Fubini-type theorems for general measure constructions

by

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Abstract. We use methods from descriptive set theory to derive Fubini-like results for the very general Method I and Method II (outer) measure constructions. Such constructions, which often lead to non-σ-finite measures, include Carathéodory and Hausdorff-type measures. We encounter several questions of independent interest, such as the measurability of measures of sections of sets, the decomposition of sets into subsets with good sectional properties, and the analyticity of certain operators over sets. We indicate applications to Hausdorff and generalised Hausdorff measures and to packing dimensions.

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1. Method I and Method II measures

Throughout this paper (X, d) and (T, ρ) will be Polish spaces, that is, complete separable metric spaces. If $B \subset T \times X$, and $t \in T$, then by B_t, the *t-section* or *fiber* of B, we mean the set $\{x \in X : (t, x) \in B\}$. It is sometimes convenient to regard B_t as the set $\{t\} \times B_t$; it should be clear from the context which interpretation is intended.

Let τ be a nonnegative set function defined on the subsets of X, such that $\tau(\emptyset) = 0$, and $\tau(E) = \infty$ if $\overline{E}$ is not compact. Let ν be a complete Borel probability measure on T. This means that Σ, the σ-algebra of subsets on which ν is defined, includes the Borel subsets of T and if $N \subset M$ and $\nu(M) = 0$, then $N \in \Sigma$. Since ν is complete, every analytic subset of T is ν-measurable and consequently every set in $\mathcal{BA}(T)$, the σ-algebra generated by the analytic subsets of T, is ν-measurable.

We first consider the Method I measure induced by τ and then the Method II measure, see Rogers [Ro] for a general treatment of such measures. Method I measures will be denoted by an asterisk, *. Thus τ^*, the usual Method I measure induced by τ, is defined by setting

$$\tau^*(E) = \inf_{E \subset \cup E_i} \sum \tau(E_i),$$

for $E \subset X$, where, as always, $\{E_i\}$ is a countable cover. (Often Method I measures are defined in terms of coverings by sets E_i from a restricted class of sets $\mathcal{C}$. However, the same definition of τ^* may be achieved by setting $\tau(E) = \infty$ for all $E \notin \mathcal{C}$, and this allows the convenience of having $\tau(E)$ defined for all E.)

We define the set function μ on $T \times X$ by setting

$$\mu(B) = \int_T^* \tau(B_t) d\nu(t), \tag{1}$$

where $\int^*$ denotes the upper integral. Let μ^* be the outer measure on $T \times X$ constructed from μ by Method I. Thus, for $B \subset T \times X$,

$$\mu^*(B) = \inf_{B \subset \cup B_i} \sum \mu(B_i). \tag{2}$$

A major aim of this paper is to establish conditions that enable $\mu^*(B)$ to be expressed as an integral of the sectional measures $\tau^*(B_t)$ with respect to ν, that is to obtain identities such as

$$\mu^*(B) = \int_T \tau(B_t) d\nu(t).$$

for certain sets B. There a basic inequality relating these set functions:

Lemma 1. Let $B \subset T \times X$. Then

$$\int_T^* \tau^*(B_t) d\nu(t) \leq \mu^*(B). \tag{3}$$

Proof.

$$\int^* \tau^*(B_t)d\nu(t) = \int^* \left(\inf_{B_t \subset \cup E_i^t} \sum \tau(E_i^t) \right) d\nu(t)$$

$$= \inf_{B \subset \cup B_i} \int^* \sum \tau((B_i)_t) d\nu(t) \le \inf_{B \subset \cup B_i} \sum_i \int^* \tau((B_i)_t) d\nu(t)$$

$$= \inf_{B \subset \cup B_i} \sum \mu(B_i) = \mu^*(B),$$

where we take the cover of B defined by $B_i = \cup_{t \in T}\{t\} \times E_i^t$. ■

We seek conditions for equality in (3). We note that for a given set $B \subset T \times X$ inequality (3) becomes an equality provided for each $\epsilon > 0$, there is a sequence of sets $\{B_i\}_{i=1}^{\infty}$ such that (i) $B \subset \cup B_i$, (ii) for each i, $t \mapsto \tau((B_i)_t)$ is ν-measurable, and (iii) for ν a.e. t, $\sum \tau((B_i)_t) \le \tau^*(B_t) + \epsilon$. These are very general conditions and it is desirable to have some more easily-checked conditions on τ that lead to equality in (3) for a reasonably large class of sets B. Thus we list below various verifable conditions on τ. We denote the space of compact subsets of X endowed with the topology inherited from the Hausdorff metric by $K(X)$.

(C1) τ is monotone,

(C2) $\tau(E) = \tau(\overline{E})$,

(C3) for each closed set F, $\tau(F) = \sup\{\tau(K) : K \in K(X) \text{ and } K \subset F\}$,

(C4) $K \mapsto \tau(K)$ is a Borel measurable map on the space $K(X)$,

(C5) For each compact set K, $\tau(K) = \inf\{\tau(V) : V \text{ is open}, \ K \subset V\}$.

The best known examples of Method I measures are the pre-Hausdorff measures. Fixing $X = \mathbf{R}^n$ and $s, \delta > 0$, we define, for $E \subset X$, $\tau(E) = |E|^s$ if $|E| \le \delta$ and $\tau(E) = \infty$ if $|E| > \delta$, where $|E|$ denotes the diameter of E. Thus only sets E with $|E| \le \delta$ provide useful covering sets. (Later on we will consider Method II which takes the limit as $\delta \to 0$ to give Hausdorff measures.) This example may be generalised, by taking an outer measure λ on X and $s, q > 0$, and setting, for $E \subset X$, $\tau(E) = |E|^s \lambda(E)^q$ if $|E| \le \delta$ and $\tau(E) = \infty$ if $|E| > \delta$. In this case τ will satisfy (C1) and (C3) if λ is regular and (C4) and (C5) if λ is outer regular.

Theorem 2. Suppose τ satisfies conditions (C2) and (C4). Let A be an analytic subset of $T \times X$ such that $\overline{A_t}$ is compact for each $t \in T$. Then the map $t \mapsto \tau(A_t)$ is ν-measurable. Indeed, this map is measurable with respect to the σ-algebra $\mathcal{BA}(T)$ of subsets of T generated by the analytic subsets of T.

Proof. Let G be the sectionwise closure of A. Thus $(t, x) \in G$ if and only if there is some sequence $\{x_n\}_{n=1}^{\infty}$ with $\{x_n\}$ converging to x and $(t, x_n) \in A$ for all n. Since $G = \pi_{T \times X}(\{(t, x, x_1, x_2, x_3, ...) \in T \times X \times X^{\mathbf{N}} : \forall n (t, x_n) \in A \text{ and } x_n \to x\})$ where π denotes projection, the set G is the projection of an analytic set and so is analytic. We check that the map $g : T \mapsto K(X)$ given by $g(t) = G_t$ is $\mathcal{BA}(T)$−measurable. Fix a nonempty open subset U of X. Let $I(U) = \{K \in K(X) : K \cap U \neq \emptyset\}$ and let $C(U) = \{K \in K(X) : K \subset U\}$. Then $g^{-1}(I(U)) = \{t : G_t \cap U \neq \emptyset\} = \pi_T((T \times U) \cap G)$ is an analytic set and $g^{-1}(C(U)) = T \setminus \{t : G_t \cap X \setminus U \neq \emptyset\}$ is a coanalytic set. Since the sets of the form $I(U)$ and $C(U)$ form a subbasis for the topology of $K(X)$, g is a $\mathcal{BA}(T)$-measurable function. Finally, $t \mapsto \tau(A_t) = \tau(G_t)$ is $\mathcal{BA}(T)$-measurable, since it is the composition of g with a Borel measurable map. ■

Theorem 3. Suppose X is locally compact and τ satisfies conditions (C1), (C2), (C3) and (C4). Let B be an analytic subset of $T \times X$, then the map $t \mapsto \tau(B_t)$ is $\mathcal{BA}(T)$-measurable.

Proof. Let G be the sectionwise closure of B. Let $\{U_n\}$ be an ascending sequence of open subsets of X such that $\overline{U_n}$ is compact for each n, and $\cup U_n = X$. For each n, let $f_n(t) = \tau(G_t \cap \overline{U_n})$ for $t \in T$. By Theorem 2, f_n is $\mathcal{BA}(T)$-measurable for each n. Note that by property (C1) and the local compactness of X, for each $t, f(t) := \lim_{n\to\infty} f_n(t) = \sup\{\tau(K) : K \in K(X) \text{ and } K \subset G_t\}$. By properties (C2) and (C3), $f(t) = \tau(G_t) = \tau(B_t)$. Since f is $\mathcal{BA}(T)$-measurable, the proof is finished. ■

We use the following theorem of Saint Raymond [Ra] in several places. Let T and X be complete separable metric spaces and let B be a Borel subset of $T \times X$ such that for each $t \in T$, the t-section of B, B_t, is σ-compact. Then $\pi_T(B)$ is a Borel set, and there exist Borel sets $B_n \subset T \times X$ such that $B = \cup_n B_n$, and $(B_n)_t$ is compact for each t.

Theorem 4. Let X be locally compact. Let B be a Borel subset of $T \times X$ such that each t-section of B is σ-compact. Let

$$F = F(B) = \{(t, (K_n)) \in T \times K(X)^{\mathbf{N}} : \cup \mathrm{int} K_n \supset B_t\}.$$

Then F is a Borel set.

Proof. Notice $T \times K(X)^{\mathbf{N}} \setminus F = \pi_{T\times K(X)^{\mathbf{N}}}(H)$, where $H = \{(t, (K_n), x) \in T \times K(X)^{\mathbf{N}} \times X : (t, x) \in B \text{ and } \forall n,\ x \notin \mathrm{int} K_n\})$. Thus, H is a Borel subset of $T \times K(X)^{\mathbf{N}} \times X$ Also, for each $(t, (K_n))$, the section $H_{(t,(K_n))} = B_t \setminus \cup \mathrm{int} K_n$ is σ-compact. So, by Saint Raymond's theorem, F is a Borel set. ■

We recall that a map $f : D \mapsto K(X)$, where D is a Borel subset of T is Borel measurable if and only if the graph of f, $\mathrm{Gr}(f) = \{(t, x) : x \in f(t)\}$, is a Borel set in $T \times X$. This fact also follows easily from Saint Raymond's theorem.

Theorem 5. Let X be locally compact and let τ satisfy conditions (C1)-(C5). Let B be a Borel subset of $T \times X$ such that each t-section of B is σ-compact. Then the map $t \mapsto \tau^*(B_t)$ is $\mathcal{BA}(T)$-measurable. Moreover, for each $\epsilon > 0$, there are Borel sets $B_i \subset T \times X$, $i = 1, 2, 3, \ldots$, with compact sections, and a Borel set $N \subset \pi_T(B)$ with $\nu(N) = 0$, such that if $t \in T \setminus N$, then $B_t \subset \cup_i (B_i)_t$ and $\Sigma_i \tau((B_i)_t) \leq \tau^*(B_t) + \epsilon$.

Proof. Since the theorem is trivially true if $\nu(\pi_T(B)) = 0$, we may assume the projection $\pi_T(B)$ has positive measure. It follows from assumption (C4) that the map $f : K(X)^{\mathbf{N}} \mapsto \mathbf{R}$ defined by $f((K_n)) = \Sigma \tau(K_n)$ is Borel measurable. It also follows from (C5) and the local compactness of X that for each $t \in T, g(t) = \tau^*(B_t) = \inf\{f((K_n)) : (t, (K_n)) \in F(B)\}$, where $F = F(B)$ is defined in Theorem 4. If q is a positive rational or ∞, $F \cap (f \leq$

$q)$ is a Borel set, where $(f \le q) = \{(t, (K_n)) : \sum \tau(K_n) \le q\}$. Therefore, by the Jankov-von Neumann Theorem [K], there is a function $s_q : D_q \to K(X)^{\mathbb{N}}$ where $D_q \equiv \pi_T(F \cap (f \le q))$ is analytic, such that s_q is a $\mathcal{BA}(T)$ measurable selector for $F \cap (f \le q)$. Let s_{qi} be the i-th coordinate function of s_q. Noting that $g(t) = \inf\{q : t \in D_q \text{ and } q \text{ is rational}\}$, it follows that g is $\mathcal{BA}(T)$-measurable. Next, fix $\epsilon > 0$ and enumerate the rationals as $\{q_n\}$. For each p, let $A_p = \{t : q_p \text{ is the first rational with } s_{q_p}(t) \le g(t) + \epsilon\}$. We may find a set $N \subset \pi_T(B)$ such that $\nu(N) = 0$ and $s_{q_p}|A_p \setminus N$ is Borel measurable for each p. Since for each p and i the map $s_{q_p i}$ is a Borel measurable map of $A_p \setminus N$ into $K(X)$, the graph $B_{pi} = \{(t, x) : x \in s_{q_p i}(t)\}$ is a Borel set. For each i, setting $B_i = \cup_{p=1}^{\infty} B_{pi}$ gives sets with the required properties. ■

Doubtless these theorems or variants hold under more general conditions on X or by relaxing some of the conditions on τ. For example, let X be σ-compact, let B be a Borel subset of $T \times X$, and suppose τ satisfies conditions (C1)-(C5). Is it true that the map $t \mapsto \tau^*(B_t)$ is $\mathcal{BA}(T)$-measurable or universally measurable? Does the second part of Theorem 5 hold for B? What is the situation if τ only satisfies conditions (C1)-(C4)?

Theorem 6. Let X be locally compact and let τ satisfy conditions (C1)-(C5). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. Then we have equality in (3):

$$\mu^*(B) = \int_T \tau^*(B_t) d\nu(t). \tag{4}$$

Proof. Let $\epsilon > 0$. Let B_i be the Borel sets and N the ν-null set given by Theorem 5. By Theorem 2, $t \mapsto \tau((B_i)_t)$ is ν-measurable, so by definition

$$\mu^*(B) \le \sum_i \mu(B_i) = \sum_i \int \tau((B_i)_t) d\nu(t) \le \int \tau^*(B_t) d\nu(t) + \epsilon,$$

where we have interchanged summation and integration and used the final conclusion of Theorem 5. Taking ϵ arbitrarily small and combining with Lemma 1 gives the result. ■

Equation (4) specialises to the following product fromula for Borel rectangles $U \times E$:

$$\mu^*(U \times E) = \nu(U)\tau^*(E).$$

Thus, the outer measure μ^* on $T \times X$, defined in terms of μ using Method I, may be regarded as a product measure of ν and τ^*. However, for τ^* non-σ-finite (as occurs in many applications) a product measure is generally far from uniquely defined by the product formula on rectangles. It also follows from (4) that a Borel set $B \subset T \times X$ is μ^*-measurable if and only if $B_t \subset X$ is τ^*-measurable for ν-almost all t, see Rogers [Ro, Chapter 1.2].

The following refinement of Theorem 6, which restricts covering sets to Borel rectangles, is required for the Method II results which follow.

Theorem 7. Let X be locally compact and let τ satisfy conditions (C1)-(C5). Let $B \subset T \times X$ be a Borel set such that each t-section B_t is σ-compact. Then

$$\mu^*(B) = \inf\left\{ \sum_i \mu(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles} \right\}. \tag{5}$$

Proof. Let $\epsilon > 0$. By Theorem 5, there are a Borel set N with $\nu(N) = 0$ and a sequence of Borel sets G_i, such that for $t \in T \setminus N$, each $(G_i)_t$ is compact, $B_t \subset \cup_i (G_i)_t$ and $\sum_i \tau((G_i)_t) \leq \tau^*(B_t) + \epsilon$. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of sets forming a base for the topology of X such that $\overline{U_n}$ is compact for each n, and with this sequence closed under finite unions. For each i, n, let $T_{in} = \{t \in T \setminus N : n$ is the first integer such that $U_n \supset (G_i)_t$ and $\tau(\overline{U_n}) < \tau((G_i)_t) + \epsilon\}$. Then $R_{in} = T_{in} \times \overline{U_n}$ is a Borel rectangle. By Theorem 6,

$$\begin{aligned}
\mu^*(B) &= \int \tau^*(B_t) d\nu(t) \geq \int (\sum_i \tau((G_i)_t) - \epsilon) d\nu(t) \\
&= \sum_i \sum_n \int_{T_{in}} \tau((G_i)_t) d\nu(t) - \epsilon \\
&\geq \sum_i \sum_n \int_{T_{in}} (\tau(\overline{U_n}) - \epsilon) d\nu(t) - \epsilon \\
&= \sum_i \sum_n \mu(R_{in}) + \mu(N \times X) - 2\epsilon,
\end{aligned}$$

as $\nu(N) = 0$. Since $\epsilon > 0$ is arbitrary, and $B \subset \cup_i \cup_n R_{in} \cup (N \times X)$ is a cover of B by rectangles,

$$\mu^*(B) \geq \inf \left\{ \sum_i \mu(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles} \right\},$$

with the opposite inequality immediate from (2). ■

We now introduce Method II constructions which by their definition depend on the metric structure of the sets. For these constructions we make the additional assumption that d is a metric on X with the property that for some $\delta_0 > 0$, if $|E| < \delta_0$, then $\overline{E}$ is compact. We work with the metric $d_0 = \max\{d, \rho\}$ on $T \times X$, and write $|\cdot|$ for the diameter of a set in any of the metric spaces.

For $\delta > 0$, define for $E \subset X$

$$\tau_\delta(E) = \tau(E) \text{ if } |E| \leq \delta \text{ and } \tau_\delta(E) = \infty \text{ if } |E| > \delta.$$

This is equivalent to seeking covers by sets of diameters at most δ. As before, we set

$$\tau_\delta^*(E) = \inf_{E \subset \cup E_i} \sum \tau_\delta(E_i) = \inf_{E \subset \cup E_i, |E_i| \leq \delta} \sum \tau_\delta(E_i).$$

The Method II measure on X constructed from the set function τ is then defined by

$$\tau^{**}(E) = \lim_{\delta \to 0} \tau_\delta^*(E).$$

This is a metric outer measure on X and thus all the Borel sets and analytic sets are measurable, see Rogers [Ro]. Proceeding as before, we set, for $B \subset T \times X$,

$$\mu_\delta(B) = \int_T \tau_\delta(B_t) d\nu(t) \qquad (6)$$

and

$$\mu_\delta^*(B) = \inf_{B\subset\cup B_i} \sum \mu_\delta(B_i). \tag{7}$$

The set function μ_δ^* may be presented in several different ways for certain Borel sets.

Lemma 8. Let X be locally compact and let τ satisfy conditions (C1)-(C5). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. For each $\delta > 0$,

$$\mu_\delta^*(B) = \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i \text{ and } B_i \text{ are Borel rectangles}\right\} \tag{8}$$

$$= \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i \text{ and } \; |B_i| \leq \delta\right\} \tag{9}$$

$$= \inf\left\{\sum \mu_\delta(B_i) : B \subset \cup_i B_i,\ |B_i| \leq \delta \text{ and } B_i \text{ are Borel rectangles}\right\}. \tag{10}$$

Proof. The equality in (8) is just Theorem 7. Certainly, the right-hand side of (8) is no greater than expression (10), and if (8) is infinite then these two expressions are equal. So suppose (8) is finite and $B_i = T_i \times E_i$ is a family of Borel rectangles covering B with $\mu_\delta(B_i) < \infty$. We may decompose $T_i = \cup_{j=1}^{\infty} T_{ij}$, where the T_{ij} are disjoint Borel subsets of T with $|T_{ij}| \leq \delta$. Then $B_i = \cup_{j=1}^{\infty} T_{ij} \times E_i$. Since B_i is a rectangle,

$$\mu_\delta(B_i) = \sum_{j=1}^{\infty} \tau_\delta(E_i)\nu(T_{ij}) = \sum_{j=1}^{\infty} \mu_\delta(T_{ij} \times E_i). \tag{11}$$

If $\tau_\delta(E_i) = \infty$, we must have $\nu(T_{ij}) = 0$ for all j, so $\mu_\delta(T_{ij} \times E_i) = \mu_\delta(B_i) = 0$ for all j. Otherwise, $\tau_\delta(E_i) < \infty$, so $|E_i| \leq \delta$ and $|T_{ij} \times E_i| \leq \delta$ for all j. It follows, using (11) that the sum in (8) is unchanged if we replace each set $B_i = T_i \times E_i$ by the countable union $\cup_j T_{ij} \times E_i$ of sets of diameter at most δ. Thus expressions (8) and (10) are equal. Finally, expression (9) lies between $\mu_\delta^*(B)$ as defined by (7) and (10). ■

We now relate the Method II measure on the sections X_t obtained from τ to the Method II measure on $T \times X$ obtained from μ. Thus we set

$$\mu^{**}(B) = \lim_{\delta\to 0} \mu_\delta^*(B); \tag{12}$$

this is the Method II measure on $T \times X$ obtained from μ by virtue of (9).

Theorem 9. Let X be locally compact and let τ satisfy conditions (C1)-(C5). Let $B \subset T \times X$ be a Borel set with each t-section B_t σ-compact. Then

$$\mu^{**}(B) = \int_T \tau^{**}(B_t)d\nu(t). \tag{13}$$

Proof. For each $\delta > 0$, applying Theorem 6 to τ_δ gives

$$\mu_\delta^*(B) = \int_T \tau_\delta^*(B_t)d\nu(t).$$

Letting $\delta \to 0$ we have $\tau^*_\delta(B_t) \to \tau^{**}(B_t)$ for all t, and $\mu^*_\delta(B) \to \mu^{**}(B)$. Identity (13) follows by the monotone convergence theorem. ∎

Since μ^{**} is a Method II measure, it is a metric measure, and all Borel and analytic subsets of $T \times X$ are μ^{**} measurable. We again have a product formula for Borel rectangles,

$$\mu^{**}(U \times E) = \nu(U)\tau^{**}(E),$$

so μ^{**} is a product of ν and τ^{**}, though once again extensions to X may be far from unique given that τ^{**} is likely to be non-σ-finite.

Example I Our principle example is Hausdorff measure. For $s \geq 0$, setting $\tau(E) = |E|^s$, where, as usual, $|\cdot|$ denotes diameter, we get that τ^{**} is the usual s-dimensional Hausdorff measure, $\mathcal{H}^s$ on X, see Rogers [Ro]. Thus, by Theorem 9, if B is a Borel set with σ-compact t-sections,

$$\mu^{**}(B) = \int_T \mathcal{H}^s(B_t)d\nu(t),$$

where μ^{**} is the Method II measure constructed from the set function $\mu(B) = \int |B_t|^s d\nu(t)$. It follows from Lemma 8 that we may use Borel rectangles $R_i = U_i \times E_i$ in covers for finding μ^*_δ and μ^{**}, so

$$\begin{aligned}\mu^{**}(B) &= \liminf_{\delta\to 0} \sum_{B\subset\cup R_i, |R_i|\leq\delta} \mu(R_i)\\ &= \liminf_{\delta\to 0} \sum_{B\subset\cup R_i, |R_i|\leq\delta} \nu(U_i)|E_i|^s.\end{aligned}$$

Now let ν be the restriction of d-dimensional Hausdorff measure $\mathcal{H}^d$ to a compact set $T \subset R^n$ with $0 < \mathcal{H}^d(T) < \infty$; we lose little by assuming that $\mathcal{H}^d(T) = 1$. By a standard result on upper densities, see [Ma], we have that $\limsup_{r\to 0} \nu(B(x,r))(2r)^{-d} \leq 1$ for ν-almost all x. Thus, if $\epsilon > 0$, we may take an increasing sequence of Borel sets $T_i \to T_0$, where $\nu(T_0) = 0$, and $\delta_i \to 0$, such that $\nu(U) \leq (1+\epsilon)2^d|U|^d$ if $|U| \leq \delta_i$ and $U \cap T_i \neq \emptyset$. Then

$$\begin{aligned}\mu^{**}(B \cap (T_i \times X)) &\leq \liminf_{\delta\to 0} \sum_{B\subset\cup R_i, |R_i|\leq\delta} (1+\epsilon)2^d|U_i|^d|E_i|^s\\ &\leq \liminf_{\delta\to 0} \sum_{B\subset\cup R_i, |R_i|\leq\delta} (1+\epsilon)2^d|R_i|^{d+s}\\ &= (1+\epsilon)2^d\mathcal{H}^{d+s}(B)\end{aligned}$$

for all i. Using Theorem 9 and taking the limit as $i \to \infty$, $\mu^{**}(B) \leq (1+\epsilon)2^d\mathcal{H}^{d+s}(B)$ so, since ϵ may be taken arbitraily small,

$$\mu^{**}(B) = \int_T \mathcal{H}^s(B_t)d\mathcal{H}^d(t) \leq 2^d\mathcal{H}^{d+s}(B).$$

The right-hand inequality is well-known, see [Ma]. Here we have given an alternative derivation of a somewhat stronger fact, that $B \mapsto \int_T \mathcal{H}^s(B_t)d\mathcal{H}^d(t)$ is itself a Method II measure on $T \times E$ constructed from the set function $\mu(B) = \int |B_t|^s d\mathcal{H}^d(t)$.

Example II Let λ be a given probability measure on $\mathbf{R}^n$ which we assume satisfies (C1)-(C5), and let $s, q \geq 0$. In connection with multifractal measures, several authors, for example Olsen [Ol], have considered measures of Hausdorff type which are Method II measures constructed from the set functions such as $\tau(E) = |E|^s \lambda(E)^q$ for $E \subset \mathbf{R}^n$. (For certain purposes, this τ may be modified so that $\tau(E) = \infty$ unless $\overline{E}$ is a ball.) This leads to Borel measures $\mathcal{H}_\lambda^{s,q}$ given by

$$\mathcal{H}_\lambda^{s,q}(E) = \tau^{**}(E) = \liminf_{\delta \to 0} \sum_{E \subset \cup E_i} |E_i|^s \lambda(E_i)^q.$$

Just as in Example I, we get a formula for the integral of sections of a Borel sets B with σ-compact sections as a Method II measure. Thus

$$\mu^{**}(B) = \int_T \mathcal{H}_\lambda^{s,q}(B_t) d\nu(t),$$

where μ^{**} is the Method II measure constructed from the set function $\mu(B) = \int^* |B_t|^s \lambda(B_t) d\nu(t)$. Such formulae may be applied to problems on sections of multifractal measures.

2. Analytic operators and packing dimensions

In this section we use properties of analytic operators to obtain some stronger results relating to packing dimensions of sections. For our purposes, it is enough to use the definition of packing dimension via upper box-counting dimension. For K a compact subset of some seperable metric space Y we set $N_r(K)$ for the least number of open balls of radius r that are needed to cover K. The *upper box-counting dimension* $\overline{\dim}_B K$ of K is defined by

$$\overline{\dim}_B K = \limsup_{r \to 0} \log N_r(K) / -\log r. \tag{14}$$

We define the *packing dimension* $\dim_P B$ of $B \subset Y$ by

$$\dim_P B = \inf \left\{ \sup_i \overline{\dim}_B K_i : B \subset \cup_i K_i \text{ with } K_i \text{ compact} \right\}. \tag{15}$$

For further properties of these dimensions, and the equivalent definition of packing dimension via packing measure, see [Fa,Ma].

We recall that the Borel operators over a Polish space X are generated in much the same way as the Borel sets [CM]. Thus, a function Δ mapping $P(X)$, the power set of X, into itself is said to be a *Borel operator* provided it is in the smallest family $\mathcal{F}$ of operators containing the following operators:

(*a*) $\Delta(K) = B$, B is a fixed Borel subset of X,

(*b*) $\Delta(K) = f^{-1}(K)$, where f is a fixed Borel map from X into X,

(*c*) $\Delta(K) = X \setminus K$,

and such that the family is closed under the operations of composition and countable unions:

$$(d)\ \ \Delta(K) = \Delta_1(\Delta_2(K)), \quad \Delta_1, \Delta_2 \in \mathcal{F}$$
$$(e)\ \ \Delta(K) = \cup_{n=0}^{\infty}\Delta_n(K), \quad \Delta_n \in \mathcal{F}.$$

An operator $\Theta : P(X) \to P(X)$ is said to be *analytic* if and only if there is a Polish space Y and a Borel operator $\Delta : P(X \times Y) \to P(X \times Y)$ such that $\Theta(M) = \pi_X(\Delta(M \times Y))$ for each $M \subset X$.

For each $d \geq 0$, let $\Gamma = \Gamma^{(d)} : P(X) \to P(X)$ be the operator defined by

$$x \in \Gamma(M) \iff \forall \epsilon > 0\ [\overline{\dim}_B(M \cap B(x, \epsilon)) \geq d].$$

Theorem 10. The operator Γ is analytic, that is Σ_1^1.

Proof. For each n, defining $\Gamma_n : P(X) \to P(X)$ by

$$x \in \Gamma_n(M) \iff \overline{\dim}_B(M \cap B(x, 1/n)) \geq d,$$

we have

$$\Gamma(M) = \bigcap_{n=1}^{\infty} \Gamma_n(M).$$

Since the intersection of a sequence of analytic operators is analytic, it suffices to show that each operator Γ_n is analytic. To this end, we consider the Polish space $Y = X^{\mathbf{N}}$.

Let $D = \{(y_p) \in X^{\mathbf{N}} : \overline{\dim}_B\{y_p : p \in \mathbf{N}\} \geq d\}$. We note that D is a Borel subset of Y. There are several ways to prove this. For example, one can easily check that $I = \{(y_p) \in Y : \{y_p : p \in \mathbf{N}\}$ is not conditionally compact$\}$ is a Borel set and the map $\phi : Y \setminus I \mapsto K(X)$, defined by $\phi((y_p)) = \overline{\{y_p : p \in \mathbf{N}\}}$ is Borel measurable. In [MM] it is shown that the map $K \mapsto \overline{\dim}_B(K)$ is Borel measurable, and composing these maps gives that B is a Borel set.

Next, define the the operator Δ over $X \times Y$ by:

$$\Delta(A) = (X \times D) \cap \bigcap_{k=1}^{\infty} (B_k \cap f_k^{-1}(A)),$$

where $B_k = \{(x, (y_p)) : d(x, y_k) < 1/n\}$ is a Borel subset of $X \times Y$ for each k, and $f : X \times Y \mapsto X \times Y$ given by $f_k(x, (y_p)) = (y_k, (y_p))$ is a Borel measurable map. Thus $\Delta : P(X \times Y) \to P(X \times Y)$ is a Borel operator, see [CM, p. 58]. Since

$$x \in \Gamma_n(M) \iff \exists (y_p) \in Y\ [(x, (y_p)) \in \Delta(M \times Y)],$$

or

$$\Gamma_n(M) = \pi_X(\Delta(M \times Y)),$$

the operator Γ_n is an analytic, that is Σ_1^1, operator [CM, p. 58]. Therefore, $\Gamma = \bigcap \Gamma_n$ is an analytic operator. ■

Let $\Gamma_0(M) = M \cap \Gamma(M)$, so $x \in \Gamma_0(M)$ if and only if $x \in M$ and, for every neighborhood U of $x, \overline{\dim}_B(U \cap M) \geq d$. As the intersection of two analytic operators, Γ_0 is an analytic, or Σ_1^1, operator and therefore the dual operator Ψ defined by

$$\Psi(M) = X \backslash \Gamma_0(X \backslash M)$$

is a coanalytic, or Π_1^1, operator. The operator Ψ is also monotone, that is $M \subset \Psi(M)$, since

$$\Psi(M) = M \cup \{x \in X \backslash M : \exists \text{ an open set } U\,[x \in U \text{ and } \overline{\dim}_B(U \cap (X \setminus M)) < d]\}.$$

Thus Ψ adds to M all points x of $X \setminus M$ at which $X \setminus M$ is small in the sense that there is some neighborhood U of x such that $U \cap (X \setminus M)$ has upper box counting dimension less than d. An important feature of this operator is that Ψ adds to M a relatively open subset of $X \setminus M$.

We next consider the effect of iterating the operator Ψ. By transfinite recursion, we set $\Psi^0(E) = E$, and $\Psi^{\alpha+1}(E) = \Psi(\Psi^\alpha(E))$ for each ordinal α, and $\Psi^\lambda(E) = \cup_{\gamma<\lambda}\Psi^\gamma(E)$ if λ is a limit ordinal. We note some properties of the operator Ψ including a simple boundedness principle or stabilization property.

Lemma 11. For each $M \subset X$, there is an ordinal $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal, such that $\Psi^\alpha(M) = \Psi^{\alpha+1}(M)$. If $X \setminus M$ is compact, then for each ordinal α, the set $X \setminus \Psi^\alpha(M)$ is compact, and if in addition $\dim_P(X \setminus M) < d$, then there is a countable ordinal α such that $\Psi^\alpha(M) = X$.

Proof. Let $(U_n)_{n\in N}$ be a base for the topology of X. Suppose that for each countable ordinal α, $\Psi^\alpha(M)$ is a proper subset of $\Psi^{\alpha+1}(M)$. For each such α choose $n(\alpha)$ such that $U_{n(\alpha)} \cap X \setminus \Psi^\alpha(M) \subset \Psi^{\alpha+1}(M)$. Thus we may choose two countable ordinals $\alpha < \beta$ such that $n = n(\alpha) = n(\beta)$, and $x \in U_n \cap (X \setminus \Psi^\beta(M))$, so that $x \in \Psi^{\beta+1}(M) \setminus \Psi^\beta(M)$. On the other hand, $x \in \Psi^{\alpha+1}(M) \subset \Psi^\beta(M)$. This contradiction establishes the first part of the lemma.

For the second part, suppose $X \setminus M$ is compact. Since Ψ adds to M a relatively open subset of $X \setminus M$, the set $X \setminus \Psi(M)$ is compact. It follows by transfinite induction that $X \setminus \Psi^\alpha(M)$ is compact for each ordinal α. Finally, suppose in addition that $\dim_P(X \setminus M) < d$, that $\Psi^\alpha(M) = \Psi^{\alpha+1}(M)$ and that $Z = X \setminus \Psi^\alpha(M) \neq \emptyset$. Since $\dim_P(Z) < d$, there is a cover of Z by compact sets K_n such that for each $n, \overline{\dim}_B(K_n) < d$. By Baire's category theorem, for some n, the set K_n has nonempty interior U with respect to Z. Then $\emptyset \neq U \subset \Psi^{\alpha+1}(M) \setminus \Psi^\alpha(M)$. This last contradiction completes the proof of the lemma. ■

We need a parametrized version of the operator Ψ. Let us define the operator Φ over $P(T \times X)$ by:

$$\Phi(M) = \bigcup_{t\in T} \{t\} \times \Psi(M_t).$$

We note some of the basic properties of this operator.

Lemma 12. The operator Φ is monotone and coanalytic. Let M be a Borel subset of $T \times X$ such that for each $t \in T, X \setminus M_t$ is compact. Then $\Phi(M)$ is a Borel set and for each $t, X \setminus (\Phi(M))_t$ is compact. Moreover, for each countable ordinal $\alpha, \Phi^\alpha(M)$ is a Borel set, and for each $t, X \setminus (\Phi^\alpha(M))_t$ is compact.

Proof. Clearly, Φ is monotone and it is shown in [CM] that such operators are coanalytic. Let (U_n) be a basis for the topology of X. We note

$$(t,x) \in \Phi(M) \iff (t,x) \in M \text{ or } \exists U_n[x \in U_n \cap X \setminus M_t \text{ and } \overline{\dim}_B(U_n \cap X \setminus M_t) < d].$$

For each n, set $S_n = ((T \times X) \setminus M) \cap (T \times U_n)$. Then S_n is a Borel subset of $T \times X$ and each t-section of S_n is σ-compact. Thus, $D_n = \pi_T(S_n)$ is a Borel set. So, $G_n = (D_n \times X) \cap T \times X \setminus M$ is a Borel set and each t-section of G_n is compact. Therefore, the map $\phi_n : D_n \mapsto K(X)$ defined by $\phi(t) = (G_n)_t$ is a Borel measurable map. Since the map $K \mapsto \overline{K \cap U_n}$ is Borel measurable and the map $K \mapsto \overline{\dim}_B(K)$ is Borel measurable [MM], the set $E_n = \{t \in D_n : \overline{\dim}_B(G_n)_t \cap U_n = \overline{\dim}_B((X \setminus M_t) \cap U_n) = \overline{\dim}_B(\overline{X \setminus M_t \cap U_n}) < d\}$ is a Borel set. Since

$$\Phi(M) = M \cup \cup_n (E_n \times U_n) \cap (T \times X \setminus M),$$

it follows that $\Phi(M)$ is a Borel set. This finishes the proof of the middle part of the lemma. The last part follows by transfinite induction using the middle conclusion of the lemma. ■

We now deduce that reasonable Borel sets $B \subset T \times X$ have a countable decomposition into subsets, such that the packing dimension of the sections of B are determined by the upper box-counting dimensions of the sections of the subsets.

Theorem 13. Let T and X be Polish spaces and let B be a Borel subset of $T \times X$ such that for all $t \in T$, the t-section B_t is σ-compact with $\dim_P(B_t) < d$. Then there is a sequence of Borel sets $\{E_n\}_{n=1}^\infty$ such that $B = \bigcup_{n \in N} E_n$, and for all $t \in T$ and $n \in \mathbf{N}$ the section $(E_n)_t$ is compact with $\overline{\dim}_B(E_n)_t < d$.

Proof. By Saint Raymond's theorem the Borel set B can be expressed as a countable union of Borel sets each of which have all t-sections compact. Thus it suffices to prove the theorem under the assumption that each t-section of B is compact. For each ordinal α, let $B_\alpha = \Phi^\alpha((T \times X) \setminus B) \subset T \times X$. For each $t \in T$, Lemma 11 implies that there is some countable ordinal $\alpha(t)$ such that $B_{\alpha(t)} = X$. In the terminology of [CM] this means $T \times X$ is the closure of the operator Φ on the Borel set $T \times X \setminus B$ which is defined to be $\cup_\alpha \Phi^\alpha((T \times X) \setminus B)$. By the boundedness principle for monotone coanalytic operators, [CM, Theorem 1.6(e)], there is a countable ordinal α such that

$$T \times X = B_\alpha,$$

so

$$B = \bigcup_{\gamma < \alpha} B_{\gamma+1} \setminus B_\gamma.$$

By Lemma 12, for each γ, the set $B_{\gamma+1} \setminus B_\gamma$ is Borel, and for each t, the set $(B_{\gamma+1} \setminus B_\gamma)_t$ is σ-compact. Also, if K is compact and $K \subset (B_{\gamma+1} \setminus B_\gamma)_t$, then $\overline{\dim}_B K < d$. Applying

Saint Raymond's theorem (see Section 1), we can express each set $B_{\gamma+1} \backslash B_\gamma$ as a countable union of Borel sets each with every t-section compact. The theorem now follows. ■

We now apply Theorem 13 to give an alternative derivation of a formula for the essential supremum of the packing dimension of sections of sets, originally presented in [FJ]. For this illustration we take $T = \mathbf{R}^m$ and $X = \mathbf{R}^n$ with ν as m-dimensional Lebesgue measure, although the results extend to other homogeneous metric spaces.

We express our results in terms of a generalised packing dimension defined analogously to the usual packing dimension, see[FJ]. For K a compact subset of $T \times X$ we set

$$N_r^*(K) = \inf \left\{ \sum_i \nu(\pi_T(K \cap U_i)) : K \subset \cup_i U_i \text{ with } |U_i| \le r \right\}.$$

The *generalised upper box-counting dimension* $\overline{\dim}_B^* K$ of K is defined by

$$\overline{\dim}_B^* K = \limsup_{r \to 0} \log N_r^*(K) / -\log r.$$

Analogously to the usual dimensions, we define the *generalised packing dimension* $\dim_P^* B$ of $B \subset T \times X$ by

$$\dim_P^* B = \inf \left\{ \sup_i \overline{\dim}_B^* K_i : B \subset \cup_i K_i \text{ with } K_i \text{ compact} \right\}.$$

For further properties and a measure approach to these dimensions see [FJ].

As in [FJ, Proposition 3.5], a straightforward integration argument establishes that for all $B \subset T \times X$ we have

$$\dim_P B_t \le \dim_P^* B \tag{16}$$

for ν-almost all t. Another integration argument gives that for B bounded and analytic,

$$\dim_P^* B \le \text{esssup}_t \overline{\dim}_B(B_t). \tag{17}$$

A much more technical argument is used in [FJ, Proposition 9] to obtain the natural and useful identity

$$\dim_P^* B = \text{esssup}_t \dim_P(B_t) \tag{18}$$

which gives an expression for the packing dimension of a typical section of a compact set B.

Equation (18) may alternatively be obtained as a simple corollary of Theorem 13. Let B be a compact subset of $T \times X$, and let $d > \text{esssup}_t \dim_P(B_t)$ so $\dim_P(B_t) < d$ for almost all t. Theorem 13 applied to a subset of T of full measure (noting that $\dim_P(B_t)$ is measurable) gives a Borel decomposition $B = \cup B_n$ with $\overline{\dim}_B B_{nt} < d$ for almost all t, for all n. By (17), $\dim_P^* B_n \le d$ for all n, so $\dim_P^* B \le \sup_n \dim_P^* B_n \le d$. This gives $\dim_P^* B \le \text{esssup}_t \dim_P(B_t)$ and the opposite inequality is immediate from (16).

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EXAMPLES ILLUSTRATING THE INSTABILITY OF PACKING DIMENSIONS OF SECTIONS

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Abstract. We shall use the "iterated Venetian blind" construction to show that the packing dimensions of plane sections of subsets of $\mathbb{R}^n$ can depend essentially on the directions of the planes. We shall also establish the instability of the packing dimension of sections under smooth diffeomorphisms.

1. Introduction and notation

Let m and n be integers with $0 < m < n$. We use the notation $\gamma_{n,m}$ for the unique orthogonally invariant Radon probability measure on the Grassmann manifold $G_{n,m}$ consisting of all m-dimensional linear subspaces of $\mathbb{R}^n$. The uniqueness of $\gamma_{n,m}$ implies that there is a positive and finite constant c depending on m and n such that for all $A \subset G_{n,m}$

$$\begin{aligned}\gamma_{n,m}(A) = c(\mathcal{H}^n \times \cdots \times \mathcal{H}^n)(\{(y_1, \dots, y_m) \in (\mathbb{R}^n)^m : \quad & |y_i| \leq 1 \text{ for all } i = 1, \dots, m \\ & \text{and } V(y_1, \dots, y_m) \in A\}) \end{aligned} \tag{1.1}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure and $V(y_1, \dots, y_m)$ is the m-dimensional linear subspace spanned by the vectors $y_1, \dots, y_m$. For $V \in G_{n,m}$ we denote by proj_V the orthogonal projection onto V, by $V^\perp$ the orthogonal complement of V, and by V_a the m-plane $\{v + a : v \in V\}$ for $a \in V^\perp$.

For Borel sets $E \subset \mathbb{R}^n$ one has the following very precise information about the Hausdorff dimension, $\dim_H$ (for the definition see [F2, Chapter 2] or [Mat3, Chapter 4]), of projections and plane sections of E (see [K], [Mar], and [Mat1]): for $\gamma_{n,m}$-almost all $V \in G_{n,m}$

$$\dim_H \text{proj}_V(E) = \min\{m, \dim_H E\} \tag{1.2}$$

and

$$\mathcal{H}^{n-m}(\{a \in V^\perp : \dim_H(E \cap V_a) = \dim_H E - (n-m)\}) > 0 \tag{1.3}$$

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provided that in (1.3) $\dim_{\mathrm{H}} E \geq n-m$ and $0 < \mathcal{H}^{\dim_{\mathrm{H}} E}(E) < \infty$.

Note that for the Hausdorff and packing dimensions, $\dim_{\mathrm{p}}$, (for the definition see [F2, Chapter 3] or [Mat3, Chapter 5]), of sections we have the following natural upper bounds: if $E \subset \mathbb{R}^n$ and $V \in G_{n,m}$, then

$$\dim_{\mathrm{H}}(E \cap V_a) \leq \max\{0, \dim_{\mathrm{H}} E - (n-m)\} \tag{1.4}$$

and

$$\dim_{\mathrm{p}}(E \cap V_a) \leq \max\{0, \dim_{\mathrm{p}} E - (n-m)\} \tag{1.5}$$

for $\mathcal{H}^{n-m}$-almost all $a \in V^{\perp}$ (see [F3, Lemma 5] and [Mat3, Chapter 10]). For the packing dimension, the formulae (1.2) and (1.3) are false, but there are weaker results for both sets and measures (see [FH1-2], [FJ], [FM], and [JM]). Although there is no formula such as (1.2) for the packing dimensions of projections, Falconer and Howroyd showed in [FH2] that given an analytic set $E \subset \mathbb{R}^n$, $\dim_{\mathrm{p}} \mathrm{proj}_V(E)$ is almost surely a constant, that is, there is a number $d_m(E)$ such that $\dim_{\mathrm{p}} \mathrm{proj}_V(E) = d_m(E)$ for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. The purpose of this paper is to show that there is no such result for plane sections. We shall prove that there exists a compact set $E \subset \mathbb{R}^n$ and compact subsets A and B of $G_{n,m}$ with $\gamma_{n,m}(A) > 0$ and $\gamma_{n,m}(B) > 0$ such that for all $V \in A$ we have $\mathcal{H}^m(\mathrm{proj}_{V^{\perp}}(E)) = 0$, that is, $E \cap V_a = \emptyset$ for $\mathcal{H}^{n-m}$-almost all $a \in V^{\perp}$, and for all $V \in B$ we have $\dim_{\mathrm{p}}(E \cap V_a) = m$ for points a in a non-empty open subset of $V^{\perp}$. Quite likely, but perhaps with considerable technical complications, it would be possible to show that given a Borel function f from the space of affine m-planes in $\mathbb{R}^n$ into the closed interval $[0, m]$ there is a Borel set $E \subset \mathbb{R}^n$ such that $\dim_{\mathrm{p}}(E \cap V) = f(V)$ for almost all affine m-planes V. This would be analogous to the results of Davies [D] and Falconer [F1] where $A_V \subset V$ is given in an arbitrary but measurable way and then E is found such that for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ $\mathrm{proj}_V(E)$ agrees with A_V up to a set of m-dimensional measure zero.

In Section 5 we shall establish the instability of the packing dimensions of sections under smooth "bending" diffeomorphisms. We shall show that given a C^2-diffeomorphism $f : A \to B$ between two plane domains A and B which does not map every line segment onto a line segment there is a compact subset E of A such that $\mathcal{H}^1(\mathrm{proj}_L(E)) = 0$ for $\gamma_{2,1}$-almost all $L \in G_{2,1}$, that is, almost all sections $E \cap L_a$ are empty, but for all $L \in G_{2,1}$ we have $\dim_{\mathrm{p}}(f(E) \cap L_a) = 1$ for all points a in some non-empty open subset of $L^{\perp}$.

2. The basic result for hyperplanes in $\mathbb{R}^n$

In this section we begin a two-stage induction process that proves the result on which our first construction is based. Here we consider hyperplanes in $\mathbb{R}^n$ and in the next section we work with general m-planes in $\mathbb{R}^n$.

Let $P \subset [0,1]^n$ be a non-degenerate closed parallelepiped. We name the edges of P such that the shortest parallel edges are called 1-edges, the second shortest parallel edges are 2-edges and so on. This numbering distinguishes edges which are not parallel, that is, if two edges have the same length but they are not parallel then they have different numbers. For all $i = 1, \ldots, n$ we call P_i^1 and P_i^2 the $(n-1)$-faces of P which are generated by the edges numbered by $1, \ldots, i-1, i+1, \ldots, n$.

For our purposes it is enough to consider a specific class of subparallelepipeds of $[0,1]^n$. Let $\{x_1, \ldots, x_n\}$ be the standard basis of $\mathbb{R}^n$. For all $i = 1, \ldots, n$ we denote by W_i the hyperplane spanned by $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. We call $P \subset [0,1]^n$ *a hyperregular parallelepiped in* $\mathbb{R}^n$ if P_i^1 and P_i^2 are parallel to W_i for all $i \neq n-1$; let P_i^1 be the one that is nearest to W_i. For a hyperregular parallelepiped P and $\delta > 0$ we define

$$\mathcal{A}_{n,n-1}(P) = \{V : V \text{ is an affine } (n-1)\text{-plane meeting both } P_i^1 \text{ and } P_i^2 \text{ for all } i \neq n \text{ but not } P_n^1 \text{ and } P_n^2\}$$

and

$$\mathcal{A}^{\delta}_{n,n-1}(P) = \{V \in \mathcal{A}_{n,n-1}(P) : \operatorname{dist}(V \cap P, P_n^2) \geq \delta\}$$

where $\operatorname{dist}(V \cap P, P_n^2) = \inf\{|a-b| : a \in V \cap P, b \in P_n^2\}$ is the distance between $V \cap P$ and P_n^2.

The following lemma from [Mat2] describes the plane case underlying the basic construction for hyperplanes in higher dimensions.

Lemma 2.1. *There are disjoint compact sets* A, $B \subset G_{2,1}$ *with* $\gamma_{2,1}(A) > 0$ *and* $\gamma_{2,1}(B) > 0$ *such that for all hyperregular parallelograms* $P \subset [0,1]^2$ *and for all* $\varepsilon > 0$ *there exists a finite family* $\mathcal{P}_\varepsilon$ *of hyperregular subparallelograms of* P *with the following properties:*

(1) $\mathcal{H}^1(\operatorname{proj}_{L^\perp}(\cup \mathcal{P}_\varepsilon)) \leq \varepsilon$ *for all* $L \in A$.
(2) *There is* $\delta > 0$ *such that if* $L \in \mathcal{A}_{2,1}(P) \cap \mathcal{A}_{2,1}([0,1]^2)$ *is parallel to some line belonging to* B, *then there exists* $Q \in \mathcal{P}_\varepsilon$ *such that* $L \in \mathcal{A}^{\delta}_{2,1}(Q)$.

Proof. See [Mat2, Lemma 2]. Note that in the plane we can parametrize the lines through the origin by the angle they make with the positive x_1-axis. Using this parametrization [Mat2, Lemma 2] gives $A = [a, a+b]$ and $B = [0, a-b] \cup [a+2b, \pi]$ where a and b are real numbers with $0 < b < a$ and $a + 2b < \pi$. □

Next we prove the higher-dimensional version of Lemma 2.1 for hyperplanes.

Lemma 2.2. *There are disjoint compact sets* A, $B \subset G_{n,n-1}$ *with* $\gamma_{n,n-1}(A) > 0$ *and* $\gamma_{n,n-1}(B) > 0$ *such that for all hyperregular parallelepipeds* $P \subset [0,1]^n$ *and for all* $\varepsilon > 0$ *there exists a finite family* $\mathcal{P}_\varepsilon$ *of hyperregular subparallelepipeds of* P *with the following properties:*

(1) $\mathcal{H}^1(\operatorname{proj}_{V^\perp}(\cup \mathcal{P}_\varepsilon)) \leq \varepsilon$ *for all* $V \in A$.
(2) *There is* $\delta > 0$ *such that if* $V \in \mathcal{A}_{n,n-1}(P) \cap \mathcal{A}_{n,n-1}([0,1]^n)$ *is parallel to some hyperplane belonging to* B, *then there exists* $Q \in \mathcal{P}_\varepsilon$ *such that* $V \in \mathcal{A}^{\delta}_{n,n-1}(Q)$.

Proof. If $n = 2$, the result is a restatement of Lemma 2.1. We assume inductively that the claim holds in $\mathbb{R}^{n-1}$ and deduce the result in $\mathbb{R}^n$.

We may restrict our consideration to hyperregular parallelepipeds P with $P_1^1 \subset W_1$. We use the notation $\gamma_{W_1,n-2}$ for the invariant measure on the Grassmann manifold $G_{W_1,n-2}$ of all $(n-2)$-dimensional linear subspaces of W_1. Applying the induction hypothesis to W_1 which is identified with $\mathbb{R}^{n-1}$ and defining $\mathcal{A}_{W_1,n-2}(\widetilde{P})$ and $\mathcal{A}^{\delta}_{W_1,n-2}(\widetilde{P})$ in the obvious way for hyperregular parallelepipeds $\widetilde{P} \subset [0,1]^{n-1}$ in W_1 and for $\delta > 0$, we find disjoint compact sets $\widetilde{A}$, $\widetilde{B} \subset G_{W_1,n-2}$ with $\gamma_{W_1,n-2}(\widetilde{A}) > 0$

and $\gamma_{W_1,n-2}(\widetilde{B}) > 0$ such that for all hyperregular parallelepipeds $\widetilde{P} \subset [0,1]^{n-1}$ and for all $\varepsilon > 0$ there exists a finite family $\widetilde{\mathcal{P}}_\varepsilon$ of hyperregular subparallelepipeds of $\widetilde{P}$ such that

$$\mathcal{H}^1(\mathrm{proj}_{V^\perp,W_1}(\cup\widetilde{\mathcal{P}}_\varepsilon)) \leq \varepsilon \tag{2.3}$$

for all $V \in \widetilde{A}$. Here $\mathrm{proj}_{V^\perp,W_1} : W_1 \to V^{\perp,W_1}$ is the orthogonal projection onto the orthogonal complement $V^{\perp,W_1} \in G_{W_1,1}$ of V. Further, there is $\delta > 0$ such that if $V \in \mathcal{A}_{W_1,n-2}(\widetilde{P}) \cap \mathcal{A}_{W_1,n-2}([0,1]^{n-1})$ is parallel to some $(n-2)$-plane belonging to $\widetilde{B}$, then

$$V \in \mathcal{A}^\delta_{W_1,n-2}(\widetilde{Q}) \tag{2.4}$$

for some $\widetilde{Q} \in \widetilde{\mathcal{P}}_\varepsilon$.

Define

$$A = \{V \in G_{n,n-1} : V \cap W_1 \in \widetilde{A}\}$$

and

$$B = \{V \in G_{n,n-1} : V \cap W_1 \in \widetilde{B}, 0 \leq \mathrm{angle}(x_1, V \cap (V \cap W_1)^\perp) \leq \pi/4\},$$

where $\mathrm{angle}(x_1, V \cap (V \cap W_1)^\perp)$ is the angle between the x_1-axis and the line $V \cap (V \cap W_1)^\perp$ measured on $(V \cap W_1)^\perp \in G_{n,2}$. Here the positivity of the angle is determined by requiring that the half-line $V \cap (V \cap W_1)^\perp \cap \{(y_1,\ldots,y_n) \in \mathbb{R}^n : y_1 \geq 0\}$ intersects the $(n-1)$-plane where $x_n = 1$. In this way we fix the positive direction of the angle for all $(V \cap W_1)^\perp \in G_{n,2}$ which are not subsets of W_n. For the rest of the 2-planes $(V \cap W_1)^\perp$ we do this in some fixed sense; it turns out that either of the two possibilities will do.

Clearly A and B are disjoint. Since $\gamma_{W_1,n-2}(\widetilde{A}) > 0$ and $\gamma_{W_1,n-2}(\widetilde{B}) > 0$, it is easy to see from (1.1) that $\gamma_{n,n-1}(A) > 0$ and $\gamma_{n,n-1}(B) > 0$.

Let $P \subset [0,1]^n$ be a hyperregular parallelepiped with $P_1^1 \subset W_1$ and let $\varepsilon > 0$. Since $\widetilde{P} = P \cap W_1$ is a hyperregular parallelepiped in W_1, there exists by the induction hypothesis a finite family $\widetilde{\mathcal{P}}_\varepsilon$ of hyperregular subparallelepipeds of $\widetilde{P}$ such that (2.3) and (2.4) hold. Let $V \in A$. Since

$$\mathrm{proj}_{V^\perp}(\cup\widetilde{\mathcal{P}}_\varepsilon) = \mathrm{proj}_{V^\perp}\mathrm{proj}_{(V\cap W_1)^\perp}(\cup\widetilde{\mathcal{P}}_\varepsilon) = \mathrm{proj}_{V^\perp}\mathrm{proj}_{(V\cap W_1)^{\perp},W_1}(\cup\widetilde{\mathcal{P}}_\varepsilon)$$

we obtain from (2.3) that

$$\mathcal{H}^1(\mathrm{proj}_{V^\perp}(\cup\widetilde{\mathcal{P}}_\varepsilon)) \leq \varepsilon. \tag{2.5}$$

Let $\mathcal{P}_\varepsilon$ be a finite family of hyperregular subparallelepipeds of P obtained by extending the parallelepipeds of $\widetilde{\mathcal{P}}_\varepsilon$ to very thin parallelepipeds to the direction of the positive x_1-axis. Then (1) holds by (2.5).

Let $\delta > 0$ be as in (2.4). If $V \in \mathcal{A}_{n,n-1}(P) \cap \mathcal{A}_{n,n-1}([0,1]^n)$ is parallel to some hyperplane belonging to B, then $V \cap W_1 \in \mathcal{A}_{W_1,n-2}(\widetilde{P}) \cap \mathcal{A}_{W_1,n-2}([0,1]^{n-1})$ is parallel to some $(n-2)$-plane belonging to $\widetilde{B}$. Using (2.4), we find $\widetilde{Q} \in \widetilde{\mathcal{P}}_\varepsilon$ such that $V \cap W_1 \in \mathcal{A}^\delta_{W_1,n-2}(\widetilde{Q})$. Since $0 \leq \mathrm{angle}(x_1, V \cap (V \cap W_1)^\perp) \leq \pi/4$ and since we may choose the length of the 1-edges of the parallelepipeds of $\mathcal{P}_\varepsilon$ to be less than $\delta/2$, we have $V \in \mathcal{A}^{\delta/2}_{n,n-1}(Q)$ where $Q \in \mathcal{P}_\varepsilon$ is the enlargement of $\widetilde{Q}$. Note that since here $V \in \mathcal{A}_{n,n-1}([0,1]^n)$ is parallel to some $V_p \in B$, the x_n-axis cannot be a subset of $V_p \cap W_1$. Thus $(V_p \cap W_1)^\perp$ is not a subset of W_n. In this case the positiveness of $\mathrm{angle}(x_1, V_p \cap (V_p \cap W_1)^\perp)$ is explicitly defined. □

3. The extension of the basic result to m-planes in $\mathbb{R}^n$

In order to extend the result of Lemma 2.2 for general m-planes in $\mathbb{R}^n$ we do a two-stage induction process: first we use the results of the previous section for hyperplanes and then we prove the general case. As before we restrict our attention to a certain class of parallelepipeds. We say that a non-degenerate closed parallelepiped $P \subset [0,1]^n$ is *an m-regular parallelepiped in* $\mathbb{R}^n$ if P is of the form $S \times [0,1]^{n-(m+1)}$ where $S \subset [0,1]^{m+1}$ is a hyperregular parallelepiped in $\mathbb{R}^{m+1}$. We number the edges of P in the same way as before and define for all $i = 1, \ldots, n$ the $(n-1)$-faces P_i^1 and P_i^2 as before. Note that for all $i \neq m$ both P_i^1 and P_i^2 are parallel to W_i. For an m-regular parallelepiped $P \subset [0,1]^n$ we set

$$\mathcal{A}_{n,m}(P) = \{V : V \text{ is an affine } m\text{-plane meeting both } P_i^1 \text{ and } P_i^2 \text{ for all } i = 1, \ldots, m \text{ but not } P_i^1 \text{ and } P_i^2 \text{ when } i = m+1, \ldots, n\}.$$

Lemma 3.1. *There are disjoint compact sets A, $B \subset G_{n,m}$ with $\gamma_{n,m}(A) > 0$ and $\gamma_{n,m}(B) > 0$ such that for all m-regular parallelepipeds $P \subset [0,1]^n$ and for all $\varepsilon > 0$ there exists a finite family $\mathcal{P}_\varepsilon$ of m-regular subparallelepipeds of P with the following properties:*

(1) *$\mathcal{H}^{n-m}(\mathrm{proj}_{V^\perp}(\cup\mathcal{P}_\varepsilon)) \leq \varepsilon$ for all $V \in A$.*
(2) *If $V \in \mathcal{A}_{n,m}(P) \cap \mathcal{A}_{n,m}([0,1]^n)$ is parallel to some m-plane belonging to B, then there exists $Q \in \mathcal{P}_\varepsilon$ such that $V \in \mathcal{A}_{n,m}(Q)$.*

Proof. If $n = m+1$, the result is a consequence of Lemma 2.2. Keeping m fixed, we assume inductively that the result holds in $\mathbb{R}^{n-1}$ and prove it in $\mathbb{R}^n$.

Identifying W_n with $\mathbb{R}^{n-1}$ and using the induction hypothesis, we find disjoint compact sets $\widetilde{A}$, $\widetilde{B} \subset G_{W_n,m}$ with $\gamma_{W_n,m}(\widetilde{A}) > 0$ and $\gamma_{W_n,m}(\widetilde{B}) > 0$ such that for all m-regular parallelepipeds $\widetilde{P} \subset [0,1]^{n-1}$ and for all $\varepsilon > 0$ there exists a finite family $\widetilde{\mathcal{P}}_\varepsilon$ of m-regular subparallelepipeds of $\widetilde{P}$ such that for all $V \in \widetilde{A}$

$$\mathcal{H}^{n-1-m}(\mathrm{proj}_{V^{\perp,W_n}}(\cup\widetilde{\mathcal{P}}_\varepsilon)) \leq \varepsilon. \tag{3.2}$$

Further, if $V \in \mathcal{A}_{W_n,m}(\widetilde{P}) \cap \mathcal{A}_{W_n,m}([0,1]^{n-1})$ is parallel to some m-plane belonging to $\widetilde{B}$, then

$$V \in \mathcal{A}_{W_n,m}(\widetilde{Q}) \tag{3.3}$$

for some $\widetilde{Q} \in \widetilde{\mathcal{P}}_\varepsilon$.

Define

$$A = \{V \in G_{n,m} : \mathrm{proj}_{W_n}(V) \in \widetilde{A}\}$$

and

$$B = \{V \in G_{n,m} : \mathrm{proj}_{W_n}(V) \in \widetilde{B}\}.$$

Clearly A and B are disjoint compact sets with $\gamma_{n,m}(A) > 0$ and $\gamma_{n,m}(B) > 0$.

Let $P \subset [0,1]^n$ be an m-regular parallelepiped and let $\varepsilon > 0$. Using the induction hypothesis for the m-regular parallelepiped $\widetilde{P} = P \cap W_n$ in W_n we find a finite family $\{\widetilde{P}_\varepsilon^1, \ldots, \widetilde{P}_\varepsilon^k\}$ of m-regular subparallelepipeds of $\widetilde{P}$ such that (3.2) and (3.3) hold. Now $\mathcal{P}_\varepsilon = \{\widetilde{P}_\varepsilon^1 \times [0,1], \ldots, \widetilde{P}_\varepsilon^k \times [0,1]\}$ is a finite family of m-regular subparallelepipeds of P. Consider $V \in A$. Note that for $W = \mathrm{proj}_{W_n}(V) \in \widetilde{A}$ we have $W^{\perp, W_n} \subset V^{\perp}$. Since $\mathcal{H}^{n-m}(\mathrm{proj}_{V^{\perp}}(\cup \mathcal{P}_\varepsilon)) \leq 2n\mathcal{H}^{n-1-m}(\mathrm{proj}_{W^{\perp},W_n}(\cup \mathcal{P}_\varepsilon))$ and $\mathrm{proj}_{W^{\perp},W_n}(\cup \mathcal{P}_\varepsilon) = \mathrm{proj}_{W^{\perp},W_n}(\cup \widetilde{\mathcal{P}}_\varepsilon)$, we obtain (1) from (3.2). Finally, if $V \in \mathcal{A}_{n,m}(P) \cap \mathcal{A}_{n,m}([0,1]^n)$ is parallel to some m-plane belonging to B, then for all $i = 1, \ldots, n-1$ we have $\mathrm{proj}_{W_n}(V \cap P_i^j) = \mathrm{proj}_{W_n}(V) \cap \widetilde{P}_i^j$ for $j = 1, 2$. Since $\mathrm{proj}_{W_n}(V) \in \mathcal{A}_{W_n,m}(\widetilde{P}) \cap \mathcal{A}_{W_n,m}([0,1]^{n-1})$ is parallel to some m-plane belonging to $\widetilde{B}$, we obtain by (3.3) that $\mathrm{proj}_{W_n}(V) \in \mathcal{A}_{W_n,m}(\widetilde{P}_\varepsilon^l)$ for some $1 \leq l \leq k$ giving $V \in \mathcal{A}_{n,m}(\widetilde{P}_\varepsilon^l \times [0,1])$. □

4. The main construction

Using Lemma 3.1 we prove our main result:

Theorem 4.1. *There exist compact sets $E \subset \mathbb{R}^n$ and A, $B \subset G_{n,m}$ with $\gamma_{n,m}(A) > 0$ and $\gamma_{n,m}(B) > 0$ such that*

1. *for all $V \in A$ we have $\mathcal{H}^{n-m}(\mathrm{proj}_{V^{\perp}}(E)) = 0$, and*
2. *for all $V \in B$ there exists a non-empty open subset U_V of $V^{\perp}$ such that $\dim_{\mathrm{p}}(E \cap V_a) = m$ for all $a \in U_V$.*

Proof. Let A, $B \subset G_{n,m}$ be as in Lemma 3.1. Setting $P_{1,1} = [0,1]^n$ and using Lemma 3.1 we find m-regular parallelepipeds $Q_{2,1}, \ldots, Q_{2,l_2} \subset P_{1,1}$ such that for all $V \in A$

$$\mathcal{H}^{n-m}(\mathrm{proj}_{V^{\perp}}(\bigcup_{q=1}^{l_2} Q_{2,q})) \leq \frac{1}{2}.$$

Further, if $V \in \mathcal{A}_{n,m}(P_{1,1})$ is parallel to some m-plane belonging to B, then $V \in \mathcal{A}_{n,m}(Q_{2,q})$ for some $1 \leq q \leq l_2$. For all $1 \leq q \leq l_2$ and $1 \leq i \leq m$ let $e_i(Q_{2,q})$ be the length of the i-edges of $Q_{2,q}$. Let k_2 be the smallest positive integer such that for all $1 \leq q \leq l_2$

$$k_2 \geq e_1(Q_{2,q})^{-2m+1}.$$

Dividing each $Q_{2,q}$ into $(k_2)^m$ m-regular parallelepipeds with all the edges parallel to the corresponding edges of $Q_{2,q}$ and with the length of the i-edges equal to $\frac{1}{k_2} e_i(Q_{2,q})$ for all $1 \leq i \leq m$, we obtain m-regular parallelepipeds $P_{2,1}, \ldots, P_{2,N_2}$ where $N_2 = l_2(k_2)^m$. Clearly

$$\mathcal{H}^{n-m}(\mathrm{proj}_{V^{\perp}}(\bigcup_{q=1}^{N_2} P_{2,q})) \leq \frac{1}{2}$$

for all $V \in A$. By Lemma 3.1 we find for all $1 \leq q \leq N_2$ m-regular parallelepipeds $Q_{3,1}^q, \ldots, Q_{3,l_3^q}^q \subset P_{2,q}$ such that for all $V \in A$

$$\mathcal{H}^{n-m}(\mathrm{proj}_{V^{\perp}}(\bigcup_{p=1}^{l_3^q} Q_{3,p}^q)) \leq \frac{1}{3N_2}. \tag{4.2}$$

Further, whenever $V \in \mathcal{A}_{n,m}(P_{2,q}) \cap \mathcal{A}_{n,m}(P_{1,1})$ is an m-plane parallel to some m-plane belonging to B, then $V \in \mathcal{A}_{n,m}(Q_{3,p}^q)$ for some $1 \leq p \leq l_3^q$. As before, divide each $Q_{3,p}^q$ into $(k_3)^m$ m-regular parallelepipeds with all edges parallel to the corresponding edges of $Q^q_{3,p}$ and with the length of the i-edges equal to $\frac{1}{k_3}e_i(Q^q_{3,p})$ for all $1 \leq i \leq m$. Here k_3 is the smallest integer such that for all $1 \leq q \leq N_2$ and $1 \leq p \leq l^q_3$

$$k_3 \geq e_1(Q^q_{3,p})^{-3m+1}.$$

This gives us m-regular parallelepipeds $P_{3,1}, \ldots, P_{3,N_3}$ where $N_3 = \sum_{q=1}^{N_2} l^q_3 (k_3)^m$. Since

$$\bigcup_{q=1}^{N_3} P_{3,q} \subset \bigcup_{q=1}^{N_2} \bigcup_{p=1}^{l^q_3} Q^q_{3,p},$$

we have by (4.2)

$$\mathcal{H}^{n-m}(\mathrm{proj}_{V^\perp}(\bigcup_{q=1}^{N_3} P_{3,q})) \leq \frac{1}{3}.$$

Continue in this way and define a compact set

$$E = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{N_p} P_{p,q}.$$

If $V \in A$, then for all positive integers p

$$\mathcal{H}^{n-m}(\mathrm{proj}_{V^\perp}(E)) \leq \mathcal{H}^{n-m}(\mathrm{proj}_{V^\perp}(\bigcup_{q=1}^{N_p} P_{p,q})) \leq \frac{1}{p}$$

giving the first claim.

Finally, let $V \in \mathcal{A}_{n,m}(P_{1,1})$ be parallel to some m-plane belonging to B. By the construction for all j we have $V \in \mathcal{A}_{n,m}(Q^q_{j,p})$ for some $1 \leq q \leq N_{j-1}$ and $1 \leq p \leq l^q_j$ and therefore $V \in \mathcal{A}_{n,m}(P_{j,i})$ for all $P_{j,i} \subset Q^q_{j,p}$. Since there are $(k_j)^m$ such parallelepipeds $P_{j,i}$ and since $E \cap V \cap P_{j,i} \neq \emptyset$ for all of them, we need at least $(\frac{k_j}{3})^m$ m-cubes with side-length

$$d_j = \frac{1}{k_j} \min_{\substack{1 \leq q \leq N_{j-1} \\ 1 \leq p \leq l^q_j}} e_1(Q^q_{j,p})$$

to cover $E \cap V$. Using the fact that

$$k_j \geq \Big(\min_{\substack{1 \leq q \leq N_{j-1} \\ 1 \leq p \leq l^q_j}} e_1(Q^q_{j,p}) \Big)^{-jm+1}$$

we have $(k_j)^{jm} \geq (d_j)^{1-jm}$ which gives $\overline{\dim}_{\mathrm{B}}(E \cap V) = m$ where $\overline{\dim}_{\mathrm{B}}$ is the upper box-counting dimension (for the definition see [F2, Chapter 3] or [Mat3, Chapter 5]). Similarly we see that $\overline{\dim}_{\mathrm{B}}(E \cap V \cap O) = m$ for all open sets $O \subset \mathbb{R}^n$ with $E \cap V \cap O \neq \emptyset$, and so [F2, Corollary 3.9] gives $\dim_{\mathrm{p}}(E \cap V) = m$. This completes the proof since in Lemma 3.1 the set B can be chosen in such a way that for all $V \in B$ the set $\{a \in V^\perp : V_a \in \mathcal{A}_{n,m}(P_{1,1})\}$ is open. □

5. Bending maps and packing dimensions of sections

In this section we shall indicate another difference between the behaviour of Hausdorff and packing dimensions of sections of sets. By (1.3), (1.4), and the preservation of Hausdorff dimension under smooth mappings, the typical Hausdorff dimension of sections of a smooth image of a set is the same as the typical Hausdorff dimension of sections of the original set. We shall show that the packing dimensions of sections can change very radically under smooth diffeomorphisms. For simplicity, we shall do this only in the plane, although the techniques of the previous sections could certainly be used to prove similar results in higher dimensions.

Theorem 5.1. *Let $f : A \to B$ be a C^2-diffeomorphism between open subsets A and B of $\mathbb{R}^2$. Suppose that f does not map every line segment of A onto a line segment. Then there is a compact subset E of A such that*

(1) $\mathcal{H}^1(\text{proj}_{L^\perp}(E)) = 0$ *for $\gamma_{2,1}$-almost all $L \in G_{2,1}$, and*
(2) *for all $L \in G_{2,1}$ we have $\dim_p(f(E) \cap L_a) = 1$ for all $a \in I_L$, where I_L is some non-empty open subinterval of $L^\perp$.*

The proof is a slight modification of the methods of Section 4 and [Mat2] and therefore we shall only sketch it. We recall some terminology and notation from [Mat2]. From now on a parallelogram will always mean a non-degenerate closed parallelogram in $\mathbb{R}^2$ whose shorter sides are parallel to the x_1-axis. Given a C^1-curve $\mathcal{C}$ and a parallelogram P, we say that $\mathcal{C} \in \Gamma(P)$ if $\mathcal{C} \cap P$ has a connected component meeting both of the longer sides of P but neither of the shorter ones. We denote by $\text{dir}(\mathcal{C}, x)$ the direction of the tangent of $\mathcal{C}$ at $x \in \mathcal{C}$. Finally, $\text{p}_\theta = \text{proj}_{l_\theta^\perp}$ where $l_\theta = \{t(\cos\theta, \sin\theta) : t \in \mathbb{R}\}$ for $\theta \in [0, \pi)$.

Lemma 5.2. *Let P be a parallelogram, $\varepsilon > 0$, $0 < s < 1$, $0 < \alpha < \frac{\pi}{10}$, and let $k_\alpha \geq 1$ be the largest integer with $5(k_\alpha + 1)\alpha < \pi$. Then there is a finite family $\mathcal{P}$ of subparallelograms of P with the following properties:*

(1) $\mathcal{H}^1(\text{p}_\theta(\cup\mathcal{P})) \leq \varepsilon$ *for $5i\alpha \leq \theta \leq (5i+1)\alpha$, $i = 1, \ldots, k_\alpha$.*
(2) *If $\mathcal{C} \in \Gamma(P)$ with $\text{dir}(\mathcal{C}, x) \notin ((5i-1)\alpha, (5i+2)\alpha)$ for all $i = 1, \ldots, k_\alpha$, $x \in \mathcal{C}$, then there are parallelograms $P_1, \ldots, P_l \in \mathcal{P}$ having the same side-length d for their shorter sides such that $ld^s > 1$ and $\mathcal{C} \in \Gamma(P_i)$ for all $i = 1, \ldots, l$.*

Proof. [Mat2, Lemma 3] gives a finite family $\mathcal{R}$ of subparallelograms of P for which (1) is satisfied and if $\mathcal{C}$ is as in (2), then $\mathcal{C} \in \Gamma(Q)$ for some $Q \in \mathcal{R}$. Subdividing each parallelogram of $\mathcal{R}$ into sufficiently many subparallelograms we get the required family $\mathcal{P}$. □

We can now use the argument in [Mat2, pp. 307–309]. First we choose a small open subset U of A such that f bends many line segments in U. We may not be able to get this for all line segments in U, but if we stay away from some exceptional directions as described in [Mat2, Lemma 1] we find a subinterval I of $[0, \pi)$ of length $\frac{1}{2}$ such that for line segments J whose direction is in I, $f(J)$ is not a line segment. Using Lemma 5.2 we construct a compact set F with the following properties:

(5.3) $F = \bigcap_{m=1}^\infty \bigcup \mathcal{P}_m$ where $(\mathcal{P}_m)$ is a nested sequence of subparallelograms of U.
(5.4) $\mathcal{H}^1(\text{p}_\theta(F)) = 0$ for almost all $\theta \in [0, \pi)$.
(5.5) For all $\theta \in I$ we have $\dim_p(f(F) \cap (l_\theta + a)) = 1$ for all $a \in I_\theta$, where $I_\theta \subset \mathbb{R}$ is some non-empty open subinterval of $l_\theta^\perp$.

The set F can be taken to be one of the sets E_n in [Mat2, p. 307] (for example, $F = E_6$ if we take $\varepsilon = \frac{1}{6}$ in the application of [Mat2, Lemma 1] when choosing the set U above). Then (5.3) and (5.4) are satisfied. To get (3.5) we choose a sequence $s_m \in (0,1)$ with $\lim_{m\to\infty} s_m = 1$ and take $s = s_m$ when applying Lemma 5.2 as in [Mat2] to obtain the family $\mathcal{P}_{m+1}$. As in the last paragraph in [Mat2, p. 308] we see that for all $\theta \in I$ we have $f(F) \cap (l_\theta + a) \neq \emptyset$ for $a \in I_\theta$ (which is an open subinterval of $l_\theta^\perp$). (There is a misprint in [Mat2, p. 308]: the first sentence of the last paragraph should read $(l_\theta + a) \cap f(E_n) \neq \emptyset$ instead of $(l_\theta + a) \cap f(E_n) = \emptyset$.) Because of the stronger formulation of Lemma 5.2 we now know more: for any open set U' with $f(F) \cap U' \neq \emptyset$ and for sufficiently large m we need at the m-th stage at least l_m intervals of length d_m with $l_m(d_m)^{s_m} > 1$ to cover $f(F) \cap U' \cap (l_\theta + a)$ for $\theta \in I$ and $a \in I_\theta$. Further, $\lim_{m\to\infty} d_m = 0$, and therefore $\dim_{\mathrm{p}}(f(F) \cap (l_\theta + a)) = 1$ for $\theta \in I$ and $a \in I_\theta$. Since I has length $\frac{1}{2}$, we can take as E the union of seven suitably rotated copies of F.

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