On full asymptotic expansion of the solutions of nonlinear periodic rapidly oscillating problems

K.D. Cherednichenko
Department of Mathematical Sciences
University of Bath, Bath BA2 7AY, U.K.
mapkdc@maths.bath.ac.uk
and
V.P. Smyshlyaev*
Department of Mathematical Sciences
University of Bath, Bath BA2 7AY, U.K.
V.P.Smyshlyaev@maths.bath.ac.uk

1 Introduction

This preprint is an overview of our attempt to construct higher order terms in the asymptotic expansion for the solution of the following nonlinear problem:

$$\operatorname{div} \boldsymbol{j}\,(\boldsymbol{x}/\varepsilon, \nabla u^\varepsilon(\boldsymbol{x})) = -f(\boldsymbol{x}), \tag{1.1}$$

where the nonlinear function $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})$ is periodic in $\boldsymbol{\xi}$ and $\varepsilon > 0$ is a small parameter.

The problem of finding the asymptotic behaviour of the solutions to rapidly oscillating equations has been considered in numerous mathematical works during the last two decades.

The case when the function $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})$ is linear with respect to $\boldsymbol{e}$ has been studied comprehensively by many authors. The asymptotic expansions for linear homogenisation problems with periodic coefficients can be found in Bensoussan *et al* (1978),

*The corresponding author

Sanchez-Palencia (1980), Bakhvalov & Panasenko (1984). Also, there have been works treating nonlinear problems, *e.g.* Suquet (1982) derives the homogenised equation and studies the principal term of the asymptotics of the solution in the nonlinear case.

Recently, there has been growing interest to the effect of the *higher order* terms of such asymptotic expansions. The higher order effects have appeared to play an important role in a number of applications (*e.g.* scale effects, strain gradient effects, *etc*). Thus, the problem of finding full asymptotic expansions of solutions to rapidly oscillating problems arises.

Bakhvalov & Panasenko (1984) constructed the full asymptotic expansions for the linear problems. In the paper by Smyshlyaev & Cherednichenko (2000) we studied in detail the structure of higher order terms and implications for construction of higher order homogenised equations in this case.

Bakhvalov & Panasenko (1984) have also discussed briefly possible extensions of their approach to the nonlinear case and the structure of the associated formal asymptotic expansion. But the problem of accurate construction, finding precise structure and rigorous justification of the full asymptotic expansion for solutions of nonlinear equations has not been addressed.

This preprint is intended to have a role of the draft of our recent achievements in this direction.

We show that the solution $u^\varepsilon(\boldsymbol{x})$ to the problem (1.1) has asymptotics (3.10), (3.11) and the "infinite order" homogenised equation has the form (3.21). In particular, the higher order terms in ε of the nonlinear homogenised equation (3.21) involve higher derivatives ("strain gradients") of the homogenised solution $v(\boldsymbol{x})$. This asymptotics is further rigorously justified when the nonlinear function $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})$ satisfies certain technical conditions (Section 4). We also discuss some further developments and prospects (Section 5).

We use the notation $\nabla_e, \nabla_x, ..., \text{div}_e, \text{div}_x, ...$ *etc* for corresponding differential operators with respect to the appropriate variables. The powers $(\nabla_e)^l, (\nabla_x)^l, ...$ denote corresponding tensors of l-th order derivatives. Also in the formulas, when indices repeat summation is implied, every time being carried out over the whole range of the index. Throughout the text, symbols '$\cdot$' and '$\otimes$' denote dot product and tensor product respectively, and angle brackets '$\langle\ \rangle$' stand for the average with respect to the variable $\boldsymbol{\xi}$.

2 Formulation of the problem

Consider the following equation

$$\text{div}\, \boldsymbol{j}\,(\boldsymbol{x}/\varepsilon, \nabla u^\varepsilon(\boldsymbol{x})) = -f(\boldsymbol{x}), \quad \varepsilon > 0. \tag{2.2}$$

Here $\boldsymbol{x} \in \mathbf{R}^d$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})$ is some nonlinear vector function, periodic in $\boldsymbol{\xi} \in \mathbf{R}^d$ with the periodicity cell $Q = [0,1]^d$ ($d = 2$ or $d = 3$ in physical applications). For example, this function can be conductivity current density or elastic stress tensor. (In the latter case $\boldsymbol{j}$ is a $d \times d$ matrix.) We will consider the case when the unknown function $u^\varepsilon(\boldsymbol{x})$ is scalar and the function $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})$ takes values in $\mathbf{R}^d$. Then the function $f(\boldsymbol{x})$ is scalar. We also assume that it is periodic with a fixed period $\mathbf{T} = [0,T]^d$ where T is a multiple of ε, *i.e.* $T/\varepsilon \in \mathbf{N}$, and the function $f(\boldsymbol{x})$ has zero mean over $\mathbf{T}$.

In this study, we restrict ourselves to the case when there exists a potential $W = W(\boldsymbol{\xi}, \boldsymbol{e})$ such that $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})$. We assume that the function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$ is infinitely smooth, satisfies a growth condition as follows

$$-A_1 + B_1|\boldsymbol{e}|^p \leq W(\boldsymbol{\xi}, \boldsymbol{e}) \leq A_2 + B_2|\boldsymbol{e}|^p \;\text{ for any } \boldsymbol{\xi}, \boldsymbol{e} \in \mathbf{R}^d \tag{2.3}$$

with some positive constants A_1, A_2, B_1, B_2 and $p > 1$. Function $W(\boldsymbol{\xi}, \boldsymbol{e})$ is required to be convex in $\boldsymbol{e}$. Moreover, we will usually require that the following inequality holds with some constant $\nu > 0$

$$\frac{\partial^2 W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_i \partial e_j}\eta_i\eta_j \geq \nu\eta_i\eta_i \tag{2.4}$$

for any $\boldsymbol{\xi}, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\eta} = (\eta_1, ..., \eta_d) \in \mathbf{R}^d$.

Having fixed $A_1, A_2, B_1, B_2, p, \nu$ and $\mathbf{T}-$periodic function $f \in C^\infty(\mathbf{R}^d) \subset L^{p'}(\mathbf{T})$, $1/p + 1/p' = 1$, with zero mean over $\mathbf{T}$ we consider the following variational problem

$$\min_{u \in W^{1,p}_{0,per}(\mathbf{T})} \int_{\mathbf{T}} \Big(W(\boldsymbol{x}/\varepsilon, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x}) \Big) \mathrm{d}\boldsymbol{x}, \tag{2.5}$$

where $W^{1,p}_{0,per}(\mathbf{T})$ is the space of all $\mathbf{T}$-periodic functions from the Sobolev space $W^{1,p}_{loc}(\mathbf{R}^d)$ having zero mean over $\mathbf{T}$, with the norm being $\|u\|_{W^{1,p}_{0,per}(\mathbf{T})} = \|\nabla u\|_{L^p(\mathbf{T})}$.

The functional $F_\varepsilon[u] = \int_{\mathbf{T}}(W(\boldsymbol{x}/\varepsilon, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$ is weakly lower semi-continuous and coercive on $W^{1,p}_{0,per}(\mathbf{T})$. Thus, the problem (2.5) has at least one solution in $W^{1,p}_{0,per}(\mathbf{T})$. Equation (2.2) is the Euler-Lagrange equation for the problem (2.5).

Normally, one also needs to impose some restriction on the function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$ to make sure that the solution is unique and depends continuously on the right-hand side $-f(\boldsymbol{x})$ of the equation (2.2). This is important for the subsequent asymptotic analysis. A typical restriction for this purpose is the requirement of strong monotonicity of the function $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})$ with respect to $\boldsymbol{e}$:

$$\Big(\nabla_e W(\boldsymbol{\xi}, \boldsymbol{e}_1) - \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e}_2)\Big) \cdot \Big(\boldsymbol{e}_1 - \boldsymbol{e}_2\Big) \geq \alpha |\boldsymbol{e}_1 - \boldsymbol{e}_2|^p, \quad \alpha > 0, \tag{2.6}$$

where p is the same as in (2.3), for every $\boldsymbol{\xi} \in \mathbf{T}$ and all $\boldsymbol{e}_1, \boldsymbol{e}_1 \in \mathbf{R}^d$. It is well-known that if (2.6) and (2.3) hold then the solution to the problem (2.5) (equivalently, to the problem (2.2)) is unique and continuously depends on $f \in L^{p'}(\mathbf{T})$. We re-derive this below for the reader's convenience.

Note that a solution $u^\varepsilon(\boldsymbol{x})$ to the problem (2.5) is a stationary point of the functional $F_\varepsilon[u]$, *i.e.* the following identity holds

$$\int_{\mathbf{T}} \nabla_e W(\boldsymbol{x}/\varepsilon, \boldsymbol{e})\Big|_{e=\nabla u^\varepsilon(x)} \cdot \nabla\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} f(\boldsymbol{x})\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{2.7}$$

for any function $\phi \in W^{1,p}_{0,per}(\mathbf{T})$.[1] Suppose $u_1^\varepsilon(\boldsymbol{x})$ and $u_2^\varepsilon(\boldsymbol{x})$ are solutions to the problem (2.2) with the right-hand sides $-f_1(\boldsymbol{x})$ and $-f_2(\boldsymbol{x})$ respectively. Then in view of (2.7)

$$\int_{\mathbf{T}}\Big(\nabla_e W(\boldsymbol{x}/\varepsilon, \boldsymbol{e})\Big|_{e=\nabla u_1^\varepsilon(x))} - \nabla_e W(\boldsymbol{x}/\varepsilon, \boldsymbol{e})\Big|_{e=\nabla u_2^\varepsilon(x))}\Big) \cdot \nabla(u_1^\varepsilon(\boldsymbol{x}) - u_2^\varepsilon(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$

$$= \int_{\mathbf{T}} (f_1(\boldsymbol{x}) - f_2(\boldsymbol{x}))(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}.$$

Using the inequality (2.6) we obtain

$$\int_{\mathbf{T}}(\nabla_e W(\boldsymbol{x}/\varepsilon, \boldsymbol{e})\Big|_{e=\nabla u_1^\varepsilon(x))} - \nabla_e W(\boldsymbol{x}/\varepsilon, \boldsymbol{e})\Big|_{e=\nabla u_2^\varepsilon(x))}) \cdot \nabla(u_1^\varepsilon(\boldsymbol{x}) - u_2^\varepsilon(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$$

$$\geq \alpha \int_{\mathbf{T}} |\nabla(u_1^\varepsilon(\boldsymbol{x}) - u_2^\varepsilon(\boldsymbol{x}))|^p \mathrm{d}\boldsymbol{x} = \alpha \|u_1^\varepsilon(\boldsymbol{x}) - u_2^\varepsilon(\boldsymbol{x})\|^p_{W^{1,p}_{0,per}(\mathbf{T})}.$$

On the other hand, the following inequality holds

$$\left|\int_{\mathbf{T}} (f_1(\boldsymbol{x}) - f_2(\boldsymbol{x}))(u_1(\boldsymbol{x}) - u_2(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right| \leq \|f_1 - f_2\|_{L^{p'}(\mathbf{T})}\|u_1^\varepsilon - u_2^\varepsilon\|_{L^p(\mathbf{T})} \leq$$

[1]Derivation of (2.7) uses the fact that for a smooth convex function $W(\boldsymbol{\xi}, e)$ satisfying (2.3) $|\nabla_e W(\boldsymbol{\xi}, e)| \leq A + B|e|^{p-1}$ with some positive constants A and B.

$$\leq c||f_1 - f_2||_{L^{p'}(\mathbf{T})}||u_1^\varepsilon - u_2^\varepsilon||_{W^{1,p}_{0,per}(\mathbf{T})}, \quad c > 0,$$

where Hölder and Poincaré inequalities have been used. Hence, we conclude that

$$||u_1^\varepsilon - u_2^\varepsilon||_{W^{1,p}_{0,per}(\mathbf{T})} \leq \left(\frac{c}{\alpha}||f_1 - f_2||_{L^{p'}(\mathbf{T})}\right)^{\frac{1}{p-1}}. \tag{2.8}$$

3 Formal asymptotical procedure

Following a sketch in Bakhvalov & Panasenko (1984), we are seeking a formal asymptotic expansion of the solution to the problem (1.1) in the following form separating the "slow" and the "fast" variables

$$u^\varepsilon(\boldsymbol{x}) \sim \sum_{l=0}^{\infty} \varepsilon^l u_l(\boldsymbol{x}/\varepsilon, \boldsymbol{x}), \tag{3.9}$$

where the functions $u_l(\boldsymbol{\xi}, \boldsymbol{x})$, $l = 0, 1, 2, \ldots$ are Q-periodic with respect to the "fast" variable $\boldsymbol{\xi} = \boldsymbol{x}/\varepsilon$ and $\mathbf{T}$-periodic with respect to the "slow" variable $\boldsymbol{x}$. The idea of the classical "ansatz" (3.9) is to seek the solution as a decomposition in sequential powers of the small parameter ε whose "coefficients", the functions u_l, are periodically oscillating with respect to the fast variable while the oscillation parameters are modulated by the dependency on the slow variable $\boldsymbol{x}$.

Substitution of the ansatz (3.9) into the original equation (2.2) leads us to a more specific structure of the functions $u_l(\boldsymbol{\xi}, \boldsymbol{x})$. Namely, further we consider the following ansatz

$$u^\varepsilon(\boldsymbol{x}) \sim v(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l u_l(\boldsymbol{x}/\varepsilon, \nabla v(\boldsymbol{x}, \varepsilon), \nabla\nabla v(\boldsymbol{x}, \varepsilon), ..., \nabla^l v(\boldsymbol{x}, \varepsilon)), \tag{3.10}$$

where

$$v(\boldsymbol{x}, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}). \tag{3.11}$$

Functions $u_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x}))$ are Q-periodic in $\boldsymbol{\xi}$ and have zero mean over Q in $\boldsymbol{\xi}$; functions $v_s(\boldsymbol{x})$ are $\mathbf{T}$-periodic with zero mean over $\mathbf{T}$, and do not depend on the fast variable $\boldsymbol{\xi} = \boldsymbol{x}/\varepsilon$.

Now, substitute the series (3.10) into the equation (2.2). After differentiation, formal application of the Taylor formula and another differentiation we end up with a formal asymptotic series in the left-hand side of the equation:

$$\sum_{l=-1}^{\infty} \varepsilon^l H_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})) = -f(\boldsymbol{x}), \tag{3.12}$$

where the functions $H_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x}))$ are Q-periodic in $\boldsymbol{\xi}$. In particular,

$$H_{-1}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x})) = \mathrm{div}_\xi \boldsymbol{j}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}) + \nabla_\xi u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))),$$

$$H_0(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})) = \mathrm{div}_x \boldsymbol{j}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}) + \nabla_\xi u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x})))$$
$$+\mathrm{div}_\xi \Big(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}) + \nabla_\xi u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x})))$$
$$\cdot(\nabla_x u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x})) + \nabla_\xi u_2(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})))\Big).$$

At this point we are going to introduce some conditions on the functions $H_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x}))$, which will later prove to be plausible in the sense that they provide us with the way on which we can find an asymptotics of the solution $u^\varepsilon(\boldsymbol{x})$, which can be justified.

First, it is natural to require that the function $H_{-1}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))$ is identically zero:

$$\mathrm{div}_\xi \boldsymbol{j}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}) + \nabla_\xi u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))) = 0 \tag{3.13}$$

This can be viewed as an equation for the function $u_1(\boldsymbol{\xi}, \boldsymbol{z})$, where $\boldsymbol{z} \in \mathbf{R}^d$ is a parameter:

$$\mathrm{div}_\xi \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z})) = 0. \tag{3.14}$$

By virtue of the conditions formulated in the previous section, the last equation has a Q-periodic solution $u_1(\boldsymbol{\xi}, \boldsymbol{z})$, which is unique up to an arbitrary constant. We impose the condition $\langle u_1(\boldsymbol{\xi}, \boldsymbol{z}) \rangle = 0$ for any $\boldsymbol{z} \in \mathbf{R}^d$, which provides a unique solution $u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))$ to the equation (3.13). Note that the function $v(\boldsymbol{x})$ is still unknown. The function $u_1(\boldsymbol{\xi}, \boldsymbol{z})$ is a nonlinear version of the solution to the periodic unit-cell problem. It minimizes the functional

$$\int_Q W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u(\boldsymbol{\xi}, \boldsymbol{z})) \mathrm{d}\boldsymbol{\xi}. \tag{3.15}$$

It is well-known (see *e.g.* Ladyzhenskaya & Uraltseva, 1968) that $u_1(\boldsymbol{\xi}, \boldsymbol{z})$ is smooth with respect to $\boldsymbol{\xi}$ as a minimizer of regular functional (3.15). In fact, it can also be shown that under the assumptions on the function $W(\boldsymbol{\xi}, \boldsymbol{e})$ listed in Section 2 the function $u_1(\boldsymbol{\xi}, \boldsymbol{z})$ is smooth with respect to the *pair* of arguments $\boldsymbol{\xi}$ and $\boldsymbol{z}$. The derivation is nontrivial and uses the implicit function theorem in functional spaces.

Proceeding further, we require that the functions $H_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x}))$ for $l \geq 0$ do not depend on $\boldsymbol{\xi}$, *i.e.*,

$$H_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})) = h_l(\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})), \quad l = 0, 1, 2, ... \tag{3.16}$$

for some functions h_l depending on the slow variable *only*. This requirement gives a set of recurrence relations for the functions $u_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x}))$, $l \geq 2$.

For example, the condition $H_0(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})) = h_0(\nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}))$ gives us the following equation for the function $u_2(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}))$:

$$\begin{aligned}
&\operatorname{div}_\xi\Big(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})|_{e=\nabla v(x)+\nabla_\xi u_1(\xi,\nabla v(x))} \cdot \nabla_\xi u_2(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}))\Big) \\
&= h_0(\nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})) - \operatorname{div}_x \boldsymbol{j}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}) + \nabla_\xi u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))) \\
&\quad -\operatorname{div}_\xi\Big(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})|_{e=\nabla v(x)+\nabla_\xi u_1(\xi,\nabla v(x))} \cdot \nabla_x u_1(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}))\Big)
\end{aligned} \tag{3.17}$$

For better understanding of the structure of the last equation, introduce parameters $\boldsymbol{z} \in \mathbf{R}^d$ and $\boldsymbol{w} \in \mathbf{R}^{d\times d}$ standing for $\nabla v(\boldsymbol{x})$ and $\nabla\nabla v(\boldsymbol{x})$ respectively.

The equation (3.17) takes the following form

$$\begin{aligned}
&\operatorname{div}_\xi\Big(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})|_{e=z+\nabla_\xi u_1(\xi,z)} \cdot \nabla_\xi u_2(\boldsymbol{\xi}, \boldsymbol{z}, \boldsymbol{w})\Big) \\
&= h_0(\boldsymbol{z}, \boldsymbol{w}) - \nabla_z \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z})) \cdot \boldsymbol{w} \\
&\quad -\operatorname{div}_\xi(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})|_{e=z+\nabla_\xi u_1(\xi,z)} \otimes \nabla_z u_1(\boldsymbol{\xi}, \boldsymbol{z})) \cdot \boldsymbol{w}.
\end{aligned} \tag{3.18}$$

Note that this equation for $u_2(\boldsymbol{\xi}, \boldsymbol{z}, \boldsymbol{w})$ with respect to $\boldsymbol{\xi}$ with parameters $\boldsymbol{z}$ and $\boldsymbol{w}$ is linear. Further, since by our assumption $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})$ and hence $\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e}) = (\nabla_e)^2 W(\boldsymbol{\xi}, \boldsymbol{e})$, in view of (2.4) the equation (3.18) is uniformly elliptic. The smoothness of $u_1(\boldsymbol{\xi}, \boldsymbol{z})$ with respect to $\boldsymbol{\xi}$ and $\boldsymbol{z}$ ensures the smoothness of the right-hand side of (3.18) and therefore the smoothness of u_2. It is well-known that for solvability of such an equation it is necessary and sufficient that the average with respect to $\boldsymbol{\xi}$ of its right-hand side is zero. This condition gives us the formula for the function $h_0(\boldsymbol{z}, \boldsymbol{w})$ as follows

$$h_0(\boldsymbol{z}, \boldsymbol{w}) = \nabla \hat{\boldsymbol{j}}(\boldsymbol{z}) \cdot \boldsymbol{w}, \tag{3.19}$$

where $\hat{\boldsymbol{j}}(\boldsymbol{z}) = \langle \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z})) \rangle$.

Define the function $h_0(\boldsymbol{z}, \boldsymbol{w})$ by the formula (3.19). Then there exists a unique solution of the equation (3.18) with zero mean over Q. It could be further shown routinely that in the same fashion the functions $u_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})), l = 3, 4, ...$ can be found. The equations for them will have the same structure

$$\operatorname{div}_\xi \Big(\nabla_e \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e})|_{e=\nabla v(x)+\nabla_\xi u_1(\xi, \nabla v(x))} \cdot \nabla_\xi u_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})) \Big)$$
$$= h_{l-2}(\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})) - F_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})), \qquad (3.20)$$

where the function $F_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x}))$ can be expressed in terms of the functions $u_{l'}(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})), l' = 1, 2, ..., l-1$, which are already known.

The equation (3.20) is obviously linear and uniformly elliptic, so the solvability condition for this equation is following

$$h_{l-2}(\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})) = \Big\langle F_l(\boldsymbol{\xi}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})) \Big\rangle$$

The last equation defines the function $h_{l-2}(\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x}))$.

The function $v(\boldsymbol{x})$ is still unknown.

Now as a result of the above construction we have, from (3.12), (3.16), a formal asymptotic equation in the following form involving only the slow variable $\boldsymbol{x}$

$$\sum_{l=0}^{\infty} \varepsilon^l h_l(\nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})) = -f(\boldsymbol{x}). \qquad (3.21)$$

This equation can be resolved formally by substituting into it the series (3.11) and performing some formal transformations, namely, a series of differentiations in the arguments of the functions h_l and then expanding slowly varying functions h_l into the Taylor series in powers of ε. On this way we obtain a sequence of equations for the functions $v_s(\boldsymbol{x})$.

The first equation of this sequence is following

$$h_0(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x})) = -f(\boldsymbol{x}). \qquad (3.22)$$

Recall that from (3.19)

$$h_0(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x})) = \operatorname{div} \hat{\boldsymbol{j}}(\nabla v_0(\boldsymbol{x})),$$

where $\hat{\boldsymbol{j}}(\boldsymbol{z}) = \langle \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z})) \rangle$. We review in the Appendix A the fact that there exists a potential function $\hat{W}(\boldsymbol{z})$ for the equation (3.22) such that

$$\hat{\boldsymbol{j}}(\boldsymbol{z}) = \nabla \hat{W}(\boldsymbol{z}). \qquad (3.23)$$

The function $\hat{W}(\boldsymbol{z}) = \langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z}))\rangle$ is the "conventional" effective, or homogenised, energy for nonlinear periodic homogenisation.

Hence, the equation (3.22) reads

$$\operatorname{div}\left(\nabla \hat{W}(\boldsymbol{z})\Big|_{z=\nabla v_0(x)}\right) = -f(\boldsymbol{x})$$

and admits the equivalent variational formulation

$$\min_{v(\boldsymbol{x})\in W^{1,p}_{0,per}(\mathbf{T})} \int_{\mathbf{T}} (\hat{W}(\nabla v(\boldsymbol{x})) - f(\boldsymbol{x})v(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \tag{3.24}$$

The homogenised energy $\hat{W}(\boldsymbol{z})$ inherits all the properties of the function $W(\boldsymbol{\xi}, \boldsymbol{e})$ that are of importance to us. In the Appendix B we show that if (2.3) is satisfied then the similar growth condition is fulfilled for $\hat{W}$:

$$-A_1 + B_1|\boldsymbol{z}|^p \leq \hat{W}(\boldsymbol{z}) \leq A_2 + B_2|\boldsymbol{z}|^p \quad \text{for any } \boldsymbol{z} \in \mathbf{R}^d, \tag{3.25}$$

and in the Appendix C we verify that the function $\hat{W}(\boldsymbol{z})$ has the property of strong monotonicity:

$$\left(\nabla\hat{W}(\boldsymbol{z}_1) - \nabla\hat{W}(\boldsymbol{z}_2)\right) \cdot \left(\boldsymbol{z}_1 - \boldsymbol{z}_2\right) \geq \alpha|\boldsymbol{z}_1 - \boldsymbol{z}_2|^p, \tag{3.26}$$

as long as (2.6) holds. Also, we prove in the Appendix A that if (2.4) holds then the similar inequality holds for $\hat{W}$:

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j}\eta_i\eta_j \geq \nu\eta_i\eta_i \tag{3.27}$$

for any $\boldsymbol{z}, \boldsymbol{\eta} \in \mathbf{R}^d$. Note that all the constants $(p, A_1, A_2, B_1, B_2, \alpha, \nu)$ entering (3.25)–(3.27) are the same as in (2.3)–(2.6).

Hence, the equation (3.22) has a solution, which is unique up to an arbitrary constant. We choose the function $v_0(\boldsymbol{x})$ to have zero mean over $\mathbf{T}$, *i.e.* $\int_{\mathbf{T}} v_0(\boldsymbol{x})\mathrm{d}x = 0$. If $\hat{W}(\boldsymbol{z})$ is smooth[2], it can be shown that the minimizer of (3.24) is smooth (see *e.g.* Ladyzhenskaya & Uraltseva, 1968).

[2]Smoothness of the function $\hat{W}(\boldsymbol{z})$ is ensured by smoothness of $u_1(\boldsymbol{\xi}, \boldsymbol{z})$, see the discussion on p.6.

The second equation of the sequence obtained by substituting the series (3.11) into the formal equation (3.21), *i.e.* the equation for $v_1(\boldsymbol{x})$, is linear and can be written in the following way

$$\left(\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial z_i}\frac{\partial v_1(\boldsymbol{x})}{\partial x_i}+\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial w_{ij}}\frac{\partial^2 v_1(\boldsymbol{x})}{\partial x_i \partial x_j}\right)\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)}$$
$$= -h_1(\nabla v_0(\boldsymbol{x}),\nabla\nabla v_0(\boldsymbol{x}),\nabla\nabla\nabla v_0(\boldsymbol{x})). \tag{3.28}$$

Note that

$$h_0(\boldsymbol{z},\boldsymbol{w}) = \nabla\hat{\boldsymbol{j}}(\boldsymbol{z})\cdot\boldsymbol{w},$$

where $\hat{\boldsymbol{j}}(\boldsymbol{z}) = \nabla\hat{W}(\boldsymbol{z})$, and so the following identities hold

$$\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial z_i}\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)} = \frac{\partial}{\partial x_j}\left(\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial w_{ij}}\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)}\right) =$$
$$= \frac{\partial}{\partial x_j}\left(\frac{\partial^2\hat{W}(\boldsymbol{z})}{\partial z_i\partial z_j}\Bigg|_{z=\nabla v_0(x)}\right), \quad i=1,...,d.$$

In view of these identities we rewrite the equation (3.28) in a divergence form as follows

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^2\hat{W}(\boldsymbol{z})}{\partial z_i\partial z_j}\Bigg|_{z=\nabla v_0(x)}\frac{\partial v_1}{\partial x_j}\right) = f_1(\boldsymbol{x}), \tag{3.29}$$

where $f_1(\boldsymbol{x}) = -h_1(\nabla v_0(\boldsymbol{x}),\nabla\nabla v_0(\boldsymbol{x}),\nabla\nabla\nabla v_0(\boldsymbol{x}))$. It can be shown that the function $f_1(\boldsymbol{x})$ has zero mean over $\mathbf{T}$. The linear equation (3.29) is uniformly elliptic by virtue of (3.27). It follows that there exists a unique solution $v_1(\boldsymbol{x})$ of (3.29) with zero mean over $\mathbf{T}$.

In the same fashion one can proceed with this recurrent procedure of finding the functions $v_s(\boldsymbol{x})$, $s=0,1,...$ and see that at the s-th step the equation for $v_s(\boldsymbol{x})$ can be obtained, which has the following form akin to (3.29)

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^2\hat{W}(\boldsymbol{z})}{\partial z_i\partial z_j}\Bigg|_{z=\nabla v_0(x)}\frac{\partial v_s}{\partial x_j}\right) = f_s(\boldsymbol{x}), \tag{3.30}$$

where the function $f_s(\boldsymbol{x})$ is expressed in terms of the functions $v_0, v_1, ..., v_{s-1}$, which are already known, and has zero mean over $\mathbf{T}$.

The equation (3.30) is linear and uniformly elliptic, thus the solution $v_s(\boldsymbol{x})$ with zero mean over $\mathbf{T}$ does exist and is unique.

This completes the procedure of constructing a formal asymptotic solution for the nonlinear equation (1.1).

4 Justification of the formal asymptotics (3.10), (3.11)

The asymptotics (3.10), (3.11) can be justified in the following sense. If we truncate both series (3.10) and (3.11) and substitute the truncation of the second series

$$v^{(K)}(\boldsymbol{x},\varepsilon)=\sum_{s=0}^{K}\varepsilon^s v_s(\boldsymbol{x}). \tag{4.31}$$

into the truncation of the first one

$$u^{(K)}(\boldsymbol{x},\varepsilon)=v^{(K)}(\boldsymbol{x},\varepsilon)+\sum_{l=1}^{K}\varepsilon^l u_l(\boldsymbol{\xi}/\varepsilon,\nabla v^{(K)}(\boldsymbol{x},\varepsilon),\nabla\nabla v^{(K)}(\boldsymbol{x},\varepsilon),\dots,\nabla^l v^{(K)}(\boldsymbol{x},\varepsilon)), \tag{4.32}$$

then the following inequality holds with some constant C_{K-1}

$$\|u^\varepsilon(\boldsymbol{x})-u^{(K)}(\boldsymbol{x},\varepsilon)\|_{W^{1,p}_{0,per}(\mathbf{T})}\le C_{K-1}\varepsilon^{K-1}. \tag{4.33}$$

We prove the inequality (4.33) in the following way. Substitute the sum (4.32) into the original equation (2.2). Using the well-known formula for the remainder of the Taylor series and formulas for the functions H_l obtained in the previous section we get

$$\text{div}\ \boldsymbol{j}\left(\boldsymbol{x}/\varepsilon,\nabla u^{(K)}(\boldsymbol{x})\right)$$

$$=-f(\boldsymbol{x})+\sum_{l=-1}^{K-2}\varepsilon^l H_l(\boldsymbol{x}/\varepsilon,\nabla v^{(K)}(\boldsymbol{x}),\nabla\nabla v^{(K)}(\boldsymbol{x}),\dots,\nabla^{l+2}v^{(K)}(\boldsymbol{x}))+\varepsilon^{K-1}R_{K-1}(\boldsymbol{x}/\varepsilon,\boldsymbol{x}),$$

where $R_{K-1}(\boldsymbol{\xi},\boldsymbol{x})$ is certain polynomial of $\nabla_x u_1+\nabla_\xi u_2,\dots,\nabla_x u_{K-1}+\nabla_\xi u_K$, $\nabla_x u_K$ and of $(\nabla_e)^l W(\boldsymbol{\xi},\boldsymbol{e})|_{e=e_l(x)}$ with $1\le l\le K+1$ and uniformly bounded vector functions $\boldsymbol{e}_l(\boldsymbol{x})$. Since the potential $W(\boldsymbol{\xi},\boldsymbol{e})$ is assumed to be infinitely smooth for $\boldsymbol{\xi}\in Q$ and $\boldsymbol{e}\in\mathbf{R}^d$, the remainder $R_{K-1}(\boldsymbol{\xi},\boldsymbol{x})$ is uniformly bounded by some constant $\hat{C}_{K-1}$. Proceeding further, we recall that $H_{-1}(\boldsymbol{\xi},\nabla v^{(K)}(\boldsymbol{x}))\equiv 0$ and also in view of (3.16)

$$H_l(\boldsymbol{\xi},\nabla v^{(K)}(\boldsymbol{x}),\dots,\nabla^{l+2}v^{(K)}(\boldsymbol{x}))=h_l(\nabla v^{(K)}(\boldsymbol{x}),\dots,\nabla^{l+2}v^{(K)}(\boldsymbol{x})),\quad l=0,1,2,\dots$$

Thus,

$$\text{div}\ \boldsymbol{j}\left(\boldsymbol{x}/\varepsilon,\nabla u^{(K)}(\boldsymbol{x})\right)=$$

$$= \sum_{l=0}^{K-2} \varepsilon^l h_l(\nabla v^{(K)}(\boldsymbol{x}), \nabla\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})) + \varepsilon^{K-1} R_{K-1}(\boldsymbol{x}/\varepsilon, \boldsymbol{x}). \quad (4.34)$$

Taking into account the recurrence relations (3.30) for the functions $v_0(\boldsymbol{x}), ..., v_K(\boldsymbol{x})$ we get (*c.f.* (3.21))

$$\sum_{l=0}^{K-2} \varepsilon^l h_l(\nabla v^{(K)}(\boldsymbol{x}), \nabla\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})) = -f(\boldsymbol{x}) + \varepsilon^{K-1} r_{K-1}(\boldsymbol{x}), \quad (4.35)$$

where $r_{K-1}(\boldsymbol{x})$ is a polynomial of $\nabla^l v_1,...,\nabla^l v_K$ with $1 \leq l \leq K$ and of the derivatives $(\nabla_{z_1}, ..., \nabla_{z_{l+2}})^l h_l(\boldsymbol{z}_1, ..., \boldsymbol{z}_{j+2})|_{z_1=z_1(x),...,z_{l+2}=z_{l+2}(x)}$ with $1 \leq l \leq K$ and some uniformly bounded vector functions $\boldsymbol{z}_1(\boldsymbol{x}),..., \boldsymbol{z}_{j+2}(\boldsymbol{x})$. Hence, the remainder $r_{K-1}(\boldsymbol{x})$ is uniformly bounded by some constant $\tilde{C}_{K-1}$.

From (4.34) and (4.35) we achieve the following equality

$$\operatorname{div} \boldsymbol{j} \left(\boldsymbol{x}/\varepsilon, \nabla u^{(K)}(\boldsymbol{x})\right) = -f(\boldsymbol{x}) + \varepsilon^{K-1}(R_{K-1}(\boldsymbol{x}/\varepsilon, \boldsymbol{x}) + r_{K-1}(\boldsymbol{x}))$$

$$= -f(\boldsymbol{x}) + \varepsilon^{K-1}\theta_{K-1}(\boldsymbol{x}, \varepsilon),$$

where $|\theta_{K-1}(\boldsymbol{x}, \varepsilon)| \leq C_{K-1} = \hat{C}_{K-1} + \tilde{C}_{K-1}$.

The inequality (4.33) now follows from (2.8).

5 Some further remarks and prospects

Infinite order homogenised solution

We execute an idea introduced in the paper by Smyshlyaev & Cherednichenko (2000) to cancel the effect of rapid oscillations in the asymptotics (3.10) by considering a family of "translated" problems of the form (2.2) with a parameter $\boldsymbol{\zeta} \in Q$:

$$\operatorname{div} \boldsymbol{j} \left(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u^{\varepsilon}(\boldsymbol{x})\right) = -f(\boldsymbol{x}), \quad \varepsilon > 0. \quad (5.36)$$

For any $\boldsymbol{\zeta} \in Q$ the problem (5.36) has a unique solution $u^{\zeta,\varepsilon}(\boldsymbol{x})$. Consider the averaging of this solution with respect to the parameter $\boldsymbol{\zeta}$:

$$\bar{u}^{\varepsilon}(\boldsymbol{x}) = \int_Q u^{\zeta,\varepsilon}(\boldsymbol{x}) \mathrm{d}\boldsymbol{\zeta}.$$

Then for any $K = 0, 1, 2, ...$ the following estimate holds with some constant $C^{(K)} > 0$

$$\int_{\mathbf{T}} \left(\bar{u}^{\varepsilon}(\boldsymbol{x}) - \sum_{s=0}^{K} \varepsilon^s v_s(\boldsymbol{x}) \right)^2 \mathrm{d}\boldsymbol{x} \leq C^{(K)} \varepsilon^{2K}. \tag{5.37}$$

The proof is completely analogous to that given in (Smyshlyaev & Cherednichenko, 2000).

Therefore, $\bar{u}^{\varepsilon}(\boldsymbol{x})$ may be called the infinite order homogenised solution and (5.37) implies that the series (3.11) is the asymptotics of $\bar{u}^{\varepsilon}$.

Higher order homogenised variational problems

In the same fashion as in the paper (Smyshlyaev & Cherednichenko, 2000) we can consider a family of variational problems with a parameter $\boldsymbol{\zeta} \in Q$:

$$I^{\boldsymbol{\zeta}}(\varepsilon, f) = \min_{u(\boldsymbol{x})} E_{\varepsilon}^{\zeta}(u, f) = \min_{u \in W^{1,p}_{0,per}} \int_{\mathbf{T}} (W(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}.$$

Introducing the $\boldsymbol{\zeta}$-averaged energy functional

$$\bar{I}(\varepsilon, f) = \int_Q I^{\zeta}(\varepsilon, f)\mathrm{d}\boldsymbol{\zeta} = \min_{u(\boldsymbol{x},\zeta)} \int_Q E_{\varepsilon}^{\zeta}(u, f)\mathrm{d}\zeta$$

$$= \min_{u(\boldsymbol{x},\zeta)} \int_Q \int_{\mathbf{T}} \Big(W(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x}) \Big) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}$$

we restrict the last minimisation to the set

$$U^{(K)} = \left\{ u(\boldsymbol{x}, \boldsymbol{\zeta}) \in W^{1,p}_{0,per}(\mathbf{T}) : u(\boldsymbol{x}, \boldsymbol{\zeta}) = v(\boldsymbol{x}) + \sum_{l=1}^{K-1} \varepsilon^l u_l \Big(\frac{\boldsymbol{x}}{\varepsilon} + \boldsymbol{\zeta}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x}) \Big) \right\}, \tag{5.38}$$

where $v(\boldsymbol{x}) \in W^{K,p}_{0,per}(\mathbf{T})$, the set of all $\mathbf{T}$-periodic functions from the Sobolev space $W^{K,p}_{loc}(\mathbf{R}^d)$. Then

$$\min_{u(\boldsymbol{x},\zeta) \in U^{(K)}} \int_{\mathbf{T}} \int_Q \Big(W(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x}) \Big) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}$$

$$= \min_{v(\boldsymbol{x})} \int_{\mathbf{T}} \Big(\hat{W}^{(K)}(\nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})) - f(\boldsymbol{x})v(\boldsymbol{x}) \Big) \mathrm{d}\boldsymbol{x}. \tag{5.39}$$

In the last formula

$$\hat{W}^{(K)}(\nabla v(\boldsymbol{x}),...,\nabla^K v(\boldsymbol{x})) = \Big\langle W(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta}, \Phi^\varepsilon_K(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta}, \nabla v(\boldsymbol{x}),...,\nabla^K v(\boldsymbol{x})))\Big\rangle,$$

where $\Phi^\varepsilon_K(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta}, \nabla v(\boldsymbol{x}),...,\nabla^K v(\boldsymbol{x}))$ is a finite sum in powers of ε akin to that in the definition of the set $U^{(K)}$.

The functional

$$\int_{\mathbf{T}} \Big(\hat{W}^{(K)}(\nabla v(\boldsymbol{x}),...,\nabla^K v(\boldsymbol{x})) - f(\boldsymbol{x})v(\boldsymbol{x})\Big)\,\mathrm{d}\boldsymbol{x} \qquad (5.40)$$

is convex in the linear case (see Smyshlyaev & Cherednichenko, 2000). It is unknown if (5.40) is convex also in the nonlinear case. If so, then there exists a solution $v_K(\boldsymbol{x})$ to the problem (5.39) and it is natural to call it the homogenised solution of order K.

Otherwise, one may need to choose differently the truncation procedure leading to (5.38) and determining the minimisation set $U^{(K)}$ in (5.39). We are going to investigate this in detail in future.

Applications to non-uniformly elliptic problems

In our analysis so far we substantially used the ellipticity condition (2.4). However, in many applications potential functions $W(\boldsymbol{\xi},\boldsymbol{e})$ arise that do not satisfy this condition at some points. One of the most well known examples is the so called power-law potential

$$W(\boldsymbol{\xi},\boldsymbol{e}) = \gamma(\boldsymbol{\xi})|\boldsymbol{e}|^p, \quad p \geq 1, \qquad (5.41)$$

where $\gamma(\boldsymbol{\xi}) \geq \gamma_0 > 0$ is some smooth function. If $p > 2$, the function (5.41) does not satisfy the inequality (2.4) in the vicinity of the point $\boldsymbol{e} = \mathbf{0}$. However, there have been a number of works providing some indication that in the case of the potential (5.41) one can expect that in the dimension two ($d = 2$) the expression $\boldsymbol{z}+\nabla_\xi u_1(\boldsymbol{\xi},\boldsymbol{z})$ (*c.f.* (3.14)) does not take zero value *i.e.* $|\boldsymbol{z}+\nabla_\xi u_1(\boldsymbol{\xi},\boldsymbol{z})| \geq \delta(\boldsymbol{z}) > 0$ for $\boldsymbol{z} \neq \mathbf{0}$ (see *e.g.* Alessandrini & Singalotti, 1999; Bauman *et al,* 1999).

Alternatively, if $d \neq 2$, the function $\boldsymbol{z}+\nabla_\xi u_1(\boldsymbol{\xi},\boldsymbol{z})$ may vanish and hence the linear operator in the equations for u_2, u_3,*etc* (see (3.20)) ceases to be uniformly elliptic. In this case methods of analysis in *weighted* spaces (see *e.g.* Zhikov, 1999a) may still be applicable. These issues need further accurate consideration and we are going to study them in more detail in the future.

On the other hand, it is known (see *e.g.* Zhikov, 1999b) that the loss of uniform ellipticity (in the linear problems) may lead to "non-classical" homogenised limits (involving *e.g. non-locality*). From this point of view, it is of interest to explore possible relations between nonlocal effects and higher order terms in homogenised equations, as well as the possibility of non-classical effects in the *nonlinear* homogenisation.

We are also going to study in more detail the structure and properties of higher order homogenised equations for particular examples (both in linear and nonlinear cases) and their implications in particular applications.

Appendix A: Proof of the inequality (3.27).

In this appendix we follow the argument of Bakhvalov & Panasenko (1984).

Lemma.

Let a function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$, $\boldsymbol{\xi}, \boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in ξ, $Q = [0,1]^d$ and satisfy the following inequality with a positive constant ν

$$\frac{\partial^2 W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_i \partial e_j} \zeta_i \zeta_j \geq \nu \zeta_i \zeta_i \tag{A.1}$$

for any $\boldsymbol{\xi}, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\zeta} = (\zeta_1, ..., \zeta_d) \in \mathbf{R}^d$.

Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(\boldsymbol{\xi})} \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla \psi(\boldsymbol{\xi})) \Big\rangle \tag{A.2}$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{\xi})$.

Then the following inequality holds

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j} \eta_i \eta_j \geq \nu \eta_i \eta_i \tag{A.3}$$

for any $\boldsymbol{z} = (z_1, ..., z_d), \boldsymbol{\eta} = (\eta_1, ..., \eta_d) \in \mathbf{R}^d$.

Proof:

Denote $\boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})$. It is easy to see that the Euler-Lagrange equation for the minimisation problem (A.2) is

$$\mathrm{div}_\xi \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi \psi(\boldsymbol{\xi}, \boldsymbol{z})) = 0. \tag{A.4}$$

It obviously coincides with the equation (3.14), *i.e.* $\psi = u_1(\boldsymbol{\xi}, \boldsymbol{z})$ is the minimiser for the problem (A.2).

Let us introduce the following notation

$$\boldsymbol{Y}(\boldsymbol{\xi}, \boldsymbol{z}) = \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z})$$

and substitute $\boldsymbol{e} = \boldsymbol{Y}(\boldsymbol{\xi}, \boldsymbol{z})$ and $\boldsymbol{\zeta} = \boldsymbol{Y}_{,z_q} \eta_q$ into the inequality (A.1).

Thus

$$\frac{\partial^2 W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_i \partial e_j}\bigg|_{e=Y(\xi,z)} \frac{\partial Y_i(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \boldsymbol{z}_q} \eta_q \frac{\partial Y_j(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \eta_r \geq \nu \frac{\partial Y_i(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \boldsymbol{z}_q} \eta_q \frac{\partial Y_i(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \eta_r.$$

Taking the average with respect to $\boldsymbol{\xi}$ over Q in the last inequality we get

$$\beta_{qr}(\boldsymbol{z})\eta_q\eta_r \geq \nu\Big\langle \frac{\partial Y_i(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_q}\eta_q \frac{\partial Y_i(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\eta_r \Big\rangle, \tag{A.5}$$

where

$$\beta_{qr}(\boldsymbol{z}) = \Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_i \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_i(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_q}\frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\Big\rangle$$

$$= \Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_i \partial e_j}\Big|_{e=Y(\xi,z)} \Big(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{\xi},\boldsymbol{z})}{\partial \xi_i \partial z_q}\Big)\frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\Big\rangle$$

$$= \Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_q \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\Big\rangle + \Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_i \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial^2 u_1(\boldsymbol{\xi},\boldsymbol{z})}{\partial \xi_i \partial z_q}\frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\Big\rangle \tag{A.6}$$

We claim that the second term in (A.6) is identically zero. To verify this consider the equation (A.4). Multiply its both sides by some arbitrary function $\phi = \phi(\boldsymbol{\xi})$, take the average with respect to $\boldsymbol{\xi}$ over Q, and integrate by parts. We get

$$\Big\langle \boldsymbol{j}(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_\xi u_1(\boldsymbol{\xi},\boldsymbol{z}))\cdot \nabla\phi(\boldsymbol{\xi})\Big\rangle = 0.$$

Differentiate the last equality with respect to z_r and note that $\boldsymbol{j}(\boldsymbol{\xi},\boldsymbol{e}) = \nabla_e W(\boldsymbol{\xi},\boldsymbol{e})$. We come to the following equality

$$\Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_i \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\frac{\partial \phi(\boldsymbol{\xi})}{\partial \xi_i}\Big\rangle = 0. \tag{A.7}$$

This equality holds for all $\boldsymbol{z} \in \mathbf{R}^d$. Now, set $\phi_q(\boldsymbol{\xi},\boldsymbol{z}) = (u_1(\boldsymbol{\xi},\boldsymbol{z}))_{,z_q}$ for every $\boldsymbol{z} \in \mathbf{R}^d$, $q = 1, ..., d$. Substituting the functions $\phi_q(\boldsymbol{\xi},\boldsymbol{z})$ instead of $\phi(\boldsymbol{\xi})$ into the identity (A.7) we successively get

$$\Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_i \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\frac{\partial^2 u_1(\boldsymbol{\xi},\boldsymbol{z})}{\partial \xi_i \partial z_q}\Big\rangle = 0.$$

for all $\boldsymbol{z} \in \mathbf{R}^d, q = 1, ..., d$. Hence,

$$\beta_{qr}(\boldsymbol{z}) = \Big\langle \frac{\partial^2 W(\boldsymbol{\xi},\boldsymbol{e})}{\partial e_q \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_j(\boldsymbol{\xi},\boldsymbol{z})}{\partial z_r}\Big\rangle.$$

To verify that the following identity holds

$$\beta_{qr}(\boldsymbol{z}) = \frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r},$$

differentiate the equality

$$\hat{W}(\boldsymbol{z}) = \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla_{\xi} u_1(\boldsymbol{\xi}, \boldsymbol{z}))\Big\rangle$$

to get

$$\frac{\partial \hat{W}(\boldsymbol{z})}{\partial z_q} = \left\langle \frac{\partial W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_j}\Big|_{e=Y(\xi,z)} \left(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q}\right)\right\rangle$$

$$= \left\langle \frac{\partial W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_q}\Big|_{e=Y(\xi,z)}\right\rangle + \left\langle \frac{\partial W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q}\right\rangle.$$

Integrate by parts in the second term of the last sum and note that

$$\operatorname{div}_{\boldsymbol{\xi}}\Big(\nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})|_{e=Y(\xi,z)}\Big) = 0$$

in view of (A.4). Thus,

$$\left\langle \frac{\partial W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q}\right\rangle = 0$$

for all $\boldsymbol{z} \in \mathbf{R}^d$ and so

$$\frac{\partial \hat{W}(\boldsymbol{z})}{\partial z_q} = \left\langle \frac{\partial W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_q}\Big|_{e=Y(\xi,z)}\right\rangle.$$

Differentiating the last equality one more time we obtain the following identity

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r} = \left\langle \frac{\partial^2 W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_q \partial e_j}\Big|_{e=Y(\xi,z)} \frac{\partial Y_j(\boldsymbol{\xi}, \boldsymbol{z})}{\partial z_r}\right\rangle,$$

which immediately implies

$$\beta_{qr}(\boldsymbol{z}) = \frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r}$$

for all $\boldsymbol{z} \in \mathbf{R}^d$.

Note finally that the following estimate holds

$$\left\langle \left(\frac{\partial Y_i}{\partial z_q} \eta_q \right)^2 \right\rangle = \left\langle \left(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q} \right)^2 \eta_q^2 \right\rangle$$

$$= \left\langle \eta_i^2 + 2 \frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_i} + \left(\frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q} \right)^2 \eta_q^2 \right\rangle$$

$$= \eta_i^2 + \left\langle \left(\frac{\partial^2 u_1(\boldsymbol{\xi}, \boldsymbol{z})}{\partial \xi_i \partial z_q} \right)^2 \eta_q^2 \right\rangle \geq \eta_i^2.$$

for any $i = 1, ..., d$.

Now, taking into account (A.5) we get

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r} \eta_q \eta_r \geq \nu \eta_i \eta_i$$

as required.

Appendix B: Growth condition for the homogenised energy.

Some ideas used in the proof of the following lemma can be found in mathematical literature (see *e.g.* Jikov *et al*, 1984).

Lemma.

Let a function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$, $\boldsymbol{\xi}, \boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in ξ, $Q = [0, 1]^d$. Suppose there exists $\nu > 0$ such that the following inequality holds

$$\frac{\partial^2 W(\boldsymbol{\xi}, \boldsymbol{e})}{\partial e_i \partial e_j} \zeta_i \zeta_j \geq \nu \zeta_i \zeta_i$$

for any $\boldsymbol{\xi}, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\zeta} = (\zeta_1, ..., \zeta_d) \in \mathbf{R}^d$ and the function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$ also satisfies the following estimates

$$-A_1 + B_1 |\boldsymbol{e}|^p \leq W(\boldsymbol{\xi}, \boldsymbol{e}) \leq A_2 + B_2 |\boldsymbol{e}|^p \quad \text{for any } \boldsymbol{\xi}, \boldsymbol{e} \in \mathbf{R}^d \tag{B.1}$$

with some positive constants A_1, A_2, B_1, B_2 and $p \geq 1$.

Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(\xi)} \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla\psi(\boldsymbol{\xi}))\Big\rangle \tag{B.2}$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{\xi})$. Then

$$-A_1 + B_1|\boldsymbol{e}|^p \le \hat{W}(\boldsymbol{z}) \le A_2 + B_2|\boldsymbol{z}|^p \ \text{ for any } \boldsymbol{z} \in \mathbf{R}^d. \tag{B.3}$$

Proof:

Taking $\psi(\boldsymbol{\xi}) \equiv 0$ we conclude that

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(\xi)} \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla\psi(\boldsymbol{\xi}))\Big\rangle \le \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z})\Big\rangle \le A_2 + B_2|\boldsymbol{z}|^p.$$

so the right-hand inequality in (B.3) is proved.

If $p = 1$ then the left inequality in (B.3) is trivial.

Suppose $p > 1$ and consider the Legendre transform (convex dual) of the function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$ with respect to $\boldsymbol{e}$:

$$W^*(\boldsymbol{\xi}, \boldsymbol{\tau}) = \sup_{\boldsymbol{e}}\{\boldsymbol{\tau}\cdot\boldsymbol{e} - W(\boldsymbol{\xi}, \boldsymbol{e})\}.$$

Taking into account the left inequality in (B.1) we get

$$\begin{aligned} W^*(\boldsymbol{\xi}, \boldsymbol{\tau}) &\le \sup_{\boldsymbol{e}}\{\boldsymbol{\tau}\cdot\boldsymbol{e} + A_1 - B_1|\boldsymbol{e}|^p\} \\ &= A_1 + B_1(p-1)\Big(\frac{1}{pB_1}|\boldsymbol{\tau}|\Big)^{\frac{p}{p-1}}. \end{aligned} \tag{B.4}$$

It is well-known that for the convex dual of the homogenised energy $\hat{W}$

$$\hat{W}^*(\bar{\boldsymbol{\sigma}}) = \inf_{\langle\sigma\rangle=\bar{\sigma}, \mathrm{div}\sigma=0} \Big\langle W^*(\boldsymbol{\xi}, \sigma(\boldsymbol{\xi}))\Big\rangle, \tag{B.5}$$

where $\sigma(\boldsymbol{\xi})$ is Q-periodic. (The proof of this fact is beyond the scope of the present preprint, see *e.g.* Toland & Willis, 1989.)

Combining (B.4) and (B.5) we come to the estimate

$$\hat{W}^*(\bar{\boldsymbol{\sigma}}) \le A_1 + B_1(p-1)\Big(\frac{1}{pB_1}|\bar{\boldsymbol{\sigma}}|\Big)^{\frac{p}{p-1}}.$$

Thus,

$$\hat{W}(\boldsymbol{z}) = \sup_{\bar{\sigma}}\{\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{z} - \hat{W}^*(\bar{\boldsymbol{\sigma}})\} \ge -A_1 + B_1|\boldsymbol{z}|^p$$

as required.

Note that in the above proof we substantially used the fact that the functions $W(\boldsymbol{\xi}, \boldsymbol{e})$ and $\hat{W}(\boldsymbol{z})$ are convex with respect to $\boldsymbol{e}$ and $\boldsymbol{z}$ respectively.

Appendix C: Strong monotonicity of the homogenised energy.

We aim here at proving the following lemma.

Lemma.

Let a function $W = W(\boldsymbol{\xi}, \boldsymbol{e})$, $\boldsymbol{\xi}, \boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in $\boldsymbol{\xi}$, $Q = [0,1]^d$ and strongly monotonic, *i.e.*

$$\Big(\nabla_e W(\boldsymbol{\xi}, \boldsymbol{e}_1) - \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e}_2)\Big) \cdot \Big(\boldsymbol{e}_1 - \boldsymbol{e}_2\Big) \geq \alpha |\boldsymbol{e}_1 - \boldsymbol{e}_2|^p, \quad \alpha > 0, \quad p > 1 \tag{C.1}$$

for every $\boldsymbol{\xi} \in \mathbf{T}$ and any $\boldsymbol{e}_1, \boldsymbol{e}_1 \in \mathbf{R}^d$. Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(\xi)} \Big\langle W(\boldsymbol{\xi}, \boldsymbol{z} + \nabla\psi(\boldsymbol{\xi}))\Big\rangle \tag{C.2}$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{\xi})$.

Then the function $\hat{W}(\boldsymbol{z})$ is also strongly monotonous:

$$\Big(\nabla\hat{W}(\boldsymbol{z}_1) - \nabla\hat{W}(\boldsymbol{z}_2)\Big) \cdot \Big(\boldsymbol{z}_1 - \boldsymbol{z}_2\Big) \geq \alpha|\boldsymbol{z}_1 - \boldsymbol{z}_2|^p. \tag{C.3}$$

Note that the parameters α and p in (C.3) are the same as in (C.1).

Proof:

Apply averaging with respect to $\boldsymbol{\xi} \in Q$ to the inequality (C.1) where $\boldsymbol{e}_1 = \boldsymbol{z}_1 + u_1(\boldsymbol{\xi}, \boldsymbol{z}_1)$ and $\boldsymbol{e}_2 = \boldsymbol{z}_2 + u_1(\boldsymbol{\xi}, \boldsymbol{z}_2)$.

Using the identity

$$\nabla\hat{W}(\boldsymbol{z}) = \Big\langle \nabla_e W|_{e=z+u_1(\xi,z)}\Big\rangle$$

proved in the Appendix A we come to the following inequality

$$\Big(\nabla\hat{W}(\boldsymbol{z}_1) - \nabla\hat{W}(\boldsymbol{z}_2)\Big) \cdot \Big(\boldsymbol{z}_1 - \boldsymbol{z}_2\Big)$$

$$+\Big\langle\Big(\nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})|_{e=z_1+\nabla_\xi u_1(\xi,z_1)} - \nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})|_{e=z_1+u_1(\xi,z_1)}\Big) \cdot \Big(\nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z}_1) - \nabla_\xi u_1(\boldsymbol{\xi}, \boldsymbol{z}_2)\Big)\Big\rangle$$

$$\geq \alpha\Big\langle|\boldsymbol{z}_1 - \boldsymbol{z}_2 + \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_1) - \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_2)|^p\Big\rangle.$$

Integrating by parts in the second term of the left-hand side of the last inequality and using the fact that

$$\mathrm{div}_\xi\Big(\nabla_e W(\boldsymbol{\xi}, \boldsymbol{e})|_{e=z+\nabla_\xi u_1(\xi,z)}\Big) = 0 \quad \text{for any } \boldsymbol{z} \in \mathbf{R}^d$$

we conclude that this term is zero.

Finally, from the periodicity of the functions $u_1(\boldsymbol{\xi}, \boldsymbol{z}_1)$ and $u_1(\boldsymbol{\xi}, \boldsymbol{z}_2)$ with respect to $\boldsymbol{\xi}$ we get (using Hölder inequality)

$$\Big\langle |\boldsymbol{z}_1 - \boldsymbol{z}_2 + \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_1) - \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_2)|^p \Big\rangle$$

$$\geq \Big| \Big\langle \boldsymbol{z}_1 - \boldsymbol{z}_2 + \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_1) - \nabla u_1(\boldsymbol{\xi}, \boldsymbol{z}_2) \Big\rangle \Big|^p = |\boldsymbol{z}_1 - \boldsymbol{z}_2|^p,$$

that gives us the required right-hand side in (C.3).

References

Alessandrini, G., Singalotti, M. 1999. Geometric properties of solutions to the inhomogeneous anisotropic $p-$Laplace equation in dimension two. *Preprint*

Bakhvalov, N.S., Panasenko, G.P. 1984. *Homogenisation: Averaging Processes in Periodic Media.* Nauka, Moscow. (in Russian). English translation in: Mathematics and Its Applications (Soviet Series) **36**, Kluwer Academic Publishers, Dordrecht-Boston-London.

Bauman, P., Marini, A., Nesi, V. 1999 Univalent solutions of an elliptic system of partial differential equations arising in homogenization. *Preprint*

Bensoussan, A., Lions, J.-L., Papanicolaou, G.C. 1978. *Asymptotic analysis for periodic structures.* North Holland, Amsterdam.

Jikov, V.V., Kozlov, S.M., Oleinik, O.A. 1994. *Homogenization of Differential Operators and Integral Functionals.* Springer-Verlag, Berlin Heidelberg.

Ladyzhenskaya, O.A., Uraltseva, N.N. 1968. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York.

Sanchez-Palencia, E. 1980. *Nonhomogeneous media and vibration theory.* Lecture Notes in Physics **127**.

Smyshlyaev, V.P., Cherednichenko, K.D. 2000. On derivation of "strain gradient" effects in the overall behaviour of periodic heterogeneous media. *To appear in J. Mech. Phys. Solids*

Suquet, P. 1982. Plasticite et homogeneisation. *Thesis*

Toland, J.F., Willis, J.R. 1989. Duality for families of natural variational principles in nonlinear electrostatics, *SIAM J. Math. Anal.* **20**, 1283-1292.

Zhikov, V.V. 1999a. On the problem of passing to the limit in non-uniformly elliptic divergence equations. *Preprint, in Russian*

Zhikov, V.V. 1999b. On an extension and application of the method of two-scale convergence. *Preprint, in Russian*

Recent Newton Institute Preprints

NI99006-APF	**KJ Falconer, M Järvenpää and P Mattila** *Examples illustrating the instability of packing dimensions of sections*
NI99007-APF	**J Kigami** *Markov property of Kusuoka-Zhou's Dirichlet forms on self-similar sets*
NI99008-APF	**RM Solovay** *A version of Ω for which ZFC can not predict a single bit*
NI99009-TRB	**BJ Geurts and A Leonard** *Is LES ready for complex flows?*
NI99010-TRB	**A Tsinober** *Vortex stretching versus production of strain/dissipation*
NI99011-TRB	**A Tsinober** *On statistics and structure(s) in turbulence*
NI99012-TRB	**AJ Young and WD McComb** *An ad hoc operational method to compensate for absent turbulence modes in an insuffiently resolved numerical simulation*
NI99013-TRB	**ND Sandham** *A review of progress on direct and large-eddy simulation of turbulence*
NI99014-NSP	**R Baraniuk** *Optimal tree approximation with wavelets*
NI99015-CCP	**HE Brandt** *Inconclusive rate with a positive operator valued measure*
NI99016-DAD	**U Torkelsson, GI Ogilvie, A Brandenburg et al** *The response of a turbulent accretion disc to an imposed epicyclic shearing motion*
NI99017-TRB	**M Kholmyansky and A Tsinober** *On the origins of intermittency in real turbulent flows*
NI99018-SFU	**T Shiromizu, K-i Maeda and M Sasaki** *The Einstein equations on the 3-brane world*
NI99019-CCP	**V Coffman, J Kundu and WK Wootters** *Distributed entabglement*
NI99020-CCP	**H Brandt** *Qubit devices*
NI99021-SFU	**O Seto, J Yokoyama and H Kodama** *What happens when the inflaton stops during inflation*
NI99022-SFU	**J Garriga and T Tanaka** *Gravity in the brane-world*
NI99023-SMM	**VA Kovtunenko** *Shape sensitivity of curvilinear cracks*
NI99024-SMM	**R Luciano and JR Willis** *Non-local constitutive response of a random laminate subjected to configuration-dependent body force*
NI99025-SMM	**GA Francfort and PM Suquet** *Duality relations for nonlinear incompressible two dimensional elasticity*
NI99026-SMM	**JC Michel, U Galvaneto and P Suquet** *Constitutive relations involving internal varianles based on a micromechanical analysis*
NI99027-APF	**N-R Shieh and SJ Taylor** *Multifractal spectra of a branching measure on a Galton-Watson tree*
NI99028-SMM	**KD Cherednichenko and VP Smyshlyaev** *On full asymptotic expansion of the solutions of nonlinear periodic rapidly oscillating problems*

Information about Newton Institute Preprints is also available at
`http://www.newton.cam.ac.uk/preprints.html`