ([DL], p. 155, 10.4) give the following formula, a "dual" to 6.1,

6.6 $$\Gamma_n = \sum_{J \subset I} (-1)^{|J|} \Delta_J,$$

where $\Delta_J = \Delta_{n_1} \circ \dots \circ \Delta_{n_b}$. By 6.2, there is, for each positive integer n, a polynomial $d_\pi(T) \in \mathbf{Z}[T]$ such that $\Delta_n\{\pi\} = (q-1)\, d_\pi(q)$ for all fields k of order q. Therefore by 6.4 there is, for each $J \subset I$ and each $\pi \vdash n$, a polynomial $d_{J,\pi}(T)$ such that $\Delta_J\{\pi\} = (\Delta_{n_1} \circ \dots \circ \Delta_{n_b})\{\pi\} = (q-1)\, d_{J,\pi}(q)$ for all fields k of order q. Then 6.6 shows that $c_\pi(T) = \sum_{J \subset I} (-1)^{|J|} d_{J,\pi}(T) \in \mathbf{Z}[T]$ has the property that $\Gamma_n\{\pi\} = (q-1)\, c_\pi(q)$, and we may use this, together with 6.4 again, to construct the polynomials $c_{J,\pi}(T)$ required by the lemma.

Remark We often write $c_{J,\pi}(T) = c_{\lambda,\pi}(T)$, if J and λ are related as in section 3.
It is clear that the property $\Gamma_J\{\pi\} = (q-1)\, c_{J,\pi}(q)$, i.e.

6.7 $$c_{\lambda,\pi}(q) = \frac{1}{q-1} (\Gamma_{n_1} \circ \dots \circ \Gamma_{n_b})\{\pi\},$$

defines the polynomial $c_{\lambda,\pi}(T)$ uniquely. Because the $\circ$ product is commutative, $c_{\lambda,\pi}(T) = c_{\lambda',\pi}(T)$ whenever $\lambda' \approx \lambda$ (see section 3, remark (2)).

7 Proof of proposition 4.1(ii)

We want to calculate the values of the character $X_{d,n}(\psi,\lambda) = \mathrm{Ind}_{H_{d,n}(k)}^{G_n(k)}(\psi.\lambda)$ defined in section 3, and we start with the special case $d = 1$. According to the definitions in section 2, $H_{1,n}(k) = H_n(k)$, $Z_{1,n}(k) = Z_n(k)$ and $P_{1,n}(k) = P_n(k)$ (we identify $\mathrm{Mat}_1(k)$ with k, so that j_1 is the identity map). Write these groups H, Z and P for short, and write $G_n(k) = G$. Observe that $H = ZP$ has order $(q-1)|P|$.

7.1 Lemma *For all partitions $\lambda, \pi \vdash n$ and for all $x \in k^\times$*

7.2 $$X_{1,n}(\psi,\lambda)\{(t-x)^\pi\} = \psi(x).c_{\lambda,\pi}(q).$$

Proof Let u be an element of the class $(t-1)^\pi$. Then $g = xI_n.u$ is an element of the class $(t-x)^\pi$, and we have $X_{1,n}(\psi,\lambda)\{(t-x)^\pi\} = \mathrm{Ind}_{ZP}^{G}(\psi.\lambda)\{g\} = \frac{1}{|ZP|}\sum_{s\in G, s^{-1}gs\in H}(\psi.\lambda)(s^{-1}gs) = \frac{\psi(x)}{q-1}\mathrm{Ind}_P^G(\theta_\lambda)\{(t-1)^\pi\}$, because $s^{-1}gs = xI_n.s^{-1}us$, hence $s^{-1}gs \in H$ if and only if $s^{-1}us \in P$, and in that case $(\psi.\lambda)(s^{-1}gs) = \psi(x).\,\theta_\lambda(s^{-1}us)$.

To calculate $\mathrm{Ind}_P^G(\theta_\lambda)\{(t-1)^\pi\}$, notice that from the definition of θ_λ (section 3), and in the notation of section 6, $\mathrm{Ind}_P^{P(J)}(\theta_\lambda)$ takes $A \in P(J)$ to $\Gamma_{n_1}(A_{11})\ldots\Gamma_{n_b}(A_{bb})$. Therefore

7.3 $$\mathrm{Ind}_P^G(\theta_\lambda) = \Gamma_{n_1}\circ\ldots\circ\Gamma_{n_b},$$

and now lemma 7.1 follows from 6.7.

Now consider the situation of proposition 4.1(ii). We have a divisor d of n, partitions λ, π of $e = n/d$, and a polynomial $f \in \Phi$ of degree m which divides d. Let $\sigma = \pi\cdot d/m$, so that $\sigma \vdash n/m$ and f^σ is a conjugacy class of $G_n(k)$. From the proof of 2.1(i) we know that f^σ contains an element $h = j_d(\zeta)_{(\pi)} \in H_{d,n}(k)$, where $\zeta \in k_m^\times$ is a zero of $f(t) = (t-\zeta)(t-\zeta^q)\ldots(t-\zeta^{q^{m-1}})$. The semisimple part of $(\zeta)_{(\pi)}$ is $(\zeta)_{(1^{n/d})}$, and its unipotent part is $(1)_{(\pi)}$. The following lemma is an elementary consequence of this, together with the discussion following 2.2.

7.4 Lemma *If $h = h_{p'}h_p$ is the Jordan decomposition of h, then (i) $h_{p'} = j_d((\zeta)_{1^{n/d}}) = C(f)_{(1^{n/m})}$ satisfies the equation $f(h_{p'}) = 0$, and the k–algebra generated by $h_{p'}$ (in $\mathrm{Mat}_n(k)$) is a field; (ii) $h_p = j_d(1)_{(\pi)}$ is unipotent of type $\sigma \cdot m = \pi \cdot d$.*

It is clear that

7.5 $$X_{d,n}(\psi,\lambda)\{f^{\sigma}\} = \mathrm{Ind}_{G_{d,n}(k)}^{G_n(k)}(\mathrm{Y})\{h\},$$

where

7.6 $$\mathrm{Y} := \mathrm{Ind}_{H_{d,n}(k)}^{G_{d,n}(k)}(\psi.\lambda).$$

From the standard definition of induced character we have

7.7 $$\mathrm{Ind}_{G_{d,n}(k)}^{G_n(k)}(\mathrm{Y})\{h\} = \frac{1}{|G_{d,n}(k)|}\sum_{s\in\Theta}\mathrm{Y}(s^{-1}hs),$$

where

7.8 $$\Theta = \{s \in G_n(k) | s^{-1}hs \in G_{d,n}(k)\}.$$

7.9 Lemma *If $s \in \Theta$, then $(s^{-1}hs)_{p'} = (h_{p'})^{q^j}$ for some $j \in \{0,\ldots,m-1\}$.*

Proof Since $s^{-1}hs \in G_{d,n}(k)$, then also $(s^{-1}hs)_{p'} \in G_{d,n}(k)$. Let $z \in G_e(k_d)$ be such that $(s^{-1}hs)_{p'} = s^{-1}h_{p'}s = j_d(z)$. By 7.4(i), z satisfies $f(z) = 0$, and the k-subalgebra (of $\mathrm{Mat}_e(k_d)$) generated by z is a field. But this means that the minimum polynomial of z over k_d is irreducible, and it divides $f(t) = (t-\zeta)(t-\zeta^q)\ldots(t-\zeta^{q^{m-1}})$. It follows that $z = a.\zeta^{q^j}$, for some $a \in k_d^{\times}$ and some $j \in \{0,\ldots,m-1\}$. But since $j_d(z)$ has the same eigenvalues as $h_{p'} = j_d(\zeta)_{1^{n/d}}$, we must have $a = 1$.

Since the elements ζ^{q^j} $(j = 0,\ldots,m-1)$ are distinct, we deduce from this lemma that $\Theta = \Theta_0 \cup \ldots \cup \Theta_{m-1}$ (disjoint union), where

7.10 $$\Theta_j = \{s \in G_n(k) | s^{-1}hs \in G_{d,n}(k), (s^{-1}hs)_{p'} = h_{p'}{}^{q^j}\}.$$

7.11 Lemma *Let* $\mathbf{F}$ *denote the Frobenius endomorphism* $(a_{ij}) \to \left(a_{ij}{}^q\right)$ *on* $G_e(k_d)$. *Then there exists a matrix* $M \in G_n(k)$ *such that*

7.12 $$M.j_d(a).M^{-1} = j_d(a^{\mathbf{F}}) \textit{ for all } a \in G_e(k_d)$$

Proof Let $\{v_1,\ldots v_d\}$ be the k-basis of k_d which was used to define the k-algebra map $j_d\colon k_d \to \mathrm{Mat}_d(k)$(section 2). Then it is easy to verify that the matrix $M^0 = (m_{ij}) \in G_d(k)$ defined by the equations $v_j^q = \sum_i m_{ij} v_i$ ($j = 1,\ldots,d$) has the property: $M^0.j_d(\alpha).(M^0)^{-1} = j_d(\alpha^q)$ for all $\alpha \in k_d$. Therefore the matrix $M = (M^0)_{(1^{n/d})} \in G_n(k)$ (i.e. M is the diagonal sum of n/d copies of M^0) has the property 7.12.

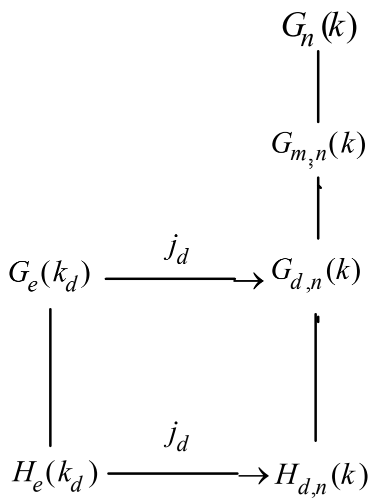
7.13 Corollary $\Theta_0 M^{-j} = \Theta_j$, *for all* $j = 0,\ldots,m-1$.

Proof 7.12 implies that M normalizes $G_{d,n}(k)$), and also that $M^j h_{p'} M^{-j} = h_{p'}^{q^j}$, because $h_{p'} = j_d((\zeta)_{(1^{n/d})})$ by 7.4. The corollary now follows from the definition 7.10.

Now suppose that $s \in \Theta_0$. Then $s^{-1}h_{p'}s = h_{p'}$. But by 7.4(i) we know that $h_{p'}$ is the diagonal sum of n/m copies of $C(f)$, and $C(f)$ is the image of $\zeta \in k_m$ under the k-algebra map $j_m\colon k_m \to \mathrm{Mat}_m(k)$ determined by the part $\{v_1,\ldots,v_m\}$ of the k-basis $\{v_1,\ldots,v_d\}$ of k_d which was used to obtain 2.2. So the centralizer of $h_{p'}$ in $G_n(k)$ consists of all non-singular matrices $B = (B_{rs})_{r,s=1,\ldots,n/m}$, in which each B_{rs} is an $m \times m$ matrix belonging to the centralizer of $C(f)$ in $\mathrm{Mat}_m(k)$. But this latter centralizer is exactly $\mathrm{Im}\, j_m$ (it consists of those matrices which correspond to elements of the k_m-endomorphism algebra of the left k_m-module k_m; but since k_m is commutative, this is the same as the algebra of all left multiplications by elements of k_m). This proves that

7.14 *The centralizer of* $h_{p'}$ *in* $G_n(k)$ *is* $G_{m,n}(k)$.

The groups which we are dealing with are shown in the diagram below.



Let R denote the conjugacy class of $G_{n,d}(k)$ which contains $h_p = j_d((1)_\pi)$; this consists of all unipotent elements of $G_{n,d}(k)$ of type $\pi \cdot d$. Clearly $R = j_d(S)$, where S is the conjugacy class of $G_e(k_d)$ which contains $(1)_{(\pi)}$.

Let $\rho \vdash N$. We shall recall in section 8 that there is a monic polynomial $a_\rho(T) \in \mathbf{Z}[T]$, such that the order of the centralizer in $G_N(K)$ of the unipotent element $(1)_{(\rho)}$ is $a_\rho(Q)$, for any field K of finite order Q (see [GL], p.409, or [M], p.181).

7.15 Lemma (i) *The order of* Θ_0 *is* $a_{\pi \cdot d/m}(q^m) \big/ a_\pi(q^d)$ *times* $|G_{n,d}(k)|$.
(ii) *For each* $s \in \Theta_0$, $s^{-1}hs$ *lies in the class* $j_d((t-\zeta)^\pi)$ *of* $G_{n,d}(k)$. *Hence*

7.16 $$\frac{1}{|G_{d,n}(k)|} \sum_{s \in \Theta_0} \Upsilon(s^{-1}hs) = \psi(\zeta) \,.\, \frac{a_{\pi \cdot d/m}(q^m)}{a_\pi(q^d)} \,.\, c_{\lambda,\pi}(q^d).$$

Proof (i) From the definition 7.10 of Θ_0, together with 7.14, we see that Θ_0 consists of all $s \in G_{m,n}(k)$ such that $s^{-1}h_p s \in G_{d,n}(k)$, i.e such that $s^{-1}h_p s \in R$. The order of R is the same as that of S, i.e. $|G_e(k_d)| / a_\pi(q^d) = |G_{d,n}(k)| / a_\pi(q^d)$. But the number of $s \in G_{m,n}(k)$ for which $s^{-1}h_p s$ has a given value, is the order of the centralizer in $G_{m,n}(k)$ of h_p. Since $h_p = j_d((1)_\pi)$ is conjugate to $j_m((1)_{\pi \cdot d/m})$, this order is $a_{\pi \cdot d/m}(q^m)$. This proves (i).

Final Version 5 March 1998

(ii) If $s \in \Theta_0$ then $s^{-1}hs$ has semisimple part $h_{p'} = j_d(\zeta)$, and unipotent part in $j_d(S)$. This proves the first statement of (ii). But $Y = \mathrm{Ind}_{H_{d,n}(k)}^{G_{d,n}(k)}(\psi.\lambda)$ maps $\{j_d((t-\zeta)^{\pi})$ to $\mathrm{Ind}_{H_e(k_d)}^{G_e(k_d)}(\psi.\lambda)\{(t-\zeta)^{\pi}\}$, which equals $\psi(\zeta).c_{\lambda,\pi}(q^d)$, by 7.2. From this, 7.16 follows.

It is now easy to prove, using 7.13, that 7.16 remains true, if we replace Θ_0 by Θ_j and ζ by ζ^j, for any $j \in \{0,\ldots,m-1\}$. Add the m equations which result; we get the

7.17 Proposition *Suppose d divides n, and $\lambda, \pi \vdash e = n/d$, and $f \in \Phi$ has degree m which divides d. Let $\sigma = \pi \cdot d/m$, so that $\sigma \vdash n/m$, and f^{σ} is a conjugacy class of $G_n(k)$. Then for any $\psi \in \hat{M}_n$*

7.18 $$X_{d,n}(\psi,\lambda)\{f^{\sigma}\} = \psi(f).\frac{a_{\sigma}(q^m)}{a_{\pi}(q^d)}.c_{\lambda,\pi}(q^d)$$

7.19 Lemma $\dfrac{a_{\pi\cdot l}(T)}{a_{\pi}(T^l)} \in \mathbf{Z}[T]$, *for all partitions π and all positive integers l.*

Proof With the notation of lemma 7.17, $a_{\sigma}(q^m)$, $a_{\pi}(q^d)$ are the orders of the centralizers of h_p in $G_{m,n}(k)$, $G_{d,n}(k)$, respectively. Since $G_{d,n}(k)$ is a subgroup of $G_{m,n}(k)$, it follows that $a_{\sigma}(q^m)/a_{\pi}(q^d) \in \mathbf{Z}$ for all prime-powers q. Now take $m = 1$, and write l for d. Since $\sigma = \pi \cdot d/m = \pi \cdot l$, this shows that $a_{\pi\cdot l}(q)/a_{\pi}(q^l) \in \mathbf{Z}$ for all prime-powers q. Now lemma 7.19 follows from lemma 5.15.

Thus 7.18 proves proposition 4.4(ii); the polynomials $x_{\lambda,\pi,l}(T)$ are given by

7.20 $$x_{\lambda,\pi,l}(T) = \frac{a_{\pi\cdot l}(T)}{a_{\pi}(T^l)}.c_{\lambda,\pi}(T^l).$$

8 Gelfand–Graev character for $G_n(k)$, II.

To prove propositions 4.4 and 4.6 (and hence complete the proof of theorem 3.2) we need an explicit formula for the polynomials $c_{\lambda,\pi}(T)$, which were defined rather indirectly in section 6. For this purpose we shall make use of the polynomials $Q_\rho^\lambda(T)$ which were introduced in [GL, p.420], and later defined in a different way by D.E. Littlewood ([Li]; see [M], p.246).

The polynomials $Q_\rho^\lambda(T) \in \mathbf{Z}[T]$ are defined for all partitions λ, ρ of n, and satisfy the following orthogonality relations:

8.1 $$\sum_{\pi \vdash n} \frac{1}{a_\pi(T)} Q_\rho^\pi(T) Q_\sigma^\pi(T) = \delta_{\rho,\sigma} \frac{z_\rho}{c_\rho(T)}, \textit{for all } \rho, \sigma \vdash n,$$

and

8.2 $$\sum_{\rho \vdash n} \frac{c_\rho(T)}{z_\rho} Q_\rho^\pi(T) Q_\rho^\tau(T) = \delta_{\pi,\tau} a_\pi(T), \textit{ for all } \pi, \tau \vdash n.$$

The coefficients appearing in 8.1 and 8.2 are defined as follows.

$$a_\rho(T) = T^{|\rho|+2n(\rho)} \prod_i (1-\frac{1}{T})(1-\frac{1}{T^2})\ldots(1-\frac{1}{T^{r_i}}),$$

$$c_\rho(T) = \prod_i (T^i - 1)^{r_i},$$

and $$z_\rho = \prod_i i^{r_i} r_i!,$$

for any partition $\rho = 1^{r_1} 2^{r_2} \ldots$ of n. Notice that $a_\rho(T) \in \mathbf{Z}[T]$ and is monic; we have already used the fact that $a_\rho(q)$ is the order of the centralizer in $G_n(k)$ of a element of the class $(t-1)^\rho$ ([GL], p.409; [M], p. 181).

The scalar product • , ® on the space of class-functions on $G_n(k)$ is defined by

8.3 $$\bullet F, F' \circledR = \sum_c \frac{1}{a(c)} F\{c\} F'\{\bar{c}\},$$

where the sum is over all classes c of $G_n(k)$, $a(c)$ denotes the order of the centralizer in $G_n(k)$ of an element of c, and $\bar{c} = c^{-1}$. Define U_n to be the space of all class-functions F of *unipotent support*, i.e. such that $F\{c\} \neq 0$ □ $c = (t-1)^\pi$ for some $\pi \vdash n$. As in section 6, we write $F\{(t-1)^\pi\} = F\{\pi\}$. Then 8.3 becomes, when at least one of F, F' belongs to U_n, 8.4 $$\bullet F, F' \circledR = \sum_{\pi \vdash n} \frac{1}{a(\pi)} F\{\pi\} F'\{\pi\}.$$

8.5 Lemma *For each* $\rho \vdash n$ *define* $\mathbf{Q}_\rho \in U_n$ *by setting* $\mathbf{Q}_\rho\{\pi\} = Q_\rho^\pi(q)$*. Then for any* $F \in U_n$ *there holds*

8.6 $$F = \sum_{\rho \vdash n} \frac{c_\rho(q)}{z_\rho} \bullet F, \mathbf{Q}_\rho \circledR \ \mathbf{Q}_\rho.$$

Proof Use 8.2 to evaluate the right-hand side of 8.4.

The following proposition provides convenient rules for calculating the coefficients $\bullet F, \mathbf{Q}_\rho \circledR$ of the "Fourier expansion" 8.6 of F in terms of the $\mathbf{Q}_\rho$. Let $\rho = 1^{r_1} 2^{r_2} \ldots$, $\sigma = 1^{s_1} 2^{s_2} \ldots$ and $\tau = 1^{t_1} 2^{t_2} \ldots$ be partitions, and recall the definition $\rho + \sigma = 1^{r_1 + s_1} 2^{r_2 + s_2} \ldots$.

8.7 Proposition (i) *If* $\rho \vdash l$, $\sigma \vdash m$ *then* $\mathbf{Q}_\rho \circ \mathbf{Q}_\sigma = \mathbf{Q}_{\rho + \sigma}$.

(ii) *If* $\rho, \sigma \vdash n$ *then* $\bullet \mathbf{Q}_\rho, \mathbf{Q}_\sigma \circledR = \delta_{\rho,\sigma} \cdot \dfrac{z_\rho}{c_\rho(q)}$.

(iii) *If* $f \in U_l$ *and* $g \in U_m$ *then for any* $\tau \vdash l + m$

8.8 $$\bullet f \circ g, \mathbf{Q}_\tau \circledR = \sum_{\rho + \sigma = \tau} \begin{bmatrix} \tau \\ \rho, \sigma \end{bmatrix} \bullet f, \mathbf{Q}_\rho \circledR \bullet g, \mathbf{Q}_\sigma \circledR,$$

where the sum is over all pairs (ρ, σ) *such that* $\rho \vdash l$, $\sigma \vdash m$ *and* $\rho + \sigma = \tau$, *and*

8.9 $$\begin{bmatrix} \tau \\ \rho, \sigma \end{bmatrix} = \prod_i \frac{t_i!}{r_i! s_i!}.$$

Proof (i) Use 6.4, together with lemma 4.4 of [GL], p.420.

(ii) Follows at once from 8.4 and 8.1.

(iii) Since both sides of 8.8 are linear in both f and g, it is enough verify that 8.8 holds when $f = \mathbf{Q}_\lambda$ and $g = \mathbf{Q}_\mu$, where $\lambda \vdash l$ and $\mu \vdash m$. This is a routine calculation using (i) and (ii).

The next proposition shows the connection between the Gelfand-Graev character Γ_n, the $Q^{\rho}_{\lambda}(q)$, and the characters of $S(n)$.

8.10 Proposition (i) $\langle \Delta_n, \mathbf{Q}_\rho \rangle = 1$, *for all* $\rho \vdash n$.

(ii) $\langle \Gamma_n, \mathbf{Q}_\rho \rangle = \varepsilon_\rho$, *where* $\varepsilon_\rho = (-1)^{r_2+r_4+\cdots}$ *is the value of the alternating character* ε *of* $S(n)$ *at the conjugacy class* ρ *of* $S(n)$.

Proof (i) By 8.4 $\langle \Delta_n, \mathbf{Q}_\rho \rangle = \sum_{\pi \vdash n} \frac{1}{a_\pi(q)} . \Delta_n\{\pi\}. Q^{\pi}_{\rho}(q)$. Using 6.2, this reduces to $\frac{1}{a_{(n)}(q)} . q^{n-1}(q-1). Q^{(n)}_{\rho}(q)$, which equals 1, because $a_{(n)}(q) = q^{n-1}(q-1)$, and $Q^{(n)}_{\rho}(q)$ = 1 for all $\rho \vdash n$ ([GL], p.445, or [M] , p.248, Ex. 1).

(ii) Let $\lambda = (n_1, \ldots, n_b)$ be the composition of n associated to a subset J of $I = \{1, \ldots, n-1\}$. We consider the function $\Delta_J = \Delta_{n_1} \circ \ldots \circ \Delta_{n_b}$, as in section 6. Using the evident extension of 8.8 to several factors, together with (i), we find that $\langle \Delta_J, \mathbf{Q}_\rho \rangle = H(J, \rho)$.where

8.11 $$H(J,\rho) = \sum_{\boldsymbol{\rho}} \prod_i \frac{r_i!}{r_i(1)! \ldots r_i(b)!},$$

the sum being over all vectors $\boldsymbol{\rho} = (\rho(1), \ldots, \rho(b))$ such that $\rho(j) = 1^{r_1(j)} 2^{r_2(j)} \ldots \vdash n_j$ for all $j = 1, \ldots, b$, and $\rho(1) + \cdots + \rho(b) = \rho$. Frobenius showed, in his classic paper [F] on the characters of $S(n)$, that $H(J,\rho)$ is the value at class ρ (of $S(n)$) of the character χ_J = $\mathrm{Ind}^{S(n)}_{S(J)}(1_{S(J)})$, where $S(J)$ is the subgroup of $S(n)$ consisting of all permutations of $\{1, \ldots, n\}$ which leave fixed each of the subsets $\{1, \ldots, n_1\}$, $\{n_1+1, \ldots, n_2\}$, …, $\{n_1 + \ldots + n_{b-1} + 1, \ldots, n\}$ (see [F], p.149, or [Le], p.103). L. Solomon has proved a formula ([S], theorem 2), which gives as special case the following equation on characters of $S(n)$

8.12 $$\sum_{J \subset I} (-1)^{|J|} \chi_J = \varepsilon.$$

If we combine 8.12 with Deligne–Lusztig's formula 6.6 for Γ_n, we get
$$\langle \Gamma_n, \mathbf{Q}_\rho \rangle = \sum_{J \subset I} (-1)^{|J|} \langle \Delta_J, \mathbf{Q}_\rho \rangle = \sum_{J \subset I} (-1)^{|J|} H(J,\rho) = \sum_{J \subset I} (-1)^{|J|} \chi_J\{\rho\} = \varepsilon_\rho.$$

8.13 Corollary *For any* $J \subset I$ *and* $\rho \vdash n$ *there holds*

$$\bullet\Gamma_J, \mathbf{Q}_\rho \text{®} = \bullet\Gamma_{n_1} \circ \ldots \circ \Gamma_{n_b}, \mathbf{Q}_\rho \text{®} = H(J,\rho).\varepsilon_\rho.$$

Proof This follows from 8.10(ii) and 8.8.

From now on we shall always write F_λ, $H(\lambda,\pi)$ and $c_{\lambda,\pi}$ instead of F_J, $H(J,\pi)$ and $c_{J,\pi}$, when $\lambda \models n$ is the composition associated to J. Notice that each of these is unchanged if λ is replaced by any $\lambda' \approx \lambda$, so we lose nothing if we assumed that $\lambda \vdash n$. From 8.13 and 8.6 we have

8.14 $$\Gamma_\lambda\{\pi\} = \sum_{\rho \vdash n} \frac{c_\rho(q)}{z_\rho}.H(\lambda,\rho).\varepsilon_\rho.Q_\rho^\pi(q),$$

for all $\lambda, \pi \vdash n$. Notice that all the polynomials $c_\rho(T)$ are divisible by $T-1$. Therefore, if we write $\hat{c}_\rho(T) = \frac{1}{T-1}.c_\rho(T)$, we have from 8.14 and 6.7

8.15 $$c_{\lambda,\pi}(T) = \sum_{\rho \vdash n} \frac{\hat{c}_\rho(T)}{z_\rho}.H(\lambda,\rho).\varepsilon_\rho.Q_\rho^\pi(T).$$

9 Proof of propositions 4.4 and 4.6

We need to connect the numbers $H(\lambda,\pi)$ with the characters χ^λ of $S(n)$. Using Macdonald's notation for symmetric functions (see [M]), we have equations $p_\rho(x) = \sum_{\lambda \vdash n} H(\lambda,\rho) m_\lambda(x)$ (see [F], p.149, or [Le], p.103). If we combine these with $p_\rho(x) = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda(x)$ and $s_\lambda(x) = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu(x)$ (see [M], p.101; the $K_{\tau\lambda}$ are the "Kostka numbers") we get

9.1 $$H(\lambda,\pi) = \sum_{\tau \vdash n} \chi_\rho^\tau K_{\tau\lambda}.$$

9.2 Proof of proposition 4.4 We want to calculate $W(\lambda,\mu) = \bullet X_{1,n}(1,\lambda), I^{\mu} \circledR$. From 7.2, $X_{1,n}(1,\lambda)$ is zero on all classes except the classes $(t-x)^{\pi}$, $x \in M_1 = k^{\times}$, $\pi \vdash n$. Moreover, for fixed π, $X_{1,n}(1,\lambda)\{(t-x)^{\pi}\} = c_{\lambda,\pi}(q)$, for all $x \in M_1$. From 4.5, $I^{\mu}\{(t-x)^{\pi}\} =$ $\sum_{\rho \vdash n} \frac{1}{z_\rho}.Q_\rho^{\pi}(q).\chi_\rho^{\mu}$ for all $x \in M_1$. Then it follows easily from 8.15 that

9.3 $$\bullet X_{1,n}(1,\lambda), I^{\mu} \circledR = (q-1)\bullet \sum_{\rho \vdash n} \frac{\hat{c}_\rho(q)}{z_\rho}.\varepsilon_\rho.H(\lambda,\rho).\mathbf{Q}_\rho, \sum_{\sigma \vdash n} \frac{1}{z_\sigma}.\chi_\sigma^{\mu}.\mathbf{Q}_\sigma \circledR,$$

and by 8.7(ii) this reduces to $\sum_{\rho \vdash n} \frac{1}{z_\rho}.\varepsilon_\rho.H(\lambda,\rho).\chi_\rho^{\mu}$. Since $\varepsilon_\rho \chi_\rho^{\mu} = \chi_\rho^{\tilde{\mu}}$, where $\tilde{\mu}$ is the conjugate of μ ([Le], p.135), we get from 9.3

9.4 $$W(\lambda,\mu) = \sum_{\rho,\tau \vdash n} \frac{1}{z_\rho}.K_{\tau\lambda}.\chi_\rho^{\tau}\chi_\rho^{\tilde{\mu}} = K_{\tilde{\mu}\lambda}.$$

But the matrix $(K_{\lambda\mu})$ has integer coefficients and is unimodular ([M], p.101), and therefore the same is true of the matrix $(W(\lambda,\mu)) = (K_{\tilde{\mu}\lambda})$.

9.5 Proof of proposition 4.6 If this proposition is false, there exist complex numbers $h_{d,\lambda}$, not all zero, such that

9.6 $$\sum_{d|n} \sum_{\lambda \vdash n/d} h_{d,\lambda} X_{d,n}(1,\lambda) = 0.$$

Let d_0 be the largest divisor of n such that $h_{d_0,\lambda} \neq 0$ for some $\lambda \vdash \frac{n}{d_0}$. Now take any $f \in \Phi(k)$ such that $d(f) = d_0$, and any $\sigma \vdash \frac{n}{d_0}$.

Let d be any divisor of n. If $d > d_0$, then $h_{d,\lambda} = 0$, by the definition of d_0. If $d < d_0$, then $d(f) = d_0$ does not divide d, hence $X_{d,n}(1,\lambda)\{f^{\sigma}\} = 0$ by 4.1(i). Therefore if we evaluate 9.6 at the class f^{σ}, we get

9.7 $$\sum_{\lambda \vdash n/d_0} h_{d_0,\lambda} X_{d,n}(1,\lambda)\{f^{\sigma}\} = 0, \text{ for all } \sigma \vdash \frac{n}{d_0}.$$

But 7.18 tells us that $X_{d_0,n}(1,\lambda)\{f^{\sigma}\} = c_{\lambda,\sigma}(q^{d_0})$ (notice that $d(f) = m = d_0$, hence $\sigma = \pi$), and by 8.16 the matrix $\left(c_{\lambda,\sigma}(q^{d_0})\right)$ is non-singular. Thus 9.7 implies that $h_{d_0,\lambda} = 0$ for all λ. This contradiction proves proposition 4.6.

Final Version 5 March 1998

Appendix: some $r_\lambda(T)$ These are found by the inductive construction in section 5. Values of the characters $X_{d,n}(\psi,\lambda)$ are calculated from formulae 7.18 and 8.15 .

λ	(1)	(2)	(1^2)	(3)	(21)	(1^3)	(4)	(31)	(2^2)	(21^2)	(1^4)
$r_\lambda(T)$	1	–1	0	$2-T$	–1	0	$T-2$	1	0	0	0

λ	(5)	(41)
$r_\lambda(T)$	$4-3T-T^3+2T^5-T^6$	$-3+2T-T^2+T^3+T^4-T^5$

λ	(32)	(31^2)	(2^21)	(21^3)	(1^5)
$r_\lambda(T)$	$-3+T+T^2+T^3-T^4$	$2-T+T^2-T^3$	$2-T^2$	–1	0

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