Anomalous scaling and regularity of the Navier-Stokes equations

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Abstract

The Navier-Stokes equations are widely believed to faithfully model turbulent fluid dynamics. Unresolved mathematical issues concerning the regularity of solutions, in particular the open question of whether divergent singularities may spontaneously appear in the solutions, remain a problem for the foundations of the theory. In this paper we suggest that anomalous scaling of small lengths in turbulent flows may contain new information about the possibility of singularities.

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Intermittency in dynamical systems and turbulence is the name given to phenomena which display localized fluctuations in time and/or space [1, 2, 3]. It is most easily characterized by probability distributions in terms of the relative likelihood of large, rare events. In turbulence the relevant events are usually identified as localized fluctuations in velocity differences or energy dissipation [4], and the phenomena are identified quantitatively via anomalous scaling of appropriate moments.

For example, moments of velocity differences $\delta u = \hat{\mathbf{r}} \cdot (\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))$ in high Reynolds number, incompressible, homogeneous, isotropic turbulence are observed to behave according to [5]

$$\langle \delta u^p \rangle \sim \varepsilon^{p/3} L^{(p/3-\zeta_p)} r^{\zeta_p} \tag{1}$$

where $r = |\mathbf{r}|$ is in the inertial range, $\hat{\mathbf{r}} = \mathbf{r}/r$, L is the integral scale in the flow, and ε is the energy dissipation rate per unit mass. The näive scaling, where there is a single velocity scale at length scale r, corresponds to $\zeta_p = p/3$ in which case

$$\langle \delta u^p \rangle \sim \varepsilon^{p/3} r^{p/3}. \tag{2}$$

The average $\langle \cdot \rangle$ is usually taken to mean an ensemble or time average. Hölder's inequality ensures ζ_p is a concave function of p for even p [5]. Apart from the exact result that $\zeta_3 = 1$ [6], these scalings have never been deduced rigorously from the Navier-Stokes equations although some rigorous relations between scaling exponents have been established [7]. Experimental evidence suggests that the ζ_p all fall on a monotonically increasing concave curve [5]. The deviation of ζ_p from $\frac{p}{3}$ is referred to as anomalous scaling and is taken as an indication of the relatively large fluctuations at small scales associated with the phenomena of intermittency.

To elucidate the connection between anomalous scaling and intermittency, let us define the "p-velocity" scale [8] at length scale r for $p > 3$ according to

$$U_p = \left(\frac{\langle \delta u^p \rangle}{\langle \delta u^3 \rangle} \right)^{\frac{1}{p-3}}. \tag{3}$$

The scaling in equation (1) then implies

$$U_p \sim U \left(\frac{r}{L} \right)^{\frac{\zeta_p - 1}{p-3}} \tag{4}$$

where $U = (\varepsilon L)^{1/3}$ is a large scale velocity. Because ζ_p is monotonically increasing from $\zeta_3 = 1$ and yet bounded above by $\frac{p}{3}$, the exponent above satisfies $0 < \frac{\zeta_p - 1}{p-3} \leq \frac{1}{3}$. Hence all the $U_p \to 0$ as $r \to 0$. The essence of anomalous scaling and intermittency, however, is that these p-velocity scales separate from each other in this limit. Higher p-velocity scales will be dominated by rare but large fluctuations, i.e., by extreme values in the probability distribution. So depending on how one chooses to measure the velocity, i.e., depending on the choice of p, different scales emerge. This is most easily seen by looking at the dimensionless ratio of sequential p-velocity scales:

$$\frac{U_{p+1}}{U_p} \sim \left(\frac{L}{r}\right)^{\gamma_p} \tag{5}$$

where

$$\gamma_p = \frac{\zeta_p - 1}{p-3} - \frac{\zeta_{p+1} - 1}{(p+1)-3}. \tag{6}$$

For the classical scaling $\zeta_p = \frac{p}{3}$, each $\gamma_p = 0$ and the scales are all the same. On the other hand, anomalous scaling and the concavity of ζ_p as a function of p for $p > 3$ means precisely that $\gamma_p > 0$ and that the ratio diverges as r descends to smaller and smaller scales in the inertial range. Again we reiterate that these are purely phenomenological observations to date; no rigorous predictions for such scaling exponents have ever been derived from the Navier-Stokes equations.

In this paper we want to consider a different hierarchy of moments and investigate the implications of anomalous scaling—should it occur—for questions of the regularity of solutions to the Navier-Stokes equations. The moments we consider are not of velocity differences, rather we focus on the $2n^{th}$ moments of the energy spectrum in wavenumber space as defined by second moments of n^{th} derivatives of the velocity field. These wavenumber moments correspond to small length scales deep in the dissipation range. We do not consider averages in a turbulent steady state, but rather the time evolution of these length scales as constrained by the Navier-Stokes equations. The question addressed here is not how these scales separate at high Reynolds number, but rather how these small length scales behave when a singular event occurs in the Navier-Stokes equations—if indeed a singular event does occur.

We begin in the context of decaying turbulence, with the incompressible 3D Navier-Stokes equations on a periodic domain $\Omega = [0, L]^3$:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u}, \tag{7}$$

where $\nabla \cdot \mathbf{u} = 0$, with the divergence-free initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$. Here are some previously established facts [9, 10, 11]. For small enough values of the dimensionless quantity

$$I_0 = \frac{\|\nabla \mathbf{u}_0\|_2^2 L}{\nu^2}, \tag{8}$$

where $\|\cdot\|_2$ denotes the L^2 norm on Ω, we can be assured of the existence of a unique smooth (C^∞ in fact) solution for all for $0 < t < \infty$. But for large values of I_0, after a finite time t_* that depends on ν, L and the magnitude of I_0, we may only assert the existence of weak, not necessarily unique, solutions [12]. Weak solutions have finite kinetic energy $\frac{1}{2}\|\mathbf{u}(\cdot,t)\|_2^2$ at all times, but may exhibit isolated integrable singularities in the instantaneous energy dissipation rate $\nu\|\nabla\mathbf{u}(\cdot,t)\|_2^2$. Such singularities, *if* they occur (which remains an open question [13]), are responsible for our inability to establish the uniqueness of solutions. Moreover, singularities of this sort would violate the assumption of separation of scales invoked in the derivation of the Navier-Stokes equations from the Boltzmann equations, raising the question of the validity of these equations in the presence of such violently turbulent dynamics.

Singular events are characterized by the vanishing of small length scales in the flow. Indeed, a divergence in the instantaneous energy dissipation rate means that the Fourier energy spectrum $E(\mathbf{k},t) = |\hat{\mathbf{u}}(\mathbf{k},t)|^2$ for the solution has no effective high wavenumber cutoff. In terms of moments, singular events correspond to the divergence of the second and all higher even moments of $E(\mathbf{k},t)$. During the initial period $0 < t < t_*$ when the solution is smooth, it is very smooth, and all the moments of $E(\mathbf{k},t)$ exist.

The moments of the energy spectrum are given by the square of the L^2-norms of derivatives of the velocity vector field, and they define small length scales. We denote

$$H_n(t) = \int_\Omega |\nabla^n \mathbf{u}|^2 \, d^3x \tag{9}$$

where $|\nabla^n\mathbf{u}|^2$ means $|\Delta^{\frac{n}{2}}\mathbf{u}|^2$ for n even and $|\Delta^m\nabla\mathbf{u}|^2$ for $n = 2m+1$ odd. For all $n \geq 1$ the ratio

$$\kappa_n(t) = \left(\frac{H_n}{H_0}\right)^{\frac{1}{2n}} = \left(\frac{\sum_{\mathbf{k}} k^{2n}|\hat{\mathbf{u}}|^2}{\sum_{\mathbf{k}} |\hat{\mathbf{u}}|^2}\right)^{\frac{1}{2n}}, \tag{10}$$

defines a high wavenumber corresponding to a small length scale. In particular $\kappa_1^{-1} = \left(\frac{H_1}{H_0}\right)^{-\frac{1}{2}}$ is the familiar Taylor microscale.

Hölder's inequality implies the ordering $\kappa_1 \leq \kappa_2 \leq \kappa_3 \ldots$, so the corresponding length scales $\kappa_1^{-1} \geq \kappa_2^{-1} \geq \kappa_3^{-1} \ldots$ are a decreasing sequence. This fact is not limited to solutions of the Navier-Stokes equations but is true for any velocity vector field on this domain. We do not expect this inequality to ever be saturated; the κ_n are all exactly equal only if the energy is concentrated in a single mode. They can be asymptotically equal as $n \to \infty$ only if there is a single strict (although not absolutely sharp) high wavenumber cutoff in the spectrum. Here are some previously established facts about the κ_n's [11]:

1. Singularities in the flow occur when κ_1 diverges, sending all the length scales (κ_n^{-1}) to zero.
2. No singularity can occur if any one of the κ_n, for $n \geq 1$, stays finite. Hence at any instant of time $t > 0$, either the κ_n are all finite or, if they diverge, they diverge together.
3. If they do diverge then the divergence is integrable in time [14].

If there is a single vanishing cutoff length scale associated with a singularity then all the wavenumbers diverge together, in which case the ratios $\frac{\kappa_{n+1}}{\kappa_n} = O(1)$ as $\kappa_n \to \infty$. We know that singularities must be localized in space because $\|\mathbf{u}(\cdot, t)\|_2$ remains finite, so they must have a broadband spectrum in wavenumber space [13].

The central result of this paper is contained in the following hierarchy of coupled differential inequalities for the evolution of the $\kappa_n(t)$ for $n \geq 3$:

$$n\frac{d\kappa_n}{dt} \leq -\nu \left[\left(\frac{\kappa_{n+1}}{\kappa_n} \right)^{2n+2} - 1 \right] \kappa_n^3 + c_n \, \kappa_n^{7/2} \|\mathbf{u}(\cdot, t)\|_2 \tag{11}$$

where the c_n are absolute constants. These differential inequalities are derived directly from the Navier-Stokes equations (see appendix) and are valid for weak as well as strong solutions. For any initial condition with finite I_0, all the κ_n are finite for $t > 0$ up until at least t_*. The restriction to $n \geq 3$ is a technical necessity in the derivation to allow us to "close" the heirarchy as we have, involving only $\|\mathbf{u}(\cdot, t)\|_2$ which we know is a monotonically decreasing (and hence *a priori* bounded) function of time.

The ordering of the wavenumbers due to Hölder's inequality ensures that the first term on the right hand side of (11) from the viscous term in the Navier-Stokes equation is not positive. The superlinear growth of the second term $\sim \kappa_n^{7/2}$, however, allows the possibility of a finite time divergence of $\kappa_n(t)$. But the broadband nature of any singularity implies that we can confidently assert that $\frac{\kappa_{n+1}}{\kappa_n} > 1$ (strictly) as $\kappa_n \geq \infty$. Hence the first term from the viscosity is likely to have more damping power than is rigorously justified by the Hölder limit. However, the fact that $\frac{7}{2} > 3$ means that it needs to be very much more effective if a large fluctuation is to be arrested before it leads to a divergence.

Sufficient anomalous scaling (in a sense to be described precisely below) associated with potential singular events could elevate the viscous damping term in (11) to a more prominant, or even dominant, role in the process. By anomalous scaling we mean the emergence of a spectrum of small length scales in the vicinity of a violent fluctuation. That is, we consider the possibility that there is not just one scale but rather a spectrum of distinct scales associated with each of the κ_n.

In analogy with the observed phenomena of intermittency and anomalous scaling for moments of velocity differences, let us suppose that in a singular event each $\kappa_n(t)$ diverges with its *own* exponent β_n with respect to $\kappa_3(t)$, the lowest one valid in (11); that is,

$$\kappa_n \sim L^{-1+\beta_n}\kappa_3^{\beta_n} \tag{12}$$

as $\kappa_3 \to \infty$ where L is an outer scale (the domain length scale or an appropriate scale in the initial data). The $\beta_n \geq 1$ must then be an increasing sequence of numbers starting from $\beta_3 = 1$ and the ratio of sequential wavenumbers would scale as

$$\frac{\kappa_{n+1}}{\kappa_n} \sim L^{\beta_{n+1}-\beta_n}\kappa_3^{\beta_{n+1}-\beta_n} \sim (\kappa_n L)^{\mu_n} \tag{13}$$

with exponents

$$\mu_n = \frac{\beta_{n+1}}{\beta_n} - 1 > 0. \tag{14}$$

Under these circumstances the viscous term in (11) has more power; now the nonlinear growth $\sim \kappa_n^{7/2}$ has to compete with dissipation $\sim \kappa_n^{3+2(n+1)\mu_n}$.

The hierarchy of differential inequalities in (11) implies that if such anomalous scaling occurs, then there is an upper bound on all the μ_n exponents if a true singularity actually occurs. For if

$$\mu_n > \frac{1}{4(n+1)} \tag{15}$$

for any $n \geq 3$, then in the course of a large fluctuation of κ_n, the magnitude of the damping term will eventually exceed the nonlinear growth so $d\kappa_n/dt$ reverses sign and no singularity is possible. The perhaps somewhat surprising conclusion is that a sufficiently divergent spectrum of small scales will regularize Navier-Stokes dynamics.

Anomalous scaling of this sort for the κ_n in potentially singular or truly singular events is quite similar in spirit to intermittency for moments of velocity differences. In the inertial range scaling of velocity differences it is the presence of ever larger velocity amplitude fluctuations at ever smaller scales that is reflected in the anomalous scaling exponents.

In the hypothesized anomalous scaling for the κ_n it is again the presence of relatively velocity larger fluctuations at ever smaller scales—now deep in the dissipation range in the vicinity of (near) singular events—that would result in positive μ_n. It is interesting that the Navier-Stokes equations place a limit on the magnitude of such fluctuations if solutions ever exhibit true singularities.

These observations lead naturally to suggestions for theoretical, computational and experimental investigations. It would be interesting to evaluate the κ_n for exact solutions, such as Lundgren's spiral solution [15], to see how spectrum behaves. The κ_n are in principle accessible from direct numerical simulations of turbulent flows. This would provide the most direct investigation of the question of scaling and anomalous scaling for the κ_n, but it is not clear how important the practical limitations on the Reynolds number will be. Most intruigingly, velocity derivatives in a turbulent flow can be measured. Although in practice it may be difficult to separate spatial from temporal averaging, some related moments of the derivatives may be extracted to probe the spectrum of small scales in the turbulence corresponding to the κ_n's. The open question of regularity of solutions of the Navier-Stokes equations remains an outstanding mathematical challenge. It is an exciting prospect that experimental observations may shed new light on this problem.

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A Appendix

The differential inequality for κ_n, displayed in (11) can easily be found using two ladder inequalities for H_n [11]

$$\frac{1}{2}\dot{H}_n \leq -\nu H_{n+1} + c_{n,r}^{(1)}\|\nabla\mathbf{u}\|_\infty H_n \qquad (16)$$

$$\frac{1}{2}\dot{H}_n \geq -\nu H_{n+1} - c_{n,r}^{(1)}\|\nabla\mathbf{u}\|_\infty H_n \qquad (17)$$

where the $c_{n,r}^{(1)}$ is an absolute constant depending only on n and r. $\|\nabla\mathbf{u}\|_\infty$ means the supremum over space of all derivatives of all components of the velocity field. Firstly we extend our definition of the κ_n to

$$\kappa_{n,r} = \left(\frac{H_n}{H_r}\right)^{\frac{1}{2(n-r)}} \qquad (18)$$

and then use the inequalities in (16) and (17) on the H_n and H_r to obtain

$$(n-r)\dot{\kappa}_{n,r} \leq -\nu\left[\left(\frac{\kappa_{n+1,r}}{\kappa_{n,r}}\right)^{2(n-r)+2}\kappa_{n,r}^{3}-\kappa_{n,r}\kappa_{r+1,r}^{2}\right]+c_{n,r}^{(2)}\,\|\nabla\mathbf{u}\|_{\infty}\kappa_{n,r}. \tag{19}$$

where $c_{n,r}^{(2)}$ is another absolute constant. To estimate $\|\nabla\mathbf{u}\|_{\infty}$ in terms of $\kappa_{n,r}$ we use a Gagliardo-Nirenberg inequality [11], valid for $n\geq 3$

$$\|\nabla\mathbf{u}\|_{\infty}\leq c_{n,r}^{(3)}\,\kappa_{n,0}^{5/2}\,\|\mathbf{u}\|_{2} \tag{20}$$

Then we note two facts; firstly $\kappa_{r+1,r}\leq\kappa_{n,r}$ and $\kappa_{n,0}\leq\kappa_{n,r}$ for $r<n$. Hence (19) becomes

$$(n-r)\dot{\kappa}_{n,r}\leq-\nu\left[\left(\frac{\kappa_{n+1,r}}{\kappa_{n,r}}\right)^{2(n-r)+2}-1\right]\kappa_{n,r}^{3}+c_{n,r}^{(4)}\,\kappa_{n,r}^{7/2}\|\mathbf{u}\|_{2}. \tag{21}$$

Finally we set $r=0$ and drop the second label and write $\kappa_{n,0}\equiv\kappa_n$ and the $c_{n,r}^{(4)}\equiv c_n$ to obtain (11).

To discuss the Navier-Stokes equations with an additive forcing function $f(\mathbf{x})$ to extend our result to a potentially turbulent statistical steady state we replace our H_n with

$$F_n(t)=\int_{\Omega}\left(|\nabla^{n}\mathbf{u}|^{2}+|\nabla^{n}\mathbf{u}_{f}|^{2}\right)\,dV \tag{22}$$

where $\mathbf{u}_f=L^2\nu^{-1}f(\mathbf{x})$. Then the F_n obey the same equations as the H_n in (16) and (17) with the respective addition and subtraction of an extra term $\nu\lambda_0^{-2}F_n$. The length λ_0 is defined by $\lambda_0^{-2}=L^{-2}+\lambda_f^{-2}$ where $\lambda_f\leq L$ is the smallest scale cut-off in $\mathbf{f}(\mathbf{x})$ where L is the domain length. The H_n in the κ_n become F_n and the only change to equation (11) is the addition of a term $\nu\lambda_0^{-2}\kappa_n$ which does not change the argument or the observation concerning anomalous scaling and regularity.

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