On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part IV. Generalized isovorticity principle for three-dimensional flows

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The equations of magnetohydrodynamics (MHD) of an ideal fluid have two families of topological invariants: the magnetic helicity invariants and the cross-helicity invariants. It is first shown that these invariants define a natural foliation (described as isomagnetovortical, or imv for short) in the function space in which solutions $\{\mathbf{u}(\mathbf{x},t),\mathbf{h}(\mathbf{x},t)\}$ of the MHD equations reside. A relaxation process is constructed whereby total energy (magnetic plus kinetic) decreases on an imv folium (all magnetic and cross helicity invariants being thus conserved). The energy has a positive lower bound determined by the global cross-helicity, and it is thus shown that a steady state exists having (arbitrarily) prescribed families of magnetic and cross helicity invariants.

The stability of such steady states is considered by an appropriate generalisation of (Arnold) energy techniques. The first variation of energy on the imv folium is shown to vanish, and the second variation $\delta^2 E$ is constructed. It is shown that $\delta^2 E$ is a quadratic functional of the first-order variations $\delta^1\mathbf{u}, \delta^1\mathbf{h}$ of $\mathbf{u}$ and $\mathbf{h}$ (from a steady state $\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})$), and that $\delta^2 E$ is an invariant of the linearised MHD equations. Linear stability is then assured provided $\delta^2 E$ is either positive-definite or negative-definite for all imv-perturbations. It is shown that the results may be equivalently obtained through consideration of the frozen-in 'modified' vorticity field introduced in Part I of this series.

Finally, in §8, the general stability criterion is applied to a variety of classes of steady states $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$, and new sufficient conditions for stability to three-dimensional imv perturbations are obtained.

1. Introduction

In Part I of this series, we have established two variational principles for steady three-dimensional solutions of the equations of magnetohydrodynamics (MHD) of an ideal incompressible fluid; in Parts II and III, the techniques developed in Part I were used to obtain stability criteria for two-dimensional and axisymmetric flows. We now return to the general case of 3D MHD flows. Our aim is first to extend the concept of 'isovortical deformations' to the MHD context, for which we shall find that a more general 'isomagnetovortical' (or imv, for short) deformation needs to be defined. This leads to the concept of an isomagnetovortical foliation of the function space in which solutions $\{\mathbf{u}(\mathbf{x},t),\mathbf{h}(\mathbf{x},t)\}$ of the MHD equations reside. The essential property of this foliation is that, under generalised deformations on an imv folium, all topological invariants associated with the MHD equations (and notably magnetic helicity and cross-helicity) are conserved. We then show that a relaxation process may be defined (a generalisation of the relaxation to *magnetostatic* equilibrium described by Moffatt 1985) in which energy decreases to a minimum while the above topological invariants are conserved. This minimum corresponds to a stable steady solution of the MHD equations.

Our second aim is to extend Arnold's (1965) variational principle to the above MHD situation. This requires consideration of perturbations of an arbitrary steady state resulting from small imv deformations. We shall show that the first-order variation of energy $\delta^1 E$ vanishes at the equilibrium (i.e. at the steady state); and that the second-order variation of energy $\delta^2 E$ is an invariant of the MHD equations linearised about this steady state. This latter result is to be expected from general theory; but its verification provides useful confirmation that the imv foliation has been correctly and completely identified.

It is known from general theory (Arnold 1965) that the steady state is stable if $\delta^2 E$ is either positive-definite or negative-definite for all admissible perturbations on the imv folium. We use this principle in §8 to obtain stability criteria for several classes of nontrivial steady MHD flows. The abstract geometric approach to the equations of ideal MHD has been developed in a number of previous publications, notably Arnold (1966), S.M. Vishik & Dolzhanskii (1978), Marsden, Ratiu & Weinstein (1984), Holm *et al* (1985), Khesin & Chekanov (1989), Zeitlin & Kambe (1993), Ono (1995), Friedlander & M. Vishik (1990,1995). The background is extensively covered in the recent book by Arnold & Khesin (1998).

2. Basic equations and their invariants

Consider an incompressible, inviscid and perfectly conducting fluid contained in a domain $\mathcal{D}$ with fixed boundary $\partial\mathcal{D}$. Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$ the vorticity field, $\mathbf{h}(\mathbf{x},t)$ the magnetic field (in Alfven velocity units), $\mathbf{j}=\nabla\wedge\mathbf{h}$ the current density, and $p(\mathbf{x},t)$ the pressure (divided by density). The governing equations are then:

$$\mathbf{u}_t = \mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{h} - \nabla(p + \tfrac{1}{2}\mathbf{u}^2)\,, \tag{2.1}$$

$$\mathbf{h}_t = \nabla\wedge(\mathbf{u}\wedge\mathbf{h})\,, \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0\,. \tag{2.3}$$

We shall use the 'commutator' notation

$$[\mathbf{u},\mathbf{h}] = \nabla\wedge(\mathbf{u}\wedge\mathbf{h})\,, \tag{2.4}$$

and we shall make frequent use of the property

$$[\mathbf{u},\mathbf{h}] = -[\mathbf{h},\mathbf{u}]\,, \tag{2.5}$$

and the Jacobi identity for any three fields

$$[\mathbf{a},[\mathbf{b},\mathbf{c}]]+[\mathbf{b},[\mathbf{c},\mathbf{a}]]+[\mathbf{c},[\mathbf{a},\mathbf{b}]]=0\,. \tag{2.6}$$

With this notation, (2.2) becomes

$$\mathbf{h}_t=[\mathbf{u},\mathbf{h}]\,, \tag{2.7}$$

and the curl of (2.1) gives

$$\boldsymbol{\omega}_t=[\mathbf{u},\boldsymbol{\omega}]+[\mathbf{j},\mathbf{h}]\,. \tag{2.8}$$

The boundary conditions on $\mathbf{u}$ and $\mathbf{h}$ are

$$\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{h}=0 \quad \text{on } \partial\mathcal{D} \tag{2.9}$$

where $\mathbf{n}$ is the unit outward normal on $\partial\mathcal{D}$.

Equation (2.7) implies that the field $\mathbf{h}$ is frozen in the fluid, its flux through any surface S bounded by a material curve C being conserved; the conditions $\nabla\cdot\mathbf{h}=0$ and $\mathbf{n}\cdot\mathbf{h}=0$ on $\partial\mathcal{D}$ are conserved under evolution governed by (2.7).

Let S_h be any closed material surface in $\mathcal{D}$ on which $\mathbf{n}\cdot\mathbf{h}=0$ (again a condition conserved by (2.7)) and let V_h be the volume inside S_h. Then it is well known that there are two invariants for each such volume, namely the magnetic helicity

$$\mathcal{H}_M(V_h)=\int_{V_h}\mathbf{h}\cdot\ \text{curl}^{-1}\mathbf{h}\;dV \tag{2.10}$$

and the cross-helicity

$$\mathcal{H}_C(V_h)=\int_{V_h}\mathbf{h}\cdot\mathbf{u}\;dV\,. \tag{2.11}$$

Note that, if there is any closed $\mathbf{h}$-line, γ_h say, then taking V_h to be a flux tube of vanishingly small cross-section centred on γ_h, we obtain the corresponding invariant

$$\Gamma_h=\oint_{\gamma_h}\mathbf{u}\cdot d\mathbf{x}=\int_{\Sigma_h}\boldsymbol{\omega}\cdot\mathbf{n}\;dS\,, \tag{2.12}$$

where Σ_h is any surface bounded by γ_h. Thus the flux of vorticity through any closed $\mathbf{h}$-line is conserved. Note however that the vorticity field is not frozen in the fluid, since the Lorentz force $\mathbf{j}\wedge\mathbf{h}$ in (2.1) is in general rotational (i.e. $[\mathbf{j},\mathbf{h}]\neq 0$).

The set of invariants (2.10) and (2.11) are topological in character (Moffatt 1969), carrying information about the linkage of $\mathbf{h}$-lines and the mutual linkage of $\boldsymbol{\omega}$-lines and $\mathbf{h}$-lines. We shall describe them as the topological invariants of the system of equations (2.1)–(2.3). Note that a possible special choice of V_h in (2.10), (2.11) is $V_h=\mathcal{D}$, in which case we may talk of the global magnetic helicity and the global cross-helicity of the flow.

We wish to study the existence, structure and stability of steady solutions of (2.1)–(2.3). The following discussion follows Moffatt (1989). Let $\mathbf{u}=\mathbf{U}(\mathbf{x})$, $\mathbf{h}=\mathbf{H}(\mathbf{x})$ be one such solution, with $\boldsymbol{\Omega}=\nabla\wedge\mathbf{U}$ and $\mathbf{J}=\nabla\wedge\mathbf{H}$. Then, from (2.2),

$$\mathbf{U}\wedge\mathbf{H}=\nabla\Phi\,, \tag{2.13}$$

and, from (2.1),

$$\mathbf{U}\wedge\boldsymbol{\Omega}+\mathbf{J}\wedge\mathbf{H}=\nabla\Psi \tag{2.14}$$

where $\Psi=p+\frac{1}{2}\mathbf{u}^2$. From (2.13), we have

$$\mathbf{U}\cdot\nabla\Phi=\mathbf{H}\cdot\nabla\Phi=0\,, \tag{2.15}$$

so that **U**-lines (i.e. the streamlines of the flow) and **H**-lines (i.e. the magnetic lines of force) lie on surfaces $\Phi = cst$. The surface $\partial\mathcal{D}$ is one such surface, i.e.

$$\Phi = \text{ cst.} \quad \text{on } \partial\mathcal{D}\,. \tag{2.16}$$

In general therefore, there is a strong topological constraint on the structure of the fields $\mathbf{U}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$. The only escape from this constraint arises if

$$\nabla\Phi \equiv 0 \quad \text{in some} \quad \mathcal{D}_1 \subseteq \mathcal{D}\,. \tag{2.17}$$

Then $\mathbf{U}\wedge\mathbf{H} = 0$ in $\mathcal{D}_1$, and so

$$\mathbf{U} = \alpha(\mathbf{x})\mathbf{H} \quad \text{with } (\mathbf{H}\cdot\nabla)\alpha = 0 \quad \text{in } \mathcal{D}_1\,. \tag{2.18}$$

Hence, again, the **U**-lines and **H**-lines are constrained to lie on surfaces $\alpha =$ cst. in $\mathcal{D}_1$. Again the only escape from this constraint arises if

$$\nabla\alpha \equiv 0 \quad \text{in some} \quad \mathcal{D}_2 \subseteq \mathcal{D}_1\,. \tag{2.19}$$

In this case

$$\mathbf{U} = \alpha\mathbf{H} \quad \text{with } \alpha = \text{ cst. in } \mathcal{D}_2\,. \tag{2.20}$$

It follows immediately that

$$\boldsymbol{\Omega} = \alpha\mathbf{J} \quad \text{in } \mathcal{D}_2 \tag{2.21}$$

and so (2.14) becomes

$$\left(1-\alpha^2\right)\mathbf{U}\wedge\boldsymbol{\Omega} = \nabla\Psi\,. \tag{2.22}$$

Here, the possibility $\alpha = 1$ is very special: it corresponds to $\mathbf{U} \equiv \mathbf{H}$ in $\mathcal{D}_2$, in which case (2.13) and (2.14) are satisfied for *any* choice of $\mathbf{H}(\mathbf{x})$. If $\alpha \neq 1$, then (2.22) implies that

$$\mathbf{U}\cdot\nabla\Psi = \boldsymbol{\Omega}\cdot\nabla\Psi = 0 \quad \text{in } \mathcal{D}_2 \tag{2.23}$$

Hence the **U**-lines (which now coincide with the **H**-lines) still lie on surfaces $\Psi =$ cst. – unless, that is,

$$\nabla\Psi \equiv 0 \quad \text{in some} \quad \mathcal{D}_3 \subseteq \mathcal{D}_2 \tag{2.24}$$

in which case

$$\boldsymbol{\Omega} = \beta(\mathbf{x})\mathbf{U} \quad \text{with} \quad \mathbf{U}\cdot\nabla\beta = 0 \quad \text{in } \mathcal{D}_3\,. \tag{2.25}$$

Now the **U**-lines lie on surfaces $\beta =$ cst. – It seems hard to escape the constraint that **U**-lines lie on surfaces! However, and finally, it may happen that

$$\nabla\beta \equiv 0 \quad \text{in some} \quad \mathcal{D}_4 \subseteq \mathcal{D}_3 \tag{2.26}$$

in which case

$$\boldsymbol{\Omega} = \beta\mathbf{U} \quad \text{with} \quad \beta = \text{ cst.} \quad \text{in } \mathcal{D}_4\,. \tag{2.27}$$

i.e. **U** is a Beltrami field in $\mathcal{D}_4$. Now, at last, we are released from the constraint that **U**-lines must lie on surfaces; under the condition (2.27), the **U**-lines may be chaotic in $\mathcal{D}_4$ (the prototype example is the ABC-flow studied by Henon 1966 and Arnold 1966). Note that in $\mathcal{D}_4$,

$$\mathbf{U} = \beta^{-1}\boldsymbol{\Omega} = \alpha\mathbf{H} = \beta^{-1}\alpha\mathbf{J}\,. \tag{2.28}$$

3. The isomagnetovortical (imv) foliation

It is known that, if we replace the field $\mathbf{u}$ in (2.7) by any other vector field $\mathbf{v}(\mathbf{x},\tau)$†, so that

$$\mathbf{h}_\tau = [\mathbf{v},\mathbf{h}]\,, \tag{3.1}$$

then the integrals $\mathcal{H}_M(V_h)$ given by (2.10) are still invariant; this is obvious because the field $\mathbf{h}$ is now frozen in the hypothetical flow $\mathbf{v}(\mathbf{x},\tau)$ and so all its topological properties are conserved. It is natural to adopt the constraint

$$\nabla\cdot\mathbf{v} = 0 \tag{3.2}$$

and the boundary condition $\mathbf{n}\cdot\mathbf{v} = 0$ on $\partial\mathcal{D}$.

In similar spirit, we may enquire what simultaneous modification of (2.8) may be made that will still guarantee conservation of the set of cross-helicities (2.11). The answer is that we must replace $\mathbf{u}$ on the right-hand side of (2.8) by the same $\mathbf{v}(\mathbf{x},\tau)$ as in (3.1); and at the same time, we may replace $\mathbf{j}$ by a vector field $\mathbf{c}(\mathbf{x},\tau)$, so that (2.8) is replaced by

$$\boldsymbol{\omega}_\tau = [\mathbf{v},\boldsymbol{\omega}] + [\mathbf{c},\mathbf{h}]\,. \tag{3.3}$$

It is again natural to impose the solenoidality condition

$$\nabla\cdot\mathbf{c} = 0\,. \tag{3.4}$$

It will not be necessary at this stage to impose any boundary condition on $\mathbf{c}$ (just as there was no boundary condition on $\mathbf{j}$ in the parent problem (2.1)–(2.3)). We may refer to $\mathbf{v}$ and $\mathbf{c}$ as 'auxiliary' velocity and current fields.

In order to verify this, we write (3.1) in the form

$$\frac{D\mathbf{h}}{D\tau} \equiv \frac{\partial\mathbf{h}}{\partial\tau} + \mathbf{v}\cdot\nabla\mathbf{h} = \mathbf{h}\cdot\nabla\mathbf{v}\,, \tag{3.5}$$

and the 'uncurled' version of (3.3) in the form

$$\frac{D\mathbf{u}}{D\tau} \equiv \frac{\partial\mathbf{u}}{\partial\tau} + \mathbf{v}\cdot\nabla\mathbf{u} = \mathbf{v}\cdot(\nabla\mathbf{u})^T + \mathbf{c}\wedge\mathbf{h} - \nabla P\,, \tag{3.6}$$

where $[\mathbf{v}\cdot(\nabla\mathbf{u})^T]_i = v_j\partial u_j/\partial x_i$. Then

$$\begin{aligned}\frac{d}{d\tau}\int_{V_h}(\mathbf{u}\cdot\mathbf{h})dV &= \int_{V_h}\left(\mathbf{u}\cdot\frac{D\mathbf{h}}{D\tau} + \mathbf{h}\cdot\frac{D\mathbf{u}}{D\tau}\right)dV\\ &= \int_{S_h}(\mathbf{h}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{v} - P)dS = 0\,,\end{aligned} \tag{3.7}$$

using standard manipulations and the condition $\mathbf{n}\cdot\mathbf{h} = 0$ on S_h.

The pair of equations (3.1), (3.3) is wider in scope than the pair (2.7), (2.8) because in the latter

$$\mathbf{u} = \operatorname{curl}^{-1}\boldsymbol{\omega}\,,\quad \mathbf{j} = \operatorname{curl}\mathbf{h}\,, \tag{3.8}$$

whereas $\mathbf{v}$ and $\mathbf{c}$ suffer no such restrictions.

Note that, if $\mathbf{c} = 0$ in (3.3), then the vortex lines are frozen (like the $\mathbf{h}$-lines) in the flow $\mathbf{v}$. When $\mathbf{c} \neq 0$, the vortex lines are *not* frozen in this flow; but the equation is such

† Throughout §§3–5, we replace t by a 'virtual time' τ, in order to distinguish the virtual processes considered from real time evolution.

that the quantities Γ_h defined by (2.12), which represent flux of vorticity through any closed **h**-line, do remain invariant. The term $[\mathbf{c},\mathbf{h}]$ in (3.3) redistributes vorticity flux within any such closed **h**-line but conserves the integrated flux.

Equations (3.1), (3.3) thus conserve all known topological invariants of the parent equations, but allow a much wider range of behaviour due to the freedom of choice of the auxiliary fields $\{\mathbf{v},\mathbf{c}\}$. Let $\mathcal{A}(\mathbf{x})$ represent the pair of fields $\{\mathbf{u}(\mathbf{x}),\mathbf{h}(\mathbf{x})\}$ (where $\mathbf{u} = \mathrm{curl}^{-1}\boldsymbol{\omega}$). Under the action of a pair of auxiliary fields $\{\mathbf{v}(\mathbf{x},\tau),\mathbf{c}(\mathbf{x},\tau)\}$ during a time interval $[\tau_1,\tau_2]$, evolution governed by (3.1) and (3.3) will convert a pair $\mathcal{A}_1(\mathbf{x})$ to a pair $\mathcal{A}_2(\mathbf{x})$ having the same set of topological invariants. Conversely, we may say that two pairs $\mathcal{A}_1(\mathbf{x})$ and $\mathcal{A}_2(\mathbf{x})$ lie on the same isomagnetovortical (or imv) folium of the function space of such pairs if and only if there exist fields $\{\mathbf{v}(\mathbf{x},\tau),\mathbf{c}(\mathbf{x},\tau)\}$ which effect the conversion $\mathcal{A}_1(\mathbf{x}) \to \mathcal{A}_2(\mathbf{x})$ in a time interval $[\tau_1,\tau_2]$. This requirement clearly defines an *imv foliation* of the function space, two pairs being on the same folium if and only if they are 'accessible' one from another via (3.1) and (3.3), and therefore certainly only if they have the same set of topological invariants. This foliation provides the required generalisation of the 'isovortical' foliation introduced by Arnold (1965).

It is of course obvious that the function space trajectory of a solution $\{\mathbf{u}(\mathbf{x},t),\,\mathbf{h}(\mathbf{x},t)\}$ of the parent equations (2.1)–(2.3) lies on an imv folium, since this evolution corresponds to the particular choice of auxiliary fields $\mathbf{v}(\mathbf{x},\tau) = \mathbf{u}(\mathbf{x},\tau)$ and $\mathbf{c}(\mathbf{x},\tau) = \mathbf{j}(\mathbf{x},\tau)$ and the restoration of real time $\tau \to t$. This may be stated as:

Proposition 3.1 *A trajectory* $\{\mathbf{u}(\mathbf{x},t),\mathbf{h}(\mathbf{x},t)\}$ *of the ideal MHD equations lies on an isomagnetovortical folium.*

4. Relaxation to minimum energy states

Let us now consider the energy functional

$$E = \tfrac{1}{2}\int_{\mathcal{D}} \left(\mathbf{h}^2 + \mathbf{u}^2\right) dV\,, \tag{4.1}$$

which is known to be an invariant of equations (2.1)–(2.3). Let us calculate the rate of change of energy under (3.1), (3.3), i.e. under a more general evolution on an imv folium. We have

$$\begin{aligned}\frac{dE}{d\tau} &= \int_{\mathcal{D}} \{\mathbf{h}\cdot[\mathbf{v},\mathbf{h}] + \mathbf{u}\cdot(\mathbf{v}\wedge\boldsymbol{\omega} + \mathbf{c}\wedge\mathbf{h} - \nabla P\} dV \\ &= -\int_{\mathcal{D}} \{\mathbf{v}\cdot(\mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{h}) + \mathbf{c}\cdot(\mathbf{u}\wedge\mathbf{h})\} dV\,. \end{aligned} \tag{4.2}$$

Here, we have used $\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{v} = 0$ on $\partial\mathcal{D}$ and have discarded surface terms. Note that if we put $\mathbf{v} = \mathbf{u}$ and $\mathbf{c} = \mathbf{j}$ in (4.2), then the integrand vanishes and $dE/d\tau = 0$ as expected.

We now exploit our freedom of choice of **v** and **c** in order to ensure monotonic decrease of E, the idea being to drive the system towards a minimum energy state. The obvious choice is

$$\mathbf{v} = \mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{h} - \nabla\alpha \tag{4.3}$$

$$\mathbf{c} = \mathbf{u}\wedge\mathbf{h} - \nabla\beta \tag{4.4}$$

where α and β are chosen so that

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{c} = 0 \ , \ \mathbf{n} \cdot \mathbf{v} = \beta = 0 \text{ on } \partial \mathcal{D} \, . \tag{4.5}$$

These requirements define a Neumann problems for α and a Dirichlet problem for β which provide unique solutions for $\nabla\alpha$ and $\nabla\beta$. Equation (4.2) now gives

$$\frac{dE}{d\tau} = -\int_{\mathcal{D}} \left(\mathbf{v}^2 + \mathbf{c}^2\right) dV \, , \tag{4.6}$$

where we have used (4.5) to discard surface contributions. Hence E does decrease monotonically as required.

In order for this result to be useful, we require a positive lower bound on E. This is provided by the cross-helicity invariant $\mathcal{H}_C$ which is conserved on the imv folium; for

$$E = \tfrac{1}{2} \int_{\mathcal{D}} \left(\mathbf{u}^2 + \mathbf{h}^2\right) dV \geqslant \left| \int_{\mathcal{D}} \mathbf{u} \cdot \mathbf{h} \, dV \right| = |\mathcal{H}_C| \, . \tag{4.7}$$

Hence, provided $\mathcal{H}_C \neq 0$, we certainly have a lower bound †. It then follows that E tends to a limit as $\tau \to \infty$, and hence, from (4.6), that‡

$$\mathbf{v} \to 0 \quad \text{and} \quad \mathbf{c} \to 0 \quad \text{as} \quad \tau \to \infty \, . \tag{4.8}$$

From (4.3) and (4.4), we then see that $\mathbf{u}$ and $\mathbf{h}$ must tend to steady (equilibrium) fields $\mathbf{U}^E(\mathbf{x}), \mathbf{H}^E(\mathbf{x})$, say, satisfying the steady state equations

$$\left[\mathbf{U}^E, \mathbf{H}^E\right] = 0 \, , \tag{4.9}$$

$$\left[\mathbf{U}^E, \mathbf{\Omega}^E\right] + \left[\mathbf{J}^E, \mathbf{H}^E\right] = 0 \, , \tag{4.10}$$

where $\mathbf{\Omega}^E = \text{ curl } \mathbf{U}^E$, $\mathbf{J}^E = \text{ curl } \mathbf{H}^E$. Moreover, since the energy E ‘goes downhill all the way’ during the relaxation process, it attains a minimum in the asymptotic state (only in the most exceptional circumstances could the process be arrested at a saddle point), and this asymptotic state must therefore be stable.

The process described above is very similar to the magnetic relaxation process described by Moffatt (1985). In that case, there was a lower bound on the magnetic energy obtained through a combination of Schwarz and Poincaré inequalities:

$$\tfrac{1}{2} \int_{\mathcal{D}} \mathbf{h}^2 dV \geqslant q^{-1} |\mathcal{H}_M| \, , \tag{4.11}$$

where q^{-1} is a constant (with dimensions of length) dependent on the geometry of the domain $\mathcal{D}$. The inequality (4.11) still holds in the present context and implies, in conjunction with (4.7), that

$$E \geqslant \max \left\{ |\mathcal{H}_C|, q^{-1} |\mathcal{H}_M| \right\} \, . \tag{4.12}$$

There is no lower bound on the kinetic energy $\frac{1}{2}\int \mathbf{u}^2 dV$.

It may well be asked why the above procedure does not work if the fluid is non-conducting, and $\mathbf{h} \equiv 0$. The answer is that then both $\mathcal{H}_C$ and $\mathcal{H}_M$ are zero, and no positive lower bound is available for E. Thus although a process may be constructed

† The technique of Freedman (1988) may presumably be adapted to demonstrate the existence of a positive lower bound on E even when $\mathcal{H}_C = 0$ provided there is a nontrivial linkage between the vorticity and magnetic fields.

‡ The appearance of mild point singularities in $\mathbf{v}$ and/or $\mathbf{c}$ seems unlikely; a formal proof however is at present lacking, and is desirable.

as above which monotonically reduces E, there is no guarantee of a limit other than $E = 0$. Thus the procedure does not yield any three-dimensional stable steady solutions of the Euler equations of ideal hydrodynamics, consistent with the conclusion of Rouchon (1991) that no such steady solutions exist satisfying Arnold's (1965) sufficient condition for stability.

5. Variational principle

Suppose now that $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$ is a steady solution of equations (2.1)–(2.3); we may think of $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$ as a fixed point in the function space. Following Moffatt (1986), we consider a virtual displacement $\mathbf{x} \rightarrow \boldsymbol{\zeta}(\mathbf{x}, \tau_0)$ on the imv folium containing this fixed point, considered as the displacement associated with a *steady* auxiliary velocity field $\mathbf{v}(\mathbf{x})$ acting over a short time interval $[0, \tau_0]$†. Thus

$$\boldsymbol{\zeta}_\tau = \mathbf{v}(\boldsymbol{\zeta}) \qquad (0 \leqslant \tau \leqslant \tau_0) \tag{5.1}$$

and, for small τ_0,

$$\boldsymbol{\zeta}(\mathbf{x}, \tau) = \mathbf{x} + \tau_0 \mathbf{v}(\mathbf{x}) + \tfrac{1}{2}\tau_0^2 \mathbf{v} \cdot \nabla \mathbf{v} + \mathrm{O}(\tau_0^3)\,. \tag{5.2}$$

We define the first-order displacement field

$$\boldsymbol{\xi}(\mathbf{x}) = \tau_0 \mathbf{v}(\mathbf{x})\,, \tag{5.3}$$

which evidently satisfies

$$\nabla \cdot \boldsymbol{\xi} = 0 \quad , \quad \mathbf{n} \cdot \boldsymbol{\xi} = 0 \quad \text{on} \quad \partial\mathcal{D}\,. \tag{5.4}$$

Under the frozen-field distortion of $\mathbf{H}(\mathbf{x})$ associated with the virtual displacement $\boldsymbol{\zeta}(\mathbf{x}, \tau_0)$, it is known (Moffatt 1986) that the first and second order variations of $\mathbf{H}$ are

$$\delta^1 \mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}] \quad , \ \delta^2 \mathbf{h} = \tfrac{1}{2}\left[\boldsymbol{\xi}, \delta^1 \mathbf{h}\right]\,, \tag{5.5}$$

In similar spirit, we may now calculate from (3.3) the first and second order variations of $\boldsymbol{\Omega}(\mathbf{x}) =$ curl $\mathbf{U}(\mathbf{x})$ consequent upon the 'application' of steady auxiliary fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ over the time interval $[0, \tau_0]$. Defining

$$\boldsymbol{\eta}(\mathbf{x}) = \tau_0 \mathbf{c}(\mathbf{x}) \quad , \quad \nabla \cdot \boldsymbol{\eta} = 0\,, \tag{5.6}$$

these are

$$\delta^1 \boldsymbol{\omega} = [\boldsymbol{\xi}, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, \mathbf{H}]\,, \tag{5.7}$$

and

$$\delta^2 \boldsymbol{\omega} = \tfrac{1}{2}\left[\boldsymbol{\xi}, \delta^1 \boldsymbol{\omega}\right] + \tfrac{1}{2}\left[\boldsymbol{\eta}, \delta^1 \mathbf{h}\right]\,. \tag{5.8}$$

The corresponding perturbations of $\mathbf{U}$ are evidently

$$\delta^1 \mathbf{u} = \boldsymbol{\xi} \wedge \boldsymbol{\Omega} + \boldsymbol{\eta} \wedge \mathbf{H} - \nabla \alpha\,, \tag{5.9}$$

$$\delta^2 \mathbf{u} = \tfrac{1}{2}\boldsymbol{\xi} \wedge \delta^1 \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\eta} \wedge \delta^1 \mathbf{h} - \nabla \beta\,. \tag{5.10}$$

By virtue of their construction, the variations (5.5), (5.9) and (5.10) are first- and second-order perturbations on the imv folium containing $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$.

The expressions (5.9) and (5.10) have been obtained previously by Friedlander & Vishik (1995) following an abstract prescription for general Hamiltonian systems of Khesin &

† It is possible to represent displacements in this way because the domain $\mathcal{D}$ is fixed.

Chekanov (1989). We believe that the above derivation is easier to understand, and that it sheds some light on the physical meaning of the displacement fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

The first order variation of energy is

$$\begin{aligned}\delta^1 E &= \int_{\mathcal{D}} \big(\mathbf{H}\cdot\delta^1\mathbf{h} + \mathbf{U}\cdot\delta^1\mathbf{u}\big)\,dV \\ &= \int_{\mathcal{D}} \Big\{\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}] + \mathbf{U}\cdot(\boldsymbol{\xi}\wedge\boldsymbol{\Omega} + \boldsymbol{\eta}\wedge\mathbf{H})\Big\}\,dV \\ &= -\int_{\mathcal{D}} \Big\{\boldsymbol{\xi}\cdot(\mathbf{U}\wedge\boldsymbol{\Omega} + \mathbf{J}\wedge\mathbf{H}) + \boldsymbol{\eta}\cdot(\mathbf{U}\wedge\mathbf{H})\Big\}\,dV \\ &= -\int_{\mathcal{D}} (\boldsymbol{\xi}\cdot\nabla\Psi + \boldsymbol{\eta}\cdot\nabla\Phi)dV \\ &= 0\,. \end{aligned} \tag{5.11}$$

Here we have used $\nabla\cdot\boldsymbol{\xi} = \nabla\cdot\boldsymbol{\eta} = 0$, $\mathbf{n}\cdot\boldsymbol{\xi} = 0$ on $\partial\mathcal{D}$ and $\Phi = cst$ on $\partial\mathcal{D}$. We have thus proved:

Proposition 5.1 *The energy functional has a stationary point at the steady state (or fixed point)* $\{\mathbf{U}(\mathbf{x}),\mathbf{H}(\mathbf{x})\}$ *with respect to all admissible perturbations on the imv folium containing the fixed point.*

Similarly, the second-order variation of energy on the imv folium is given by

$$\delta^2 E = \tfrac{1}{2}\int \big[(\delta^1\mathbf{u})^2 + (\delta^1\mathbf{h})^2 + 2\mathbf{U}\cdot\delta^2\mathbf{u} + 2\mathbf{H}\cdot\delta^2\mathbf{h}\big]\,dV\,. \tag{5.12}$$

Substituting for $\delta^2\mathbf{h}$, $\delta^2\mathbf{u}$ from (5.5) and (5.10) and rearranging (using integration by parts) gives

$$\delta^2 E = \tfrac{1}{2}\int \Big[(\delta^1\mathbf{u})^2 + (\delta^1\mathbf{h})^2 - \boldsymbol{\xi}\cdot\Big(\mathbf{U}\wedge\delta^1\boldsymbol{\omega} + \mathbf{J}\wedge\delta^1\mathbf{h}\Big) - \boldsymbol{\eta}\cdot\Big(\mathbf{U}\wedge\delta^1\mathbf{h}\Big)\Big]\,dV\,. \tag{5.13}$$

This result is as previously stated (allowing for change of notation) by Friedlander & Vishik (1995).

The expression (5.13) apparently depends on the choice of fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. However $\delta^2 E$ in fact depends only on the perturbations $\delta^1\mathbf{u}$ and $\delta^1\mathbf{h}$ and is a quadratic functional of these fields. This follows from:

Proposition 5.2 *Let* $\{\boldsymbol{\xi}_1,\boldsymbol{\eta}_1\}$, $\{\boldsymbol{\xi}_2,\boldsymbol{\eta}_2\}$ *be two distinct pairs of displacement fields giving the same* $\delta^1\mathbf{u}$*,* $\delta^1\mathbf{h}$*, i.e.*

$$\delta^1\mathbf{h} = [\boldsymbol{\xi}_1,\mathbf{H}] = [\boldsymbol{\xi}_2,\mathbf{H}]\,, \tag{5.14}$$

$$\delta^1\mathbf{u} = \boldsymbol{\xi}_1\wedge\boldsymbol{\Omega} + \boldsymbol{\eta}_1\wedge\mathbf{H} - \nabla\alpha_1 = \boldsymbol{\xi}_2\wedge\boldsymbol{\Omega} + \boldsymbol{\eta}_2\wedge\mathbf{H} - \nabla\alpha_2\,. \tag{5.15}$$

Then

$$\delta^2 E(\boldsymbol{\xi}_1,\boldsymbol{\eta}_1) = \delta^2 E(\boldsymbol{\xi}_2,\boldsymbol{\eta}_2)\,. \tag{5.16}$$

This result is a natural generalisation of that of Arnold (1966). Its proof involves long vector manipulation and is relegated to Appendix A.

Arnold's general stability principle is now applicable: *the flow* $\{\mathbf{U}(\mathbf{x}),\mathbf{H}(\mathbf{x})\}$ *is stable provided* $\delta^2 E$ *is either positive definite or negative definite for all admissible* $\{\boldsymbol{\xi}(\mathbf{x}),\boldsymbol{\eta}(\mathbf{x})\}$.

Before exploiting this principle, we shall first establish the relationship between $\delta^2 E$

and the linear stability problem, and then we shall provide an alternative formulation of the variational principle in terms of the frozen-in 'modified vorticity field' introduced in Part I of this series.

6. Linearised stability problem

Consider now an infinitesimal perturbation of the state $\{\mathbf{U}(\mathbf{x}), \mathbf{H}(\mathbf{x}), P(\mathbf{x})\}$ in the form

$$\{\mathbf{U} + \mathbf{u}(\mathbf{x}, t), \mathbf{H} + \mathbf{h}(\mathbf{x}, t), P + p(\mathbf{x}, t)\} \; . \tag{6.1}$$

We shall identify the perturbation fields $\mathbf{u}, \mathbf{h}$ with the first-order variations $\delta^1\mathbf{u}$, $\delta^1\mathbf{h}$ introduced in §5; now however we are concerned with the actual time (t) evolution of the system. Substituting (6.1) in (2.1)–(2.3) and linearizing in the perturbations, we obtain the linear evolution equations

$$\mathbf{u}_t + (\mathbf{U}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{U} = -\nabla p + \mathbf{J}\wedge\mathbf{h} + \mathbf{j}\wedge\mathbf{H}\,, \tag{6.2}$$

$$\mathbf{h}_t = [\mathbf{u}, \mathbf{H}] + [\mathbf{U}, \mathbf{h}]\,, \tag{6.3}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0\,. \tag{6.4}$$

We may assume that at time $t = 0$, the perturbations $\{\mathbf{u}, \mathbf{h}\}$ lie on the imv folium containing $\{\mathbf{U}, \mathbf{H}\}$; then by Proposition 3.1, $\{\mathbf{u}(\mathbf{x}, t), \mathbf{h}(\mathbf{x}, t)\}$ remains on this imv folium for all $t > 0$. We may then infer the existence of time-dependent displacement fields $\boldsymbol{\xi}(\mathbf{x}, t)$, $\boldsymbol{\eta}(\mathbf{x}, t)$ such that

$$\mathbf{h}(\mathbf{x}, t) = [\boldsymbol{\xi}(\mathbf{x}, t), \mathbf{H}(\mathbf{x})] \tag{6.5}$$

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\xi}(\mathbf{x}, t)\wedge\boldsymbol{\Omega}(\mathbf{x}) + \boldsymbol{\eta}(\mathbf{x}, t)\wedge\mathbf{H}(\mathbf{x}) - \nabla\alpha\,. \tag{6.6}$$

The corresponding vorticity perturbation is

$$\boldsymbol{\omega}(\mathbf{x}, t) = [\boldsymbol{\xi}, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, \mathbf{H}]\,. \tag{6.7}$$

The evolution equations (6.2)–(6.4) imply corresponding evolution equations for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. These may be found as follows. First, from (6.5),

$$\mathbf{h}_t = [\boldsymbol{\xi}_t, \mathbf{H}] \tag{6.8}$$

so that, substituting in (6.3) and using the Jacobi identity,

$$\begin{aligned} [\boldsymbol{\xi}_t, \mathbf{H}] &= [\mathbf{u}, \mathbf{H}] + [\mathbf{U}, [\boldsymbol{\xi}, \mathbf{H}]] \\ &= [\mathbf{u}, \mathbf{H}] - [\boldsymbol{\xi}, [\mathbf{H}, \mathbf{U}]] - [\mathbf{H}, [\mathbf{U}, \boldsymbol{\xi}]] \; . \end{aligned} \tag{6.9}$$

Hence, using $[\mathbf{H}, \mathbf{U}] = 0$ (from (2.13)), we have

$$[\boldsymbol{\xi}_t, \mathbf{H}] = [(\mathbf{u} + [\mathbf{U}, \boldsymbol{\xi}])\, , \mathbf{H}] \; , \tag{6.10}$$

and so (up to an arbitrary additive vector field which commutes with $\mathbf{H}$ and therefore does not contribute the $\mathbf{h}$ given by (6.5))

$$\boldsymbol{\xi}_t = \mathbf{u} + [\mathbf{U}, \boldsymbol{\xi}]\,. \tag{6.11}$$

This is the required evolution equation for $\boldsymbol{\xi}$. It implies that

$$(\mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}])_t = [\mathbf{U}, \mathbf{h} - [\boldsymbol{\xi}, \mathbf{H}]] \tag{6.12}$$

thus verifying that, if (6.5) is satisfied at $t = 0$, then it is also satisfied for all $t > 0$.

Equation (6.11) implies the following interpretation for $\boldsymbol{\xi}(\mathbf{x},t)$ (Chandrasekhar 1987): let $\mathbf{X}(\mathbf{x},t)$ be the position of a particle in the undisturbed flow $\mathbf{U}(\mathbf{x})$ with initial position $\mathbf{x}$; then $\mathbf{X}(\mathbf{x},t)+\boldsymbol{\xi}(\mathbf{x},t)$ is the position of the particle in the (linearly) disturbed flow $\mathbf{U}(\mathbf{x})+\mathbf{u}(\mathbf{x},t)$.

Secondly, from (6.7) we have

$$\boldsymbol{\omega}_t = [\boldsymbol{\xi}_t, \boldsymbol{\Omega}] + [\boldsymbol{\eta}_t, \mathbf{H}]\ , \tag{6.13}$$

while from the curl of (6.2) we have

$$\boldsymbol{\omega}_t = [\mathbf{U}, \boldsymbol{\omega}] + [\mathbf{u}, \boldsymbol{\Omega}] + [\mathbf{j}, \mathbf{H}] + [\mathbf{J}, \mathbf{h}]\,. \tag{6.14}$$

Hence

$$[\boldsymbol{\xi}_t, \boldsymbol{\Omega}] + [\boldsymbol{\eta}_t, \mathbf{H}] = [\mathbf{U}, [\boldsymbol{\xi}, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, \mathbf{H}]] + [\mathbf{u}, \boldsymbol{\Omega}] + [\mathbf{j}, \mathbf{H}] + [\mathbf{J}, [\boldsymbol{\xi}, \mathbf{H}]]\ . \tag{6.15}$$

This may be rearranged in the form

$$[\boldsymbol{\xi}_t - \mathbf{u} - [\mathbf{U}, \boldsymbol{\xi}], \boldsymbol{\Omega}] + [\boldsymbol{\eta}_t - \mathbf{j} - [\mathbf{U}, \boldsymbol{\eta}] - [\mathbf{J}, \boldsymbol{\xi}], \mathbf{H}] = 0\,. \tag{6.16}$$

The first term vanishes by (6.11); and we then have (again up to an arbitrary additive term commuting with $\mathbf{H}$ which does not contribute to $\boldsymbol{\omega}$ in (6.7))

$$\boldsymbol{\eta}_t = \mathbf{j} + [\mathbf{U}, \boldsymbol{\eta}] + [\mathbf{J}, \boldsymbol{\xi}]\,. \tag{6.17}$$

This is the required evolution equation for $\boldsymbol{\eta}$. It is the same as that previously obtained by Vladimirov & Moffatt (1995). Equation (6.17) implies that

$$(\boldsymbol{\omega} - [\boldsymbol{\xi}, \boldsymbol{\Omega}] - [\boldsymbol{\eta}, \mathbf{H}])_t = [\mathbf{U}, \boldsymbol{\omega} - [\boldsymbol{\xi}, \boldsymbol{\Omega}] - [\boldsymbol{\eta}, \mathbf{H}]] \tag{6.18}$$

(provable by now standard manipulations), thus verifying that if (6.7) is satisfied at $t=0$, then it is also satisfied for all $t>0$.

Invariance of $\delta^2 E$ under linearized evolution

We know that the energy E is an exact invariant of the equations (2.1)–(2.3). Moreover, we know that under an imv perturbation from the state $(\mathbf{U},\mathbf{H})$, $\delta^1 E = 0$, and so

$$E = E_0 + \delta^2 E + \ldots\ , \tag{6.19}$$

where $\delta^2 E$ is a quadratic functional of the perturbation fields $(\mathbf{u},\mathbf{h})$. It is to be expected therefore that $\delta^2 E$ should be an invariant of the linearised evolution equations (6.2)–(6.4), since if it were not, then E could not be an invariant of the exact equations. We thus have the following MHD counterpart of Arnold's (1966) result for ideal hydrodynamics:

Proposition 6.1 *The second variation of energy (5.13) is an invariant of the linearised equations (6.2)–(6.4).*

The direct verification of this result, which again involves long manipulations, is given in Appendix B.

7. Alternative form of variational principle

The theory developed in §§3–6 above is based on the need to consider perturbations which conserve the cross-helicity invariants (2.11), and the associated circulation invariants (2.12) when closed $\mathbf{h}$-lines exist. These integrals are evidently invariant under the replacement

$$\mathbf{u} \longrightarrow \tilde{\mathbf{u}} = \mathbf{u} + \mathbf{h} \wedge \mathbf{m} + \nabla\alpha\,, \tag{7.1}$$

where $\mathbf{m}$ is an arbitrary vector field and α is chosen so that $\nabla \cdot \tilde{\mathbf{u}} = 0$ and $\mathbf{n} \cdot \tilde{\mathbf{u}} = 0$ on $\partial\mathcal{D}$.

It is natural to ask whether $\mathbf{m}$ may be chosen in such a way that the circulation of $\tilde{\mathbf{u}}$ round *any* contour moving with the velocity field $\mathbf{u}$ (and not only those which coincide with closed $\mathbf{h}$-lines) is conserved. If this is so, then the 'modified vorticity field'

$$\tilde{\boldsymbol{\omega}} = \text{ curl } \tilde{\mathbf{u}} = \boldsymbol{\omega} + [\mathbf{h}, \mathbf{m}] \tag{7.2}$$

must satisfy the frozen-field equation

$$\tilde{\boldsymbol{\omega}}_t = [\mathbf{u}, \tilde{\boldsymbol{\omega}}] \,. \tag{7.3}$$

It follows from (7.2) that

$$\boldsymbol{\omega}_t + [\mathbf{h}_t, \mathbf{m}] + [\mathbf{h}, \mathbf{m}_t] = [\mathbf{u}, \boldsymbol{\omega}] + [\mathbf{u}, [\mathbf{h}, \mathbf{m}]] \ . \tag{7.4}$$

Substituting for $\mathbf{h}_t$ and $\boldsymbol{\omega}_t$ from (2.2), (2.8) and using the Jacobi identity, we obtain

$$[\mathbf{h}, \mathbf{m}_t - \mathbf{j} - [\mathbf{u}, \mathbf{m}]] = 0 \tag{7.5}$$

and so, up to a field commuting with $\mathbf{h}$, $\mathbf{m}$ must satisfy the evolution equation

$$\mathbf{m}_t = [\mathbf{u}, \mathbf{m}] + \mathbf{j} \,. \tag{7.6}$$

This equation is compatible with imposition of the subsidiary condition

$$\nabla \cdot \mathbf{m} = 0 \,. \tag{7.7}$$

Equations (7.2), (7.6) are the same as those obtained by a different procedure by Vladimirov & Moffatt (1995). Here, we view $\mathbf{m}$ as the generator of transformations (7.1) that leave cross-helicities invariant; equation (7.6) is then a consequence of the requirement (7.3) that the circulation of the modified velocity $\tilde{\mathbf{u}}$ round any material circuit (in the flow $\mathbf{u}$) be conserved.

The imv folium introduced in §3 now admits reformulation, putting the emphasis on the frozen-in fields $\mathbf{h}$ and $\tilde{\boldsymbol{\omega}}$. If, as in §3, $\mathbf{u}$ is replaced by $\mathbf{v}$ in both (2.2) and (7.3) and t by τ, giving

$$\mathbf{h}_\tau = [\mathbf{v}, \mathbf{h}] \ , \tag{7.8}$$

$$\tilde{\boldsymbol{\omega}}_\tau = [\mathbf{v}, \tilde{\boldsymbol{\omega}}] \ , \tag{7.9}$$

then both $\mathbf{h}$ and $\tilde{\boldsymbol{\omega}}$ are frozen-in in the flow $\mathbf{v}$ (with conservation of fluxes through all material circuits). The appropriate modification of (7.6) is

$$\mathbf{m}_\tau = [\mathbf{v}, \mathbf{m}] + \mathbf{c} \,, \tag{7.10}$$

where $\mathbf{c}$ is the 'modified current field' of §3. It may easily be verified that (7.10) is compatible with (3.3), (7.8), (7.9) and (7.2). Since $\mathbf{c}(\mathbf{x}, \tau)$ is an arbitrary solenoidal field, we may equally regard $\mathbf{m}(\mathbf{x}, \tau)$ as the arbitrary field (with $\mathbf{c}$ then given by (7.10)), if this proves convenient.

Consider now the basic state $\mathbf{U}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ satisfying (2.13), (2.14), or equivalently

$$[\mathbf{U}, \mathbf{H}] = 0 \quad , \quad [\mathbf{U}, \boldsymbol{\Omega}] + [\mathbf{J}, \mathbf{H}] = 0 \,. \tag{7.11}$$

Let $\mathbf{M}(\mathbf{x})$ be the associated steady field satisfying

$$[\mathbf{U}, \mathbf{M}] + \mathbf{J} = 0 \,, \tag{7.12}$$

and let

$$\tilde{\mathbf{U}} = \mathbf{U} + \mathbf{H} \wedge \mathbf{M} + \nabla \alpha \,, \tag{7.13}$$

$$\tilde{\mathbf{\Omega}} = \mathbf{\Omega} + [\mathbf{H}, \mathbf{M}] \,. \tag{7.14}$$

Then, from (7.3), we have

$$[\mathbf{U}, \tilde{\mathbf{\Omega}}] = 0 \,. \tag{7.15}$$

As in §5, we may now consider imv perturbations described by displacement fields $\{\boldsymbol{\xi}(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x})\}$ where $\boldsymbol{\xi} = \tau \mathbf{v}(\mathbf{x})$, $\boldsymbol{\eta} = \tau \mathbf{c}(\mathbf{x})$ with τ small. The corresponding perturbation of the field $\mathbf{M}$ is given from (7.10) by $\mathbf{M} \to \mathbf{M} + \boldsymbol{\mu}(\mathbf{x})$ where

$$\boldsymbol{\mu}(\mathbf{x}) = [\boldsymbol{\xi}, \mathbf{M}] + \boldsymbol{\eta} \,, \tag{7.16}$$

and we may use the pair $\{\boldsymbol{\xi}, \boldsymbol{\mu}\}$ instead of $\{\boldsymbol{\xi}, \boldsymbol{\eta}\}$ to describe imv perturbations.

In terms of these fields the first and second order variations of $\tilde{\boldsymbol{\omega}}$ are given by the frozen field relations

$$\delta^1 \tilde{\boldsymbol{\omega}} = \left[\boldsymbol{\xi}, \tilde{\mathbf{\Omega}}\right] \tag{7.17}$$

$$\delta^2 \tilde{\boldsymbol{\omega}} = \tfrac{1}{2} \left[\boldsymbol{\xi}, \delta^1 \tilde{\boldsymbol{\omega}}\right] \,. \tag{7.18}$$

Hence, from (7.2),

$$\delta^1 \boldsymbol{\omega} = \left[\boldsymbol{\xi}, \tilde{\mathbf{\Omega}}\right] - \left[\delta^1 \mathbf{h}, \mathbf{M}\right] - [\mathbf{H}, \boldsymbol{\mu}] \tag{7.19}$$

$$\delta^2 \boldsymbol{\omega} = \tfrac{1}{2} \left[\boldsymbol{\xi}, \delta^1 \tilde{\boldsymbol{\omega}}\right] - \left[\delta^2 \mathbf{h}, \mathbf{M}\right] - \left[\delta^1 \mathbf{h}, \boldsymbol{\mu}\right] \,. \tag{7.20}$$

The corresponding first variation of energy is

$$\begin{aligned} \delta^1 E &= \int_{\mathcal{D}} \left\{ \mathbf{U} \cdot (\mathbf{M} \wedge [\boldsymbol{\xi}, \mathbf{H}] + \boldsymbol{\mu} \wedge \mathbf{H} + \boldsymbol{\xi} \wedge \tilde{\mathbf{\Omega}} - \nabla \alpha) + \mathbf{H} \cdot [\boldsymbol{\xi}, \mathbf{H}] \right\} dV \\ &= \int_{\mathcal{D}} \left\{ (\boldsymbol{\xi} \wedge \mathbf{H}) \cdot ([\mathbf{U}, \mathbf{M}] + \mathbf{J}) + \boldsymbol{\mu} \cdot (\mathbf{H} \wedge \mathbf{U}) + \boldsymbol{\xi} \cdot (\tilde{\mathbf{\Omega}} \wedge \mathbf{U}) \right\} dV \\ &= 0 \end{aligned} \tag{7.21}$$

using (7.11)–(7.15). Thus we have re-established Proposition 5.1 from the alternative point of view.

Similarly the second variation of energy is now given by

$$\delta^2 E = \frac{1}{2} \int_{\mathcal{D}} \left\{ (\delta^1 \mathbf{u})^2 + (\delta^1 \mathbf{h})^2 + \delta^1 \boldsymbol{\omega} \cdot (\mathbf{U} \wedge \boldsymbol{\xi}) + \delta^1 \mathbf{h} \cdot (\mathbf{U} \wedge (\boldsymbol{\mu} - [\boldsymbol{\xi}, \mathbf{M}]) + \mathbf{J} \wedge \boldsymbol{\xi}) \right\} dV \,. \tag{7.22}$$

The relationship (7.16) guarantees that this expression is the same as (5.13); and Proposition 5.2 may be reformulated in an obvious way in terms of the fields $\{\boldsymbol{\xi}, \boldsymbol{\mu}\}$. We shall not labour the details; it is evident that we have merely reformulated the results of §5 from the alternative viewpoint.

8. Sufficient conditions for stability

Let us now consider in more detail the expression (5.13) for $\delta^2 E$; with the change of notation (as in §6) $\delta^1 \mathbf{u} \to \mathbf{u}$, $\delta^1 \mathbf{h} \to \mathbf{h}$, this may be written

$$\delta^2 E = \tfrac{1}{2} \int_{\mathcal{D}} \left\{ \mathbf{u}^2 + \mathbf{h}^2 - \mathbf{u} \cdot [\boldsymbol{\xi}, \mathbf{U}] - \mathbf{h} \cdot (\boldsymbol{\eta} \wedge \mathbf{U} + \boldsymbol{\xi} \wedge \mathbf{J}) \right\} dV \,. \tag{8.1}$$

We restrict attention to the class of steady MHD flows for which

$$\mathbf{U}(\mathbf{x}) = \lambda(\mathbf{x})\mathbf{H}(\mathbf{x}) \quad , \quad \mathbf{H}\cdot\nabla\lambda = 0\,. \tag{8.2}$$

For this class of flows, the expression (8.1) may be reduced (see Appendix C) to the form

$$\delta^2 E = \tfrac{1}{2}\int_{\mathcal{D}} (\mathbf{u} - \lambda\mathbf{h} + \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda))^2\, dV + W\,, \tag{8.3}$$

where

$$W = \tfrac{1}{2}\int_{\mathcal{D}} \left\{ \left(1-\lambda^2\right)\left(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\right) - 2\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})\right\} dV\,, \tag{8.4}$$

with, as usual, $\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}]$. Note that W is a quadratic functional of $\boldsymbol{\xi}$ alone; the dependence of $\delta^2 E$ on $\boldsymbol{\eta}$ in (8.3) is contained in the first integral (via the dependence of $\mathbf{u}$ on $\boldsymbol{\eta}$).

The flow (8.2) is stable if $\delta^2 E$ is sign-definite; hence from (8.3) we have immediately:

Proposition 8.1 *Steady MHD flows satisfying (8.2) are linearly stable to three-dimensional perturbations provided*

$$W \geqslant 0 \textit{ for all admissible } \boldsymbol{\xi}\,. \tag{8.5}$$

(The possibility $W = 0$ is included to cover those displacements $\boldsymbol{\xi}$ for which $\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}] = 0$.) Let us consider some particular cases.

(i) Flows with constant λ

If $\lambda =$ cst. in $\mathcal{D}$, (8.4) reduces to

$$W = (1-\lambda^2)W_0 \quad , \quad W_0 = \tfrac{1}{2}\int_{\mathcal{D}} \left\{\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\right\} dV\,. \tag{8.6}$$

If $|\lambda| < 1$ (i.e. the flow is *sub-Alfvenic*) then it is stable provided

$$W_0 \geqslant 0 \quad (\text{all admissible } \boldsymbol{\xi})\,. \tag{8.7}$$

This is as found previously by Friedlander & Vishik (1995). The criterion (8.7) applies equally to the case of magnetostatic equilibrium ($\lambda = 0$) and is the well-known criterion of Bernstein et al (1958). Sub-Alfvenic flows with constant λ have the same structure as 'equivalent' magnetostatic equilibria, and are evidently governed by the same stability criterion.

If $|\lambda| > 1$, then the flow would be stable provided $W_0 \leqslant 0$ (all admissible $\boldsymbol{\xi}$). However, as shown by Friedlander & Vishik (1995), this condition is never satisfied: perturbations $\boldsymbol{\xi}$ of sufficiently small length-scale can always be found such that $\mathbf{h}^2 > |\mathbf{h}\cdot(\mathbf{J}\wedge\xi)|$ in (8.6).

(ii) Parallel flow and field

Let $\mathcal{D}$ be an infinite cylinder of arbitrary cross-section with axis parallel to Oz, and suppose that

$$\mathbf{H} = H_0(x,y)\mathbf{e}_z \quad , \quad \mathbf{U} = \lambda(x,y)\mathbf{H}\,, \tag{8.8}$$

a particular case of (8.2). Then $(\mathbf{H}\cdot\nabla)\mathbf{H} = 0$, and

$$\mathbf{h} = \nabla\wedge(\boldsymbol{\xi}\wedge\mathbf{H}) = H_0(\mathbf{e}_z\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla H_0)\mathbf{e}_z\,. \tag{8.9}$$

After some algebra, (8.4) reduces to

$$W = \tfrac{1}{2}\int_{\mathcal{D}} (1-\lambda^2)H_0^2\,((\mathbf{e}_z\cdot\nabla)\boldsymbol{\xi})^2\, dV\,. \tag{8.10}$$

We assume that the fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ decay sufficiently rapidly as $|z| \to \infty$, to that the integrals in (8.3), (8.4) converge. [Alternatively, we may consider perturbations periodic in z, and take $\mathcal{D}$ to cover one period.] Thus we obtain:

Proposition 8.2 *The state (8.8) is linearly stable to imv perturbations provided* $|\lambda(x, y)| \leqslant 1$ *in* $\mathcal{D}$.

(iii) Annular basic state

Let $\mathcal{D}$ be an annular region between two cylinders C_1, C_2 of arbitrary cross-sections, and let

$$\mathbf{H} = -\mathbf{e}_z \wedge \nabla A\ ,\ \mathbf{U} = \lambda\mathbf{H}\ ,\ \lambda = \lambda(A)\,, \tag{8.11}$$

where $A(x, y)$ is the flux-function of $\mathbf{H}$. We shall suppose that $|\nabla A| \neq 0$ in $\mathcal{D}$, i.e. $\mathbf{H}$ has no neutral points in $\mathcal{D}$. The function A satisfies the Grad-Shafranov equation, which may be written

$$(1 - \lambda^2)\nabla^2 A - \lambda\lambda'\mathbf{H}^2 = G(A) \tag{8.12}$$

where $\lambda' = d\lambda/dA$. Also, in the state (8.11),

$$\mathbf{J} = -\nabla^2 A\mathbf{e}_z\ ,\ (\mathbf{H}\cdot\nabla)\mathbf{H} = \nabla(\tfrac{1}{2}\mathbf{H}^2) - (\nabla^2 A)\nabla A\,. \tag{8.13}$$

Defining the unit vector

$$\boldsymbol{\nu} = \nabla A/|\nabla A|\,, \tag{8.14}$$

the expression for W may be reduced (see Appendix D) to the form

$$W = W_1 + W_2\,, \tag{8.15}$$

where

$$W_1 = \tfrac{1}{2}\int_{\mathcal{D}} (1 - \lambda^2)(\mathbf{h} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\wedge\boldsymbol{\nu})^2 dV\,, \tag{8.16}$$

$$W_2 = -\int_{\mathcal{D}} \left\{(1 - \lambda^2)(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \lambda\lambda'|\mathbf{H}|(\boldsymbol{\nu}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H})(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\right\} dV\,. \tag{8.17}$$

Further transformation of W_2, using (8.13), (8.14) yields

$$W_2 = \tfrac{1}{2}\int_{\mathcal{D}} (1 - \lambda^2)\left(\nabla^2 A - \frac{\lambda\lambda'}{1 - \lambda^2}\mathbf{H}^2\right)\left(-\nabla^2 A + \frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2}\right)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 dV\,. \tag{8.18}$$

From (8.15)–(8.18) we conclude:

Proposition 8.3 *The state (8.11) is linearly stable to imv perturbations provided* $|\lambda| < 1$ *(i.e. the flow is sub-alfvenic) and*
either

$$\frac{\lambda\lambda'}{1 - \lambda^2}\mathbf{H}^2 \leqslant \nabla^2 A \leqslant \frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2} \tag{8.19}$$

or

$$\frac{\nabla A\cdot\nabla(\mathbf{H}^2)}{2\mathbf{H}^2} \leqslant \nabla^2 A \leqslant \frac{\lambda\lambda'}{1 - \lambda^2}\mathbf{H}^2 \tag{8.20}$$

throughout $\mathcal{D}$.

Note that for the limiting case in which $\mathbf{H} = H_0(y)\mathbf{e}_x$, $\mathbf{U} = \lambda(y)\mathbf{H}$, the integral W_2 given by (8.18) vanishes, and the stability criterion of Proposition 8.3 reduces simply to $|\lambda| < 1$, in conformity with Proposition 8.2.

To clarify the conditions (8.19), (8.20), consider the simple case for which $\mathcal{D}$ is the annular domain $a < r < b$, and

$$\mathbf{H} = H_0(r)\mathbf{e}_\theta \ , \ H_0(r) = -A'(r) \ , \ \mathbf{U} = \lambda(r)\mathbf{H} \, . \tag{8.21}$$

Then, defining

$$\chi(r) = r\lambda(r)\lambda'(r)/(1-\lambda^2) \, , \tag{8.22}$$

(8.19), (8.20) are equivalent to the inequalities
either

$$H_0(r) \geqslant 0 \text{ and } H_0'(r) + \frac{1-\chi(r)}{r} H_0(r) \leqslant 0 \tag{8.23}$$

or

$$H_0(r) \leqslant 0 \text{ and } H_0(r) + \frac{1-\chi(r)}{r} H_0(r) \geqslant 0 \, . \tag{8.24}$$

Both (8.23), (8.24) may be combined in the single inequality

$$H_0(r) \left(H_0'(r) + \frac{1-\chi(r)}{r} H_0(r) \right) \leqslant 0 \quad \text{for } a < r < b \tag{8.25}$$

or equivalently

$$\frac{d}{dr} \left[(1-\lambda^2)(rH_0)^2 \right] \leqslant 0 \, . \tag{8.26}$$

We conclude that if (8.26) is satisfied, then the state (8.21) is linearly stable to arbitrary imv perturbations.

Note that, for the case of magnetostatic equilibrium ($\lambda = 0$), (8.26) reduces to

$$\frac{d}{dr}(rH_0)^2 \leqslant 0 \, . \tag{8.27}$$

This criterion for stability to arbitrary 3D imv perturbations is, as might be expected, more restrictive than the criterion

$$\frac{d}{dr}(H_0/r)^2 \leqslant 0 \tag{8.28}$$

obtained by Moffatt (1986) for stability to axisymmetric perturbations.

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Appendix A. Proof of Proposition 5.2

For simplicity we consider here only a simply connected domain $\mathcal{D}$; more complicated geometry can be similarly treated. Let $\hat{\boldsymbol{\xi}} \equiv \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1$, $\hat{\boldsymbol{\eta}} \equiv \boldsymbol{\eta}_2 - \boldsymbol{\eta}_1$. Then, from (5.14) and (5.15),

$$[\hat{\boldsymbol{\xi}}, \mathbf{H}] = 0 \, , \quad [\hat{\boldsymbol{\xi}}, \boldsymbol{\Omega}] + [\hat{\boldsymbol{\eta}}, \mathbf{H}] = 0 \, . \tag{A 1}$$

Hence

$$\Delta E \equiv \delta^2 E(\boldsymbol{\xi}_2, \boldsymbol{\eta}_2) - \delta^2 E(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1) = \tfrac{1}{2} \int_{\mathcal{D}} \left(\delta^1 \boldsymbol{\omega} \cdot (\mathbf{U} \wedge \hat{\boldsymbol{\xi}}) + \delta^1 \mathbf{h} \cdot (\mathbf{U} \wedge \hat{\boldsymbol{\eta}} + \mathbf{J} \wedge \hat{\boldsymbol{\xi}}) \right) dV \, . \tag{A 2}$$

Using integration by parts and the Jacobi identity, we obtain

$$
\begin{aligned}
I_1 \equiv \int_{\mathcal{D}} \delta^1\boldsymbol{\omega}\cdot(\mathbf{U}\wedge\hat{\boldsymbol{\xi}})dV &= \int_{\mathcal{D}} \delta^1\mathbf{u}\cdot[\mathbf{U},\hat{\boldsymbol{\xi}}]dV \\
&= \int_{\mathcal{D}} (\boldsymbol{\xi}_1\wedge\boldsymbol{\Omega}+\boldsymbol{\eta}_1\wedge\mathbf{H})\cdot[\mathbf{U},\hat{\boldsymbol{\xi}}]dV \\
&= \int_{\mathcal{D}} \Big(\boldsymbol{\Omega}\cdot(\boldsymbol{\xi}_1\wedge[\hat{\boldsymbol{\xi}},\mathbf{U}])+(\boldsymbol{\eta}_1\wedge\mathbf{H})\cdot[\mathbf{U},\hat{\boldsymbol{\xi}}]\Big)\,dV \\
&= \int_{\mathcal{D}} \Big(\mathbf{U}\cdot[\boldsymbol{\xi}_1,[\hat{\boldsymbol{\xi}},\mathbf{U}]]+(\boldsymbol{\eta}_1\wedge\mathbf{H})\cdot[\mathbf{U},\hat{\boldsymbol{\xi}}]\Big)\,dV \\
&= \int_{\mathcal{D}} \Big(-\mathbf{U}\cdot[\hat{\boldsymbol{\xi}},[\mathbf{U},\boldsymbol{\xi}_1]]-\mathbf{U}\cdot[\mathbf{U},[\boldsymbol{\xi}_1,\hat{\boldsymbol{\xi}}]]+(\boldsymbol{\eta}_1\wedge\mathbf{H})\cdot[\mathbf{U},\hat{\boldsymbol{\xi}}]\Big)\,dV\,. \qquad \text{(A 3)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 \equiv \int_{\mathcal{D}} \delta^1\mathbf{h}\cdot(\mathbf{J}\wedge\hat{\boldsymbol{\xi}})dV &= \int_{\mathcal{D}} \mathbf{J}\cdot(\hat{\boldsymbol{\xi}}\wedge\delta^1\mathbf{h})dV = \int_{\mathcal{D}} \mathbf{H}\cdot[\hat{\boldsymbol{\xi}},\mathbf{h}]dV \\
&= \int_{\mathcal{D}} \mathbf{H}\cdot[\hat{\boldsymbol{\xi}},[\boldsymbol{\xi}_1,\mathbf{H}]]dV = \int_{\mathcal{D}} \mathbf{H}\cdot\Big(-[\boldsymbol{\xi}_1,[\mathbf{H},\hat{\boldsymbol{\xi}}]]-[\mathbf{H},[\hat{\boldsymbol{\xi}},\boldsymbol{\xi}_1]]\Big)\,dV \\
&= -\int_{\mathcal{D}} \mathbf{H}\cdot[\mathbf{H},[\hat{\boldsymbol{\xi}},\boldsymbol{\xi}_1]]dV = -\int_{\mathcal{D}} (\mathbf{J}\wedge\mathbf{H})\cdot[\hat{\boldsymbol{\xi}},\boldsymbol{\xi}_1]dV\,, \qquad \text{(A 4)}
\end{aligned}
$$

where we have used (A1). Substituting (A3) and (A4) in (A2) and taking account of (2.14), we obtain

$$
2\Delta E = \int_{\mathcal{D}} \Big(\delta^1\mathbf{h}\cdot(\mathbf{U}\wedge\hat{\boldsymbol{\eta}})+(\hat{\boldsymbol{\xi}}\wedge\boldsymbol{\Omega})\cdot[\mathbf{U},\boldsymbol{\xi}_1]\Big)\,dV\,. \qquad \text{(A 5)}
$$

Hence, in view of (A1),

$$
\begin{aligned}
2\Delta E &= \int_{\mathcal{D}} \Big(\delta^1\mathbf{h}\cdot(\mathbf{U}\wedge\hat{\boldsymbol{\eta}})-(\hat{\boldsymbol{\eta}}\wedge\mathbf{H})\cdot[\mathbf{U},\boldsymbol{\xi}_1]\Big)\,dV \\
&= \int_{\mathcal{D}} \hat{\boldsymbol{\eta}}\cdot\Big(\mathbf{H}\wedge[\boldsymbol{\xi}_1,\mathbf{U}]-\mathbf{U}\wedge[\boldsymbol{\xi}_1,\mathbf{H}]\Big)\,dV\,. \qquad \text{(A 6)}
\end{aligned}
$$

Since $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\eta}}=0$ in $\mathcal{D}$, it may be expressed in the form: $\hat{\boldsymbol{\eta}}=\nabla\wedge\mathbf{g}$ for some vector field $\mathbf{g}$. From (A6), we then obtain

$$
\begin{aligned}
2\Delta E &= \int_{\mathcal{D}} (\nabla\wedge\mathbf{g})\cdot\Big(\mathbf{H}\wedge[\boldsymbol{\xi}_1,\mathbf{U}]-\mathbf{U}\wedge[\boldsymbol{\xi}_1,\mathbf{H}]\Big)\,dV \\
&= \int_{\mathcal{D}} \mathbf{g}\cdot([\mathbf{H},[\boldsymbol{\xi}_1,\mathbf{U}]]+[\mathbf{U},[\mathbf{H},\boldsymbol{\xi}_1]])\,dV = -\int_{\mathcal{D}} \mathbf{g}\cdot[\boldsymbol{\xi}_1,[\mathbf{U},\mathbf{H}]]dV = 0\,, \qquad \text{(A 7)}
\end{aligned}
$$

and this completes the proof.

Appendix B. Proof of Proposition 6.1

The proof is a direct verification that $\delta^2 E$ is conserved by the linearized equations (6.2)-(6.4). First, we identify $\delta^1\mathbf{u}$ and $\delta^1\mathbf{u}$ with infinitesimal perturbation $\mathbf{u}$ and $\mathbf{h}$ whose evolution is governed by (6.2)-(6.4), so that

$$
\delta^2 E = \tfrac{1}{2}\int_{\mathcal{D}} \Big\{\mathbf{u}^2+\mathbf{h}^2-\mathbf{u}\cdot[\boldsymbol{\xi},\mathbf{U}]-\mathbf{h}\cdot(\boldsymbol{\eta}\wedge\mathbf{U}+\boldsymbol{\xi}\wedge\mathbf{J})\Big\}\,dV\,. \qquad \text{(B 1)}
$$

Then we have

$$\frac{d}{dt}\left(\delta^2 E\right) = \int_{\mathcal{D}} \Big\{ \ \mathbf{u}\cdot\mathbf{u}_t + \mathbf{h}\cdot\mathbf{h}_t + \tfrac{1}{2}\boldsymbol{\omega}_t\cdot(\mathbf{U}\wedge\boldsymbol{\xi}) + \tfrac{1}{2}\mathbf{h}_t\cdot(\mathbf{U}\wedge\boldsymbol{\eta} + \mathbf{J}\wedge\boldsymbol{\xi}) + \tfrac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{U}\wedge\boldsymbol{\xi}_t) + \tfrac{1}{2}\mathbf{h}\cdot(\mathbf{U}\wedge\boldsymbol{\eta}_t + \mathbf{J}\wedge\boldsymbol{\xi}_t)\Big\} dV\,.$$

Now we show that

$$I \equiv \int_{\mathcal{D}} \Big\{\boldsymbol{\omega}\cdot(\mathbf{U}\wedge\boldsymbol{\xi}_t) + \mathbf{h}\cdot(\mathbf{U}\wedge\boldsymbol{\eta}_t + \mathbf{J}\wedge\boldsymbol{\xi}_t)\Big\} dV = \int_{\mathcal{D}} \Big\{\boldsymbol{\omega}_t\cdot(\mathbf{U}\wedge\boldsymbol{\xi}) + \mathbf{h}_t\cdot(\mathbf{U}\wedge\boldsymbol{\eta} + \mathbf{J}\wedge\boldsymbol{\xi})\Big\} dV\,. \quad \text{(B 2)}$$

Let

$$I_1 \equiv \int_{\mathcal{D}} \boldsymbol{\omega}\cdot(\mathbf{U}\wedge\boldsymbol{\xi}_t) dV\,, \quad I_2 \equiv \int_{\mathcal{D}} \mathbf{h}\cdot(\mathbf{U}\wedge\boldsymbol{\eta}_t + \mathbf{J}\wedge\boldsymbol{\xi}_t) dV\,. \quad \text{(B 3)}$$

We have

$$\begin{aligned} I_1 &= \int_{\mathcal{D}} \Big([\boldsymbol{\xi},\boldsymbol{\Omega}] + [\boldsymbol{\eta},\mathbf{H}]\Big)\cdot(\mathbf{U}\wedge\boldsymbol{\xi}_t) dV = \int_{\mathcal{D}} \Big(\boldsymbol{\xi}\wedge\boldsymbol{\Omega} + \boldsymbol{\eta}\wedge\mathbf{H}\Big)\cdot[\mathbf{U},\boldsymbol{\xi}_t] dV \\ &= \int_{\mathcal{D}} \Big\{\boldsymbol{\Omega}\cdot(\boldsymbol{\xi}\wedge[\boldsymbol{\xi}_t,\mathbf{U}]) + \boldsymbol{\eta}\cdot(\mathbf{H}\wedge[\mathbf{U},\boldsymbol{\xi}_t])\Big\} dV = \int_{\mathcal{D}} \Big\{\mathbf{U}\cdot[\boldsymbol{\xi},[\boldsymbol{\xi}_t,\mathbf{U}]] + \mathbf{g}\cdot[\mathbf{H},[\mathbf{U},\boldsymbol{\xi}_t]]\Big\} dV \\ &= -\int_{\mathcal{D}} \Big\{\mathbf{U}\cdot\Big([\boldsymbol{\xi}_t,[\mathbf{U},\boldsymbol{\xi}]] + [\mathbf{U},[\boldsymbol{\xi},\boldsymbol{\xi}_t]]\Big) + \mathbf{g}\cdot\Big([\mathbf{U},[\boldsymbol{\xi}_t,\mathbf{H}]] + [\boldsymbol{\xi}_t,[\mathbf{H},\mathbf{U}]]\Big)\Big\} dV \\ &= \int_{\mathcal{D}} \Big\{(\boldsymbol{\xi}_t\wedge\boldsymbol{\Omega})\cdot[\mathbf{U},\boldsymbol{\xi}] - (\boldsymbol{\Omega}\wedge\mathbf{U})\cdot[\boldsymbol{\xi},\boldsymbol{\xi}_t] + (\mathbf{U}\wedge\boldsymbol{\eta})\cdot[\boldsymbol{\xi}_t,\mathbf{H}]\Big\} dV\,. \end{aligned} \quad \text{(B 4)}$$

Here the property that $\nabla\cdot\boldsymbol{\eta} = 0$ has been used: we have introduced the vector field $\mathbf{g}$ such that $\boldsymbol{\eta} = \nabla\wedge\mathbf{g}$.

Similar manipulations result in

$$I_2 = \int_{\mathcal{D}} \Big\{(\boldsymbol{\eta}_t\wedge\mathbf{H})\cdot[\mathbf{U},\boldsymbol{\xi}] + (\mathbf{J}\wedge\boldsymbol{\xi})\cdot[\boldsymbol{\xi}_t,\mathbf{H}] + (\mathbf{J}\wedge\mathbf{H})\cdot[\boldsymbol{\xi},\boldsymbol{\xi}_t]\Big\} dV\,. \quad \text{(B 5)}$$

It follows from (B4), (B5) that

$$I = I_1 + I_2 = \int_{\mathcal{D}} \Big\{ \ [\mathbf{U},\boldsymbol{\xi}]\cdot\Big(\boldsymbol{\xi}_t\times\boldsymbol{\Omega} + \boldsymbol{\eta}_t\wedge\mathbf{H}\Big) + [\boldsymbol{\xi},\boldsymbol{\xi}_t]\cdot\Big(\mathbf{U}\wedge\boldsymbol{\Omega} + \mathbf{J}\wedge\mathbf{H}\Big) + (\mathbf{J}\wedge\boldsymbol{\xi})\cdot[\boldsymbol{\xi}_t,\mathbf{H}]\Big\} dV\,. \quad \text{(B 6)}$$

Finally, using (6.5), (6,6) and (2.14), we arrive at the formula (B2).

It follows from (B2) that

$$\frac{d}{dt}\Big(\delta^2 E\Big) = \int_{\mathcal{D}} \Big\{\mathbf{u}\cdot\mathbf{u}_t + \mathbf{h}\cdot\mathbf{h}_t + \boldsymbol{\omega}_t\cdot(\mathbf{U}\wedge\boldsymbol{\xi}) + \mathbf{h}_t\cdot(\mathbf{U}\wedge\boldsymbol{\eta} + \mathbf{J}\wedge\boldsymbol{\xi})\Big\} dV\,. \quad \text{(B 7)}$$

After substitution of $\mathbf{u}_t$, $\mathbf{h}_t$ from (6.2)-(6.3) and some manipulations, this may be written in the form

$$\frac{d}{dt}\Big(\delta^2 E\Big) = I_3 + I_4 + I_5 \quad \text{(B 8)}$$

where

$$I_3 = \int_{\mathcal{D}} \Big\{\mathbf{u}\cdot(\mathbf{U}\wedge\boldsymbol{\omega}) + (\mathbf{U}\wedge\boldsymbol{\xi})\cdot\Big([\mathbf{u},\boldsymbol{\Omega}] + [\mathbf{U},\boldsymbol{\omega}]\Big)\Big\} dV\,, \quad \text{(B 9)}$$

$$I_4 = \int_{\mathcal{D}} \Big\{ \mathbf{u} \cdot \Big(\mathbf{j} \wedge \mathbf{H} + \mathbf{J} \wedge \mathbf{h} \Big) + \mathbf{h} \cdot [\mathbf{u}, \mathbf{H}] + [\mathbf{u}, \mathbf{H}] \cdot (\mathbf{J} \wedge \boldsymbol{\xi}) \Big\} dV \,, \tag{B 10}$$

$$I_5 = \int_{\mathcal{D}} \Big\{ (\mathbf{U} \wedge \boldsymbol{\xi}) \cdot \Big([\mathbf{j}, \mathbf{H}] + [\mathbf{J}, \mathbf{h}] \Big) + \mathbf{h} \cdot [\mathbf{U}, \mathbf{h}] \tag{B 11}$$

$$+ [\mathbf{u}, \mathbf{H}] \cdot (\mathbf{U} \wedge \boldsymbol{\eta}) + [\mathbf{U}, \mathbf{h}] \cdot \Big(\mathbf{U} \wedge \boldsymbol{\eta} + \mathbf{J} \wedge \boldsymbol{\xi} \Big) \Big\} dV \,. \tag{B 12}$$

Consider first I_3. We have

$$\begin{aligned}
I_3 &= \int_{\mathcal{D}} \Big\{ \boldsymbol{\omega} \cdot (\mathbf{u} \wedge \mathbf{U}) + [\mathbf{U}, \boldsymbol{\xi}] \cdot (\mathbf{u} \wedge \boldsymbol{\Omega}) + (\mathbf{U} \wedge \boldsymbol{\xi}) \Big([\mathbf{U}, [\boldsymbol{\xi}, \boldsymbol{\Omega}]] + [\mathbf{U}, [\boldsymbol{\eta}, \mathbf{H}]] \Big) \Big\} dV \\
&= \int_{\mathcal{D}} \Big\{ \Big([\boldsymbol{\xi}, \boldsymbol{\Omega}] + [\boldsymbol{\eta}, \mathbf{H}] \Big) \cdot (\mathbf{u} \wedge \mathbf{U}) + \boldsymbol{\Omega} \cdot (\mathbf{u} \wedge [\boldsymbol{\xi}, \mathbf{U}]) \\
&\qquad - (\mathbf{U} \wedge \boldsymbol{\xi}) \cdot \Big([\boldsymbol{\xi}, [\boldsymbol{\Omega}, \mathbf{U}]] + [\boldsymbol{\Omega}, [\mathbf{U}, \boldsymbol{\xi}]] + [\boldsymbol{\eta}, [\mathbf{H}, \mathbf{U}]] + [\mathbf{H}, [\mathbf{U}, \boldsymbol{\eta}]] \Big) \Big\} dV \\
&= \int_{\mathcal{D}} \Big\{ (\boldsymbol{\xi} \wedge \boldsymbol{\Omega}) \cdot [\mathbf{u}, \mathbf{U}] + [\boldsymbol{\eta}, \mathbf{H}] \cdot (\mathbf{u} \wedge \mathbf{U}) + \mathbf{U} \cdot [\mathbf{u}, [\boldsymbol{\xi}, \mathbf{U}]] \\
&\qquad - [\mathbf{U}, \boldsymbol{\xi}] \cdot (\boldsymbol{\xi} \wedge [\boldsymbol{\Omega}, \mathbf{U}]) - [\mathbf{U}, \boldsymbol{\xi}] \cdot (\mathbf{H} \wedge [\mathbf{U}, \boldsymbol{\eta}]) \Big\} dV \\
&= \int_{\mathcal{D}} \Big\{ (\boldsymbol{\xi} \wedge \boldsymbol{\Omega}) \cdot [\mathbf{u}, \mathbf{U}] + [\boldsymbol{\eta}, \mathbf{H}] \cdot (\mathbf{u} \wedge \mathbf{U}) - \mathbf{U} \cdot [\boldsymbol{\xi}, [\mathbf{U}, \mathbf{u}]] - \mathbf{U} \cdot [\mathbf{U}, [\mathbf{u}, \boldsymbol{\xi}]] \\
&\qquad - (\boldsymbol{\Omega} \wedge \mathbf{U}) \cdot [\boldsymbol{\xi}, [\boldsymbol{\xi}, \mathbf{U}]] + (\mathbf{U} \wedge \boldsymbol{\eta}) \cdot [\mathbf{H}, [\mathbf{U}, \boldsymbol{\xi}]] \Big\} dV \\
&= \int_{\mathcal{D}} \Big\{ [\boldsymbol{\eta}, \mathbf{H}] \cdot (\mathbf{u} \wedge \mathbf{U}) - (\mathbf{U} \wedge \boldsymbol{\eta}) \cdot [\mathbf{U}, \mathbf{h}] - (\boldsymbol{\Omega} \wedge \mathbf{U}) \Big([\mathbf{u}, \boldsymbol{\xi}] + [\boldsymbol{\xi}, [\boldsymbol{\xi}, \mathbf{U}]] \Big) \Big\} dV \,.
\end{aligned} \tag{B 13}$$

It can be shown by similar calculations that

$$I_4 = - \int_{\mathcal{D}} (\mathbf{J} \wedge \mathbf{H}) \cdot [\boldsymbol{\xi}, \mathbf{u}] dV \,, \tag{B 14}$$

$$I_5 = \int_{\mathcal{D}} \Big\{ (\mathbf{J} \wedge \mathbf{H}) \cdot [\boldsymbol{\xi}, [\boldsymbol{\xi}, \mathbf{U}]] + \Big([\mathbf{u}, \mathbf{H}] + [\mathbf{U}, \mathbf{h}] \Big) \cdot (\mathbf{U} \wedge \boldsymbol{\eta}) \Big\} dV \,. \tag{B 15}$$

From (B13)–(B15) and (2.14), we obtain

$$\frac{d}{dt} \Big(\delta^2 E \Big) = \int_{\mathcal{D}} \Big\{ [\boldsymbol{\eta}, \mathbf{H}] \cdot (\mathbf{u} \wedge \mathbf{U}) + [\mathbf{u}, \mathbf{H}] \cdot (\mathbf{U} \wedge \boldsymbol{\eta}) \Big\} dV \,. \tag{B 16}$$

Finally, since

$$\begin{aligned}
\int_{\mathcal{D}} [\mathbf{u}, \mathbf{H}] \cdot (\mathbf{U} \times \boldsymbol{\eta}) dV &= - \int_{\mathcal{D}} \boldsymbol{\eta} \cdot (\mathbf{U} \wedge [\mathbf{u}, \mathbf{H}]) dV \\
&= - \int_{\mathcal{D}} \mathbf{g} \cdot [\mathbf{U}, [\mathbf{u}, \mathbf{H}]] dV \\
&= \int_{\mathcal{D}} \mathbf{g} \cdot \Big([\mathbf{u}, [\mathbf{H}, \mathbf{U}]] + [\mathbf{H}, [\mathbf{U}, \mathbf{u}]] \Big) dV \\
&= \int_{\mathcal{D}} \boldsymbol{\eta} \cdot (\mathbf{H} \wedge [\mathbf{U}, \mathbf{u}]) = \int_{\mathcal{D}} [\boldsymbol{\eta}, \mathbf{H}] \cdot (\mathbf{U} \wedge \mathbf{u}) dV \,,
\end{aligned}$$

we obtain

$$\frac{d}{dt} \Big(\delta^2 E \Big) \; = \; 0 \,. \tag{B 17}$$

Note that while obtaining this formula we have only used the linearized equations (6.2)-(6.4) and the relations (6.5), (6.6) between fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and infinitesimal perturbations of the velocity and the magnetic field (we have not explicitly used the evolution equations for $\boldsymbol{\xi}$, $\boldsymbol{\eta}$).

Appendix C. Derivation of equation (8.3)

First we note that for the state (8.2) the relationship (6.6) takes the form

$$\mathbf{u} = \boldsymbol{\gamma} + \boldsymbol{\eta}\wedge\mathbf{H} - \nabla\alpha\,, \quad \boldsymbol{\gamma} = \lambda\boldsymbol{\xi}\wedge\mathbf{J} + \nabla\lambda(\boldsymbol{\xi}\cdot\mathbf{H}) - \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda)\,. \tag{C 1}$$

From (8.1), (8.2), we have

$$2\delta^2 E = \int_{\mathcal{D}} \Big\{\mathbf{u}^2 + \mathbf{h}^2 - \mathbf{u}\cdot[\boldsymbol{\xi},\mathbf{U}] - \mathbf{h}\cdot(\lambda\boldsymbol{\eta}\wedge\mathbf{H} + \boldsymbol{\xi}\wedge\mathbf{J})\Big\} dV\,. \tag{C 2}$$

Since

$$[\boldsymbol{\xi},\mathbf{U}] = \lambda[\boldsymbol{\xi},\mathbf{H}] + \nabla\lambda\wedge(\boldsymbol{\xi}\wedge\mathbf{H}) = \lambda\mathbf{h} - \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda)\,, \tag{C 3}$$

we obtain

$$2\delta^2 E = \int_{\mathcal{D}} \Big\{\mathbf{u}^2 + \mathbf{h}^2 - \lambda\mathbf{u}\cdot\mathbf{h} + (\mathbf{u}\cdot\mathbf{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi}) - \lambda\mathbf{h}\cdot(\boldsymbol{\eta}\wedge\mathbf{H})\Big\} dV\,. \tag{C 4}$$

Further, using (C1), we get

$$2\delta^2 E = \int_{\mathcal{D}} \Big\{\mathbf{u}^2 + \mathbf{h}^2 - 2\lambda\mathbf{u}\cdot\mathbf{h} + (\mathbf{u}\cdot\mathbf{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi}) + \lambda\,\mathbf{h}\cdot\boldsymbol{\gamma} - \lambda\nabla\alpha\cdot\mathbf{h}\Big\} dV\,. \tag{C 5}$$

Now

$$\begin{aligned}\int_{\mathcal{D}} \lambda\nabla\alpha\cdot\mathbf{h}\,dV &= \int_{\mathcal{D}} \lambda\nabla\alpha\cdot[\boldsymbol{\xi},\mathbf{H}]\,dV = \int_{\mathcal{D}} (\nabla\lambda\wedge\nabla\alpha)\cdot(\boldsymbol{\xi}\wedge\mathbf{H})\,dV \\ &= \int_{\mathcal{D}} \boldsymbol{\xi}\cdot(\mathbf{H}\wedge(\nabla\lambda\wedge\nabla\alpha))\,dV = \int_{\mathcal{D}} (\boldsymbol{\xi}\cdot\nabla\lambda)(\mathbf{H}\cdot\nabla\alpha))\,dV\,.\end{aligned}$$

Substitution of this in (C5) and some manipulations yield

$$\begin{aligned}2\delta^2 E = \int_{\mathcal{D}} \Big\{&\Big(\mathbf{u} + \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda) - \lambda\mathbf{h}\Big)^2 + (1-\lambda^2)\Big(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\Big) \\ &+ \lambda(\mathbf{h}\cdot\mathbf{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \lambda(\mathbf{h}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\mathbf{H}) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\Big(\boldsymbol{\xi}\cdot(\mathbf{J}\wedge\mathbf{H})\Big)\Big\} dV\,.\end{aligned} \tag{C 6}$$

It may be shown that

$$\begin{aligned}X &\equiv \lambda(\mathbf{h}\cdot\mathbf{H})(\boldsymbol{\xi}\cdot\nabla\lambda) + \lambda(\mathbf{h}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\mathbf{H}) \\ &= (\mathbf{H}\cdot\nabla)\Big(\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\mathbf{H})\Big) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)(\boldsymbol{\xi}\cdot\nabla)\Big(\tfrac{1}{2}\mathbf{H}^2\Big) - \lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\Big(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big)\,.\end{aligned}$$

With help of this identity (C6) may be transformed to the equation

$$\begin{aligned}2\delta^2 E = \int_{\mathcal{D}} \Big\{&\Big(\mathbf{u} + \mathbf{H}(\boldsymbol{\xi}\cdot\nabla\lambda) - \lambda\mathbf{h}\Big)^2 \\ &+ (1-\lambda^2)\Big(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\Big) - 2\lambda(\boldsymbol{\xi}\cdot\nabla\lambda)\Big(\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big)\Big\} dV\,,\end{aligned} \tag{C 7}$$

which evidently coincides with (8.3).

Appendix D. Derivation of formula (8.15)

Consider the integral

$$I = \tfrac{1}{2}\int_{\mathcal{D}} (1-\lambda^2)\Big(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\Big)dV\,, \tag{D 1}$$

which enters the expression (8.4). We start with some transformations of this integral similar to those of Bernstein et al (1958). First, following Bernstein et al (1958), we shall prove the identity

$$2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} = \mathbf{J}^2 + (\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H})) + \mathbf{H}\cdot(\mathbf{J}\wedge\boldsymbol{\nu})\mathrm{div}\boldsymbol{\nu}\,, \tag{D 2}$$

where $\boldsymbol{\nu}$ is defined by (8.14).

It follows from (8.11), (8.14) that $\mathbf{H}\cdot\boldsymbol{\nu}=0$. Therefore,

$$0 = \nabla(\mathbf{H}\cdot\boldsymbol{\nu}) = (\boldsymbol{\nu}\cdot\nabla)\mathbf{H} + (\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\wedge(\nabla\wedge\boldsymbol{\nu}) + \boldsymbol{\nu}\wedge\mathbf{J}\,, \tag{D 3}$$

and so

$$\begin{aligned}\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H}) + \mathbf{H}\mathrm{div}\boldsymbol{\nu} &= (\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\nabla)\mathbf{H}\\ &= 2(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\wedge(\nabla\wedge\boldsymbol{\nu}) + \boldsymbol{\nu}\wedge\mathbf{J}\,.\end{aligned}$$

Further, we have

$$\begin{aligned}Y &\equiv \mathbf{J}^2 + (\mathbf{J}\wedge\boldsymbol{\nu})\cdot\Big(\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H}) + \mathbf{H}\mathrm{div}\boldsymbol{\nu}\Big)\\ &= \mathbf{J}^2 + (\mathbf{J}\wedge\boldsymbol{\nu})\cdot\Big(2(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\wedge(\nabla\wedge\boldsymbol{\nu}) - \mathbf{J}\wedge\boldsymbol{\nu}\Big)\\ &= 2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \mathbf{H}\cdot\Big((\nabla\wedge\boldsymbol{\nu})\wedge(\mathbf{J}\wedge\boldsymbol{\nu})\Big) = 2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu}\,,\end{aligned} \tag{D 4}$$

whence the identity (D2) immediately follows.

Now we rewrite the integral (D1) in the form

$$2I = \int_{\mathcal{D}} (1-\lambda^2)\Big\{\Big(\mathbf{h} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\wedge\boldsymbol{\nu}\Big)^2 - 2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu}\Big\}dV + I_1\,, \tag{D 5}$$

where

$$\begin{aligned}I_1 = \int_{\mathcal{D}} (1-\lambda^2)\Big\{&2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - 2\mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\\ &- (\mathbf{J}\wedge\boldsymbol{\nu})^2(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\Big\}dV\,.\end{aligned} \tag{D 6}$$

Substitution of (D2) in (D6) yields

$$\begin{aligned}I_1 = \int_{\mathcal{D}} (1-\lambda^2)\Big\{&\Big((\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H})) - J_0|\mathbf{H}|\mathrm{div}\boldsymbol{\nu}\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\\ &- 2\mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) + \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\xi})\Big\}dV\,.\end{aligned} \tag{D 7}$$

Here $J_0 \equiv -\nabla^2 A$ and we have used eqns. (8.11), (8.13), (8.14).

Let

$$\boldsymbol{\xi} = (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \tilde{\boldsymbol{\xi}}\,,\quad \tilde{\boldsymbol{\xi}} = b\,\mathbf{e}_z + c\,\mathbf{H}\,, \tag{D 8}$$

i.e. $b = \boldsymbol{\xi}\cdot\mathbf{e}_z$, $c = (\boldsymbol{\xi}\cdot\mathbf{H})/\mathbf{H}^2$. Substitution of (D8) in (D7) results in

$$I_1 = \int_{\mathcal{D}} (1-\lambda^2)\Big\{\Big((\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H})) - J_0|\mathbf{H}|\mathrm{div}\boldsymbol{\nu}\Big)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2 + \mathbf{h}\cdot(\mathbf{J}\wedge\tilde{\boldsymbol{\xi}}) - \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big\}dV\,. \quad \text{(D 9)}$$

Using (6.5) and (D8), we obtain

$$\begin{aligned} Y_1 &\equiv (\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H})) - \mathbf{h}\cdot(\mathbf{J}\wedge\boldsymbol{\nu})(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) \\ &= (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{J}\wedge\boldsymbol{\nu})\cdot\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H}) - \nabla\wedge(\boldsymbol{\xi}\wedge\mathbf{H})\Big) \\ &= (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{J}\wedge\boldsymbol{\nu})\cdot\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\nabla\wedge(\boldsymbol{\nu}\wedge\mathbf{H}) - \nabla\wedge\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu}\wedge\mathbf{H} + b(\mathbf{e}_z\times\mathbf{H})\Big)\Big) \\ &= -J_0|\mathbf{H}|\Big((\boldsymbol{\nu}\cdot\nabla)\frac{(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2}{2} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{e}_z\cdot\nabla b)\Big)\,. \end{aligned} \quad \text{(D 10)}$$

From (D8) and the condition $\mathrm{div}\boldsymbol{\xi} = 0$, we have

$$\mathbf{e}_z\cdot\nabla b = -\boldsymbol{\nu}\cdot\nabla(\boldsymbol{\xi}\cdot\boldsymbol{\nu}) - (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathrm{div}\boldsymbol{\nu} - \mathbf{H}\cdot\nabla c\,, \quad \text{(D 11)}$$

whence

$$Y_1 = J_0|\mathbf{H}|\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\mathrm{div}\boldsymbol{\nu} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{H}\cdot\nabla c)\Big)\,. \quad \text{(D 12)}$$

Substituting this in (D9), we get

$$I_1 = \int_{\mathcal{D}} (1-\lambda^2)\Big\{J_0|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})(\mathbf{H}\cdot\nabla c) + \mathbf{h}\cdot(\mathbf{J}\wedge\tilde{\boldsymbol{\xi}})\Big\}dV\,. \quad \text{(D 13)}$$

Consider now the integral

$$I_2 \equiv \int_{\mathcal{D}} (1-\lambda^2)\mathbf{h}\cdot(\mathbf{J}\wedge\tilde{\boldsymbol{\xi}})dV\,. \quad \text{(D 14)}$$

Since $\mathbf{J}\wedge\tilde{\boldsymbol{\xi}} = J_0 c\nabla A$, we have

$$\begin{aligned} I_2 &= \int_{\mathcal{D}} (1-\lambda^2)J_0 c\nabla A\cdot(\nabla\wedge(\boldsymbol{\xi}\wedge\mathbf{H}))dV \\ &= \int_{\mathcal{D}} \Big(\nabla((1-\lambda^2)J_0 c)\wedge\nabla A\Big)\cdot(\boldsymbol{\xi}\wedge\mathbf{H})dV \\ &= \int_{\mathcal{D}} (1-\lambda^2)\boldsymbol{\xi}\cdot\Big(\mathbf{H}\wedge\Big(\nabla(J_0 c)\wedge\nabla A\Big)\Big)dV \\ &= -\int_{\mathcal{D}} (1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})|\mathbf{H}|\Big(\mathbf{H}\cdot(J_0 c)\Big)dV\,, \end{aligned}$$

whence,

$$I_1 = \int_{\mathcal{D}} (1-\lambda^2)|\mathbf{H}|c(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big(\mathbf{H}\cdot\nabla(\nabla^2 A)\Big)dV\,. \quad \text{(D 15)}$$

Now let us recall equation (8.12). Using (8.11) and (8.12), we obtain

$$(1-\lambda^2)(\mathbf{H}\cdot\nabla)\nabla^2 A = \lambda\lambda'\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)\,. \quad \text{(D 16)}$$

Eqn. (D15) can therefore be written as

$$I_1 = \int_{\mathcal{D}} \lambda\lambda'|\mathbf{H}|\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)c(\boldsymbol{\xi}\cdot\boldsymbol{\nu})dV\,. \quad \text{(D 17)}$$

Now, taking account of eqns. (D5) and (D17), we can rewrite (8.4) in the form

$$W = W_1 + R\,, \tag{D 18}$$

where

$$W_1 = \tfrac{1}{2}\int_{\mathcal{D}} (1-\lambda^2)\Big(\mathbf{h} + (\boldsymbol{\xi}\cdot\boldsymbol{\nu})\mathbf{J}\wedge\boldsymbol{\nu}\Big)^2 dV$$

$$R = \tfrac{1}{2}\int_{\mathcal{D}} \Big\{\lambda\lambda'|\mathbf{H}|\Big(\mathbf{H}\cdot\nabla(\mathbf{H}^2)\Big)c(\boldsymbol{\xi}\cdot\boldsymbol{\nu})$$

$$-2(1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} - 2\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big\}dV\,. \tag{D 19}$$

All that remains to be done is to show that R in (D19) coincides with the integral W_2 given by (8.17). We have

$$\begin{aligned} Y_3 &\equiv -\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\xi}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H} \\ &= -\lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\Big((\boldsymbol{\xi}\cdot\boldsymbol{\nu})\boldsymbol{\nu} + c\,\mathbf{H}\Big)\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\,. \end{aligned} \tag{D 20}$$

Substitution of this in R yields

$$R = -\int_{\mathcal{D}} \Big\{(1-\lambda^2)(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2(\mathbf{J}\wedge\boldsymbol{\nu})\cdot(\mathbf{H}\cdot\nabla)\boldsymbol{\nu} + \lambda\lambda'|\mathbf{H}|(\boldsymbol{\xi}\cdot\boldsymbol{\nu})^2\boldsymbol{\nu}\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\Big\}dV\,. \tag{D 21}$$

After comparison of this formula with (8.17) we conclude that R is indeed the same as W_2.

REFERENCES

Arnold, V.I. 1965 Variational principle for three dimensional steady flows of an ideal fluid. *Prikl. Matem. i Mekh.*, **29**, N. 5, p. 846-851 (English transl.: *J. Appl. Math. & Mech.,* **29**, 5).

Arnold, V.I. 1966 Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, **16**, 316–361.

Arnold, V.I. & Khesin, B.A. 1998 Topological methods in hydrodynamics. *Appl. Math. Ser.*, **125**, Springer-Verlag, 388 pp.

Bernstein, I.B., Frieman, E.A., Kruskal, M.D. & Kulsrud, R.M. 1958 An energy principle for hydromagnetic stability theory. *Proc. Roy. Soc. London* A**244**, 17–40.

Chandrasekhar, S. 1987 *Ellipsoidal figures of equilibrium.* New York: Dover.

Freedman, M.H. 1988 A note on topology and magnetic energy in incompressible perfectly conducting fluids. *J. Fluid Mech.* **194**, 549-551.

Friedlander, S. & Vishik, M. 1990 Nonlinear stability for stratified magnetohydrodynamics. *Geophys. Astrophys. Fluid Dynamics*, **55**, 19–45.

Friedlander, S. & Vishik, M. 1995 On stability and instability criteria for magnetohydrodynamics. *Chaos* **5**(2), 416–423.

Hénon, M. 1966 Sur la topologie des lignes de courant dans un cas particulier. *C.R. Acad. Sci. Paris* **262**, 312–314.

Holm D.D., Marsden J.E., Ratiu T., & Weinstein A. 1985 Nonlinear stability of fluid and plasma equilibria. *Physics Reports* **123**, Nos.1 & 2, 1–116.

Khesin, B.A. & Chekanov, Yu.V. 1989 Invariants of the Euler equations for ideal or barotropic hydrodynamics and superconductivity in D dimensions. *Physica* D **40**, 119–131.

Marsden, J., Ratiu, T. & Weinstein, A. 1984 Semidirect product and reduction in mechanics. *Trans. Am. Math. Soc.* **281**, 147–177.

Moffatt H.K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.*, **35**, 117–129.

Moffatt H.K. 1985 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals. *J. Fluid Mech.* **159**, 359–378.

Moffatt H.K. 1986 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations. *J. Fluid Mech.*, **166**, 359–378.

Moffatt H.K. 1989 On the existence, structure and stability of MHD equilibrium states. In: *Turbulence and Nonlinear Dynamics in MHD Flows, (Eds. M.Meneguzzi et al)*, Elsevier, 185–195.

Ono, T. 1995 Riemannian geometry of the motion of an ideal incompressible magnetohydrodynamical fluid. *Physica* D **81**, 207–220.

Rouchon, P. 1991 On the Arnol'd stability criterion for steady-state flows of an ideal fluid. *Eur. J. Mech. , B/Fluids*, **10**, 651–661.

Vishik, S.M. & Dolzhanskii F.V. 1978 Analogs of the Euler-Lagrange equations and magnetohydrodynamics equations related to Lie groups. *Sov. Math. Doklady*, **19**, 149–153.

Vladimirov, V.A. & Ilin, K.I. 1998 Generalized isovorticity principle for ideal magnetohydrodynamics. In: *Recent advances in differential equations, (Eds. H.-H. Dai and P.L.Sachdev)*, Longman, 161–177.

Vladimirov, V.A. & Moffatt, H.K. 1995 On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part I. Fundamental principles. *J. Fluid Mech.* **283**, 125–139.

Vladimirov, V.A., Moffatt, H.K. & Ilin, K.I. 1996 On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part II. Stability criteria for two-dimensional flows. *J. Fluid Mech.* **329**, 187–205.

Vladimirov, V.A., Moffatt, H.K. & Ilin, K.I. 1997 On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part III. Stability criteria for axisymmetric flows. *J. Plasma Physics* **57**, part 1, 89–120.

Zeitlin, V. & Kambe, T. 1993 Two-dimensional ideal magnetohydrodynamics and differential geometry. *J. Phys. A: Math. Gen.* , **26**, 5025–5031.