

On Nonlinear Dynamics of 3D Astrophysical Disks. ¹

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Nonlinear processes concerned with different aspects of nonlinear dynamics of astrophysical disks – structures, flows, turbulence – are reviewed. The special attention is paid to the influence of the three dimensionality of the disks on their nonlinear behaviour.

INTRODUCTION

Among different astrophysical objects disks have the most various dynamical structures and different kind of turbulence. So far the origin of many observed structures in disks is puzzle as well as turbulence mechanisms of different kinds of disks. The problem of the galactic spiral structure is waiting for own solution more than one and a half century. The origin of narrow Uranian rings and Cassini division with its complex inner structure, the cause of the turbulent viscosity many orders greater than the molecular one in the accretion disks,

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non-Kolmogorov turbulence spectrum of the Milky Way and the problem of double galactic nuclei not connected with merging are still unsolved problems.

In the present paper we review several aspects of nonlinear dynamics of astrophysical disks with the special emphasis on the crucial role of the three dimensionality of disks. In the most detail we consider two cases when small but finite disk thickness can be never neglected. It is convenient to split the non-linear dynamics of astrophysical disks onto the wave and vortex dynamics, which in turn could be subdivided onto structures and turbulence. The main topics of the paper are summarized in the Table 1.

Nonlinear Dynamics of Astrophysical Disks	Stationary Structures	Turbulence
wave dynamics	solitons	turbulence of Rossby waves
application	spiral arms	observed spectrum of turbulence
vortex dynamics	stationary vortices: solitary and double	turbulence of Rossby vortices
application	single vortices, double objects	enhanced star formation in areas with solid-body rotation
non-linear wave-medium	dissipativeless	large-scale convection
interaction	dissipative	inflow-outflow processes

Table 1: Brief summary of the main topics of the paper.

**NONLINEAR DYNAMICS OF marginally
UNSTABLE SELF-GRAVITATING DISK**

Special investigations¹ have shown that the gaseous disks of galaxies are near the boundary

of own gravitational instability. This fact is an expected one as the instability increases the velocity dispersion and the disk is coming to the boundary of the instability.

If a rotation velocity curve of the gaseous disk has jump or kink then hydrodynamical instability can be developed^{2,3}. In this case disc also lies near the boundary of own hydrodynamical instability⁴. The reason is similar. As a result of the instability the smearing of the jump begins to grow until the disk reaches the marginal stability.

The nonlinear dynamics of the marginally unstable self-gravitating disk was analyzed in works of Mikhailovskii, Petviashvili and Fridman^{5,6,7} (see also Fridman and Polyachenko⁸). As the eigen frequency of marginally unstable disk in co-rotating system of reference $\hat{\omega}$ is small ($\hat{\omega} \ll \Omega$), the consideration of the problem in 2D approximation is valid (see below conditions (10) and discussion therein). Under condition that only the small region of wave vectors is unstable $\Delta k \ll k_0$, where k_0 is the wave vector of the most unstable perturbations, the non-linear dynamical equation was derived:

$$\frac{\partial^2 \varepsilon}{\partial \tau^2} = -\nu_0^2 \varepsilon + \frac{3}{2}(2 - \gamma_S) \left(\gamma_S - \frac{5}{3} \right) |\varepsilon|^2 \varepsilon. \quad (1)$$

Here ε is the dimensionless amplitude of the wave, τ is non-dimensional time,

$$\nu_0^2 = \frac{(\pi G \sigma_0 / c)^2 - \kappa^2}{\Omega_0^2} \ll 1 \quad (2)$$

determines the dimensionless grow rate of the most unstable perturbations, G – gravitational constant, σ_0 – unperturbed surface density of the disk, κ – epicyclic frequency, Ω_0 – unperturbed angular velocity of the disk. Eq. (1) leads to the non-linear dispersion relation in the form

$$\nu^2 = \nu_0^2 + \frac{3}{2}(2 - \gamma_S) \left(\gamma_S - \frac{5}{3} \right) |\varepsilon|^2. \quad (3)$$

It is easily seen that this relation describes either a soliton propagation or explosive instability depending on the value of the "surface" polytropic index γ_S .

According to Hunter⁹ the surface polytropic index for self-gravitating disk can be expressed through the real 3D polytropic index γ_V as $\gamma_S = 3 - 2/\gamma_V$. Thus in the region

$$5/3 < \gamma_S < 2, \quad (4)$$

which is equivalent to $3/2 < \gamma_V < 2$, the nonlinear stabilization of the instability is possible at the certain level of the wave amplitude

$$\varepsilon^2 = \frac{2\nu_0^2}{3(2 - \gamma_S)(\gamma_S - 5/3)}. \quad (5)$$

Under condition of marginally unstable disk ($\nu_0^2 \ll 1$) the stabilization is achieved at low wave amplitude. The soliton solution is possible in the region (4) in the form of envelope soliton (FIGURE 1). In absence of the viscosity the soliton has a classical symmetrical form like a normal distribution function (FIGURE 1a). But in the presence of the small viscosity a soliton transforms into the shock wave with oscillating front¹⁰ (FIGURE 1b). The shock wave which was predicted in the rotating stellar disk – collisionless shock wave – has a similar form¹¹.

If $\gamma_S < 5/3 - \gamma_V < 3/2$ – then the explosive instability occurs:

$$\frac{\varepsilon}{\varepsilon(\tau = 0)} = \frac{1}{\tau - 1/(A\varepsilon(0))}, \quad (6)$$

where $A^2 = 3(2 - \gamma_S)(5/3 - \gamma_S)/4$.

SPECTRUM OF TURBULENCE OF CLOUDY POPULATION OF THE MILKY WAY

Up to now there are numerous investigations devoted to measurements of the turbulent spectrum of cloudy population of the Milky Way in both the every gaseous cloud and of the ensemble of clouds. Kaplan in 1955 found the correlation function $B_{rr} \sim r^{0.71}$ what is very close to the Kolmogorov spectrum. The systematic observations and the construction of

correlation functions began from 1964. Larson result¹² was close to that of Kaplan: $\Delta v \sim l^{0.38}$. But later more accurate investigations resulted in more steeper spectrum. Mayers¹³, Henriksen & Turner¹⁴, and Vereschagin & Solov'ev¹⁵ obtained the spectrum $\Delta v \sim l^{0.5}$. Sanders, Scovill and Solomon¹⁶ obtained $\Delta v \sim l^{0.62}$. These spectrums are different from the Kolmogorov one and to explain them we should take into account the anisotropy.

The attempt to explain the observed spectrum as a result of the turbulence of the Rossby waves in galactic gaseous disk was made in the work by Dolotin and Fridman¹⁷ (see also¹⁸).

As the basis observed correlations it was adopted: 1) correlation for the velocity fluctuations $\Delta v \sim l^{0.5}$; 2) correlation for the density fluctuations¹³ $\rho \sim l^{-1}$; and 3) correlation for mass spectrum¹⁹ $N(m)/N(m_0) \sim (m/m_0)^{-1.5}$. These correlations mutually agree if we adopt the assumption of virial equilibrium on all scales.

The following general nonlinear equation can be derived in the low-frequency approximation, $\omega \ll \Omega_0$, for the solid body rotating selfgravitating cloud¹⁷:

$$\left(\frac{\partial}{\partial t} + \frac{1}{2\Omega_0} [\nabla_{\perp} \tilde{\chi}, \nabla_{\perp}]_z \right) \left(\Delta_{\perp} \tilde{\Psi} - \frac{\omega_0^2}{4\Omega_0^2} \Delta_{\perp} \tilde{\chi} \right) - \frac{(\omega_0^2)'}{2\Omega_0} \frac{1}{r} \frac{\partial \tilde{\chi}}{\partial \varphi} = 0. \quad (7)$$

Here $\omega_0^2 \equiv 4\pi G\rho_0$, $(\omega_0^2)' \equiv d(\omega_0^2)/dr$; $\chi \equiv \mathcal{P} + \Psi$, Ψ is the gravitational potential; \mathcal{P} is the 'pressure' function determined by relation

$$\mathcal{P} \equiv \int dP_v/\rho; \quad (8)$$

P_v is the usual 'volume' pressure, ρ is the volume density.

For the small-scale perturbations corresponding to the case $\tilde{\Psi} \ll \tilde{\mathcal{P}}$ when Eq.(7) can be reduced to

$$\left(\frac{\partial}{\partial t} + \frac{1}{2\Omega_0} [\nabla_{\perp} \tilde{\mathcal{P}}, \nabla_{\perp}]_z \right) \Delta_{\perp} \tilde{\mathcal{P}} - 2\Omega_0 \frac{\rho_0'}{\rho_0} \frac{1}{r} \frac{\partial \tilde{\mathcal{P}}}{\partial \varphi} = 0. \quad (9)$$

The latter equation is similar to well known in hydrodynamics Charney-Obukhov's equation^{20, 21, 22} and in plasma physics Hasegawa-Mima equation²³. This analogy allows to

use results from the well elaborated fields to describe the dynamics of the perturbations in gravitating molecular clouds. Particularly in accordance with Sazontov²⁴ and Mikhailovskii *et al.*²⁵ the nonstationary solution of Eq.(9) describes Rossby waves turbulence with energy spectrums $w_k^{(1)} \sim k_y^{-3/2} k_x^{-2}$, and $w_k^{(2)} \sim k_y^{-3/2} k_x^{-3}$.

At the limit of the theory application, $k_x \simeq k_y \simeq k_\perp$, and taking into account characteristic property of the Rossby waves $k_\perp \simeq k$ we have $w_k^{(1)} \sim k^{-3.5}$, and $w_k^{(2)} \sim k^{-4.5}$. According to Hasegawa *et al.* numerical result²⁶ $w_k \sim k^{-4}$. The last relation gives²⁷ $v_k^2 \sim \int_k^\infty E_k dk = \int_k^\infty w_k k^2 dk \sim k^{-1} \sim \lambda$, that is 1) $v_\lambda \sim \lambda^{0.5}$. On the other hand $v \nabla v \simeq \nabla \Psi$ that is $v_\lambda^2 / \lambda \simeq \Psi / \lambda \simeq \lambda 4\pi G \rho_\lambda$. This gives for the density spectrum 2) $\rho_\lambda \sim \lambda^{-1}$. Finally as $P_\lambda \simeq n_\lambda m_\lambda v_\lambda^2 \simeq \rho_\lambda v_\lambda^2 = \text{const}$ and $m_\lambda \simeq \rho_\lambda \lambda^3 \simeq \lambda^2$ one obtains $n_\lambda \sim m_\lambda^{-1} v_\lambda^{-2}$. This gives for the mass spectrum 3) $n_\lambda \sim m_\lambda^{-1.5}$. We see that the obtained turbulent spectrums corresponds to the observed spectrums what is the evidence of a weak turbulence of the Rossby waves in cloudy population of the Milky Way.

ON THE POSSIBILITY TO STUDY THE DYNAMICS OF THIN DISKS IN 2D APPROXIMATION

Before the consideration of some essentially 3D nonlinear effects in thin astrophysical disks let us discuss the problem of 2D approximation and clarify, why in some cases we should take into account finite – while small – disk thickness.

Traditional conditions for 2D description of the dynamics of astrophysical disks

The seldom cases of the use of the 3D description of astrophysical disk dynamics are restricted by the study of the processes of bending and warping of the disk and by the formation of bars and bulges.

Due to the fact that the semithickness H of the majority of astrophysical disks is much

less than their radius R , $H \ll R$, the dynamical processes and structures in the disks were as a rule studied in the frame of 2D approximation. In doing so two conditions assumed to be fulfilled.

First: the structures and processes are symmetric about the disk plane.

Second: the typical scales of the processes and structures L are much greater than the semithickness of the disk, $L \gg H$.

But these conditions were never obtained in a rigorous manner as a complete set of sufficient conditions. Consequently our first question is: What is the complete set of sufficient conditions to describe the dynamics of astrophysical disks by 2D dynamical equations?

What are the initial equations we have to write?

The 2D dynamical equations for astrophysical disks should be derived from general 3D equations. Analysis of this problem results in the following conclusions²⁸.

If the equation of state of the gaseous disk has

1) the general form, $P_v = P_v(\rho, S)$, where P , ρ and S are pressure, density and entropy, respectively,, then the derivation of 2D equations is problematic;

2) the barotropic form, $P_v = P_v(\rho)$, then we come to the closed 2D set of integro-differential equations, the solution of which is complicated;

3) the polytropic form, $P_v = A \cdot \rho^\gamma$, where A and γ are constants, γ is polytropic index, then we come to the system of 2D partial differential equations, but with additional terms, which were not written before.

On the sufficient conditions of the correctness of 2D approximation

These conditions can be written in the following simple form^{28,29}:

$$\frac{H^2}{L^2} \ll 1, \quad \frac{H^2 R}{\zeta^2 L} \ll 1, \quad \omega^2 \ll \frac{c^2}{H^2}. \quad (10)$$

Here L and ζ are typical radial and azimuthal scales of perturbations; ω is the typical frequency of the process.

First two strong inequalities are well-known conditions of large-scale approximation.

The third condition was not used before. In most cases this condition is reduced to

$$\omega^2 \ll \Omega^2, \quad (11)$$

where Ω is the angular velocity of a disk. As a rule this condition was not fulfilled in previous 2D studies of the disk dynamics. Particularly, as it is easy to see, the majority of 2D theories of spiral structure do not fit this condition!

Typical mistakes in works devoted to 2D dynamics of disks

Schematically the reason of *a mistake* can be illustrated by the *simplest example*. Let us consider the 3D equations of motion

$$\frac{dv_{\perp}}{dt} = - \nabla_{\perp} \chi, \quad (12)$$

$$\frac{dv_z}{dt} = - \frac{\partial}{\partial z} \chi, \quad (13)$$

where the notations are given in previous Section.

For low-frequency perturbations (11) the term dv_z/dt in equation (13) may be omitted, and the χ does not depend on z . Hence the right-hand side of equation (12) does not depend on z also. As a result the left-hand side does not depend on z and the equation of motion (12) can be reduced to the 2D form.

The latter argumentation is not true for the high-frequency perturbations, $\omega \sim \Omega$. In this case 2D approximation can be correct only in some special cases²⁹:

- 1) isothermal disk in a strong outer gravitational field, and
- 2) self-gravitating disk with polytropic index $\gamma = 2$.

The case of low-frequency perturbations, $\omega \ll \Omega$

In this case 2D approximation is correct if the 2D equations are derived from 3D ones. Yet as a rule the incorrect 2D dynamical equations are used where all functions are integrated over z . Let us illustrate it by the simplest example of 2D equation of motion.

A polytropic disk in outer gravitational field

- a) Traditional form

$$\frac{d\vec{v}}{dt} = -\frac{1}{\sigma}\nabla P_s - \nabla\Psi_c = -A_s\gamma_s\sigma^{\gamma_s-2}\nabla\sigma - \nabla\Psi_c, \quad (14)$$

where $P_s = A_s\sigma^{\gamma_s}$ is "flat" pressure, Ψ_c is gravitational potential in the plane $z = 0$, A_s and γ_s are constants, γ_s is a "flat" polytropic index.

- b) Correct form³⁰

Substituting in (8) the expression for the 3D polytropic equation of state $P_v = \gamma_v$ are constants, γ_v is a "volume" polytropic index, we obtain $\mathcal{P} = A_v\rho^{\gamma_v-1}\gamma_v/(\gamma_v - 1)$. Finding hence ρ and integrating it over z from $-\infty$ to $+\infty$, we obtain the expression for the surface density

$$\sigma(r, \varphi, t) = \left(\frac{\gamma_v - 1}{A_v\gamma_v}\right)^{\frac{1}{\gamma_v-1}} \int_{-\infty}^{+\infty} [\chi(r, \varphi, t) - \Psi(r, \varphi, z, t)]^{\frac{1}{\gamma_v-1}} dz. \quad (15)$$

The condition of the infinitesimal thickness of a disk lets to write a relationship between σ and χ for an arbitrary function Ψ , using its expansion in the neighborhood of the plane $z = 0$:

$$\Psi(r, \varphi, z, t) = \Psi_c(r, \varphi, t) + \Psi'_c(r, \varphi, t)z + \frac{1}{2}\Psi''_c(r, \varphi, t)z^2. \quad (16)$$

Substituting (16) to (15) and integrating over z we obtain

$$\sigma(r, \varphi, t) = \sqrt{\pi} \left(\frac{\Psi_c''}{2} \frac{\gamma_v - 1}{A_v \gamma_v} \right)^{\frac{1}{\gamma_v - 1}} \frac{\Gamma\left(\frac{\gamma_v}{\gamma_v - 1}\right)}{\Gamma\left(\frac{\gamma_v}{\gamma_v - 1} + \frac{1}{2}\right)} \cdot \left[\left(\frac{\Psi_c'}{\Psi_c''} \right)^2 + \frac{2(\chi - \Psi_c)}{\Psi_c''} \right]^{\frac{\gamma_v + 1}{2(\gamma_v - 1)}}. \quad (17)$$

Whence

$$\chi = C \sigma^{2\lambda} + \Psi_c - \frac{(\Psi_c')^2}{2\Psi_c''}, \quad (18)$$

where $\lambda \equiv (\gamma_v - 1)/(\gamma_v + 1)$, Γ is gamma-function,

$$C \equiv \left[\frac{\Psi_c'' \Gamma^2\left(\frac{\gamma_v}{\gamma_v - 1} + \frac{1}{2}\right)}{2\pi \Gamma^2\left(\frac{\gamma_v}{\gamma_v - 1}\right)} \right]^\lambda \left(\frac{A_v \gamma_v}{\gamma_v - 1} \right)^{2/(\gamma_v + 1)}, \quad (19)$$

As the C depends on the external parameter (Ψ_c'') the actual "surface" equation of state is of more general class than the pure 2D equation of state, which is adopted as a rule.

Eq.(12) after the substitution in it (18) has a desired form of 2D equation of motion for 2D functions

$$\frac{d\vec{v}}{dt} = - \underline{2\lambda C \sigma^{2\lambda-1} \nabla \sigma} - \nabla \Psi_c - \underline{\sigma^{2\lambda} \nabla C} - \underline{\nabla \left[\frac{(\Psi_c')^2}{\Psi_c''} \right]_c}, \quad (20)$$

The terms in (20) different from (14) are underlined.

A self-gravitating disk

In this case 2D dynamics can be described by the equation similar to (14). The difference lies in the form of expression for A_s .

a) Traditional form⁹

$$(A_s)_H = \frac{\pi^{(3/2-1/\gamma_v)} \Gamma(2-1/\gamma_v)}{2^{(2-1/\gamma_v)} \Gamma(5/2-1/\gamma_v)} A_v^{1/\gamma_v} G^{1-1/\gamma_v}, \quad (21)$$

b) Correct form³⁰

$$(A_s)_{corr} = \frac{2^{1/\gamma_v} \pi^{(1-1/\gamma_v)}}{3 - 2/\gamma_v} A_v^{1/\gamma_v} G^{1-1/\gamma_v}, \quad (22)$$

Their ratio is not equal to unity:

$$\frac{(A_s)_H}{(A_s)_{corr}} = \frac{\pi^{1/2} \Gamma(\gamma_s/2 + 1/2)}{2 \Gamma(\gamma_s/2)}, \quad \gamma_s \equiv 3 - \frac{2}{\gamma_v}. \quad (23)$$

The reason of the demonstrated difference is very simple: in right-hand side of the correct 2D equation of motion stands the gradient of the function of the volume pressure, $\nabla \mathcal{P} = \nabla \int dP_v/\rho$, but not the gradient of a "flat" pressure, $\sigma^{-1} \nabla P_s \equiv \sigma^{-1} \nabla \int P_v dz$, as in equation (14). The latter term has not the physical sense of the force.

Let us consider a tube of a conic section. Let be $P_v = \text{const}$. Then a gas in the tube must be *unmoved*. But P_s in different sections is different. If we compare two positions with different cross-sections $z_1 > z_2$ we obtain

$$P_{S_1}(x_1) \equiv 2 \int_0^{z_1} P_v(x_1) dz > P_{S_2}(x_2) \equiv 2 \int_0^{z_2} P_v(x_2) dz. \quad (24)$$

Hence, according to equation (14) the gas must move from the section S_1 to the section S_2 !

We come to the nonsense.

SOLITARY VORTICES IN ASTROPHYSICAL DISKS

Using above-written vector equation of motion (20) for disk in outer gravitational field for slow-frequency perturbations (11) we derive³⁰ the following nonlinear dynamical equation for a solid-body rotating part of a disk, $\Omega = \text{const}$,

$$\frac{\partial}{\partial t} (\tilde{\chi} - a_R^2 \Delta \tilde{\chi}) + U_R \frac{\partial \tilde{\chi}}{\partial y} - \frac{c_s^2}{8\Omega^3} J(\tilde{\chi}, \Delta \tilde{\chi}) + \frac{(\ln C)'_x}{4\Omega} \frac{\partial \tilde{\chi}^2}{\partial y} = 0. \quad (25)$$

Here $\tilde{\chi}$ is perturbation of χ (see (18)); $a_R \equiv c_s/2\Omega$ is Rossby radius; $U_R \equiv -2a_R^2 \Omega \cdot (\ln \sigma_0)'_x$

is Rossby velocity; c_s is the sound speed determined by the "flat" functions

$$c_s^2 \equiv \sigma \left(\frac{\partial \chi}{\partial \sigma} \right)_0; \quad (26)$$

$$J(A, B) \equiv \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x} \quad (27)$$

is Jacobian; x, y are Cartesian coordinates with x along radius and y along azimuth; the function C is determined above in (19).

Equation (25) contains the vector, the third term, and scalar, the last term, nonlinearities. It should be emphasised that the scalar nonlinearity is a consequence of the dependence of the "surface" equation of state from the external parameter (through the dependence on external gravitational potential Ψ_c). If we start from the usual pure 2D hydrodynamical equations this term would be overlooked.

Derived for astrophysical disks nonlinear dynamical equation (25) is similar to well-known in hydrodynamics the Charney-Obukhov equation^{20,21,22}. In plasma physics the similar equation was derived by Hasegawa and Mima²³. The use of well worked out theory and laboratory modelling of these equations in hydrodynamics and plasma physics leads to the following results³⁰.

Equation (25) has two kinds of *stationary solutions*, which describes, respectively, two types of solitary vortices: single and double vortices. $1 < a/H < (R/H)^{1/3}$, where H is the disk thickness, R is the typical scale of the density inhomogeneity.

Single vortices: cyclones and anticyclones

Due to the presence of the scalar nonlinearity the existence of single Rossby vortices in galactic disks is possible. Under the definite sign of the scalar nonlinearity $((\ln C)'_x)$ it is possible to form only one kind of solitary vortices: cyclones or anticyclones. Cyclone $((\ln C)'_x < 0)$ is characterized by a minimum of the surface density. Anticyclone $((\ln C)'_x >$

0) is characterized by a maximum of the surface density.

Double vortices: modons

Due to the presence of the vector nonlinearity the existence of double Rossby vortices in galactic disks is possible. Double vortex — modon represents the cyclone–anticyclone pair and has one minimum and one maximum of perturbed surface density¹⁷. The astronomical application can be connected, for example, with the problem of double galactic nuclei formation and, may be, other pairs of nearby objects scale of which is extremely different from the galactic nuclei.

On the strong vortex turbulence

Generation of several modons results in their interaction and in the formation of vortex turbulence³¹. The vortex turbulence is distinct from the wave one in principle. The wave turbulence is strong if the amplitude of perturbed value \bar{A} is closed to the stationary one A_0 or more. The vortex turbulence can be strong under the condition as the time of vortex–vortex interaction is much longer than that for wave–wave interaction.

On the connection of enhanced SFR with large areas of solid–body rotation

Enhanced star formation is found by Keel³² for interacting galaxies with large areas of solid-body rotation. The paired Seyfert galaxies show specially a striking number of solid-body. One of the conclusions of Keel's work³² is: "The frequent presence of large areas of solid-body rotation, extending to a medial radius of 2.1 kpc, is not closely linked to the presence of optical bars."

It can be assumed that the reason of the dynamical activity of a gaseous disk in the absence of a bar, but with a large solid-body rotation area is the development of strong vortex turbulence.

NONLINEAR RADIAL LAMINATED FLOW AS A MANIFESTATION OF 3D DYNAMICS OF ASTROPHYSICAL DISKS

In this section we will show that in presence of the spiral density wave there is an observable manifestation of the three dimensional nature of astrophysical disks in the form of a large-scale quasi-stationary radial flow laminated in z direction. The flow velocity has an opposite direction in the central plane of the disk, $z = 0$, and on the disk periphery, near the planes $z = \pm H$. The streamlines are closed by the vertical motions of less magnitude, $v_z \simeq v_r H/r$. As a whole, the flow has a form of four vortices separated by the vertical surface $r = r_c$, where r_c is a corotation radius, and the central plane of the disk (FIGURE 2). Therefore, the observation of this flow could provide a direct indication of the position of corotation circle.

The characteristic velocity in the flow is about the velocity in the density wave. For the galactic disks it can be as much as several tens kilometers per second. The characteristic scale of the flow is about the disk radius. Therefore the existence of such kind of the convection can play a significant role in the overall dynamics of a disk.

By its nature the flow discussed above is a special kind of an acoustic streaming. It is caused by the quasi-stationary component of nonlinear Reynolds stresses induced by the density wave. Classical acoustic streaming is caused by the Reynolds stresses in strong acoustic waves (for a review see *e. g.*³⁴). It is the phenomenon observed in up to hundreds different laboratory experiments starting with Faraday's discovery³⁵ and described in numerous theoretical papers, starting with the pioneer work by Rayleigh³⁶.

The main difference in comparison with classical case is a drift nature of any quasi-stationary flow in a rotating disk. It is a consequence of the dominant role of the Coriolis forces, and means that the direction of the flow is perpendicular to the direction of the

applied force. If the system is two-dimensional or uniform in z direction, the conservation of the orbital momentum in absence of dissipation should require the mutual annihilation of separate nonlinear azimuthal forces such as $\langle \text{div}(\rho v_\varphi \vec{v}) \rangle$, $\langle \rho \partial \Phi / r \partial \varphi \rangle$, etc. But in the disk, which is nonuniform in z direction, the annihilation is possible on the average only. It means that locally the azimuthal force does not vanish and gives zero only after integration over disk thickness. As a result the radial flow with zero averaged over disk thickness mass flux is generated.

The method

The method used (for details see³⁷) is based on the subdivision of each value and the system of dynamical equations as a whole onto two parts

- "slow" quasi-stationary part;
- "fast" pulsating part.

The former represents the dynamics of the system "on the average" over time period much longer than the period of pulsations. The latter represents the short-time dynamics of the system, but it vanishes after averaging over time. This approach is well known in the theory of the turbulence, in the nonlinear optics and acoustics, but it was never used earlier to analyze the nonlinear self-action of the astrophysical density waves.

Particularly this approach implies the following representations. For each value (physical parameter) f we can write:

$$f = \langle f \rangle + f_1, \quad (28)$$

with

$$\langle f \rangle \equiv \frac{1}{\Delta} \int_0^{\Delta} f dt, \quad T \ll \Delta \ll t_{ev}, \quad (29)$$

where T is a characteristic period of pulsations, t_{ev} is a characteristic time of the "slow" evolution, and

$$\langle f_1 \rangle = 0. \quad (30)$$

For the product of two values one has

$$fg = \langle f \rangle \langle g \rangle + \langle f_1 g_1 \rangle + \langle f \rangle g_1 + f_1 \langle g \rangle. \quad (31)$$

In the case of the wave of a small but finite amplitude we can restrict our consideration to the quadratic nonlinearities only. Simultaneously we can use the representation of the pulsations as a monochromatic wave

$$f_1(r, \varphi, z, t) = \bar{f}_1(r, z, t) \exp[i(m\varphi - \omega t)] + \bar{f}_1^*(r, z, t) \exp[-i(m\varphi - \omega t)]. \quad (32)$$

with the amplitude $\bar{f}_1(r, z, t)$ being a slow function of the time.

Consequently quasi-stationary part of any nonlinear term has a form

$$\langle f_1 g_1 \rangle = \bar{f}_1 \bar{g}_1^* + \bar{f}_1^* \bar{g}_1. \quad (33)$$

It is independent of the azimuth. As a result all effects described below are axisymmetrical.

Nonlinear quasi-stationary response of the disk on the density wave

The total nonlinear quasi-stationary response of the disk on the density wave can be subdivided onto the following three effects. Below we presented the final results only – their derivation is a subject of the separate paper³⁸.

1. Nonlinear additives to the potential function (enthalpy plus gravitational potential) of the disk. They are similar to the well known "high frequency pressure" in plasma physics.

$$\frac{\partial \langle \chi \rangle}{\partial z} = - \frac{\partial}{\partial z} \left(\bar{v}_{z1} \bar{v}_{z1}^* + \frac{D}{\bar{\omega}^2} \bar{v}_{r1} \bar{v}_{r1}^* + \frac{m^2}{r^2 \bar{\omega}^2} \bar{\chi}_1 \bar{\chi}_1^* \right) \simeq$$

$$\simeq -\frac{1}{D} \frac{\partial}{\partial z} \left(\frac{\partial \bar{\chi}_1}{\partial r} \frac{\partial \bar{\chi}_1^*}{\partial r} \right). \quad (34)$$

Here

$$\hat{\omega} \equiv \omega - m\Omega, \quad D \equiv \hat{\omega}^2 - \kappa^2. \quad (35)$$

Functions $\langle \chi \rangle$ and χ_1 are determined according to (28).

Nonlinear additives to $\langle \chi \rangle$ could influence on the vertical equilibrium and rotation curve of the gaseous disk. The former influence can be significant only for short-wave perturbations with $kh \simeq 1$. The contribution to the rotation law is of the order of the contribution of the gaseous pressure.

2. Nonlinear corrections to the rotation law.

$$\begin{aligned} \frac{\langle \rho \rangle V_\varphi^2}{r} &= \frac{\partial \langle \chi \rangle}{\partial r} + \langle \rho_1 \frac{\partial \chi_1}{\partial r} \rangle + \frac{\partial}{\partial z} (\langle \rho \rangle \langle v_{r1} v_{z1} \rangle) + \\ &+ \frac{1}{r} \frac{\partial}{\partial r} (r \langle \rho \rangle \langle v_{r1}^2 \rangle) - \langle \rho \rangle \frac{\langle v_{\varphi 1}^2 \rangle}{r} \end{aligned} \quad (36)$$

All nonlinear corrections to the rotation law are of the order of the gaseous pressure term. The calculation of the total effect requires the account for the additives to $\langle \chi \rangle$ and thus can be performed in the frame of the 3D problem only.

3. The excitation of the quasi-stationary "acoustic streaming".

From the theory of a classical acoustic streaming it is well known that the time-average flow of the medium is characterized by average values of the mass flux components

$$\Pi_i \equiv \langle \rho v_i \rangle, \quad (37)$$

instead of the average value of the velocity itself.

The radial Π_r and vertical Π_z components of the nonlinear quasi-stationary flow are the most interesting for the description of the overall dynamics of a disk. It can be shown that

the components of the quasi-stationary flow obeys the continuity equation

$$\operatorname{div} \vec{\Pi} = \frac{1}{r} \frac{\partial(r\Pi_r)}{\partial r} + \frac{\partial\Pi_z}{\partial z} = 0. \quad (38)$$

It means that the flow has closed streamlines.

Our analysis shows that the exact expression for the radial flow can be obtained only taking into account the real three-dimensional structure of the disk and density wave:

$$\begin{aligned} \Pi_r &= - \frac{\partial}{\partial z} \left\{ \frac{i\langle\rho\rangle}{\hat{\omega}D} \left[\frac{\partial\bar{\chi}_1}{\partial z} \frac{\partial\bar{\chi}_1^*}{\partial r} - \frac{\partial\bar{\chi}_1^*}{\partial z} \frac{\partial\bar{\chi}_1}{\partial r} - \right. \right. \\ &\quad \left. \left. - \frac{2m\Omega}{r\hat{\omega}} \left(\frac{\partial\bar{\chi}_1}{\partial z} \bar{\chi}_1^* - \frac{\partial\bar{\chi}_1^*}{\partial z} \bar{\chi}_1 \right) \right] \right\} = \\ &= - \frac{\partial}{\partial z} \left(\frac{\langle\rho\rangle}{\hat{\omega}^2} \left\langle v_{r1} \frac{\partial\bar{\chi}_1}{\partial z} \right\rangle \right). \end{aligned} \quad (39)$$

The same is valid for the vertical flow:

$$\begin{aligned} \Pi_z &= - \int_0^z [1/r \partial(r\Pi_r)/\partial r] dz = \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r\langle\rho\rangle}{\hat{\omega}^2} \left\langle v_{r1} \frac{\partial\bar{\chi}_1}{\partial z} \right\rangle \right). \end{aligned} \quad (40)$$

It should be noted that in two-dimensional and cylindrical systems the effect vanishes.

In former case $\partial/\partial z \equiv 0$. In the latter case the perturbations are proportional to $\exp(ikz)$.

As a result terms like $f_1 g_1^*$ do not depend on z , and Π_r is equal to zero..

The method of the iteration

For the accurate analysis of the vertical structure of the density waves the iteration method was proposed²⁹. It is based on the reduction of the system of the linearized dynamical equations to equation for potential function χ_1 of integro-differential type:

$$\chi_1(z) = \chi_1(z=0) - \hat{\omega} \int_0^z I(z) \frac{dz}{\rho}, \quad (41)$$

where

$$I(z) \equiv -\hat{\omega} \int_0^z \left[\frac{m^2 \rho \chi_1}{r^2 D} - \rho_1 - 1r \frac{\partial}{\partial r} \left(\frac{r \rho}{D} \frac{\partial \chi_1}{\partial r} \right) + \frac{2m}{r \hat{\omega}} \frac{d}{dr} \left(\frac{\Omega \rho}{D} \right) \chi_1 \right] dz. \quad (42)$$

We use as a first approximation the solution obtained neglecting motion in the z -direction, which is equivalent to $I(z) = 0$, as $v_z \sim I$. Determining in this way the functions $\chi_1(r, z)$ and $\rho_1(r, z)$ of the first approximation we can substitute them into the integral (42) and from (41) obtain these functions in the second approximation, and so on.

It can be shown that the iterations are converged if the dispersion relation $I(\infty) = 0$ is fulfilled.

The rate of the convergence is determined by the parameter³⁹

$$N = \hat{\omega}^2 h^2 / c^2. \quad (43)$$

It characterizes the relation between time-scale of the wave $1/\hat{\omega}$ and the time of the vertical equilibrium establishment h/c . The greater will be the value N the more complicated will be the vertical structure of the density wave.

If N is much less than unity, that is for the thin disks, combining the expressions above one can obtain

$$\begin{aligned} \Pi_r &= \langle \rho_1(z=0) v_{r1}(z=0) \rangle \\ &\times \frac{1}{\rho_1(z=0)} \left[\rho_1 + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r \rho}{D} \frac{\partial \chi_1}{\partial r} \right) - \frac{m^2 \rho \chi_1}{r^2 D} - \frac{2m}{r \hat{\omega}} \frac{d}{dr} \left(\frac{\Omega \rho}{D} \right) \chi_1 \right]. \end{aligned} \quad (44)$$

In such representation the first factor characterizes the magnitude of the flow and the second factor its dependence on z .

It is easily seen that the mass flux averaged over disk thickness is zero

$$\int_0^h \Pi_r = 0. \quad (45)$$

and the characteristic velocity in the radial direction is about the peculiar velocity in the wave.

Structure of the flow

The method described allows to determine the vertical structure of the density wave and to calculate the structure of the nonlinear laminated flow. To illustrate the phenomenon let us consider a particular case of the tightly wound spiral density wave generated in a disk with the Toomre parameter⁴⁰ $Q \gg 1$.

In this case the expression for the radial flow has a form

$$\Pi_r = - \langle \rho_1 v_{r1} \rangle_{z=0} \frac{\gamma + 1}{2} \left(1 - \frac{z^2}{h^2} \right)^{\frac{2-\gamma}{\gamma-1}} \left(\frac{z^2}{h^2} - \frac{\gamma - 1}{\gamma + 1} \right), \quad (46)$$

and

$$\langle \rho_1 v_{r1} \rangle_{z=0} = A^2 \rho_0 \frac{\hat{\omega}}{k_r}. \quad (47)$$

Here $A = \rho_1(z=0)/\rho_0$ is dimensionless amplitude of the density wave.

Consequently the flow has a form of four vortices separated by the vertical cylindrical surface $r = r_c$ and the central plane of the disk $z = 0$ (FIGURE 2).

For trailing spirals the radial velocity in the central plane of the disk is negative inside the corotation circle and is positive outside. For leading spirals the situation is opposite. Therefore the observational discovery of this flow could provide also an indication of the position of the corotation circle and the type of the spiral.

Finally we would like to note that the situation described above corresponds to the dissipativeless disk. In the presence of the dissipation the additional nonlinear accretion – accretion drift – is caused by the density wave. This flow has a nonzero mass flux and can play a significant role in the redistribution of the surface density of a disk.

Particularly in Fridman, Khoruzhii and Gor'kavii⁴¹ it was shown that the excitation

of such flow can give a reasonable explanation of the formation of gaps and ringlets in planetary rings. The radial mass flux in this accretion is negative inside the corotation and positive outside. Therefore, the accretion drift can also explain the secular flows found in majority of numerical simulations of gaseous galactic disks (see *e. g.*⁴²).

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FIGURE CAPTIONS

FIGS. 1 Schematic view of the envelope solitones generating in a marginally unstable disks.

(a) The case of dissipativeless disk; (b) Disk with small viscosity.

FIGS. 2 Schematic view of nonlinear radial flow induced by a quasi-stationary density wave. The radial cut of the disk is presented. The radial scale is squeezed. The flow is azimuthally symmetrical and has a form of four tori. (a) Structure of streamlines in the flow;

(b) The vertical profile of the flow radial velocity.

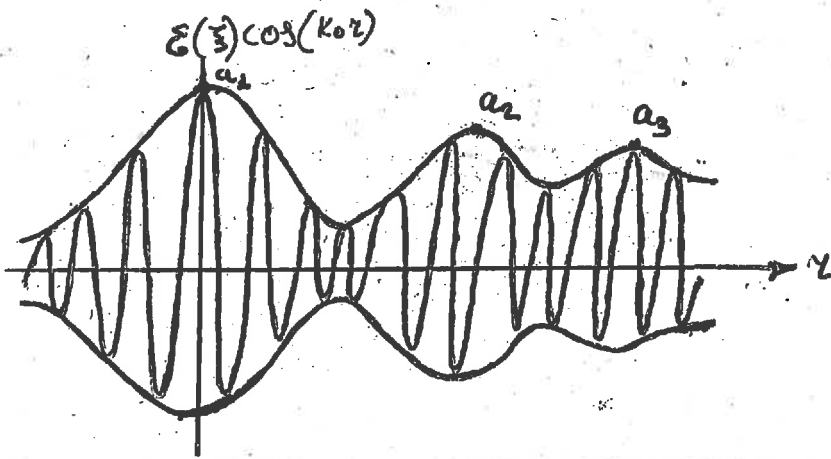
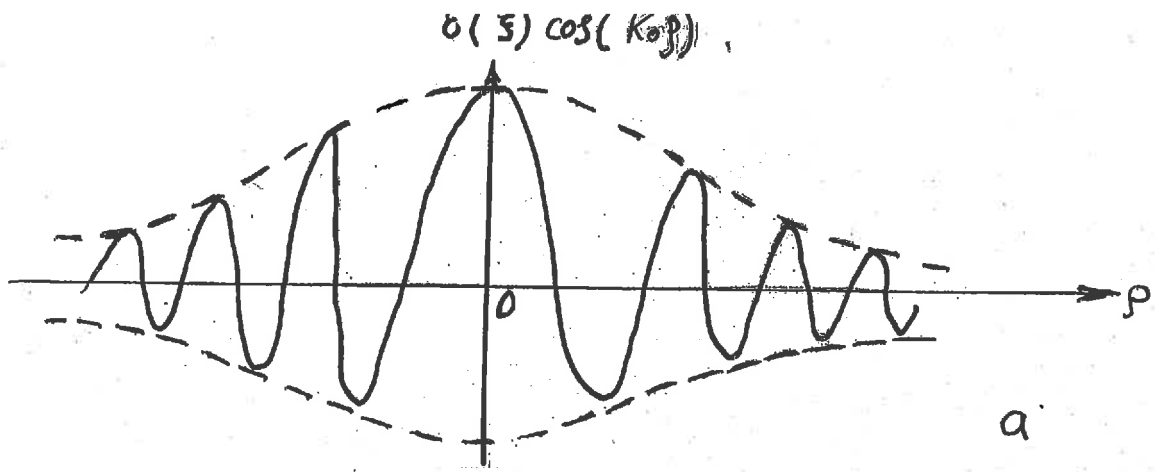


Fig. 1

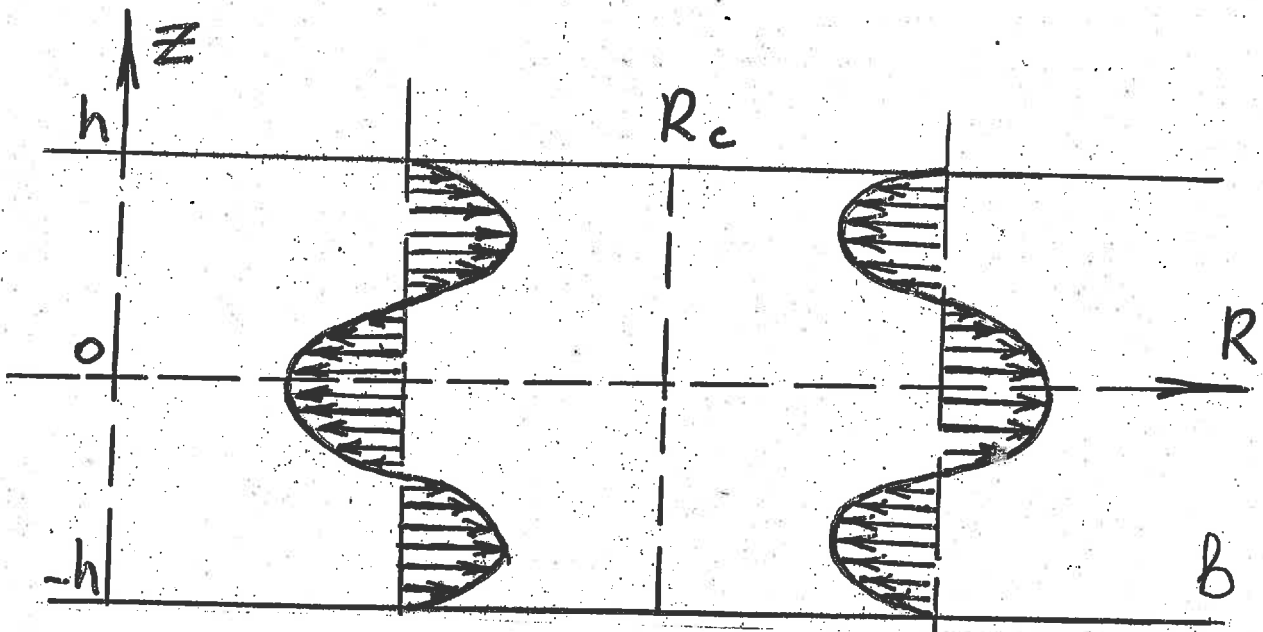
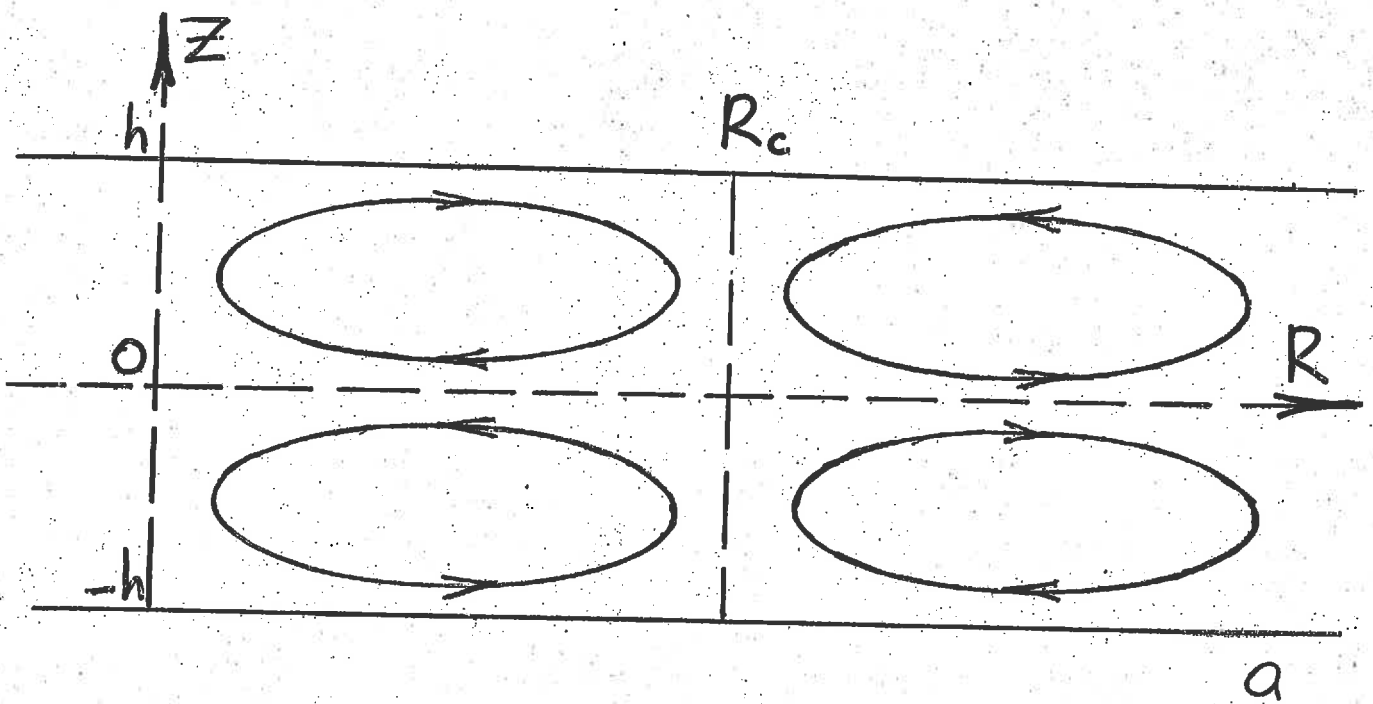


Fig. 2

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