

Monge Distance between Quantum States

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We define a metric in the space of quantum states taking the Monge distance between corresponding Husimi distributions (Q-functions). This quantity fulfills the axioms of a metric and satisfies the following *semiclassical property*: the distance between two coherent states is equal to the Euclidean distance between corresponding points in the classical phase space. We compute analytically distances between certain states (coherent, squeezed, Fock and thermal) and discuss a scheme for numerical computation of Monge distance for two arbitrary quantum states.

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I. INTRODUCTION

The state space of an n -dimensional quantum system is the set of all $n \times n$ positive semidefinite complex matrices of trace 1 called *density matrices*. The density matrices of rank one (*pure states*) can be identified with nonzero vectors in a complex Hilbert space of dimension n . However, one has to take into account that the same state is described by a vector ψ and $\lambda\psi$, where $\lambda \neq 0$. Hence pure states are in one-to-one correspondence with rays $\{\lambda\psi : 0 \neq \lambda \in \mathbf{C}\}$. The rays form a smooth manifold called complex projective space \mathbf{CP}^{n-1} . In the infinite-dimensional case we have to consider density operators instead of density matrices, and the space of pure states is the complex projective space over the infinite-dimensional Hilbert space.

The problem of measuring a distance between two quantum states with a suitable metric attracts a lot of attention in recent years. The Hilbert-Schmidt norm of an operator $\|A\|_2 = \sqrt{\text{Tr}(A^\dagger A)}$ induces a natural distance between two density operators $d_{HS}(\rho_1, \rho_2) = \sqrt{\text{Tr}[(\rho_1 - \rho_2)^2]}$. This distance has been recently used to describe the dynamics of the field in Jaynes-Cummings model [1] and to characterize the distance between certain states used in quantum optics [2]. Another distance generated by the *trace norm* $\|A\|_1 = \text{Tr}\sqrt{A^\dagger A}$ was used by Hillery [3,4] to measure the non-classical properties of quantum states.

A concept of *statistical distance* was introduced by Wootters [5] in the context of measurements which optimally resolve neighboring pure quantum states. This distance, leading to the Fubini-Study metric in the complex projective space, was later generalized to density matrices by Braustein and Caves [6] and its dynamics for a two-state system was studied by Braustein and Milburn [7]. It was shown [6] that for neighboring density matrices the statistical distance is proportional to the distance introduced by Bures in late sixties [8]. The latter was studied by Uhlmann [9] and Hübner [10], who found an explicit formula for the Bures distance between two density operators $d_B^2(\rho_1, \rho_2) = 2(1 - \text{tr}[(\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2}])$. Note that various Riemannian metrics on the spaces of quantum states were also considered by many other authors (see [11–15]).

In the present paper we introduce yet another distance in the space of density operators, which fulfills a following *semiclassical property*: the distance between two coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ localized at points a_1 and a_2 of the classical phase space Ω endowed with a metric d is equal to the distance of these points

$$D(|\alpha_1\rangle, |\alpha_2\rangle) = d(a_1, a_2). \quad (1.1)$$

This condition is quite natural in the semiclassical regime, where the quasi-probability distribution of a quantum state tends to be strongly localized in the vicinity of the corresponding classical point. A motivation to study such a distance stems from the search for quantum Lyapunov exponent, where a link between distances in Hilbert space and in the classical phase space is required [16].

In order to find a metric satisfying condition (1.1) it is convenient to represent a quantum state ρ in the Q -representation (called also the Husimi function) [17]

$$H_\rho(\alpha) := \langle \alpha | \rho | \alpha \rangle, \quad (1.2)$$

defined with the help of a family of coherent states $|\alpha\rangle$, $\alpha \in \Omega$, which fulfill the identity resolution $\mathbf{I} = \int_\Omega |\alpha\rangle\langle\alpha| dm(\alpha)$, where m is the natural measure on Ω . For a pure state $|\psi\rangle$ one has $H_\psi(\alpha) = |\langle\psi|\alpha\rangle|^2$. The choice of coherent states is somewhat arbitrary and in fact one may work with different systems of coherent states [18]. In the present paper we use the classical *harmonic oscillator (field) coherent states*, where $\Omega = \mathbf{C}$ and $dm(\alpha) = d^2\alpha/\pi$. For convenience we will use in the sequel the renormalized version of the Husimi function putting $H(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$ and integrating over $d^2\alpha$. Note that the Husimi representation of a given density operator determines it uniquely [19]. Since Husimi distributions are non-negative and normalized, i.e., $\int_\Omega H(\alpha) dm(\alpha) = 1$ and $H \geq 0$, it follows that a metric in the space of probability densities $Q : \Omega \rightarrow R_+$ induces a metric in the state space.

In this work we propose to measure the distance between density operators by the Monge distance between the corresponding Husimi distributions. The original Monge problem consists in finding an optimal way of transforming a pile of sand into a new location. The Monge distance between two piles is given by the ‘path’ traveled by all grains under the optimal transformation [20,21].

This paper is organized as follows. In sect. II we give a definition of the Monge metric, present an explicit solution for the one-dimensional case and discuss some methods of tackling the two-dimensional problem. Sect. III contains some examples of computing the Monge distance between certain states often encountered in quantum optics. Concluding remarks are given in sect. IV. Variational approach to the Monge problem is briefly presented in the Appendix.

II. MONGE METRIC

A. Monge transport problem

The original Monge problem, formulated in 1781 [22], emerged from studying the most efficient way of transporting soil [20]:

Split two equally large volumes into infinitely small particles and then associate them with each other so that the sum of products of these paths of the particles to the volume is least. Along what paths must the particles be transported and what is the smallest transportation cost?

Fig. 1 represents a scheme for this problem. Here we give a general definition of the Monge distance between two probability densities Q_1 and Q_2 defined in $S = R^n$. Let Ω_1 and Ω_2 , determined by Q_i , describe the initial and the final location of 'sand': $\Omega_i = \{(x, y) \in S \times R^+ : 0 \leq y \leq Q_i(x)\}$. Due to normalization of Q_i the integral $\int_{\Omega_i} d^n x dy$ is equal to unity. Consider C^1 maps $T : S \rightarrow S$ which generate volume preserving transformations Ω_1 into Ω_2 , i.e.,

$$Q_1(x) = Q_2(Tx) |T'(x)| \quad (2.1)$$

for all $x \in S$, where $T'(x)$ denotes the Jacobian of the map T at point x . We shall look for a transformation giving the minimal displacement integral and define the Monge distance [20,21]

$$D_M(Q_1, Q_2) := \inf \int_S |x - T(x)| Q_1(x) d^n x, \quad (2.2)$$

where the infimum is taken over all T as above. The optimal transformation (if exists) T_M is called a *Monge plan*. Note that in this formulation of the problem the "vertical" component of the sand movement is neglected.

It is easy to see that Monge distance fulfills all the axioms of a metric. This allows us to define a 'classical' distance between two quantum states ρ_1 and ρ_2 via the Monge distance between the corresponding Husimi distributions:

$$D_{cl}(\rho_1, \rho_2) := D_M(H_1(\alpha), H_2(\alpha)). \quad (2.3)$$

The Monge distance satisfies the semiclassical property: it is shown below that the distance between two coherent states, represented by Gaussian Husimi distributions localized at points a_1 and a_2 , equals to the classical distance $|a_1 - a_2|$.

It is sometimes useful to generalize the notion of the Monge metric and to define a family of distances D_{M_p} , labeled by a continuous index p ($0 < p \leq \infty$) in an analogous way:

$$[D_{M_p}(Q_1, Q_2)]^p := \inf \int_S |x - T(x)|^p Q_1(x) d^n x. \quad (2.4)$$

For $p = 1$ one recovers the Monge distance $D_{M_1} = D_M$, while the Fréchet distance D_{M_2} is obtained for $p = 2$. A more general approach to the transport problem was proposed by Kantorovitch [23] and further developed by Sudakov [24]. In contrast with the definition of Monge discussed in this work, the Kantorovitch distance between Q_1 and Q_2 is explicitly symmetric with respect to exchange of both distributions. For a comprehensive review of metrics in the space of probability measures and other generalizations of the Monge distance see the monograph of Rachev [21].

B. Salvemini solution for 1D problem

For $S = R$ the Monge distance can be expressed explicitly with the help of distribution functions $F_i(x) = \int_{-\infty}^x Q_i(t) dt$, $i = 1, 2$. Salvemini [25] and Dall'Aglio [26] obtained the following solution of the problem for $p = 1$:

$$D_M(Q_1, Q_2) = \int_{-\infty}^{+\infty} |F_1(x) - F_2(x)| dx, \quad (2.5)$$

represented schematically in Fig. 2.

This result was generalized in the fifties to all $p \geq 1$ by several authors (see [20,21]). We have

$$[D_{M_p}(Q_1, Q_2)]^p = \int_0^1 |F_1^{-1}(t) - F_2^{-1}(t)|^p dt. \quad (2.6)$$

C. Poisson-Ampere-Monge equation for 2D case

Consider smooth densities $Q_1, Q_2 : R^2 \rightarrow R^+$. We are looking for a transformation field $w = (w_1, w_2) : R^2 \rightarrow R^2$ fulfilling $w(x_1, x_2) = T_M(x_1, x_2) - (x_1, x_2)$, where T_M is an optimal Monge transformation minimizing the right-hand side of (2.4). Restricting ourselves to smooth transformations T we may apply the standard variational search for the optimal field w . In Appendix it is shown that $\text{rot}(w) = 0$ if $p = 2$, and the potential $\varphi : w = \text{grad}(\varphi)$ satisfies the following partial differential equation

$$\varphi_{x_1 x_1} + \varphi_{x_2 x_2} + \varphi_{x_1 x_1} \varphi_{x_2 x_2} - (\varphi_{x_1 x_2})^2 = \frac{Q_1(x_1, x_2)}{Q_2(x_1 + \varphi_{x_1}, x_2 + \varphi_{x_2})} - 1. \quad (2.7)$$

Solving this *Laplace-Ampere-Monge (LAM) equation* for the potential φ we get the Fréchet distance D_{M_2} from (2.4) computing the minimal displacement

$$(D_{M_2}(Q_1, Q_2))^2 = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 (\varphi_{x_1}^2 + \varphi_{x_2}^2) Q_1(x_1, x_2). \quad (2.8)$$

Though it is hardly possible to solve equation (2.7) in a general case, it might be used to check, whether a given transformation can be a solution of the Monge problem. Let us remark that an optimal transformation field w (if exists) must not be unique. Moreover, (2.7) provides only a sufficient condition for w being optimal. It is important to note that the two dimensional Monge problem posed two hundreds years ago has not been solved in a general case until now [20,21].

D. Estimation of the Monge distance via transport problem

One of major tasks of linear programming is the optimization of the following transport problem. Consider N suppliers producing a_i ($i = 1, \dots, N$) pieces of a product per a time period and M customers requiring b_j ($j = 1, \dots, M$) pieces of the product at the same time. Let (c_{ij}) be a $N \times M$ cost matrix, representing for example the distances between sites. Find the optimal transporting scheme, minimizing the total transport costs $C = \sum_{i=1}^N \sum_{j=1}^M c_{ij} x_{ij}$. The non-negative elements of the unknown matrix (x_{ij}) denote the number of products moved from i -th supplier to j -th customer. The optimization problem is subjected to the following constrains: $\sum_{j=1}^M x_{ij} = a_i$ and $\sum_{i=1}^N x_{ij} = b_j$. In the simplest case the total supply equals the total demand and $\sum_{i=1}^N a_i = \sum_{j=1}^M b_j$.

The *transport problem* described above gave, with a related *assignment problem*, an impulse to develop methods of linear programming already half a century ago [23], [27]. Since then several algorithms for solving the transport problem numerically have been proposed. Some of them, as the *northwest corner procedure* and *Vogel's approximation* [28], are available in specialized software packages. It is worth to add that the transport methods are widely used to solve a variety of problems in business and economy such as, for instance, market distribution, production planning, plant location and scheduling problems.

It is easy to see that the transport problem is a discretized version of the Monge problem defined for continuous distribution functions. One can, therefore, approximate the Monge distance between two distributions $Q_1(x)$ and $Q_2(x)$ (where x stands for a two dimensional vector), by solving the transport problem for delta peaks approximation of the continuous distributions: $q_1 = \sum_{i=1}^N Q_1(x_i) \delta(x - x_i)$ and $q_2 = \sum_{j=1}^M Q_2(x_j) \delta(x - x_j)$. The quality of this approximation depends on the numbers N and M of peaks representing the initial and the final distribution, respectively, and also on their location with respect to the shape of both distributions. Numerical study performed with the *northwest corner* algorithm for some analytically soluble examples of the 2-D Monge problem shows [29] that for reasonably smooth distributions one obtains Monge distance with fair accuracy for a number of peaks of the order of hundreds.

III. MONGE DISTANCE BETWEEN SOME STATES OF QUANTUM OPTICS

In this section we compute Monge distances between certain quantum states. Even though our considerations are valid in the general framework of quantum mechanics, for the sake of concreteness we will use the language of quantum optics. Let us recall briefly the necessary definitions and properties.

Let a and a^\dagger be the annihilation and creation operators satisfying the commutation relation $[a, a^\dagger] = 1$. The 'vacuum' state $|0\rangle$ is distinguished by the relation $a|0\rangle = 0$. Standard harmonic-oscillator *coherent states* $|\alpha\rangle$ can be defined as

the eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$ or by the Glauber translation operator $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ as $|\alpha\rangle = D(\alpha)|0\rangle$. Coherent states, determined by an arbitrary complex number $\alpha = x_1 + ix_2$, enjoy a minimum uncertainty property. They are not orthogonal and do overlap. The Husimi distribution of a coherent state $|\beta\rangle$ is Gaussian

$$H_\beta(\alpha) = \frac{1}{\pi} |\langle\beta|\alpha\rangle|^2 = \frac{1}{\pi} \exp(-|\alpha - \beta|^2). \quad (3.1)$$

Squeezed states $|\gamma, \alpha\rangle$ also minimize the uncertainty relation, however the variances of both canonically coupled variables are not equal. They are defined as

$$|\gamma, \alpha\rangle := D(\alpha)S(\gamma)|0\rangle, \quad (3.2)$$

where the squeezing operator is $S(\gamma) = \exp[\frac{1}{2}(\gamma^* a^2 - \gamma a^{\dagger 2})]$. The modulus g of the complex number $\gamma = ge^{2i\theta}$ determines the strength of squeezing, $s = e^g - 1$, while the angle θ orients the squeezing axis. The Husimi distribution of a squeezed state $|\gamma, \beta\rangle$ is a non-symmetric Gaussian localized at point β and for $\theta = 0$ reads

$$H_{\gamma, \beta}(x_1, x_2) = \frac{1}{\pi} \exp[-(\text{Re}(\beta) - x_1)^2/(s+1)^2 - (\text{Im}(\beta) - x_2)^2(s+1)^2] \quad (3.3)$$

Each pure state can be expressed in the Fock basis consisting of n -photon states $|n\rangle$, $n = 0, 1, 2, \dots$. Each coherent state can be expanded in the Fock basis as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (3.4)$$

The known scalar product $\langle\alpha|n\rangle$ allows one to write the Husimi function of a *Fock state*

$$H_{|n\rangle}(\alpha) = \frac{1}{\pi} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}. \quad (3.5)$$

The *vacuum state* $|0\rangle$ can be thus regarded as the single Fock state being simultaneously coherent.

In contrast to the above mentioned pure states, the *thermal mixture* of states with the mean number of photon equal to \bar{n} is represented by the density operator

$$\rho_{\bar{n}} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}} |n\rangle\langle n|. \quad (3.6)$$

Its Husimi distribution is the Gaussian centered at zero with the width depending on the mean photon number,

$$H_{\rho_{\bar{n}}}(\alpha) = \frac{1}{\pi(\bar{n} + 1)} \exp\left(-\frac{|\alpha|^2}{\bar{n} + 1}\right). \quad (3.7)$$

A. Two coherent states

Let us consider an arbitrary two dimensional distribution function $Q_1(x_1, x_2)$ and a shifted one $Q_2(x_1, x_2) = Q_1(x_1 - x_1^*, x_2 - x_2^*)$. It is intuitive to expect that the simple translation given by $w(x_1, x_2) = \text{const} = (x_1^*, x_2^*)$ solves the corresponding Monge problem. Since for the respective potential $\phi(x_1, x_2) = x_1^* x_1 + x_2^* x_2$ the second derivatives vanish, the both two sides of the LAM equation (2.7) are equal to zero and the maximization condition is fulfilled.

It follows from this observation that for two coherent states defined on the complex plane the Monge plan reduces to translation. Integration in (2.2) is trivial and provides the Monge distance between two arbitrary coherent states $|\alpha\rangle$ and $|\beta\rangle$

$$D_{cl}(|\alpha\rangle, |\beta\rangle) = |\alpha - \beta|. \quad (3.8)$$

This is exactly the *semiclassical property* we demanded from the metric in the state space. The distance of the coherent state $|\alpha\rangle$ from the vacuum state $|0\rangle$ is equal to $|\alpha|$, which simply is the square root of the mean number of photons in the state $|\alpha\rangle$. The classical property is fulfilled by the generalized Monge metric D_{M_p} for any positive parameter p .

B. Coherent and squeezed states

We compute the distance between a coherent state $|\alpha\rangle$ and the corresponding state squeezed $|\gamma, \alpha\rangle$. Due to invariance of the Monge optimization with respect to translations this distance is equal to the distance between vacuum $|0\rangle$ and the squeezed vacuum $|\gamma, 0\rangle$. For simplicity we will assume that squeezing is performed along the x_1 axis, i.e., the complex squeezing parameter is real $\gamma = g \in \mathbb{R}$.

The corresponding Monge problem consists in finding the optimal transformation of the symmetric Gaussian $Q_1(x_1, x_2) = \exp(-x_1^2 - x_2^2)/\pi$ into an asymmetric one $Q_2(x_1, x_2) = \exp(-x_1^2/(s+1)^2 - x_2^2/(s+1)^2)/\pi$. Considering contours of the Husimi distribution, often used to represent a state in quantum optics, a circle has to be transformed into an ellipse. If $p = 2$ then the following affine transformation $T(x_1, x_2) = (x_1/(s+1), x_2/(s+1))$ corresponds to the irrotational transport field $w(x_1, x_2) = (-sx_1/(s+1), sx_2)$. It can be obtained as the gradient of the potential $\varphi(x_1, x_2) = -sx_1^2/(2s+2) + sx_2^2/2$, for which both sides of the LAM equation (2.7) vanish. Hence field w provides a Monge plan for this problem and the distance is given by its length $|w|$ integrated over the volume of Q_1

$$D_{cl}(|0\rangle, |g, 0\rangle) = \frac{s}{\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp(-x_1^2 - x_2^2) \sqrt{x_1^2/(s+1)^2 + x_2^2} = \frac{s}{\sqrt{\pi}} E\left(\frac{s^2 + 2s}{s^2 + 2s + 1}\right), \quad (3.9)$$

where $E(x)$ stands for the complete elliptic integral of the second kind.

A simpler result may be obtained in this case for the Fréchet distance D_{M_2}

$$[D_{M_2}(|0\rangle, |g, 0\rangle)]^2 = \frac{s^2}{\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp(-x_1^2 - x_2^2) (x_1^2/(s+1)^2 + x_2^2) = \frac{s^2}{2} \left(1 + \frac{1}{(s+1)^2}\right). \quad (3.10)$$

Eventually, the ‘Chebyshev’ distance D_{M_∞} , characterizing the maximal dislocation length, is equal to s . For large squeezing all three distances grow proportionally to the squeezing parameter s , with slopes $k_1 = 1/\sqrt{\pi}$, $k_2 = 1/\sqrt{2}$ and $k_\infty = 1$ ordered according to the index p . This agrees with an observation that $p_1 < p_2$ implies $D_{M_{p_1}} \leq D_{M_{p_2}}$ [21].

C. Vacuum and thermal states

Since Husimi distributions of both states is rotationally invariant, it is convenient to use radial components of the distribution $R_i(r) = 2\pi r H_i(r, \phi)$. One may then reduce the problem to one dimension and find the Monge distance via radial distribution functions $F_i(r) = \int_0^r R_i(r') dr'$. Taking Husimi distributions (3.1) and (3.7) we get the corresponding distribution functions $F_1(r) = 1 - \exp(-r^2)$ and $F_2(r) = 1 - \exp(-r^2/(\bar{n} + 1))$. Using the Salvemini formula (2.5) we obtain

$$D_{cl}(|0\rangle\langle 0|, \rho_{\bar{n}}) = \int_0^\infty |F_1(r) - F_2(r)| dr = \frac{\sqrt{\pi}}{2} (\sqrt{\bar{n} + 1} - 1). \quad (3.11)$$

In the same way we get a more general formula for the Monge distance between two thermal states

$$D_{cl}(\rho_{\bar{n}_1}, \rho_{\bar{n}_2}) = \frac{\sqrt{\pi}}{2} |\sqrt{\bar{n}_1 + 1} - \sqrt{\bar{n}_2 + 1}|. \quad (3.12)$$

D. Two Fock states

As in the previous example the rotational symmetry of the Fock states allows us to use the 1D formula. Integrating (3.5) for a Fock state $|n\rangle$ we can express the radial distribution function in terms of the incomplete Gamma function $\Gamma(x, r)$ as

$$F_n(r) = 1 - \frac{\Gamma(n+1, r^2)}{\Gamma(n+1)}. \quad (3.13)$$

Since for different Fock states the distribution functions do not cross, applying Salvemini formula (2.5) we obtain the Monge distance

$$D_{cl}(|m\rangle, |n\rangle) = \left| \int_0^\infty F_n(r) dr - \int_0^\infty F_m(r) dr \right| = \sqrt{\pi} |C_m - C_n|, \quad (3.14)$$

where the integrals C_i can be found analytically: $C_0 = 1/2$, $C_1 = 3/4$, $C_2 = 15/16$, $C_3 = 35/32, \dots$

IV. DISCUSSION

We have presented a definition of distance between quantum states (i.e. elements of a Hilbert space) which possesses a certain classical property, natural for investigation of the semiclassical limit of quantum mechanics. Monge distance between the corresponding Husimi functions fulfills the axioms of a metric and induces a 'classical' topology in the Hilbert space. It is worth to emphasize that the Monge distance between two given density matrices depends on the topology of the corresponding classical phase space.

Consider a quantum state prepared as a superposition of two coherent states separated in the phase space by x . It is known [30–32] that such a state interacting with an environment evolves quickly toward a mixture of the two localized states. The decoherence time decreases with the classical distance x , equal just to the Monge distance between both coherent states. We expect therefore that the Monge distance between two arbitrary quantum states might be useful to determine the rate with which the superposition of these two states suffers the decoherence.

It is possible to generalize our approach in several directions. Instead of the standard Husimi distributions used throughout this paper, one may study Monge distances between generalized Husimi distributions $\tilde{H}_\rho(\alpha) = \langle \tilde{\alpha} | \rho | \tilde{\alpha} \rangle$, where $|\tilde{\alpha}\rangle$ are generalized coherent states [33]. For example, one may use for this purpose squeezed states [34], or the spin coherent states [35,36], if the classical space is the two-dimensional sphere.

Moreover, our considerations based on the Husimi representation of quantum states, may be in fact extended to the Wigner function. In spite of the fact that the Wigner function can take on negative values, it is normalized and the Monge problem of finding an optimal way to transport one Wigner function into another might also be considered. The concept of the classical distance between two Husimi (Wigner) functions is not only of theoretical interest, since novel methods of measuring Husimi and Wigner distributions have been recently developed [37–39].

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APPENDIX A: VARIATIONAL APPROACH TO MONGE PROBLEM

1. 1D case

Let Q_1 and Q_2 be smooth densities. In 1D case there is only one map T fulfilling (2.1). It can be described with the aid of distribution functions as $T(x) = F_2^{-1}(F_1(x))$, where $F_i(x) = \int_{-\infty}^x Q_i(t)dt$ for $x \in R$. This allows us to express the generalized Monge distance as an integral (2.4)

$$[D_{M_p}(Q_1, Q_2)]^p := \int_{-\infty}^{\infty} Q_1(x) |F_2^{-1}[F_1(x)] - x|^p dx. \quad (\text{A1})$$

In the simplest case $p = 1$ (A1) reduces to the Salvemini formula (2.5) and for $p > 1$ to formula (2.6).

2. 2D case

Consider two smooth densities $Q_1(x_1, x_2)$ and $Q_2(x_1, x_2)$. We are looking for a map $T(x_1, x_2) = (x + w_1(x_1, x_2), x_2 + w_2(x_1, x_2))$ transforming Q_1 into Q_2 (i.e. such that (2.1) is fulfilled) and minimizing the quantity

$$I_p = \int_{-\infty}^{+\infty} Q_1(x_1, x_2) |w_1^2(x_1, x_2) + w_2^2(x_1, x_2)|^{p/2} dx_1 dx_2. \quad (\text{A2})$$

The index p , labeling the generalized distance, is equal to one for the Monge metric and to two for the Fréchet metric. Introducing a Lagrange factor λ we write the Lagrange function in the form

$$L_p = Q_1(w_1^2 + w_2^2)^{p/2} + \lambda(Q_1 - Q_2(T))[(1 + w_{1x_1})(1 + w_{2x_2}) - w_{1x_2}w_{2x_1}]. \quad (\text{A3})$$

The Lagrange–Euler equations for variations of L_p allow us to obtain the partial derivatives of λ

$$\begin{aligned}\lambda_{x_1} &= 2pC_p(w_1(1 + w_{1x_1}) + w_2w_{2x_1}) , \\ \lambda_{x_2} &= 2pC_p(w_2(1 + w_{2x_2}) + w_1w_{1x_2}) ,\end{aligned}\tag{A4}$$

where $C_p = (w_1^2 + w_2^2)^{(p-2)/2}$. Using the equality $\lambda_{x_1x_2} = \lambda_{x_2x_1}$ we get

$$w_{1x_2} (w_1^2(p-1) + w_2^2) - w_{2x_1} (w_2^2(p-1) + w_1^2) + (p-2)(w_{2x_2} - w_{1x_1})w_1w_2 = 0 .\tag{A5}$$

If $p = 2$ we deduce from (A5) that $w_{1x_2} = w_{2x_1}$, i.e., $\text{rot}(w) = 0$. Taking the potential $\varphi : w = \text{grad}(\varphi)$ we see that φ fulfills (2.7) and formula (2.8) holds.

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- [1] Knöll L and Orłowski A 1995 *Phys. Rev. Lett.* **A 51**, 1622
- [2] Wünsche A 1995 *Appl. Phys.* **B 60**, S119
- [3] Hillery M 1987 *Phys. Rev.* **A 35**, 725
- [4] Hillery M 1989 *Phys. Rev.* **A 39**, 2994
- [5] Wootters W K 1981 *Phys. Rev.* **D 23**, 357
- [6] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72**, 3439
- [7] Braunstein S L and Milburn G J 1995 *Phys. Rev.* **A 51**, 1820
- [8] Bures D J C 1969 *Trans. Am. Math. Soc.* **135**, 199
- [9] Uhlmann A 1976 *Rep. Math. Phys.* **9**, 273
- [10] Hübner M 1992 *Phys. Lett.* **A 163**, 239
- [11] Uhlmann A 1995 *Spheres and hemispheres as quantum state spaces* Preprint ESI - 102
- [12] Dittmann J 1995 *Rep. Math. Phys.* **36**, 309
- [13] Petz D and Sudár C 1995 *Geometries of quantum states* Preprint ESI - 204
- [14] Uhlmann A 1995 *Rep. Math. Phys.* **36**, 461
- [15] Slater P B 1996 *J. Phys.* **A 29**, L271
- [16] Życzkowski K, Słomczyński W and Wiedemann H 1993 *Vistas in Astronomy* **37**, 153
- [17] Husimi K 1940 *Proc. Phys. Math. Soc. Jpn.* **22**, 264
- [18] Życzkowski K 1987 *Phys. Rev.* **A 66**, 3546
- [19] Takahashi K 1986 *J. Phys. Soc. Jap.* **55**, 762
- [20] Rachev S T 1984 *Theory Prob. Appl.* **29**, 647
- [21] Rachev S T 1991 *Probability Metrics and the Stability of Stochastic Models* (Wiley: New York)
- [22] Monge G 1781 Mémoire sur la théorie des déblais et des remblais. *Hist. de l'Academie des Sciences de Paris*, p. 666
- [23] Kantorovich L V 1939 *Mathematical Methods in the Organization and Planning of Production*. (Leningrad State University: Leningrad); translated in: 1990 *Management Science* **6**, 366
- [24] Sudakov V N 1976 *Tr. Mat. Inst. V. A. Steklov, Akad. Nauk SSSR* **141** (in Russian).
- [25] Salvemini T 1943 Sul calcolo degli indici di concordanza tra due caratteri quantitativi. *Atti della VI Riunione della Soc. Ital. di Statistica* Roma
- [26] Dall'Aglio G 1956 *Ann. Sc. Norm. Sup. di Pisa*, ser. Cl. Sci. **3**, 33
- [27] Hitchcock F L 1941 *J. Math. Phys.* **20**, 224
- [28] Wu N and Coppins R 1981 *Linear Programming and Extensions* (McGraw-Hill: New York)
- [29] Wiedemann H, Życzkowski K and Słomczyński W 1998 Proc. Conf. *Frontiers in Quantum Physics, Kuala Lumpur 1997*, ed. Lim S C (Springer-Verlag: Berlin)
- [30] Żurek W H 1986 in: *Frontiers of Nonequilibrium Statistical Mechanics*, ed. Moore G T and Scully M O (Plenum: New York)
- [31] Haake F 1991 *Quantum Signatures of Chaos* (Springer: Berlin)
- [32] Żurek W H and Paz J P 1994 *Phys. Rev. Lett.* **72**, 2508
- [33] Perelomov A 1986 *Generalized Coherent States and Their Applications*. (Springer: Berlin)
- [34] Haake F and Wilkens M 1988 *J. Stat. Phys.* **53**, 345
- [35] Radcliffe J M 1971 *J. Phys.* **A 4**, 313
- [36] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* **A 6**, 2211
- [37] Vogel K and Risken H 1989 *Phys. Rev.* **A 40**, 2847
- [38] Smithey D T, Beck M, Raymer M G and Faridani A 1993 *Phys. Rev. Lett.* **70**, 1244
- [39] D'Ariano G M 1995 *Quantum Semiclass. Opt.* **7**, 693

FIG. 1. Monge transport problem: How to shovel a pile of sand $Q_1(x_1, x_2)$ into a new location $Q_2(x_1, x_2)$ minimizing the work done?

FIG. 2. Monge distance between 1D functions $Q_1(x)$ and $Q_2(x)$ may be represented as the area between graphs of the corresponding distribution functions $F_1(x)$ and $F_2(x)$

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