Multifractal Spectra of Branching Measure on a Galton-Watson Tree

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Abstract

If Z is the branching mechanism for a supercritical Glaton–Watson tree with a single progenitor and $E[Z \log Z] < \infty$, there is a branching measure μ defined on $\partial\Gamma$ the set of all path ξ which has a unique node $\xi|n$ at each generation n. We use the natural metric $\rho(\xi, \eta) = e^{-n}$ where $n = \max\{k : \xi|k = \eta|k\}$ and observe that the local dimension index

$$d(\mu.\xi) = \lim_{n\to\infty} \frac{\log \mu(B(\xi|n))}{-n} = \alpha = \log m, \quad \mu - a.e. \ \ \xi.$$

Our objective is to consider the exceptional points where the above display may fail. There is a non–trivial "thin" spectrum for $\overline{d}(\mu, \xi)$ when $p_1 = P\{Z = 1\} > 0$ and Z has finite moments of all positive orders. Because $\underline{d}(\mu, \xi) = \alpha$ for all ξ, we obtain a "thick" spectrum by introducing the "right" power of a logarithm. In both cases we find the Hausdorff dimension of the exceptional sets.

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1 Introduction

We are interested in supercritical Galton-Watson trees with a single progenitor. Let $Z = \{p_0, p_1, \cdots\}$ be the offspring distribution of a Galton–Watson branching process, defined on a probability space (Ω, P), with mean $m = \sum jp_j > 1$ and $\Gamma = \Gamma(\omega)$ denote the associated family tree. Let μ be the branching measure on the boundary $\partial\Gamma$; see Section 2 for more detailed descriptions. We remark that μ is a random measure on the (random) tree Γ and our object in this paper is to find properties of the multifractal spectra of μ which are true with probability one. Put

$$\alpha = \log m > 0.$$

It is already known that, with probability one,

$$d(\mu, \xi) = \lim_{n\to\infty} \frac{\log \mu B(\xi|n)}{n} = -\alpha \quad \mu - a.e.\ \ \xi \in \partial\Gamma,$$

which can be translated, using the natural metric in $\partial\Gamma$, to

$$d(\mu, \xi) = \lim_{r\to 0} \frac{\log \mu B(\xi, r)}{\log r} = \alpha, \quad \mu - a.e.\ \ \xi.$$

The above display is the usual starting point for multifractal analysis of a locally finite Borel measure.

In a recent paper Liu[L3] showed that, if $p_0 = p_1 = 0$ and Z has finite moments of all orders then, with probability one,

$$d(\mu, \xi) = \alpha, \quad \forall \xi.$$

Thus the ordinary multifractal spectrum is trivial in this case. However, even in this case, as was shown for the occupation measure of stable subordinator in Shieh–Taylor[ST] and of Brownian motion in Dembo–Peres–Rosen–Zeitouni[DPRZ], one can observe a non-trivial spectrum for "thick" points, by introducing an appropriate power of a logarithm. Results of this type are obtained in Section 4.

In [L3], Liu also observed that, if Z has finite moments of all positive orders, then

$$\underline{d}(\mu, \xi) = \liminf_{n\to\infty} \frac{-\log \mu B(\xi|n)}{n} = \alpha \quad \forall \xi \in \partial\Gamma,$$

which implies that for $\beta \neq \alpha$ the set

$$E_\beta = \{\xi \in \partial\Gamma : \quad d(\mu, \xi) = \beta\} \text{ is empty,}$$

so that the standard multifractal formalism cannot yield a spectrum. However, as in Perkins–Taylor[PT] for super–Brownian motion and Hu–Taylor[HT] for stable occupation measure, we prove that, when $0 < p_1 < 1$, there is a non–trivial spectrum for

$$\overline{d}(\mu, \xi) = \limsup_{n\to\infty} \frac{-\log \mu B(\xi|n)}{n}.$$

In Section 5, we obtain the Hausdorff dimension of

$$C_\beta = \{\xi : \quad \overline{d}(\mu, \xi) \geq \beta\},$$
$$D_\beta = \{\xi : \quad \overline{d}(\mu, \xi) = \beta\},$$

for an interval of values of β in which these sets are non–empty. In both Sections 4 and 5, our method also yields the packing dimension of the relevant sets.

In this paper we again make use of the strong spherical porosity conditions first defined in [PT]. Section 2 defines this condition and its meaning on $\partial\Gamma$; in addition we recall the necessary preliminary definitions and results for Galton–Watson trees. In Dembo–Peres–Rosen–Zeitouni[DPRZ], and Khoshnevisan–Peres–Xiao[KPX], it is pointed out that many exceptional sets examined in random phenomena are of limsup type; they provide a useful theorem giving a lower bound for the Hausdorff dimension of such sets The exceptional sets which we examine now on $\partial\Gamma$ are again of limsup type. We therefore develop a version of the main theorem in [DPRZ], proved there for Euclidean cubes, which is valid in the context of a Galton–Watson tree. This is done in Section 3, and is used in both Sections 4 and 5.

2 Preliminaries

We begin with notation and results for Galton–Watson trees which we need in this paper; these are adapted from Pemantle–Peres[PP]. Let $Z = \{p_0, p_1, \cdots\}$ be the offspring distribution of a Galton–Watson branching process. We assume that $m := \sum_j jp_j < \infty$ and that $p_0 = 0, p_1 < 1$; thus $1 < m < \infty$, that is we are in the supercritical case(As pointed out in [PP], if $p_0 > 0$ and $m > 1$ there is positive probability that Γ is finite. All our results remain true, condtioned on the event that Γ is infinite. However we impose $p_0 = 0$ to eliminate the complication of conditioning). We also assume that Z is not a constant, that is $p_j < 1, \quad \forall j$. Associated with each realization of the process, we have a (random) family tree, called a *Galton–Watson tree*(GWT), which we denote by $\Gamma = \Gamma(\omega)$. Let $\Gamma_n, n = 0, 1, 2, \cdots$ be the n–th level(generation) so that $\Gamma = \cup_n \Gamma_n$. Let Z_n denote the cardinal number of Γ_n and we assume that $Z_0 = \{\emptyset\}$(single progenitor). Assume moreover that $E[Z \log Z] < \infty$, then the limit

$$W := \lim_{n\to\infty} \frac{Z_n}{m^n}$$

exists and is finite and positive a.s., see Artheya–Ney[AN]. For a GWT Γ there is associated a natural *boundary* $\partial\Gamma$, which is defined as the set of infinite self–avoiding paths from $\emptyset$ through the tree; we denote by ξ a generic point in $\partial\Gamma$. For $\sigma \in \Gamma_n$, $|\sigma| = n$ denotes its length and $B(\sigma) = B(\sigma, r), \quad r = e^{-n}$, denotes the "ball" $\{\xi \in \partial\Gamma : \sigma \text{ is the ancestor of } \xi \text{ in } \Gamma_n\}$. We also use $\xi|n$ to denote the ancestor in Γ_n of an $\xi \in \partial\Gamma$. We remark that $\partial\Gamma$ is a compact metric space under the metric $d(\xi_1, \xi_2) = e^{-n}$, where $n = \max\{k : \xi_1|k = \xi_2|k\}$. In this metric, the subtree consisting of the vertex $\sigma \in \Gamma_n$ and all its descendants is a ball in $\partial\Gamma$ of diameter of e^{-n}, as denoted above. Let $W(\sigma)$ be the shifted W at the vertex σ, that is

$$W(\sigma) := \lim_{n\to\infty} \frac{\text{card}\{\eta \in \Gamma_n : \sigma \text{ is the ancester of } \eta\}}{m^{n-|\sigma|}}.$$

Since Γ is a countable set, $W(\sigma)$ exists for all $\sigma \in \Gamma$ with probability one. By assuming the existence of $W(\sigma)$, we define *branching measure* on $\partial\Gamma$ as the unique (random) Borel measure μ on $\partial\Gamma$ such that

$$\mu B(\sigma) = m^{-n} W_\sigma, \quad \sigma \in \Gamma_n.$$

Note that $W(\sigma), \sigma \in \Gamma_n$, are iid with the same distribution as W, conditional on $Z_j, j \leq n$. Thus, the above display reflects the statistical self-similarity of the measure μ.

In Perkins–Taylor[PT], the notion of a $\gamma -$ *thin* set for $\gamma > 1$ was defined. The definition was for $\mathbb{R}^d$ but it translates to any metric space, so we define it for $\partial\Gamma$.

Definition 2.1 *Fix $\gamma > 1$. We say $E \subset \partial\Gamma$ is $\gamma -$ thin at ξ if there is a sequence $r_i \downarrow 0$ such that*

$$E \cap [B(\xi, r_i) \backslash B(\xi, r_i^\gamma)] = \emptyset \quad \forall i.$$

We say E is a $\gamma -$ thin set if E is $\gamma -$ thin at each $\xi \in E$.

The nature of our metric on $\partial\Gamma$ then asserts that $\partial\Gamma$ is $\gamma - thin$ at ξ if and only if

$$B(\xi|n_i) = B(\xi|[\gamma n_i])$$

for an increasing sequence of positive integers n_i([$\cdot$]: the greatest integer part). This is equivelent to saying that $\xi|n \in \Gamma_n$ has exactly one descendant in Γ_{n+1} for $n_i \le n \le [\gamma n_i], \quad i = 1, 2, \ldots$, This shows that $\partial\Gamma$ can have $\gamma-$thin points only if $p_1 > 0$, and in this case we denote the set of all $\gamma-$thin points for $\partial\Gamma$ by S_γ.

We now recall some facts first proved in [PT] for Euclidean space. It is easy to check that the results remain true for $\partial\Gamma$.

Lemma 2.1 *For any $\gamma - thin$ set $E \subset \partial\Gamma$, we have*

$$\mathrm{Dim}E \ge \gamma \mathrm{dim}E.$$

Here. Dim stands for packing dimension and dim stands for Hausdorff dimension. One can refer to [PP] for the detailed definitions and properties of Dim and dim on GWT's.

Lemma 2.2 *Let ν be any Borel measure on $\partial\Gamma$ and its support be $S = S(\nu)$. If $A \subset S$ is $\gamma - thin$, and*

$$\underline{d}(\nu, x) \ge a \quad \forall x \in A,$$

then,

$$\overline{d}(\nu, x) \ge \gamma a \quad \forall x \in A.$$

We note that, in the case of branching measure μ, with probability one, every ball $B(\sigma)$ has positive μ measure. Hence $S(\mu) = \partial\Gamma$. We will see that there is a range of values of γ for which $S_\gamma \ne \emptyset$, provided some simple conditions are satisfied.

We mention that the metric space $\partial\Gamma$ has fractal dimension α: with probability one,

$$\mathrm{dim}\partial\Gamma = \mathrm{Dim}\partial\Gamma = \alpha.$$

3 Limsup fractals on Galton–Watson Trees

The following two propositions are a version of Theorem 2.1 of [DPRZ] for GWT. Firstly we note that $\partial\Gamma$ can be regarded as a random subset of the infinite sequence space $\mathbb{N}^\infty$.

We define a random mapping Ψ on $\mathbb{N}^{\infty} \times \Omega$ so that $\Psi(B,\omega) = 0$, whenver $B \not\subset \partial\Gamma(\omega)$, and that $Z(B,\omega)$ is $\{0,1\}-$ valued, when $B \subset \partial\Gamma(\omega)$. Set

$$A = \limsup_n A(n),$$

where

$$A(n) = \cup_{\Psi(B(\sigma))=1, \ \ \sigma\in\Gamma_n} B(\sigma).$$

Proposition 3.1 *Assume that*
(i) the random variables $\Psi(B(\sigma))$, $\sigma \in \Gamma_n$, *are independent;*
(ii)the expectation

$$q_n := E[\Psi(B(\sigma))] = P\{\Psi(B(\sigma)) = 1\}$$

is the same for all $\sigma \in \Gamma_n$, *and*
(iii) there is a sequence of poitive integers n_k *which increase to* ∞ *rapidly enough so that* $m^{2n_k} \leq n_{k+1}$, $\forall k$, *such that* q_n *satisfies the following lower bound estimate,*

$$cm^{-\delta n_k} \leq q_{n_k} \quad \forall k,$$

where c is some absolute constant and $0 < \delta < 1$. *Then, with probability one, the limsup set A defined above has infinite Hausdorff* $\phi-$*measure,* $\phi-m(A) = +\infty$, *where the gauge function* ϕ *is defined by*

$$\phi(x) = x^{(1-\delta)\alpha}(\log(1/x))^b, \quad 0 < x < 1, \quad b > 2.$$

In particular, $\dim A \geq (1-\delta)\alpha$.

Proposition 3.2 *Under the conditions of Proposition 3.1, with probability one,*

$$\text{Dim}A = \alpha.$$

Propositions 3.1 and 3.2 can be proved using the methods of [DPRZ]. We remark that we do not need a condition which bounds the correlation since the sub-trees $B(\sigma), \sigma \in \Gamma_n$, are completely independent. We need modifications because we are now working on a random tree, rather than the binary tree; these can be made by suitable use of conditional independence. Moreover, the third condition in Proposition 3.1 is stated as required for all large n in [DPRZ], yet in fact it is only needed for a sufficiently rapidly increasing sequence. We will leave the details to readers. We also mention that Theorem 2.1 of [DPRZ] has been further refined in [KPX].

4 Dimension Spectrum for thick behavior

From Section 1, we know that the "typical behavior" of the branching measure μ on $B(\sigma), \sigma \in \Gamma_n$, is m^{-n}, for all n large enough. To describe the behavior of μ which is "thicker", we introduce the following two (random) sets

$$A_\theta := \Big\{\xi \in \partial\Gamma : \ \limsup_n \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} \geq \theta\Big\},$$

and

$$B_\theta := \Big\{\xi \in \partial\Gamma : \ \limsup_n \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} = \theta\Big\}.$$

In the above, λ is defined by

$$\lambda := 1 - \frac{\alpha}{\log ||Z||_\infty},$$

where $||\cdot||_\infty$ is the sup norm of the underlying probability space. Note that $0 < \lambda \leq 1$, and $\lambda < 1$ if and only if Z is a finite distribution. To describe the dimension spectrum of A_θ and B_θ, we need the following two parameters

$$\begin{aligned} r_0 &:= \liminf_{x\to\infty} \frac{-\log P[W > x]}{x^{1/\lambda}}, \\ r &:= \text{the one such that } \ P[W > x] \approx e^{-rx^{1/\lambda}}, \quad \text{as } x \uparrow \infty, \end{aligned}$$

here $a \approx b$ means that there are c_1, c_2 such that $c_1 a \leq b \leq c_2 a$. To discuss the dimension spectrum of A_θ, we assume that r_0 is finite and positive, which is a quite mild assumption as we can see from Lemma 4.1 below. To discuss the dimension spectrum of B_θ, we need to impose the stronger assumption that r exists, and is finite and positive. This is a strong condition, yet it holds for the interesting case that Z has a geometric distribution, which makes W have an exponential distribution and then $\lambda = r = 1$. It holds also for the case in which W has a gamma distribution, see Harris[H, p17]. Now we state our dimension spectrum separately for A_θ and B_θ.

Theorem 4.1 *Assume that r_0 is finite and positive, then, with probability one,*

$$\dim A_\theta = \alpha - r_0\theta^{1/\lambda}, \quad 0 \leq \theta \leq \Big(\frac{\alpha}{r_0}\Big)^\lambda.$$

Theorem 4.2 *Assume that r exists, and is finite and positive, then, with probability one,*

$$\dim B_\theta = \alpha - r\theta^{1/\lambda}, \quad 0 \le \theta \le \left(\frac{\alpha}{r}\right)^\lambda.$$

Moreover, under the assumption of Theorem 4.1, resp. Theorem 4.2,

$$\begin{aligned} \mathrm{Dim}A_\theta &= \alpha, \quad 0 < \theta < \left(\frac{\alpha}{r_0}\right)^\lambda, \\ \textit{resp. } \mathrm{Dim}B_\theta &= \alpha, \quad 0 < \theta < \left(\frac{\alpha}{r}\right)^\lambda. \end{aligned}$$

Remark 1. Since $B_\theta \subset A_\theta$ and r is necessarily equal to r_0 under the existence assumption, Theorem 4.2 has a stronger assertion under stronger assumption, compared with Theorem 4.1.

Remark 2. By the following uniform law for μ proved in Liu–Shieh[LS],

$$\limsup_n \sup_{\xi\in\partial\Gamma} \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} = \left(\frac{\alpha}{r_0}\right)^\lambda,$$

the set $B_\theta = \emptyset$ for $\theta > \left(\frac{\alpha}{r_0}\right)^\lambda$. Thus, the range for θ in Theorems 4.1 and 4.2 is exact.

Firstly we state a lemma giving conditions which imply that r_0 is finite and positive. The lemma is a direct consequence of Liu[L1 Theorem 3.1 and L2], however it can be deduced from earlier results.

Lemma 4.1 *The parameter r_0 is finite and positive, when $\lambda < 1$, or when $\lambda = 1$ and $E[e^{tZ}] < \infty$ for some, but not for all, $t > 0$.*

Proofs of Theorems. We concentrate first on the Hausdorff dimension, dim. We begin with the discussion of the extreme cases $\theta = 0$ and $\theta = \left(\frac{\alpha}{r_0}\right)^\lambda$. For $\theta = 0$, the asserton $\dim A_\theta = \dim B_\theta = \alpha$ is merely a consequence of the well-known result that $\dim\mu = \log m$ a.s.; see Hawkes[H] and Lyons–Pemantle–Peres[LPP]. For $\theta = \left(\frac{\alpha}{r_0}\right)^\lambda$, it is a consequence of letting a sequence θ_k strictly increase to θ and proving the spectrum for θ_k. Therefore, henceafter we assume that θ is not at the endpoints of the range in the theorems.

To prove the upper bound of dim it suffices to show that, for any $b > \alpha - r_0\theta^{1/\lambda}$, the Hausdorff b–dimensional measure, $b - m$, of A_θ is zero. This proof is standard and was given in [LS, Section 3]; we include the proof here for completeness. We observe that, for $\epsilon : 0 < \epsilon < \theta$ and positive integer k,

$$A_\theta \subset \cup_{n\ge k}\left\{\xi \in \partial\Gamma : \frac{\mu(B(\xi|n))}{m^{-n}n^\lambda} > (\theta - \epsilon)C\right\},$$

where $C = \left(\frac{\alpha}{r_0}\right)^{\lambda}$. We consider the pre-Hausdorff b-dimensional measure at level k,

$$(b-m)_k(A_\theta) = \inf\Big\{\sum_\sigma |B(\sigma)|^b : \ A \subset \cup B(\sigma), \ \ |\sigma| \geq k, \ \ \forall \sigma\Big\}.$$

Recall that $|B(\sigma)| = e^{-k}$ when $\sigma \in \Gamma_k$. Let I_k denote the random variable defined by

$$I_k = \sum_{|\sigma|=n} |B_\sigma| 1\Big\{\frac{\mu(B(\xi|n))}{m^{-n}n^{\lambda}} > (\theta - \epsilon)C\Big\},$$

then, using the definition of r_0 we see that

$$EI_k \leq \sum_{n \geq k} e^{-(b-\alpha)n} e^{-(r_0-\epsilon)(\theta-\epsilon)^{1/\lambda}C^{1/\lambda}n},$$

when $k = k(\epsilon)$ is large enough. The series in the above display is convergent, so that I_k tends to 0 a.s. as $k \uparrow \infty$. Since ϵ is arbitrary, we conclude that $b - m(A_\theta) = 0$.

To obtain the lower bound of dim, let r_1 be such that $r_0 < r_1$ and $\theta < \left(\frac{\alpha}{r_1}\right)^{\lambda}$. We prove that $\dim A_\theta \geq \alpha - r_1\theta^{1/\lambda}$ by proving that A_θ has infinite Hausdorff ϕ-measure, where

$$\phi(x) = x^{\alpha - r_1\theta^{1/\lambda}}(\log(1/x))^b, \ \ 0 < x < 1, \ \ b > 2.$$

We apply Proposition 3.1 in the following way. Define the random mapping $\Psi(B,\omega)$, $B \subset \mathbb{N}^\infty$ and $\omega \in \Omega$, to be 1, only when $B = B(\sigma), \sigma \in \partial\Gamma_n(\omega)$ and when $W(\sigma,\omega) > n^\lambda\theta$; otherwise Ψ takes value 0. Thus $q_n = E[\Psi(B(\sigma))] = P[W > n^\lambda\theta]$. By our definition of r_0 as a liminf and our choice of r_1 there exists a sequence $n_k \uparrow \infty$ such that $q_n \geq e^{-r_1 n\theta^{1/\lambda}}$, $\forall n = n_k$. We may well assume that n_k satisfies the rapid growth condition in Proposition 3.1. Therefore Theorem 4.1 is an application of Proposition 3.1 with $\delta = \alpha^{-1}r_1\theta^{1/\lambda}$ there, and that r_1 can be arbitrarily close to r_0. To prove the lower bound for B_θ, we need to use a strategy first used in [PT]. Under the stronger assumption on the existence of r, let now

$$\phi(x) = x^{\alpha - r\theta^{1/\lambda}}(\log(1/x))^b, \ \ 0 < x < 1, \ \ b > 2.$$

Then the above arguments assert that Hausdorff ϕ-measure of A_θ is infinite while, by the upper bound proof given in the above, that of $A_{\theta+1/k}$ is 0 for all $k = 1, 2, \cdots$. Thus, $B_\theta = A_\theta \setminus \cup_k A_{\theta+1/k}$ is also of infinite Hausdorff ϕ-measure; in particular we get the desired lower bound for B_θ.

Application of Proposition 3.2 gives the assertion for Dim. □

Remark. We believe that the stronger assumption in Theorem 4.2 for B_θ is not needed; a proof would require a version of the limsup theorem based on Corollary 3.3 of [KPX], in which the target set satidfies a regularity condition so that we can use the value of q_n in Proposition 3.1 on a subsequence. We have not formulated a precise theorem, so in this paper we use the methods of [DPRZ].

5 Multifractal spectrum for thin behavior

The reader is reminded that smaller than usual branching behavior is reflected in large values of $\overline{d}(\mu,\xi)$ defined in Section 1. As we have seen in Section 1 that $d(\mu,\xi)$, and hence so is $\overline{d}(\mu.\xi)$, is equal to α for all ξ whenever $p_0 = p_1 = 0$; thus there is only a trivial mulftifractal spectrum in this case. In this section, we prove that, whenever $0 < p_1$ there is an interesting spectrum for $\overline{d}(\mu,\xi)$(we always assume that $p_0 = 0$ and that $p_j < 1, \quad \forall j$). We need the following lemma which is in Liu[L3].

Lemma 5.1 ([L3, Theorem 4.1(ii)) *If Z has finite moments of all positive orders, then, with probability one,*

$$\underline{d}(\mu,\xi) = \alpha, \quad \forall \xi \in \partial\Gamma.$$

We introduce a new paremeter needed in this section, *assuming that* $p_1 > 0$.

$$\tau = -\frac{\log p_1}{\alpha}.$$

Note that

$$p_1 = e^{-\tau\alpha}.$$

The following small tail distribution of W is known from Bingham[B, p. 217].

$$P[W \le x] \approx x^\tau, \quad x \downarrow 0. \tag{5.1}$$

We first observe that the probability of a long string of vertices giving rise to a single branch leads to the same estimate for $P[W \le x]$. For, if $k > n$ and $\sigma \in \Gamma_n$, then the event E that there is only one $\eta \in \Gamma_k$ descended from σ is

$$P(E) = p_1^{k-n}. \tag{5.2}$$

Now conditional on E, $W_\eta = m^{k-n} W_\sigma$, so that

$$P[W_\sigma \leq m^{-(k-n)} | E] = P[W_\eta \leq 1].$$

Thus, we have

Lemma 5.2 *Under the above conditions, if $x \approx m^{-(k-n)}$, and E is the event that each vertex starting with $\sigma \in \Gamma_n$ has only one descendant up to $\eta \in \Gamma_k$, then*

$$1 \geq \frac{P[\{W_\sigma \leq x\} \cap E]}{P(E)} \geq c_3.$$

We now see how to obtain an efficient cover for points in the thin spectrum.

Lemma 5.3 *Suppose $\gamma > 1, 0 < \epsilon < (\gamma - 1)/3$. Then, with probability one, there is an $n_0 = n_0(\omega)$ such that every vertex $\sigma \in \Gamma_n$ with $n \geq n_0$ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$ has fewer than $e^{\epsilon \alpha n}$ descendants at the level $k = [(\gamma - \epsilon)n]$.*

Proof. For each $\eta \in \Gamma_k$ descended from $\sigma \in \Gamma_n$ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$, it is seen that

$$W_\eta \leq e^{-\epsilon \alpha n}.$$

By (5.1), the above has probability $\leq c_2 e^{-\epsilon \alpha \tau n}$. Putting N_σ as the number of descendants of σ at level k, we have then

$$P[N_\sigma > e^{\epsilon \alpha n} | W_\sigma \leq e^{-(\gamma-1)\alpha n}] \leq [c_2 e^{-\epsilon \alpha \tau n}]^{e^{\epsilon \alpha n}},$$

in which we have used the fact that W_η for distinct $\eta \in \Gamma_k$ are iid. Recall that Z_n counts the vertices $\sigma \in \Gamma_n$, thus the expected number of σ such that $W_\sigma \leq e^{-(\gamma-1)\alpha n}$ and $N_\sigma > e^{\epsilon \alpha n}$ is

$$\leq E[Z_n] \cdot c_2 e^{-\epsilon \alpha \tau n \cdot e^{\epsilon \alpha n}}.$$

Since $E[Z_n] = e^{\alpha n}$ we deduce that the probability that there is at least one such vertex is bounded by $c_2 e^{\alpha n(1 - \epsilon \tau \cdot e^{\epsilon \alpha n})}$, which is the general term of a convergent series. By the Borel–Cantelli Lemma we have proved the lemma. □

Lemma 5.4 *If $0 < p_1 < 1$, then, with probability one,*

$$\text{dim} C_\beta \leq \alpha[\frac{\alpha}{\beta}(1 + \tau) - \tau], \quad \alpha \leq \beta, \tag{5.3}$$

where $C_\beta = \{\xi \in \partial\Gamma : \ \overline{d}(\mu, \xi) \geq \beta\}$.
When $\beta > \alpha(1 + 1/\tau)$, the right hand side of (5.3) is negative, and we interpret this as stating that $C_\beta = \emptyset$.

Proof. (i) When $\beta = \alpha$, (5.3) is immediate.

(ii) When $\beta > \alpha(1+1/\tau)$, we will prove that $C_\beta = \emptyset$ a.s. Take ζ such that $\beta > \zeta > \alpha(1+1/\tau)$, then $\overline{d}(\mu,\xi) \geq \beta$ implies that $\mu B(\xi|n) < e^{-\zeta n}$ for infinitely many integers n. The expected number of those $\sigma \in \Gamma_n$ for which $W_\sigma < e^{(\alpha-\zeta)}$ is

$$E[Z_n] \cdot P[W < e^{(\alpha-\zeta)}] = e^{(\alpha+(\alpha-\zeta)\tau))n},$$

whcih is a negative power of e^n. By Borel–Cantelli Lemma, we deduce that, for $n \geq n_1 = n_1(\omega)$

$$\mu B(\xi|n) \geq e^{-\zeta n}, \quad \forall \xi.$$

By letting $\beta \downarrow \alpha(1+1/\tau)$ through a countable set, we deduce that, with probability one,

$$\overline{d}(\mu,\xi) \leq \alpha(1+1/\tau), \quad \forall \xi \in \partial\Gamma.$$

(iii) Now suppose that $\alpha < \beta < \alpha(1+1/\tau)$. Put $\gamma = \frac{\beta}{\alpha} > 1$. Instead of covering the vertices $\sigma \in \Gamma_n$ where $\mu B(\xi|n) < e^{-\beta n}$ by balls of diameter e^{-n} we use the descendant vertices at level $k = [(\gamma-\epsilon)n]$ which can be covered by balls of diameters e^{-k}. By Lemma 5.3, when n is large enough the number of such vertices is less than $e^{\epsilon\alpha n}$ so that the total number needed will be at most $e^{\epsilon\alpha n} \cdot T_n$ where T_n is the number of those $\sigma \in \Gamma_n$ for which $W_\sigma < e^{-(\beta-\alpha)n}$. Now $E[T_n] = m^n \cdot e^{-(\beta-\alpha)\tau n}$, so that we obtain

$$E[s^\delta - m(C_\beta)] \leq \sum_{n=n_1}^{\infty} e^{-[(\gamma-\epsilon)n]\delta+\epsilon\alpha n-(\beta-\alpha)\tau n+\alpha n},$$

where n_1 is arbitrary. If the power of of e^n in this series is negative, we deduce that $s^\delta - m(C_\beta) = 0$ a.s. This will be true if

$$\delta > \delta_\epsilon := \frac{1+\epsilon-(\gamma-1)\tau}{\gamma-\epsilon} \cdot \alpha.$$

Letting $\epsilon \downarrow 0$ through a countable set, we see that $s^\delta - m(C_\beta) = 0$ a.s. for $\delta \geq \frac{(1-(\gamma-1)\tau)\alpha}{\gamma}$. Substituing $\gamma = \beta/\alpha$ we prove the assertion. $\square$.

We are now ready to prove that (5.3) gives the right answer for $\dim C_\beta$. However, if we are to obtain the same answer for $\dim D_\beta$, as in Section 4, we need to find a gauge function $\phi(s) = s^\Delta L(s)$ with $L(s)$ slowly varying, such that $\phi - m(C_\beta) = \infty$. We will prove this by applying Proposition 3.1, and the strategy is the same as that used in [PT]: we find a random Cantor–like subset T_γ which is γ–thin and use Proposition 3.1 to find its Hausdorff $\phi-$ measure. This set $T_\gamma \subset C_\beta$ by Lemma 2.2 and Lemma 5.1. In order to apply Proposition 3.1, fix a $\gamma > 1$, we define the random mapping Ψ there by defining $\Psi(B,\omega) = 1$ if and only if $B = B(\sigma)$, $\sigma \in \Gamma_n(\omega)$ is such that its ancestor in $\Gamma_{[n/\gamma]-1}(\omega)$ has a string of single branches stretching to σ. Denote the limsup set A there now by T_γ. By (5.2), the probability q_n in Proposition 3.1 is now

$$q_n \geq c \cdot p_1^{(1-1/\gamma)n} \cdot e^{\alpha(1-1/\gamma)n} = c \cdot e^{\alpha n(1-1/\gamma)(\tau+1)}.$$

We remark that the third factor in the middle term of the above display comes from the expected number of all the possible ancestors in $\Gamma_{[n/\gamma]-1}$, given an element in Γ_n. We can now calculate the δ in Proposition 3.1. Note that T_γ is clearly a $\gamma-$thin subset of $\partial\Gamma$ by the construction. Thus we have

Lemma 5.5 *Assume that $0 < p_1 < 1$ and Z has finite moments of all positive orders; let β be fixed, $\alpha < \beta < (1+1/\tau)\alpha$, and define $\gamma = \beta/\alpha$. Then the Hausdorff measure of the $\gamma-$thin set T_γ defined above satisfies $\phi - m(T_\gamma) = +\infty$,*
where the gauge function ϕ is $\phi(x) = x^\Delta(\log(1/x))^3$, with

$$\Delta = \alpha[\frac{1}{\gamma}(1+\tau) - \tau]$$

We can now state our main decomposition.

Theorem 5.1 *If $0 < p_1 < 1$ and Z has finite moments of all positive orders, then the branching meassure μ has the following propoerties, with probability one. Set*

$$C_\beta = \{\xi : \ \overline{d}(\mu,\xi) \geq \beta\}, D_\beta = \{\xi : \ \overline{d}(\mu,\xi) = \beta\},$$

then

(a) C_β and therefore D_β is empty for $\beta > \alpha(1+\frac{1}{\tau})$.

(b) D_β is non-empty for $\alpha \leq \beta \leq \alpha(1+\frac{1}{\tau})$, and in this range

$$\begin{aligned} \text{dim}C_\beta = \text{dim}D_\beta &= \alpha[\frac{\alpha}{\beta}(1+\tau) - \tau] \\ \text{Dim}C_\beta = \text{Dim}D_\beta &= \alpha. \end{aligned}$$

Proof. By Lemmas 2.2 and 5.1, $T_\gamma \subset C_\beta$, where $\gamma = \beta/\alpha$. By Lemma 5.5, the Hausdorff ϕ_Δ- measure of C_β is $+\infty$, where ϕ_Δ is the gauge function there in Lemma 5.5. Regard Δ as a function of β, it is strictly monotone. Lemma 5.4 then tells that ϕ_Δ measure of $C_{\beta+1/k}$ is 0. Thus, arguing as in the proofs of Theorems 4.1 and 4.2, we see that $\dim D_\beta \geq \Delta$. Since $D_\beta \subset C_\beta$, we have completed the proof for $\alpha \leq \beta < \alpha(1+1/\tau)$. In the case where $\beta = \alpha(1+1/\tau)$ we only need to show that $D_\beta = C_\beta$ is non-empty. This will follow if we can construct T_γ for $\gamma = 1+1/\tau$ by requiring the string of single branches to stretch from the level $[n/\gamma - \log n]-1$ to n. This condition forces $\overline{d}(\mu,\xi) \geq \alpha(1+1/\tau)$, on using Lemmas 2.2 and 5.1. $\square$

Remark. In Theorem 5.1 we assume that Z has finite moments of all positive orders is mainly to apply Lemma 5.1. It seems possible that we may weaken the condition in Theorem 5.1 to, say, that Z has finite moments up to a certain order k_0 greater than one and get a spectrum involving $p_+ = \sup\{a \geq 1 : EZ^a < \infty\}$(one critical value in [L3]).

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