DERIVED EQUIVALENCES AND DADE'S INVARIANT CONJECTURE

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1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring and assume that the residue field $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed of characteristic $p > 0$, the quotient field $\mathcal{K}$ has characteristic 0 and it is also a splitting field for all the $\mathcal{K}$-algebras considered below. Let further K be a normal subgroup of a finite group H, and let $G = H/K$.

Consider a block b of $\mathcal{O}K$ with defect group $P \leq K$ and let c be the block of $\mathcal{O}N_K(P)$ corresponding to b. E. Dade stated in [D3] several forms of a conjecture and one of them, the Invariant Conjecture, involves the number of irreducible $\mathcal{K}$-characters belonging to b and having a given defect and a given stabilizer in H/K.

The aim of this note is to provide a structural look to this conjecture, which means, to find equivalences of categories preserving these invariants.

Assume for the moment that b is G-invariant and let $\mathcal{O}Hb$ and $S = \mathcal{O}N_H(P)c$. It follows that R and S are strongly G-graded $\mathcal{O}$-algebras with $R_1 = \mathcal{O}Kb$ and $S_1 = \mathcal{O}N_K(P)c$. Then the group G acts on the category of (R_1, S_1)-bimodules. If C is a tilting complex of (R_1, S_1)-bimodules, then this equivalence is called G-equivariant if

$$R_g \otimes_{R_1} C \otimes_{S_1} S_{g^{-1}} \simeq C$$

in the bounded derived category of $R_1 \otimes_{\mathcal{O}} S_1^{op}$-mod. We investigate equivariant derived equivalences in Section 2 in a slightly more general setting, and we show that they preserve stabilizers of simple $\mathcal{K}R_1$. It is well-known that derived equivalences also preserve defects of simple modules.

In section 3 we restrict to the case of group algebras and we show that an equivariant splendid derived equivalence between the blocks b and c of two groups K and K' having

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the same Brauer category induces, by applying the Brauer functor to the tilting complex, another equivariant derived equivalence between suitable blocks $kC_K(Q)$ and $kC_{K'}(Q)$, for any subgroup Q of P.

As a consequence of this discussion, we conclude in Section 4 that an "invariant" form of Broué's Conjecture implies Dade's Invariant Conjecture in the case of blocks with abelian cyclic defect groups. The invariant conjecture was checked in many cases, one of them being that of blocks with cyclic defect groups in [D4]. We shall verify that Rouquier's construction of a splendid derived equivalence between $\mathcal{O}Kb$ and $\mathcal{O}N_K(P)c$ actually yields an equivariant derived equivalence.

Let me just mention that the "equivariance" of a derived equivalence is a weaker condition than the "gradedness" considered in [M1] and [M2], and we expect that graded derived equivalences preserve the invariants involved in the stronger forms of Dade's conjecture.

Concerning the terminology, rings will always be associative with unit element and modules are unitary, finitely generated and left, unless otherwise specified. The main reference for modular representation theory is [T]. We also refer to [Br], [L1] and [L2] for general facts on various types of equivalences between blocks, and to [D1] and [D2] for Clifford theory and representations of strongly graded algebras.

2. Equivariant derived equivalences

2.1. We fix a complete discrete valuation ring $\mathcal{O}$ with algebraically closed residue field $k = \mathcal{O}/J(\mathcal{O})$ and quotient field of characteristic 0. We shall assume that all $\mathcal{O}$-modules are free of finite rank.

Fix also a finite group G and let $R = \bigoplus_{g\in G} R_g$ and $S = \bigoplus_{g\in G} S_g$ be two strongly G-graded $\mathcal{O}$-algebras. We assume that R and S are symmetric $\mathcal{O}$-algebras, such that the symmetrizing forms of R and S are G-invariant symmetrizing forms for R_1 and S_1 respectively.

We denote $\mathcal{K}R = \mathcal{K}\otimes_{\mathcal{O}} R$, $kR = k\otimes_{\mathcal{O}} R$ and we assume that $\mathcal{K}R$ and $\mathcal{K}S$ (or equivalently $\mathcal{K}R_1$ and $\mathcal{K}S_1$) are semisimple $\mathcal{K}$-algebras, and that $\mathcal{K}$ is a splitting field for all the $\mathcal{K}$-subalgebras of $\mathcal{K}R$ and $\mathcal{K}S$.

By an (R,S)-bimodule we mean a module over $R\otimes_{\mathcal{O}} S^{op}$. Observe that $R\otimes_{\mathcal{O}} S^{op}$ is a strongly $G\times G$-graded $\mathcal{O}$-algebra and it has a strongly G-graded subalgebra

$$\Delta = \Delta(R,S) = \bigoplus_{g\in G} R_g \otimes_{\mathcal{O}} S_g^{op},$$

where $S_1^{op} = S_{g^{-1}}$. Remark that and (R_1,S_1)-bimodule is the same as a Δ_1-module. Similarly, the enveloping algebras $R^{en} = R\otimes_{\mathcal{O}} R^{op}$ and $S^{en} = S\otimes_{\mathcal{O}} S^{op}$ are $G\times G$-graded $\mathcal{O}$-algebras.

2.2. Let $\mathcal{D}^b(\mathcal{K}R_1)$ be the bounded derived category of the category $\mathcal{K}R_1$-mod. Since for any $g \in G$, the functor

$$\mathcal{K}R_g \otimes_{\mathcal{K}R_1} -\colon \mathcal{K}R_1\text{-mod} \to \mathcal{K}R_1\text{-mod}$$

is an equivalence with inverse $\mathcal{K}R_{g^{-1}} \otimes_{\mathcal{K}R_1} -$, we have that G acts on R_1-mod, on $\mathcal{D}^b(\mathcal{K}R_1)$ and also on the corresponding Grothendieck groups $\mathcal{R}(\mathcal{K}R_1)$ and $\mathcal{R}(\mathcal{D}^b(\mathcal{K}R_1))$. By [H, III, Lemma 1.2], the canonical embedding of $\mathcal{K}R_1$-mod into $\mathcal{D}^b(\mathcal{K}R_1)$ induces an isomorphism

$$\mathcal{R}(\mathcal{K}R_1) \simeq \mathcal{R}(\mathcal{D}^b(\mathcal{K}R_1)),$$

which is clearly a G-isomorphism. We identify these two groups and we denote by $[X] \in \mathcal{R}(\mathcal{K}R_1)$ the class of an object $X \in \mathcal{R}(\mathcal{K}R_1)$. By definition, the stabilizer of X is the subgroup

$$G_X = \{g \in G \mid \mathcal{K}R_g \otimes_{\mathcal{K}R_1} X \simeq X \text{ in } \mathcal{D}^b(\mathcal{K}R_1)\}$$

of G. The group $\mathcal{R}(\mathcal{K}R_1)$ is endowed with a scalar product defined by

$$\langle [X], [X'] \rangle = \sum_i \dim_{\mathcal{K}} \operatorname{Hom}_{\mathcal{D}^b(\mathcal{K}R_1)}(X[i], X').$$

This scalar product is clearly G-invariant, and the G-set $\operatorname{Irr}(\mathcal{K}R_1)$ of isomorphism classes of simple $\mathcal{K}R_1$-modules is an orthonormal $\mathbb{Z}$-basis of $\mathcal{R}(\mathcal{K}R_1)$.

2.3. Let further $\mathcal{D}^b(kR_1)$ be the bounded derived category of kR_1-mod and let $\mathcal{D}^b_{\text{perf}}(kR_1)$ be the full subcategory of $\mathcal{D}^b(kR_1)$ consisting of perfect complexes (that is, complexes isomorphic to complexes of projective kR_1-modules), and denote by kR_1-proj the subcategory of finitely generated projective kR_1-modules.

Again, G acts on these categories, since for any $g \in G$ the functor

$$kR_g \otimes_{kR_1} -\colon kR_1\text{-mod} \to kR_1\text{-mod}$$

is an autoequivalence. Consequently, the canonical embedding induces a G-isomorphism between the Grothendieck groups

$$\mathcal{R}(kR_1) \simeq \mathcal{R}(\mathcal{D}^b(kR_1))$$

and

$$\mathcal{R}^{pr}(kR_1) \simeq \mathcal{R}(\mathcal{D}^b_{\text{perf}}(kR_1)).$$

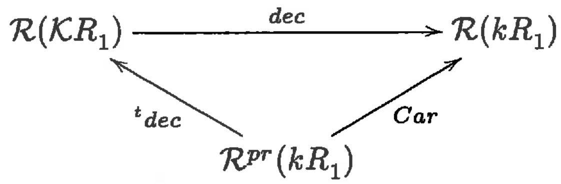
The G-set $\operatorname{Irr}(kR_1)$ of isomorphism classes of simple kR_1-modules is a $\mathbb{Z}$-basis of $\mathcal{R}(kR_1)$, while the G-set $\operatorname{Pim}(kR_1)$ of isomorphism classes of projective indecomposable kR_1-module is a $\mathbb{Z}$-basis of $\mathcal{R}^{pr}(kR_1)$.

There is a G-invariant duality between $\mathcal{R}(kR_1)$ and $\mathcal{R}^{pr}(kR_1)$ defined by

$$\langle [P],[X]\rangle = \sum_i (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}^b(kR_1)}(P[i],X),$$

where $P \in \mathcal{D}^b_{\mathrm{perf}}(kR_1)$ and $X \in \mathcal{D}^b(kR_1)$

2.4. The Cartan-Decomposition triangle $\mathcal{T}(R_1)$ of R_1 is the commutative diagram



where the maps are defined as follows.

If P is a perfect complex of kR_1-modules, then the Cartan matrix $Car\colon \mathcal{R}^{pr}(kR_1) \to \mathcal{R}(kR_1)$ sends the class $[P]$ of P to its class in $\mathcal{R}(kR_1)$.

The decomposition map $dec\colon \mathcal{R}(\mathcal{K}R_1) \to \mathcal{R}(kR_1)$ sends the class $[X]$ of a $\mathcal{K}R_1$-module X to the class of $k \otimes_{\mathcal{O}} X_0 \in kR_1$-mod, where X_0 is an R_1-lattice such that $X \simeq \mathcal{K} \otimes_{\mathcal{O}} X_0$.

Reduction modulo $J(\mathcal{O})$ is an isomorphism between $\mathcal{R}^{pr}(R_1)$ and $\mathcal{R}^{pr}(kR_1)$. Using this, the adjoint

$$^t dec\colon \mathcal{R}^{pr}(R_1) \to \mathcal{R}(\mathcal{K}R_1)$$

sends $[P]$ to $[\mathcal{K} \otimes_{\mathcal{O}} P]$.

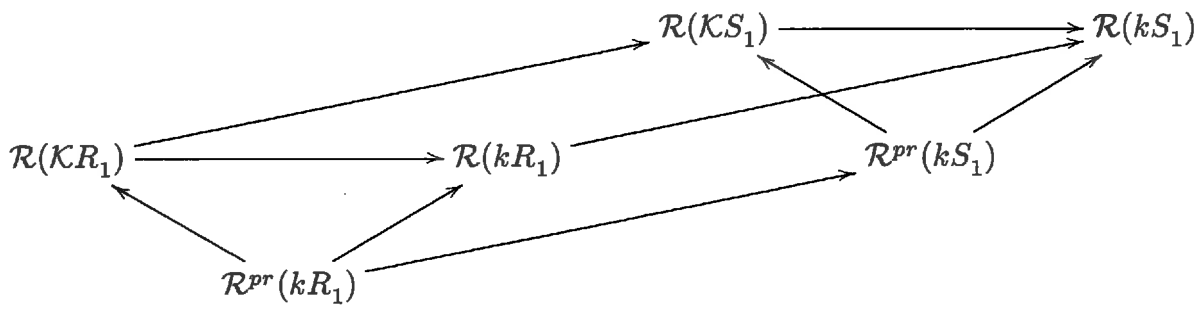
These maps are clearly compatible with the metric structure of $\mathcal{T}(R_1)$, and since $kR_g \otimes_{kR_1} -$, $R_g \otimes_{R_1} -$ and $\mathcal{K}R_g \otimes_{\mathcal{K}R_1} -$ are autoequivalences of kR_1-mod, R_1-mod and $\mathcal{K}R_1$-mod respectively, we have that Car, dec and $^t dec$ are G-maps, that is, $Car\,{}^g[P] = {}^g Car[P]$, $dec({}^g[X])$ and $^t dec\,{}^g[X] = {}^g({}^t dec[X])$ for any $g \in G$.

We shall always consider $\mathcal{T}(R_1)$ endowed with the metric structure and the G-structure.

2.5. Definition. A derived equivalence induced by the complex C of (R_1, S_1)-bimodules is called G-equivariant if C is G-invariant, that is,

$$R_g \otimes_{R_1} C \otimes_{R_1} S_{g^{-1}} \simeq C \quad \text{in } \mathcal{D}^b(\Delta_1).$$

2.6. Theorem. *Assume that the complex C of (R_1, S_1)-bimodules induces an equivariant derived equivalence between R_1 and S_1. Then there is a G-isomorphism between the Cartan-decomposition triangles $\mathcal{T}(R_1)$ and $\mathcal{T}(S_1)$.*



Proof. The inverse equivalence is induced by the complex $\mathrm{Hom}_{R_1}(C, R_1)$ of (S_1, R_1)-bimodules, which is naturally isomorphic to the $\mathcal{O}$-dual C^* of C. We first show that C^* is G-invariant too.

We have that $\mathrm{Hom}_R(R \otimes_{R_1} C, R)$ is a complex of G-graded (S_1, R)-bimodules with

$$\mathrm{Hom}_R(R \otimes_{R_1} C, R)_g \simeq \mathrm{Hom}_{R_1}(R_h \otimes_{R_1} C, R_{hg})$$

as complexes of (S_1, R_1)-bimodules for any $g, h \in G$. Since R is strongly graded, it follows that

$$\mathrm{Hom}_R(R \otimes_{R_1} C, R) \simeq \mathrm{Hom}_{R_1}(C, R_1) \otimes_{R_1} R$$

as complexes of G-graded (S_1, R)-bimodules, and consequently,

$$\mathrm{Hom}_{R_1}(R_g \otimes_{R_1} C, R_1) \simeq \mathrm{Hom}_{R_1}(C, R_1) \otimes_{R_1} R_{g^{-1}}$$

as complexes of (S_1, R_1)-bimodules.

On the other hand, $\mathrm{Hom}_{R_1}(C \otimes_{S_1} S, R_1)$ is a complex of G-graded (S, R_1)-bimodule with

$$\begin{aligned}\mathrm{Hom}_{R_1}(C \otimes_{S_1} S, R_1)_g &= \{f \mid f((C \otimes_{S_1} S_h) = 0 \text{ for } h \neq g^{-1}\} \\ &\simeq \mathrm{Hom}_{R_1}(C \otimes_{S_1} S_{g^{-1}}, R_1).\end{aligned}$$

Since S is strongly graded, we have that

$$\mathrm{Hom}_{R_1}(C \otimes_{S_1} S, R_1) \simeq S \otimes_{S_1} \mathrm{Hom}_{R_1}(C, R_1)$$

$$\mathrm{Hom}_{R_1}(C \otimes_{S_1} S_{g^{-1}}, R_1) \simeq S_g \otimes_{S_1} \mathrm{Hom}_{R_1}(C, R_1)$$

as (S_1, R_1)-bimodules. Finally, we obtain that for any $g \in G$,

$$\mathrm{Hom}_{R_1}(R_g \otimes_{R_1} C \otimes_{S_1} S_{g^{-1}}, R_1) \simeq S_g \otimes_{S_1} \mathrm{Hom}_{R_1}(C, R_1) \otimes_{R_1} R_{g^{-1}},$$

which proves the claim.

If $X \in \mathcal{D}^b(R_1)$, then for any $g \in G$,

$$\begin{aligned}C^* \overset{\mathbf{L}}{\otimes}_{R_1} (R_g \otimes_{R_1} X) &\simeq (S_g \otimes_{S_1} C^* \otimes_{R_1} R_{g^{-1}}) \overset{\mathbf{L}}{\otimes}_{R_1} (R_g \otimes_{R_1} X) \\ &\simeq S_g \otimes_{S_1} (C^* \overset{\mathbf{L}}{\otimes}_{R_1} X).\end{aligned}$$

By [Br, Proposition 4.2], we have that $\mathcal{K} \otimes_{\mathcal{O}} C$ and $\mathcal{K} \otimes_{\mathcal{O}} C^*$ induce a derived equivalence between $\mathcal{K}R_1$ and $\mathcal{K}S_1$, while $k \otimes_{\mathcal{O}} C$ and $k \otimes_{\mathcal{O}} C^*$ induces a derived equivalence between kR_1 and kS_1, and these equivalences induce an isomorphism between the triangles $\mathcal{T}(R_1)$ and $\mathcal{T}(S_1)$.

Since $\mathcal{K} \otimes_{\mathcal{O}} C$ and $k \otimes_{\mathcal{O}} C$ are G-invariant complexes, we see that the above isomorphism between $\mathcal{T}(R_1)$ and $\mathcal{T}(S_1)$ is a G-isomorphism, and since $\mathcal{K}R_1$ and $\mathcal{K}S_1$ are semisimple, we have a G-isomorphism "with signs" between $\mathrm{Irr}(\mathcal{K}R_1)$ and $\mathrm{Irr}(\mathcal{K}S_1)$.

2.7. The group $G \times G$ acts on the category of (R_1, R_1)-bimodules by

$$X \mapsto {}^gX = R_g \otimes_{R_1} X \otimes_{R_1} R_{g^{-1}},$$

and G acts on the set of ideals of R_1 by

$$I \mapsto {}^gI = R_gIR_{g^{-1}}.$$

If B is a block of R_1 and $G_B = \{g \in G \mid R_gBR_{g^{-1}} = B\}$ be the stabilizer of B, then

$$\mathrm{Tr}_{G_B}^G(B) = \sum_{g\in[G/G_B]} {}^gB$$

is a G-invariant bimodule summand of R_1.

Using [Br, Proposition 4.3] and the theorem above we obtain

2.8. Proposition. *Under the assumptions of Theorem 2.6, we have:*

a) $C \otimes_{\mathcal{O}} C^*$ induces a $G \times G$-equivariant derived equivalence between R_1^{en} and S_1^{en}.

b) If B is a block of R_1, then $B' = C^* \otimes_{R_1} B \otimes_{R_1} C$ is a block of S_1, $G_B = G_{B'}$ and BCB' induces a G_B-equivariant derived equivalence between B and B', while C induces a G-equivariant derived equivalence between $Tr_{G_B}^G(B)$ and $Tr_{G_B}^G(B')$

2.9. Remark. As in definition 2.5, one can say that, by definition, the bimodule ${}_{R_1}M_{S_1}$ induce an equivariant stable Morita equivalence between R_1 and S_1 if for any $g \in G$,

$$\Delta_g \otimes_{\Delta_1} M \simeq M$$

in the stable category $\Delta_1\text{-}\overline{\mathrm{mod}}$ of Δ_1-mod. (Recall that our assumptions force that the inverse equivalence is induced by the $\mathcal{O}$-dual M^* of M.)

By adapting the proof of [Ri, Corollary 5.5], one can easily see that if C induces an equivariant derived equivalence between R_1 and C_1, then there is a bimodule M, projective as a left R_1-module and as a right S_1-module, inducing an equivariant stable Morita equivalence between R_1 and S_1.

Indeed, truncating a projective resolution (over Δ_1) of C, we obtain for some degree n a bounded complex

$$C' = (\cdots \to 0 \to Q^{-n} \to P^{-n+1} \to P^{-n+2} \to \dots)$$

isomorphic to C in $\mathcal{D}^b(\Delta_1)$ with P^i projective for $i > -n$, such $M = \Omega^{-n}(Q^{-n})$ induces a stable Morita equivalence between R_1 and S_1. This construction is functorial, and applied to $\Delta_g \otimes_{\Delta_1} C$, we obtain the Δ_1-module $\Delta_g \otimes_{\Delta_1} M$.

The following observation will be needed in Section 4.

2.10. Lemma. *Let H be a normal p'-subgroup of G and C a complex of (R_1, S_1)-bimodules inducing a G-equivariant derived equivalence between R_1 and S_1. Assume also that C extends to Δ_H, and that the isomorphism between $\Delta_g \otimes_{\Delta_1} C$ and C holds in $\mathcal{D}^b(\Delta_H)$.*

If $D = (R_H \otimes_{\mathcal{O}} S_H^{op}) \otimes_{\Delta_H} C$, then D induces a G/H-equivariant derived equivalence between R_H and S_H.

Proof. Since H is a normal subgroup of G, [M2, Lemma 2.9 and Remark 2.10 c)] implies that $\Delta_g \otimes_{\Delta_1} C$ is naturally a Δ_H-module. It follows by [M2, Theorem 4.8] that D induces a derived equivalence between R_H and S_H.

For $g \in G$, we denote ${}^gD = R_{gH} \otimes_{R_H} D \otimes_{S_H} S_{Hg^{-1}}$, and we have to show that

$$ {}^gD \simeq D \text{ in } \mathcal{D}^b(R_H\text{-mod-}S_H). $$

Observe first that ${}^gD \simeq R_g \otimes_{R_1} D \otimes_{S_1} S_{g^{-1}}$ is an H-graded (R_H, S_H)-bimodule, where for $x \in H$,

$$ ({}^gD)_x = R_g \otimes_{R_1} D_{g^{-1}xg} \otimes_{S_1} S_{g^{-1}}. $$

By assumption, there is an isomorphism in $\mathcal{D}^b(\Delta_1)$ between $D_1 = C$ and $({}^gD)_1 \simeq R_g \otimes_{R_1} C \otimes_{S_1} S_{g^{-1}}$, and using [M2, Lemma 2.6] we conclude that ${}^gD \simeq D$.

3. Local structure

We recall from [L1] and [L2] the definition of a splendid equivalence, and we shall adapt it to our situation.

3.1. Let K be a normal subgroup of a finite group H and K' a normal subgroup of H' such that $G = H/K \simeq H'/K'$, and let $\alpha\colon H/K \to H'/K'$ be an isomorphism. It follows that $\mathcal{O}H$ and $\mathcal{O}H'$ are strongly G-graded $\mathcal{O}$-algebras. Let also b be a block idempotent of $\mathcal{O}K$ and c a block idempotent of $\mathcal{O}K'$.

Let H_b and H'_c be the stabilizers of b and c respectively. If $G_b = H_b/K$ and $G_c = H'_c/K'$, then $R := b\mathcal{O}Hb = \mathcal{O}H_b b$ is a strongly G_b-graded algebra, and $S := c\mathcal{O}H'_c = \mathcal{O}H'_c c$ is a strongly G_c-graded algebra.

We assume that b and c have a common defect group $P \leq K$, and that for each subgroup Q of P, α indices an isomorphism

$$ \alpha(Q)\colon C_H(Q)/C_K(Q) \to C_{H'}(Q)/C_{K'}(Q). $$

Denoting $G_Q = C_H(Q)/C_K(Q)$, we have that $kC_H(Q)$ and $kC_{H'}(Q)$ are strongly G_Q-graded k-algebras.

3.2. Let $i \in (\mathcal{O}Kb)^P$ and $j \in (\mathcal{O}K'c)^P$ be primitive idempotents such that $\mathrm{Br}_P^K(i) \neq 0$ and $\mathrm{Br}_P^{K'}(j) \neq 0$, where Br_P is the Brauer map. For every subgroup Q of P there are unique block idempotents $e_Q \in kC_K(Q)$ and $f_Q \in kC_{K'}(Q)$ such that $\mathrm{Br}_P^K(i)e_Q \neq 0$ and $\mathrm{Br}_P^{K'}(j)f_Q \neq 0$.

Define as above the stabilizer

$$G_{(Q,e_Q)} = \{hC_K(Q) \mid h \in C_H(Q),\ {}^he_Q = e_Q\},$$

so $e_QkC_H(Q)e_Q = e_QkC_H(Q)_{G_{(Q,e_Q)}}$ is a strongly G_{Q,e_Q}-graded k-algebra. Similarly, $f_QkC_{H'}(Q)f_Q = f_QkC_{H'}(Q)_{G_{(Q,f_Q)}}$ is a strongly $G_{(Q,f_Q)}$-graded k-algebra.

By definition, for two subgroups Q and R of P, let $E_{H,G}((Q,e_Q),(R,e_R))$ be the set of equivalence classes modulo inner automorphisms of R of group homomorphisms $Q \to R$ of the form $u \mapsto {}^xu = xux^{-1}$ for some $x \in H$, satisfying $xQx^{-1} \subseteq R$ and $xe_Qx^{-1} = e_{xQx^{-1}}$. As in [KP, 2.8], these maps are considered together with the action of H on G by left translation; it follows that homomorphisms induced by $x, y \in H$ such that $xK \neq yK$ are different.

We assume further that $\mathcal{O}Kb$ and $\mathcal{O}K'c$ have *G-equivalent Brauer categories*. We understand by this that for any subgroups Q and R of P, there is an equality

$$E_{H,G}((Q,e_Q),(R,e_R)) = E_{H',G}((Q,f_Q),(R,f_R))$$

which is compatible with the isomorphism $\alpha\colon H/K \to H'/K'$. Compatibility with α means that whenever $x \in H$ induces by conjugation a homomorphism $Q \to R$ such that ${}^xe_Q = e_{{}^xQ}$, there is $x' \in H'$ inducing the same homomorphism, such that $\alpha(xK) = x'K'$.

3.3. The indecomposable (R_1, S_1)-bimodule M is called *splendid*, if it is a direct summand of the bimodule $\mathcal{O}Ki \otimes_{\mathcal{O}P} j\mathcal{O}K'$.

Let $M(Q)$ be the $(kC_K(Q), kC_{K'}(Q)$-bimodule

$$M(Q) = M^Q/(J(\mathcal{O})M^Q + \sum_{R<Q} \mathrm{Tr}_R^Q(M^R)).$$

Then by [L1, Theorem 1.1], the $(kC_K(Q)e_Q, kC_{K'}(Q)f_Q$ bimodule $e_QM(Q)f_Q$ is also splendid.

By definition, a tilting complex of (R_1, S_1)-bimodules is *splendid* if the indecomposable summands of its components X^i are splendid.

3.4. Proposition. *With the above notations, assume that the Brauer categories of b and c are G-equivalent. Then*

a) $G_b = G_c$ and $G_{(Q,e_Q)} = G_{(Q,f_Q)}$ for any subgroup Q of P.

b) If X is a splendid complex of (R_1, S_1)-bimodules inducing a G_b-equivariant derived equivalence between R_1 and S_1, then $e_QX(Q)f_Q$ is a splendid complex inducing a $G_{(Q,e_Q)}$-equivariant derived equivalence between $kC_K(Q)e_Q$ and $kC_{K'}(Q)f_Q$.

Proof. a) By the Frattini argument, (see also [D1, (0.3a)], we have that

$$G_b = N_H(P, e_P)K/K$$

and

$$G_c = N_{H'}(P, f_P)K'/K'.$$

For any $Q \le P$ we have by (3.2) that

$$N_H(Q, e_Q)/QC_K(Q) \simeq E_{H,G}(Q, e_Q) = E_{H',G}(Q, f_Q) \simeq N_{H'}(Q, f_Q)/QC_{K'}(Q).$$

Taking $Q = P$ we obtain that $G_b = G_c$.

Since $\alpha\colon H/K \to H'/K'$ restricts to the isomorphism $\alpha(Q)$, we deduce that $G_{(Q,e_Q)} = G_{(Q,f_Q)}$ for any Q.

b) The fact that the complex $e_Q X(Q) f_Q$ induces a splendid derived equivalence between $kC_K(Q)e_Q$ and $kC_{K'}(Q)f_Q$ follows from [L1, Theorem 1.1]. We have to show that $e_Q X(Q) f_Q$ is $G_{(Q,e_Q)}$-invariant.

Indeed, let $h \in C_H(Q)$ and $h' \in C_{H'}(Q)$ such that ${}^h e_Q = e_Q$, ${}^{h'} f_Q = f_Q$, and $\alpha(hK) = h'K'$. Then we have

$$\begin{aligned} hC_K(Q)e_Q &\otimes_{kC_K(Q)e_Q} e_Q X(Q) f_Q \otimes_{kC_{K'}(Q)f_Q} kC_{K'}(Q) f_Q h'^{-1} \\ &\simeq e_Q(h \otimes_k X(Q) \otimes_k h'^{-1}) f_Q \\ &\simeq e_Q(h \otimes_k X \otimes_k h'^{-1})(Q) f_Q \\ &\simeq e_Q X(Q) f_Q. \end{aligned}$$

4. Conjectures for abelian and cyclic defect groups

4.1. Let H, K, G, b and P be as in the preceding section and let $H' = N_H(P)$, $K' = N_K(P)$, $G' = H'/K'$ and c be the Brauer correspondent of b. Let also $\alpha\colon G' \to G$ be the map induced by the inclusion of H' in H.

As we already have seen, we have

$$G_b = N_H(P, e_P)K/K \simeq N_{H'}(P, e_P)K'/K' = G_c.$$

Moreover, if P is abelian, then by slightly adapting the proof of [T, Proposition 4.9.6], one obtains that the Brauer categories of $\mathcal{O}Kb$ and $\mathcal{O}Kb'$ are G_b-equivalent.

Therefore, it makes sense to formulate the following equivariant version of Broué's conjecture:

4.1.1. *If b is a block of $\mathcal{O}K$ with abelian defect group P, then there is a G_b-equivariant splendid derived equivalence between $\mathcal{O}Kb$ and $\mathcal{O}N_K(P)c$, where c is the Brauer correspondent of b.*

4.2. Denote by

$$C: \qquad (P_0 < P_1 < \cdots < P_n)$$

a p-chain of K and by $|C| = n$ the *length* of C. The groups K and H act on the set of p-chains of K, and let $N_K(C)$ and $N_H(C)$ be the normalizers of C in K and H respectively. Denote also $N_G(C) = N_H(C)/N_K(C)$. We have that

$$C_K(P_n) \leq N_K(C) \leq N_K(P_n),$$

and if β is a block of $\mathcal{O}N_K(C)$, then by [KR, Lemma 3.2], the induced block β^K of $\mathcal{O}K$ is defined.

The group $N_G(C)$ acts on the set $\mathrm{Irr}(\mathcal{K}N_K(C))$ of simple $\mathcal{K}N_K(C)$-modules, so if $\chi \in \mathrm{Irr}(\mathcal{K}N_K(C))$, then the stabilizer $N_G(C,\chi)$ is defined. Denote further by $\mathrm{def}(\chi)$ the *defect* of χ and by $\beta(\chi)$ the block of $\mathcal{O}N_K(C)$ to which χ belongs.

If $d \leq 0$ is an integer and F is a subgroup of $N_G(C)$, then denote by $\mathrm{k}(C,b,d,F)$ the number of characters $\chi \in \mathrm{Irr}(\mathcal{K}N_K(C))$ satisfying

$$\mathrm{def}(\chi) = d, \qquad \beta(\chi)^K = b \qquad \text{and} \qquad N_G(C,\chi) = F,$$

and by $\mathrm{k}(\beta,d,F)$ the number of characters $\chi \in \mathrm{Irr}(\mathcal{K}\beta)$ satisfying

$$\mathrm{def}(\chi) = d \qquad \text{and} \qquad N_G(\beta,\chi) = F.$$

We have that $\mathrm{k}(C,b,d,F)$ depends only on the K-conjugacy class of C, and Dade's Invariant Conjecture [D3, 2.5] states that

4.2.1 If $O_p(K) = 1$ and $\mathrm{def}(b) > 0$ then

$$\sum_{C\in\mathcal{F}/K} (-1)^{|G|}\mathrm{k}(C,b,d,F) = 0,$$

where $\mathcal{F}$ is one of the families $\mathcal{P}$, $\mathcal{U}$, $\mathcal{N}$ or $\mathcal{E}$ of p-chains of K introduced in [KR].

The argument of [KR, Proposition 5.5] now gives

4.3. Proposition. *Conjecture* (4.1.1) *implies conjecture* (4.2.1).

Proof. Let b be a block of $\mathcal{O}K$ with abelian defect group $P > 1$, let $= (P_0 < \cdots < P_n)$ be an element of $\mathcal{N}$ and let β be a block of $\mathcal{O}N_G(C)$ inducing b.

If P_n is a defect group of β , then, denoting $C' = (P_0 < \cdots < P_{n-1})$, we have that $\beta^{N_K(C')}$ is defined, it has defect group P_n, and $(\beta^{N_K(C')})^K = b$.

Then by hypothesis, there is an $N_G(C)_\beta$-equivariant derived equivalence between β and $\beta^{N_K(C')}$. By [Br, Proposition 4.5], defects of irreducible characters are preserved. It follows that

$$\mathrm{k}(\beta,d,F) = \mathrm{k}(\beta^{N_K(C')},d,F).$$

If P_n is not a defect group of β, then the defect group P_{n+1} of β satisfies $P_n < P_{n+1}$, and let $C' = (1 < P_1 < \cdots < P_n < P_{n+1}) \in \mathcal{N}$. Then $N_K(C') = N_{N_K(C)}(P_{n+1})$ and there is a unique block β' of $N_K(C')$ inducing β. Again we have that

$$\mathrm{k}(\beta', d, F) = \mathrm{k}(\beta, d, F).$$

Consequently, (4.2.1) holds.

4.4. Assume from now on that P is cyclic and b is G-invariant. R. Rouquier proved in [Rou] that there is a splendid derived equivalence between $\mathcal{O}Kb$ and $\mathcal{O}K'c$. This result was a consequence of a series of derived equivalences, and in the remaining part of this paper we show that all those equivalences are equivariant. This verification, together with the fact that the composition of equivariant derived equivalences is equivariant too, will imply that Rouquier's inductive proof can be used to conclude that (4.1.1.) holds for blocks with cyclic defect groups.

We denote $R = O_p(K)$, and (under the hypothesis that P is non-normal, let Q be the subgroup of P such that $[Q : R] = p$. Let also $I = N_H(P, e_P)/C_K(P)$, $L = N_H(Q)$, $I_1 = N_K(P, e_P)/C_K(P)$, $L_1 = N_K(Q)$, and let c_1 be the block of $\mathcal{O}L_1$ corresponding to b.

We have that Q is a weakly closed subgroup of K with respect to H, and by the results of [D4, Section3] we have that c_1 is L-invariant, hence $\mathcal{O}Lc_1$ is a strongly G-graded algebra. Recall also that $I/I_1 \simeq G$.

4.5. Consider the block algebra $\mathcal{O}K'c$. Then we are in the situation of [Rou, Proposition 2.15]: c is a block of $C_K(P)$ and it has a unique simple module V. Starting with V, one constructs an $(\mathcal{O}C_K(P)c, \mathcal{O}P)$-bimodule M_1 inducing a Morita equivalence between $\mathcal{O}C_K(P)c$ and $\mathcal{O}P$. Further, the I-graded $\mathcal{O}H'c$-module $\mathcal{O}H'c \otimes_{\mathcal{O}C_K(P)c} M_1$ induces a graded Morita equivalence between $\mathcal{O}H'c$ and an I-graded crossed product of $\mathcal{O}P$ and I. By the proof of the same [Rou, Proposition 2.15], $\mathcal{O}K'c \otimes_{\mathcal{O}C_K(P)c} M_1$ induces an I_1-graded Morita equivalence between $\mathcal{O}K'c$ and $\mathcal{O}P \rtimes I_1$ (this amounts to the extendibility of M_1 to a certain "diagonal" subalgebra), which is then G-invariant (since $G \simeq I/I_1$).

4.6. By [Rou, Lemma 4.2], if $R = O_p(K) = 1$, then the bimodule $b\mathcal{O}Kc_1$ induces a Morita stable equivalence between $\mathcal{O}Kb$ and $\mathcal{O}L_1c_1$. Since $K \trianglelefteq H$ and b, c_1 are G-invariant, we have that $b\mathcal{O}Kc_1$ is a G-invariant bimodule. Moreover, its unique nonprojective bimodule summand M_1 is G-invariant too.

4.7. We show that if M is a G-invariant $(\mathcal{O}Kb, \mathcal{O}(P \rtimes I_1)$-bimodule inducing a stable Morita equivalence between $\mathcal{O}Kb$ and $\mathcal{O}(P \rtimes I_1)$, then the construction of [Rou, Section 3] provides an equivariant derived equivalence between $\mathcal{O}Kb$ and $\mathcal{O}(P \rtimes I_1)$. Indeed, let $\epsilon = |I_1|$ be the inertial index of b, $S = M \otimes_{\mathcal{O}(P\rtimes I_1)} \mathcal{O}$, and denote by Ω the Heller operator. Let v_i be the vertex of the Brauer tree corresponding to the character of $\Omega^i S$ and let l_I be the edge connecting v_i and v_{i+1}. Then

$$\bigoplus_{0\le i\le \epsilon-1} (P_{\Omega^{2i}S} \otimes_{\mathcal{O}} P^*_{\Omega^{2i}\mathcal{O}})$$

is a projective cover of M (as a bimodule). We have that S, $\Omega^{2i}S$ and $P_{\Omega^{2i}S}$ are G-invariant $\mathcal{O}Kb$-modules. One can easily see that $P_{\Omega^{2i}S} \otimes P^*_{\Omega^{2i}\mathcal{O}}$ is a G-invariant bimodule, and it follows that the tilting complex C defined in [Rou, 3.4] is G-invariant.

4.8. It remains to examine the case when $R = O_P(K)$ is not trivial.

Assume first that $1 \neq R \le Z(K)$, and denote by "$^-$" the canonical projection from K to $\bar{K} = K/R$ and from L_1 to $\bar{L}_1 = L_1/R$.

By the inductive assumption, there is an equivariant derived equivalence between $\mathcal{O}\bar{K}\bar{b}$ and $\mathcal{O}\bar{L}_1\bar{c}_1$. then the tilting complex of $(\mathcal{O}\bar{K}\bar{b}, \mathcal{O}\bar{L}_1\bar{c}_1)$-bimodules constructed in [Rou, 4.2.1] is clearly G-invariant.

Assume finally that $1 \neq R \not\le Z(K)$, and we again show that the construction of [Rou, 4.2.2] gives a G-invariant tilting complex of $(\mathcal{O}Kb, \mathcal{O}L_1c_1)$-bimodules.

We have that $\mathcal{O}Kb$ and $\mathcal{O}L_1c_1$ are I_1-graded algebras with 1-components $\mathcal{O}C_K(R)b$ and $\mathcal{O}C_{L_1}(R)c_1$ respectively, where $\mathcal{O}C_{L_1}(R)c_1$ is the Brauer correspondent of the block $\mathcal{O}C_K(R)b$.

Since b and c_1 are G-invariant, we have that $N_{H\times L}(\delta(R))$ acts by conjugation on $b\mathcal{O}C_K(R)c_1$ (where $\delta(R) = \{(u,v) \mid u \in R\}$. Regarded as an $\mathcal{O}(C_K(R)\times C_{L_1}(R))/\delta(R)$-module, $b\mathcal{O}C_K(R)c_1$ has an indecomposable direct summand M with vertex $\delta(P/R)$, which extends to an $\mathcal{O}N_{K\times L_1}(R)/\delta(R)$-module. Then $\mathrm{Ind}^{K\times L_1/\delta(R)}_{N_{K\times L_1}(R)/\delta(R)} M$ is an I_1-graded $(\mathcal{O}Kb, \mathcal{O}L_1c_1)$-bimodule. Inspecting further Rouquier's arguments and using Lemma 2.10, we see that it is enough to show that the $\mathcal{O}N_{K\times L_1}(R)/\delta(R)$-module M is I-invariant.

This holds, since I_1 is a p'-group, so M is an $\mathcal{O}N_{K\times L_1}(R)/\delta(R)$-summand of $b\mathcal{O}C_K(R)c_1$ and it still has vertex $\delta(P/R)$, while the $\mathcal{O}(C_K(R)\times C_{L_1}(R))/\delta(R)$-summands of $(b\mathcal{O}C_K(R)c_1)/M$ have smaller vertices.

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