# Qutrit Entanglement*

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## Abstract

We consider the separability of various joint states for $N$ qutrits. We derive two results: (i) the separability condition for a two-qutrit state that is a mixture of the maximally mixed state and a maximally entangled state (such a state is a generalization of the Werner state for two qubits); (ii) upper and lower bounds on the size of the neighborhood of separable states surrounding the maximally mixed state for $N$ qutrits.

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# I. INTRODUCTION

Quantum entanglement provides a powerful physical resource for new kinds of communication protocols and computation [1], which achieve results that cannot be achieved classically. Quantum entanglement refers to correlations between the results of measurements made on component subsystems of a larger physical system that cannot be explained in terms of correlations between local classical properties inherent in those same subsystems. Alternatively, an entangled state cannot be prepared by local operations and local measurements on each subsystem. While the nonclassical nature of quantum entanglement was recognized by Einstein, Schrödinger, and other pioneers of quantum mechanics, it is only recently that a full appreciation of the surprising and complex properties of entanglement has begun to emerge.

We now have a good understanding of entanglement for a pair of qubits [2], a qubit being a system with a two-dimensional Hilbert space. Moreover, we have a criterion, the partial transposition condition of Peres [3], which determines whether a general state of two qubits is entangled and whether a general state of a qubit and a qutrit is entangled [4], a qutrit being a system whose states live in a three-dimensional Hilbert space. The partial-transposition condition fails, however, to provide a criterion for entanglement in other cases, where the constituents have higher Hilbert-space dimensions [5,6] or where there are more than two constituents. In this paper we consider the entanglement of composite systems made of qutrits. In particular, we investigate the separability of various joint states of two or more qutrits. A joint state of a composite system is defined to be *separable* if it can be decomposed into a mixture of product states for the constituents. A separable state has no quantum entanglement.

In Sec. II, we review an efficient operator representation of the states of a qutrit [7], particularly the pure states. This operator representation is analogous to the familiar Pauli or Bloch-sphere representation of qubit states. The representation is applied to an analysis of qutrit entanglement in Secs. III and IV. In Sec. III, we consider states of two qutrits that are a mixture of the maximally mixed state and a maximally entangled state. Such states are a generalization of the Werner state for two qubits [8]. We show that the two-qutrit mixture is separable if and only if the probability for the maximally entangled state does not exceed 1/4. This result is a special case of a general result obtained by Vidal and Tarrach [9], which gives, for any bipartite system, the separability boundary for any mixture of the maximally mixed state with a pure state of the bipartite system. In Sec. IV, we consider the separability of $N$-qutrit states near the maximally mixed state. We find both lower and upper bounds on the size of the neighborhood of separable states around the maximally mixed state. Our analysis follows closely that of Braunstein *et al.* [10], who analyzed the separability of $N$-qubit states near the maximally mixed state.

## II. OPERATOR REPRESENTATIONS OF QUTRIT STATES

In this section we review an efficient operator representation for qutrit states, analogous to the Pauli or Bloch-sphere representation for qubits. The qutrit representation uses the (Hermitian) generators of SU(3) as an operator basis. Our discussion is taken from

Ref. [7]; the reader is referred there for more details (see also Refs. [11] and [12]). This operator representation—and its generalization to higher dimensions—has been used by Acín, Latorre, and Pascual [13] to explore optimal generalized measurements on collections of identically prepared spin systems.

Let $|1\rangle$, $|2\rangle$, and $|3\rangle$ be an orthonormal basis for a qutrit. To denote these states, we use Latin letters from the beginning of the alphabet, which take on the values 1, 2, and 3. A pure state $|\psi\rangle$ is a superposition of the states $|a\rangle$. Normalization and a choice for the arbitrary overall phase allow us to write any pure state as

$$|\psi\rangle = e^{i\chi_1} \sin\theta \cos\phi \, |1\rangle + e^{i\chi_2} \sin\theta \sin\phi \, |2\rangle + \cos\theta \, |3\rangle \ . \tag{2.1}$$

One obtains all pure states as the four coördinates vary over the ranges

$$0 \le \theta, \phi \le \pi/2 \ , \tag{2.2a}$$

$$0 \le \chi_1, \chi_2 \le 2\pi \ . \tag{2.2b}$$

To develop an operator representation of qutrit states, we begin with the eight (Hermitian) generators of SU(3). In the basis $|a\rangle$, these generators have the matrix representations

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \tag{2.3}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

We use Latin indices from the middle of the alphabet, ranging from 1 to 8, to label these generators and related quantities, and we use the summation convention to indicate a sum on repeated indices. The generators (2.3) are traceless and satisfy

$$\lambda_j \lambda_k = \frac{2}{3} \delta_{jk} + d_{jkl} \lambda_l + i f_{jkl} \lambda_l \ . \tag{2.4}$$

The coefficients $f_{jkl}$, the structure constants of the Lie group SU(3), are given by the commutators of the generators and are completely antisymmetric in the three indices. The coefficients $d_{jkl}$ are given by the anti-commutators of the generators and are completely symmetric. Values of these coefficients can be found in Ref. [7].

By supplementing the eight generators with the operator

$$\lambda_0 \equiv \sqrt{\frac{2}{3}} \, 1 \ , \tag{2.5}$$

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we obtain a Hermitian operator basis for the space of linear operators in the qutrit Hilbert space. This basis is an orthogonal basis, satisfying

$$\text{tr}(\lambda_\alpha \lambda_\beta) = 2\delta_{\alpha\beta} . \tag{2.6}$$

Here and throughout Greek indices run over the values 0 to 8.

Any density operator can be expanded uniquely as

$$\rho = \frac{1}{3} c_\alpha \lambda_\alpha , \tag{2.7}$$

where the (real) expansion coefficients are given by

$$c_\alpha = \frac{3}{2} \text{tr}(\rho \lambda_\alpha) . \tag{2.8}$$

Normalization implies that $c_0 = \sqrt{3/2}$, so the density operator takes the form

$$\rho = \frac{1}{3}(1 + c_j \lambda_j) = \frac{1}{3}(1 + \mathbf{c} \cdot \boldsymbol{\lambda}) . \tag{2.9}$$

Here $\mathbf{c} = c_j \mathbf{e}_j$ can be regarded as a vector in a real, eight-dimensional vector space, and $\boldsymbol{\lambda} = \lambda_j \mathbf{e}_j$ is an operator vector. Using Eq. (2.4), one finds that

$$\rho^2 = \frac{1}{9}\left(1 + \frac{2}{3} \mathbf{c} \cdot \mathbf{c}\right) 1 + \frac{1}{3} \boldsymbol{\lambda} \cdot \left(\frac{2}{3} \mathbf{c} + \frac{1}{3\sqrt{3}} \mathbf{c} \star \mathbf{c}\right) , \tag{2.10}$$

where the "star" product is defined by

$$\mathbf{c} \star \mathbf{d} \equiv \mathbf{e}_j d_{jkl} c_k d_l . \tag{2.11}$$

For a pure state, $\rho^2 = \rho$, so we must have $\mathbf{c} \cdot \mathbf{c} = 3$ and $\mathbf{c} \star \mathbf{c} = \sqrt{3}\,\mathbf{c}$. Defining the eight-dimensional unit vector $\mathbf{n} = \mathbf{c}/\sqrt{3}$, we find that any pure state of a qutrit can be written as

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}(I + \sqrt{3}\,\mathbf{n} \cdot \boldsymbol{\lambda}) \equiv P_{\mathbf{n}} , \tag{2.12}$$

where $\mathbf{n}$ satisfies

$$\mathbf{n} \cdot \mathbf{n} = 1 , \tag{2.13a}$$
$$\mathbf{n} \star \mathbf{n} = \mathbf{n} . \tag{2.13b}$$

We introduce the notation $P_{\mathbf{n}}$ for a pure state with unit vector $\mathbf{n}$ for use later in the paper.

Equation (2.8) implies that

$$n_j = \frac{\sqrt{3}}{2} \text{tr}(\rho \lambda_j) = \frac{\sqrt{3}}{2} \langle\psi|\lambda_j|\psi\rangle . \tag{2.14}$$

Applied to the pure state (2.1), this gives a unit vector

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$$\mathbf{n} = \sqrt{3}\bigg(\sin^2\theta\sin\phi\cos\phi\cos(\chi_2 - \chi_1)\,\mathbf{e}_1 + \sin^2\theta\sin\phi\cos\phi\sin(\chi_2 - \chi_1)\,\mathbf{e}_2$$

$$+ \frac{1}{2}\sin^2\theta(\cos^2\phi - \sin^2\phi)\,\mathbf{e}_3 + \sin\theta\cos\theta\cos\phi\cos\chi_1\,\mathbf{e}_4$$

$$- \sin\theta\cos\theta\cos\phi\sin\chi_1\,\mathbf{e}_5 + \sin\theta\cos\theta\sin\phi\cos\chi_2\,\mathbf{e}_6$$

$$- \sin\theta\cos\theta\sin\phi\sin\chi_2\,\mathbf{e}_7 + \frac{1}{2\sqrt{3}}(1 - 3\cos^2\theta)\,\mathbf{e}_8\bigg)\,. \tag{2.15}$$

The pure qutrit states lie on the unit sphere in the eight-dimensional vector space, but not all operators on the unit sphere are pure states. For example, of the unit vectors $\pm\mathbf{e}_j$, $j = 1,\ldots,8$, only $-\mathbf{e}_8$ satisfies the star-product condition (2.13b) and thus is the unit vector for a pure state. It is easy to show that the unit vectors that do not satisfy the star-product condition do not specify any state, pure or mixed, for they all give operators that have negative eigenvalues. The star-product condition (2.13b) places three constraints on the unit vector for a pure state, thus reducing the number of real parameters required to specify a pure state from the seven parameters needed to specify an arbitrary eight-dimensional unit vector to four parameters, which can be taken to be the four coördinates of Eq. (2.1).

It is useful to notice that

$$|\langle\psi|\psi'\rangle|^2 = \mathrm{tr}(\rho\rho') = \frac{1}{3}(1 + 2\mathbf{n}\cdot\mathbf{n}')\,. \tag{2.16}$$

Orthogonal pure states have unit vectors that satisfy $\mathbf{n}\cdot\mathbf{n}' = -\frac{1}{2}$ and are thus $120°$ apart. The states in an orthonormal basis have unit vectors that lie in a plane at the vertices of an equilateral triangle. The density operators that are diagonal in the orthonormal basis are the operators on the triangle or in its interior.

The orthonormal states $|a\rangle$, $a = 1, 2, 3$, of the original basis have unit vectors $\mathbf{n}_a$ that lie in the plane spanned by $\mathbf{e}_3$ and $\mathbf{e}_8$:

$$\mathbf{n}_1 = \frac{\sqrt{3}}{2}\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_8 \quad (\theta = \pi/2,\ \phi = 0,\ \chi_1\text{ and }\chi_2\text{ arbitrary}), \tag{2.17a}$$

$$\mathbf{n}_2 = -\frac{\sqrt{3}}{2}\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_8 \quad (\theta = \pi/2,\ \phi = \pi/2,\ \chi_1\text{ and }\chi_2\text{ arbitrary}), \tag{2.17b}$$

$$\mathbf{n}_3 = -\mathbf{e}_8 \quad (\theta = 0,\ \phi,\ \chi_1,\text{ and }\chi_2\text{ arbitrary}). \tag{2.17c}$$

These unit vectors correspond to the following operator relations for the projectors onto the original basis:

$$|1\rangle\langle 1| = \frac{1}{3}\left[1 + \sqrt{3}\left(\frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8\right)\right]\,, \tag{2.18a}$$

$$|2\rangle\langle 2| = \frac{1}{3}\left[1 + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8\right)\right]\,, \tag{2.18b}$$

$$|3\rangle\langle 3| = \frac{1}{3}(1 - \sqrt{3}\lambda_8)\,. \tag{2.18c}$$

For later use, we note here the operator expansions for the transition operators in the original basis:

$$|1\rangle\langle 2| = \frac{1}{2}(\lambda_1 + i\lambda_2) , \qquad |2\rangle\langle 1| = \frac{1}{2}(\lambda_1 - i\lambda_2) , \tag{2.18d}$$

$$|1\rangle\langle 3| = \frac{1}{2}(\lambda_4 + i\lambda_5) , \qquad |3\rangle\langle 1| = \frac{1}{2}(\lambda_4 - i\lambda_5) , \tag{2.18e}$$

$$|2\rangle\langle 3| = \frac{1}{2}(\lambda_6 + i\lambda_7) , \qquad |3\rangle\langle 2| = \frac{1}{2}(\lambda_6 - i\lambda_7) . \tag{2.18f}$$

The geodesic distance between pure states, according to the unitarily invariant Fubini-Studi metric [14], is given by the Hilbert-space angle between the states, i.e., $\cos s_{FS} = |\langle\psi|\psi'\rangle|$. For infinitesimally separated pure states, $|\psi\rangle$ and $|\psi'\rangle = |\psi\rangle + |d\psi\rangle$, the distance gives the Fubini-Studi line element

$$ds_{FS}^2 = 1 - |\langle\psi|\psi'\rangle|^2 = \langle d\psi|d\psi\rangle - |\langle\psi|d\psi\rangle|^2 . \tag{2.19}$$

We choose to rescale all lengths by a factor of $\sqrt{3}$, giving a line element

$$ds^2 = 3ds_{FS}^2 = d\mathbf{n}\cdot d\mathbf{n} , \tag{2.20}$$

so that the metric on the submanifold of pure states is the metric induced by the natural metric on the unit sphere in eight dimensions. In the coördinates of Eq. (2.1), the rescaled line element becomes

$$ds^2 = 3\Big(d\theta^2 + \sin^2\theta\, d\phi^2 + \sin^2\theta\cos^2\phi(1 - \sin^2\theta\cos^2\phi)\, d\chi_1^2$$
$$+ \sin^2\theta\sin^2\phi(1 - \sin^2\theta\sin^2\phi)\, d\chi_2^2 - 2\sin^4\theta\sin^2\phi\cos^2\phi\, d\chi_1\, d\chi_2\Big) . \tag{2.21}$$

The corresponding unitarily invariant volume element on the space of pure states is

$$d\Omega_{\mathbf{n}} = 9\sin^3\theta\cos\theta\sin\phi\cos\phi\, d\theta\, d\phi\, d\chi_1\, d\chi_2 , \tag{2.22}$$

which gives a total volume

$$\mathcal{V} = \int d\Omega_{\mathbf{n}} = \frac{9\pi^2}{2} \tag{2.23a}$$

(notice that the volume relative to the original Fubini-Studi scaling is $\mathcal{V}_{FS} = \mathcal{V}/9 = \pi^2/2$). We can also show that the components of $\mathbf{n}$ satisfy

$$\int d\Omega_{\mathbf{n}}\, n_j = 0 , \tag{2.23b}$$

$$\int d\Omega_{\mathbf{n}}\, n_j n_k = \frac{9\pi^2}{16}\delta_{jk} = \frac{\mathcal{V}}{8}\delta_{jk} . \tag{2.23c}$$

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## III. MIXTURES OF MAXIMALLY MIXED
## AND MAXIMALLY ENTANGLED STATES

In this section we deal with two qutrits, labeled $A$ and $B$. We consider a class of two-qutrit states, specifically mixtures of the maximally mixed state, $M_9 = \frac{1}{9}1 \otimes 1$, with a maximally entangled state, which we can choose to be

$$|\Psi\rangle = \frac{1}{\sqrt{3}}\Big(|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle + |3\rangle \otimes |3\rangle\Big) . \tag{3.1}$$

Such mixtures have the form

$$\rho_\epsilon = (1 - \epsilon)M_9 + \epsilon|\Psi\rangle\langle\Psi| , \tag{3.2}$$

where $0 \leq \epsilon \leq 1$. A state of the two qutrits is *separable* if it can be written as an ensemble of product states. In this section we show that the state (3.2) is separable if and only if $\epsilon \leq 1/4$.

In analogy to Eq. (2.7), any state $\rho$ of two qutrits can be expanded uniquely as

$$\rho = \frac{1}{9}c_{\alpha\beta}\lambda_\alpha \otimes \lambda_\beta , \tag{3.3}$$

where the expansion coefficients are given by

$$c_{\alpha\beta} = \frac{9}{4}\text{tr}(\rho\lambda_\alpha \otimes \lambda_\beta) . \tag{3.4}$$

Normalization requires that $c_{00} = 3/2$. Using the relations (2.18), we can find the operator expansion for the maximally entangled state (3.1):

$$
\begin{aligned}
|\Psi\rangle\langle\Psi| &= \sum_{a,b} |a\rangle\langle b| \otimes |a\rangle\langle b| \\
&= \frac{1}{9}\bigg(1 \otimes 1 + \frac{3}{2}\Big(\lambda_1 \otimes \lambda_1 - \lambda_2 \otimes \lambda_2 + \lambda_3 \otimes \lambda_3 \\
&\qquad + \lambda_4 \otimes \lambda_4 - \lambda_5 \otimes \lambda_5 + \lambda_6 \otimes \lambda_6 - \lambda_7 \otimes \lambda_7 + \lambda_8 \otimes \lambda_8\Big)\bigg) .
\end{aligned} \tag{3.5}
$$

Hence the operator expansion for the mixed state (3.2) is

$$
\begin{aligned}
\rho_\epsilon &= \frac{1}{9}\bigg(1 \otimes 1 + \frac{3\epsilon}{2}\Big(\lambda_1 \otimes \lambda_1 - \lambda_2 \otimes \lambda_2 + \lambda_3 \otimes \lambda_3 \\
&\qquad + \lambda_4 \otimes \lambda_4 - \lambda_5 \otimes \lambda_5 + \lambda_6 \otimes \lambda_6 - \lambda_7 \otimes \lambda_7 + \lambda_8 \otimes \lambda_8\Big)\bigg) ,
\end{aligned} \tag{3.6}
$$

from which we can read off the expansion coefficients (3.4) for the state $\rho_\epsilon$:

$$c_{0j} = c_{j0} = 0 , \quad c_{jk} = 0 \quad \text{for } j \neq k, \tag{3.7a}$$

$$c_{11} = -c_{22} = c_{33} = c_{44} = -c_{55} = c_{66} = -c_{77} = c_{88} = \frac{3\epsilon}{2} . \tag{3.7b}$$

The product pure states for two qutrits, $P_{\mathbf{n}_A} \otimes P_{\mathbf{n}_B}$, constitute an overcomplete operator basis. Thus we can expand any two-qutrit density operator in terms of them:

$$\rho = \int d\Omega_{\mathbf{n}_A} \, d\Omega_{\mathbf{n}_B} \, w(\mathbf{n}_A, \mathbf{n}_B) P_{\mathbf{n}_A} \otimes P_{\mathbf{n}_B} \, . \tag{3.8}$$

Because of the overcompleteness, the expansion function $w(\mathbf{n}_A, \mathbf{n}_B)$ is not unique. Notice that the expansion coefficients $c_{\alpha\beta}$ of Eq. (3.4) can be written as integrals over the expansion function,

$$c_{\alpha\beta} = 3 \int d\Omega_{\mathbf{n}_A} \, d\Omega_{\mathbf{n}_B} \, w(\mathbf{n}_A, \mathbf{n}_B)(\tilde{n}_A)_\alpha (\tilde{n}_B)_\beta \, , \tag{3.9}$$

where $\tilde{n}_0 \equiv 1/\sqrt{2}$ and $\tilde{n}_j \equiv n_j$.

A two-qutrit state $\rho$ is *separable* if there exists an expansion function that is everywhere nonnegative. For a separable qutrit state, the expansion function $w(\mathbf{n}_A, \mathbf{n}_B)$ can be thought of as a normalized classical probability distribution for the unit vectors $\mathbf{n}_A$ and $\mathbf{n}_B$, and the integral for $c_{\alpha\beta}$ in Eq. (3.9) can be interpreted as a classical expectation value over this distribution, i.e.,

$$c_{\alpha\beta} = 3E[(\tilde{n}_A)_\alpha (\tilde{n}_B)_\beta] \, . \tag{3.10}$$

If the state $\rho_\epsilon$ is separable, we have from Eqs. (3.7b) and (3.10) that

$$\frac{\epsilon}{2} = \frac{1}{3}|c_{jj}| = \left| E[(n_A)_j (n_B)_j] \right| \leq \frac{1}{2} \left( E[(n_A)_j^2] + E[(n_B)_j^2] \right) \, . \tag{3.11}$$

Adding over the eight values of $j$ gives

$$4\epsilon \leq \frac{1}{2} \left( E[\mathbf{n}_A \cdot \mathbf{n}_A] + E[\mathbf{n}_B \cdot \mathbf{n}_B] \right) = 1 \, . \tag{3.12}$$

We can conclude that if $\rho_\epsilon$ is separable, then $\epsilon \leq 1/4$.

To prove the converse, we need to construct a product ensemble for $\epsilon = 1/4$. For each pair $a$, $b$, with $b > a$, define four pure product states

$$|\Phi_z^{(a,b)}\rangle \equiv \frac{1}{\sqrt{2}} \big( |a\rangle + z|b\rangle \big) \otimes \frac{1}{\sqrt{2}} \big( |a\rangle + z^*|b\rangle \big) \, , \tag{3.13}$$

where $z$ takes on the values $\pm 1$ and $\pm i$, so that $0 = \sum_z z = \sum_z z^2$ and $4 = \sum_z |z|^2$. It is easy to show that an ensemble of the twelve states $|\Phi_z^{(a,b)}\rangle$, all states contributing with the same probability, gives the state $\rho_{\epsilon=1/4}$, i.e.,

$$\frac{1}{12} \sum_{\substack{a,b \\ b>a}} \sum_z |\Phi_z^{(a,b)}\rangle\langle\Phi_z^{(a,b)}| = \frac{3}{4}M_9 + \frac{1}{4}|\Psi\rangle\langle\Psi| \, , \tag{3.14}$$

thus concluding the proof that $\rho_\epsilon$ is separable if and only if $\epsilon = 1/4$.

This result should be contrasted with that for the corresponding qubit state, a Werner state [8], which is separable if and only if $\epsilon \leq 1/3$ [15]. This indicates that maximally

entangled states of two qutrits are more entangled than maximally entangled states of two qubits.

The separability boundary obtained in this section is a special case of a general result obtained by Vidal and Tarrach [9], who showed the following: for a pair of systems, one with dimension $d_1$ and the other with dimension $d_2$, a mixture, $(1 - \epsilon)M_{d_1 d_2} + \epsilon|\Psi\rangle\langle\Psi|$, of the maximally mixed state $M_{d_1 d_2}$ with any pure state $|\Psi\rangle$ is separable if and only if $\epsilon \leq (1 + d_1 d_2 a_1 a_2)^{-1}$, where $a_1^2$ and $a_2^2$ are the two largest eigenvalues of the marginal density operators obtained from $|\Psi\rangle\langle\Psi|$. The reason for presenting the much more limited result of this section is, first, that the proof of necessity, culminating in Eq. (3.12), has a nice physical interpretation in terms of the correlation coefficients $E[(n_A)_j(n_B)_j]$ and, second, that the ensemble (3.14) is different from the one used by Vidal and Tarrach.

## IV. SEPARABILITY OF STATES NEAR THE MAXIMALLY MIXED STATE

This section deals with $N$-qutrit states of the form

$$\rho_\epsilon = (1 - \epsilon)M_{3^N} + \epsilon\rho_1 , \tag{4.1}$$

where $M_{3^N} \equiv 1\otimes\cdots\otimes1/3^N$ is the maximally mixed state for $N$ qutrits and $\rho_1$ is any $N$-qutrit density operator. In this section we establish upper and lower bounds on the size of the neighborhood of separable states surrounding the maximally mixed state. In particular, we show, first, that for $\epsilon \leq (1 + 3^{2N-1})^{-1}$, all states of the form (4.1) are separable and, second, that for $\epsilon > (1 + 3^{N/2})^{-1}$, there are states of the form (4.1) that are not separable. The approach we take in this section follows slavishly the corresponding qubit analysis presented by Braunstein *et al.* [10].

Since the product pure states form an overcomplete operator basis, any $N$-qutrit state can be expanded as

$$\rho = \int d\Omega_{\mathbf{n}_1}\cdots d\Omega_{\mathbf{n}_N}\, w(\mathbf{n}_1,\ldots,\mathbf{n}_N)P_{\mathbf{n}_1}\otimes\cdots\otimes P_{\mathbf{n}_N} . \tag{4.2}$$

The expansion function $w(\mathbf{n}_1,\ldots,\mathbf{n}_N)$ is not unique, because of the overcompleteness of the pure product states. An $N$-qutrit state is separable if there exists an expansion function that is everywhere nonnegative.

We can find a particular expansion function in the following way. Any $N$-qutrit density operator can be expanded uniquely as

$$\rho = \frac{1}{3^N}c_{\alpha_1\ldots\alpha_N}\lambda_{\alpha_1}\otimes\cdots\otimes\lambda_{\alpha_N} , \tag{4.3}$$

with the expansion coefficients given by

$$c_{\alpha_1\ldots\alpha_N} = \left(\frac{3}{2}\right)^N \mathrm{tr}(\rho\lambda_{\alpha_1}\otimes\cdots\otimes\lambda_{\alpha_N}) . \tag{4.4}$$

Using Eqs. (2.23), we can write

$$\lambda_\alpha = \frac{16}{3\sqrt{3}\pi^2}\int d\Omega_{\mathbf{n}}\,\bar{n}_\alpha P_{\mathbf{n}} , \tag{4.5}$$

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where the barred components are defined by $\bar{n}_0 = 1/4\sqrt{2}$ and $\bar{n}_j = n_j$. Substituting this expression for $\lambda_\alpha$ into Eq. (4.3), we find that one choice for the expansion function is

$$w_\rho(\mathbf{n}_1, \ldots, \mathbf{n}_N) = \left(\frac{16}{9\sqrt{3}\pi^2}\right)^N c_{\alpha_1 \ldots \alpha_N} (\bar{n}_1)_{\alpha_1} \cdots (\bar{n}_N)_{\alpha_N}$$

$$= \left(\frac{2}{9\pi^2}\right)^N \mathrm{tr}\Big(\rho(1 + 4\sqrt{3}\,\mathbf{n}_1 \cdot \boldsymbol{\lambda}) \otimes \cdots \otimes (1 + 4\sqrt{3}\,\mathbf{n}_N \cdot \boldsymbol{\lambda})\Big). \qquad (4.6)$$

Now consider the product operator in Eq. (4.6). The pure-state density operator $P_\mathbf{n} = \frac{1}{3}(1 + \sqrt{3}\,\mathbf{n} \cdot \boldsymbol{\lambda})$ has eigenvalues 1, 0, and 0. This implies that $\mathbf{n} \cdot \boldsymbol{\lambda}$ has eigenvalues $2/\sqrt{3}$, $-1/\sqrt{3}$, and $-1/\sqrt{3}$ and that each term in the product operator, $1 + 4\sqrt{3}\,\mathbf{n} \cdot \boldsymbol{\lambda}$, has eigenvalues 9, -3, and -3. Hence the smallest eigenvalue of the product operator is $9^{N-1}(-3) = -3^{2N-1}$. The result is a lower bound on the expansion function (4.6):

$$w_\rho(\mathbf{n}_1, \ldots, \mathbf{n}_N) \geq \left(\frac{2}{9\pi^2}\right)^N \times \left(\begin{array}{c}\text{smallest eigenvalue of}\\ \text{product operator}\end{array}\right) = -\left(\frac{2}{9\pi^2}\right)^N 3^{2N-1}. \qquad (4.7)$$

We can use this lower bound to place a similar lower bound on the expansion function for a state of the form (4.1). Since the expansion function for the maximally mixed state $M_{3^N}$ is the uniform distribution $(2/9\pi^2)^N$, we have

$$w_{\rho_\epsilon}(\mathbf{n}_1, \ldots, \mathbf{n}_N) = (1 - \epsilon)\left(\frac{2}{9\pi^2}\right)^N + \epsilon w_{\rho_1} \geq \left(\frac{2}{9\pi^2}\right)^N \Big(1 - \epsilon(1 + 3^{2N-1})\Big). \qquad (4.8)$$

We conclude that if

$$\epsilon \leq \frac{1}{1 + 3^{2N-1}}, \qquad (4.9)$$

$w_{\rho_\epsilon}$ is nonnegative, and thus the qutrit state $\rho_\epsilon$ of Eq. (4.1) is separable. This establishes a lower bound on the size of the separable neighborhood surrounding the maximally mixed state.

We turn now to obtaining an upper bound on the size of the separable neighborhood. Consider two particles, each with spin $(3^{N/2} - 1)/2$ ($N$ even) and thus each having a $(d = 3^{N/2})$-dimensional Hilbert space. We can think of each particle as being an aggregate of $N/2$ spin-1 particles (qutrits). We consider the following joint density operator for the two particles (or of the $N$ qutrits),

$$\rho_\epsilon = (1 - \epsilon)M_{d^2} + \epsilon|\phi\rangle\langle\phi|, \qquad (4.10)$$

where $M_{d^2} = 1/d^2$ is the maximally mixed state in $d^2 = 3^N$ dimensions, and

$$|\phi\rangle \equiv \frac{1}{\sqrt{d}}\Big(|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle + \cdots + |d\rangle \otimes |d\rangle\Big) \qquad (4.11)$$

is a maximally entangled state for the two particles.

Now project each particle onto the subspace spanned by $|1\rangle$, $|2\rangle$, and $|3\rangle$. For each particle this subspace can be thought of as the Hilbert space of a single qutrit; thus the

projection leaves a joint state of two qutrits. The projection is effected by the projection operator

$$\Pi = \sum_{a=1}^{3} |a\rangle\langle a| \otimes \sum_{b=1}^{3} |b\rangle\langle b| , \qquad (4.12)$$

which is the unit operator in the projected two-qutrit space, i.e., $\Pi = 1 \otimes 1 = 9M_9$. The normalized state after projection is

$$\tilde{\rho}_\epsilon = \frac{\Pi\rho_\epsilon\Pi}{\mathrm{tr}(\rho_\epsilon\Pi)} = \frac{1}{A}\left((1-\epsilon)\frac{9}{d^2}M_9 + \frac{3\epsilon}{d}|\Psi\rangle\langle\Psi|\right) = (1-\epsilon')M_9 + \epsilon'|\Psi\rangle\langle\Psi| , \qquad (4.13)$$

where

$$A = \mathrm{tr}(\rho_\epsilon\Pi) = (1-\epsilon)\frac{9}{d^2} + \frac{3\epsilon}{d} = \frac{9}{d^2}\Big(1 + \epsilon(d/3 - 1)\Big) \qquad (4.14)$$

is a normalization factor, $|\Psi\rangle$ is the maximally entangled state of two qutrits given in Eq. (3.1), and

$$\epsilon' = \frac{3\epsilon/d}{A} = \frac{\epsilon d/3}{1 + \epsilon(d/3 - 1)} . \qquad (4.15)$$

The projected state $\tilde{\rho}_\epsilon$ is the state (3.2) considered in Sec. III, a mixture of the maximally mixed state for two qutrits, $M_9$, and the maximally entangled state $|\Psi\rangle$. As shown in Sec. III, this state is nonseparable for $\epsilon' > 1/4$, which is equivalent to

$$\epsilon > \frac{1}{1+d} = \frac{1}{1 + 3^{N/2}} . \qquad (4.16)$$

Moreover, since the local projections on the two particles cannot create entanglement from a separable state, we can conclude that the state (4.10) of $N$ qutrits is nonseparable under the same conditions. This result establishes an upper bound, scaling like $3^{-N/2}$, on the size of the separable neighborhood around the maximally mixed state.

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