Random Matrix Approaches in Number Theory

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Background

The connection between random matrix theory and the Riemann zeta function was established in 1973 when Montgomery, who had conjectured the 2-point correlations of the Riemann zeros, and Dyson, who was interested in similar statistics of the eigenvalues of ensembles of unitary matrices, realized that the formulae they had discovered independently were in fact identical in a natural asymptotic limit. Further attempts at the verification of this coincidence of Riemann zero and eigenvalue statistics were then produced from various fronts: overwhelming numerical evidence was afforded by the mammoth computations of Andrew Odlyzko (1989); the heuristic work of Bogomolny and Keating (1995) pointed towards the agreement of not just the 2-point correlation function, but all the n-point statistics as well; and Rudnick and Sarnak (1996) proved that the Riemann zeros and the eigenvalues of this random matrix ensemble have the same n-point statistics in a restricted range.

This programme was designed to draw on the expertise of number theorists, probabilists and physicists to further the recent successes of random matrix theory in predicting results about the Riemann zeta function and other L-functions, thus pointing the way to answering long-standing questions in number theory. Initial results in this direction concerned the mean values of the Riemann zeta function, and with the suggestion of Katz and Sarnak (1999) that by grouping L-functions into families one finds that the distribution of their zeros averaged over the family show random matrix statistics as well, average values of L-functions over families have also been studied using random matrix theory.

The programme has resulted in the formulation of precise and far reaching conjectures about the behaviour of zeta functions and families of zeta functions in terms of their associated (random) matrix symmetry type. These conjectures now have a firm numerical and theoretical basis. Moreover the techniques, developed in part during the programme, to establish some of these conjectures rigorously have proven to be very useful in establishing subconvexity and density theorems for such families of zeta functions. (The latter are often decisive in applications to classical number theoretic and quantum chaos problems.)

Programme Overview

From the beginning we sought to maximise the interaction between participants, especially those coming from different fields. The layout of the Institute was very helpful in this, it being easy to meet over coffee in the central area, but we also organised more structured interaction in the form of two seminars each week, after which all participants were encouraged to socialise over dinner. There was a stress on keeping the seminars accessible to all, so all speakers were asked to include the necessary background material in their talk. On the evidence of the participants' reports it appears that the programme was successful in germinating collaborations that reached beyond the usual pool of researchers' colleagues. As the programme started during term time and facilities were not available for a large initial workshop, the opening event was a focused working week, which was very productive and set the tone for the rest of the programme. A two-week school was held at the end of March to cover, in a pedagogical manner, the pertinent points of random matrix theory and number theory and the connection between them. Time and again during the remainder of the programme these lectures were referred to as common ground from which to build a mathematical discussion. In May all the current participants relocated to Warwick for a satellite workshop on the areas of probability that are relevant to the programme theme. During the final three weeks of the programme in June and July the number of participants increased, as those who had been unable to get away from university or teaching commitments arrived to take part in the meeting on Random Matrix Theory and Arithmetic Aspects of Quantum Chaos and the concluding workshop on Matrix Ensembles and L-Functions. The choice of workshops proved to be very popular; all were filled to capacity. They are discussed in more detail below.

Workshops

Ranks of Elliptic Curves and Random Matrix Theory, Clay Mathematics Institute Special Week,
9-13 February 2004
Organisers: B Conrey, D Ellwood, D Farmer, F Mezzadri and NC Snaith
The workshop began with a London Mathematical Society Spitalfields Day, a day which traditionally brings together researchers from all over the UK to listen to expository talks about a field of significant current interest. Random Matrix Theory has proven to be an important tool for modelling the value distribution of families of L-functions. The Birch and Swinnerton-Dyer conjecture relates the value of the L-functions associated with an elliptic curve to the arithmetic structure of the elliptic curve, and this relation serves as the starting point for our investigations.

For the Spitalfields Day talks were given by Bryan Birch and Peter Swinnerton-Dyer, Alice Silverberg, Christophe Delaunay, Michael Rubinstein and Chantal David, introducing topics on elliptic curves and the connection with random matrix theory.

The workshop that followed in the next four days was a little unusual in that the intention was to bring together a group of people to try to understand some very specific problems. Consequently, the schedule of the workshop had very few announced talks. The rest of the time was set aside for discussion sessions, work sessions and talks that could be given as needed or as requested by the thirty-eight participants.

The goals of the week were to better understand the frequency of rank two elliptic curves occurring in a family of quadratic twists of a given elliptic curve, as well as to discuss the occurrence of higher ranks and to find a way to compute twists of higher weight cusp forms. Working groups were formed on the Thursday to attack very specific questions relating to these issues.

The workshop seemed to be unusually successful in that we were able to make inroads in all of our stated goals. In summary, having specific aims in mind, allowing for plenty of discussion time, work time, and unscheduled time that could be planned during the course of the week, seems to have created an opportunity to accomplish these specific goals.

Recent Perspectives in Random Matrix Theory and Number Theory, EC Research Training Network School,
29 March-8 April 2004
Organisers: F Mezzadri and NC Snaith
This two-week school, funded by the European Commission Research Training Network "Mathematical Aspects of Quantum Chaos", the Mathematical and Theoretical Physics group of the Institute of Physics and the National Science Foundation, was one of the most important events of the programme.

The workshop was designed with the interdisciplinary nature of this programme in mind. Teaching young researchers in mathematical physics mathematical techniques from random matrix theory and number theory, which are not a natural part of their education, is essential to introduce a new generation of scientists to this field, which is important and rapidly developing but is hampered by the problem that very few researchers are well versed both in number theory and methods in mathematical physics.

There were twelve lecture series, comprising between three and six lectures each an hour long, plus two special lectures by Oriol Bohigas and Michael Berry giving a historical perspective of the subject. The lectures were arranged so as to start from the basics in random matrix theory and number theory separately and to progress to absolutely the most recent work utilising the connection between these two fields.

At the request of the students, a session was held to discuss open problems in the field of a type that might be suitable for a graduate student's thesis project. A volume of proceedings from this school is being created and should prove to be a valuable text for students in this field.

As well as attending lectures over the 11 days of the school, the 97 participants, from professors to graduate students, made new contacts that bridged the usual boundaries of their disciplines. In addition, many of the young researchers participated in later events in the programme, thus putting the knowledge they gained at the school to immediate use.

Random Matrices and Probability, Satellite Conference at the University of Warwick,
18-21 May 2004
Organisers: F Mezzadri, N O'Connell and NC Snaith
This meeting was a satellite workshop held at the University of Warwick and partly sponsored by the London Mathematical Society. It was focused on aspects of random matrices that appear in the theory of probability and on their connections with number theory.

The themes discussed included: Brownian motion and the Riemann zeta function; eigenvalues of non-Hermitian random matrices; universality, sparse random matrices, transition matrices and stochastic unitary matrices; free probability and free stochastic calculus; matrix-valued diffusion and Brownian motion on symmetric spaces; and intertwining relationships in random matrix theory and Markov processes.

About 40 researchers participated in the workshop, half of whom were attending the programme at the Newton Institute. The audience had quite a diverse background and included number theorists, physicists and probabilists. Indeed one of the goals of the meeting was to bring together mathematicians with different areas of expertise to produce new ideas on how recent developments in random matrix theory can have an impact in number theory. The meeting integrated very well also with the activities in the Department of Mathematics at the University of Warwick and the departmental colloquium was part of the workshop, given by one of the participants, Professor Kurt Johansson.

Random Matrix Theory and Arithmetic Aspects of Quantum Chaos
Euroconference, 28 June-2 July 2004
Organisers: J Marklof, F Mezzadri and Z Rudnick
The main open questions in quantum chaos concern the statistical properties of spectra and eigenfunctions of quantum systems whose classical limit manifests chaotic behaviour. This workshop addressed these problems by focusing on quantum systems in a natural number-theoretical setting, as these are the only models for which rigorous results have been obtained up to now.

The main themes discussed in the meeting concerned: the proof of quantum unique ergodicity; the proof of the existence of scars in quantum cat maps; the analysis of the value distribution of eigenstates on hyperbolic manifolds; the proof of the Berry-Tabor conjecture for spectral two-point statistics for certain classes of integrable systems; the analysis of spectral correlations in pseudo-integrable systems; and heuristic analysis of the spectral form factor for systems which exhibit random matrix correlations.

A successful problem session in which the most important unsolved challenges in the field were discussed was held as part of the conference programme. Approximately 60 researchers, among whom were world leading experts in the subject, attended the workshop.

Matrix Ensembles and L-functions
Euroworkshop, 12-16 July 2004
Organisers: B Conrey, P Sarnak and NC Snaith
As the concluding workshop of the programme one of the goals was to provide an opportunity to relay some of the results accomplished during the programme to a wider interested audience. Perhaps a more important aim, however, was to discuss some of the questions and ideas that arose during the programme but that were still unsolved. For this purpose two discussion sessions were timetabled which lead to very active debates and the tabulation of a list of open problems and the means by which they might be tackled. The workshop was also very important for researchers who were not able to attend the rest of the programme. For those who were not able to leave their home institution for an extended period, this workshop served as an opportunity to find out about progress in this field and also to present their own results to the rest of the community.

The main topics discussed were moments and ratios of zeta and L-functions as well as the related question of extreme values of these functions and breaking convexity bounds on the size of L-functions. Also discussed were the correlations of the zeros of L-functions, zeta functions on function fields, computing values of L-functions and applying random matrix theory to elliptic curve L-functions.

Outcomes, Achievements and New Directions

The goal of the programme was to identify important questions in number theory that can be answered using random matrix theory and to discover new areas of interest in the connection between these two subjects.

This was achieved successfully starting from the very first month of the programme. The Special Week in February raised many questions concerning ranks of elliptic curves. This will be a topic that will be pursued in diverse directions long after this programme has ended. As an example of a specific result inspired by this workshop, M Watkins used a combination of random matrix theory and elegant number theory to examine the distribution, amongst residue classes, of integers that can be written as the sum of two cubes of rational numbers. Other new results were the computation of special values of twists of L-functions of weight 4 and 6 forms and a conjecture on the distribution of the derivatives of elliptic curve L-functions and the relative size of various components of the Birch and Swinnerton-Dyer formula for the derivative of these L-functions at the critical point.
Another theme that surfaced early on during the programme was that of function field zeta functions. For these functions the Riemann Hypothesis has long been proved, and in 1999 Katz and Sarnak showed that for families of function field zeta functions the type of random matrix symmetry displayed by the zeros can also be calculated. Discussions throughout the programme indicate that this is an area that deserves more attention from number theorists and random matrix theorists alike.

The flexibility of the Newton Institute allowed a focused week on random polynomials to be organised midway through the programme when links with both random matrix theory and number theory began to interest the participants. This lead to comparisons of the statistics of zeros on the unit circle of random polynomials with those of eigenvalues of random matrices, as well as new results by C Hughes and A Nikeghbali on the clustering round the unit circle of zeros of random polynomials.

The block of uninterrupted research time afforded by the programme has also meant that several long-running collaborations have been brought to a conclusion and prepared for publication, such as the paper on integral moments of L-functions by B Conrey, D Farmer, J Keating, M Rubinstein and NC Snaith and the extension of this to ratios of L-functions by B Conrey, D Farmer and M Zirnbauer.

A development at the end of the programme, resulting from the collaborations of D Farmer, S Gonek and C Hughes, all of whom were present for the entire six months, was a new approach toward the important and controversial question of the maximal order of the Riemann zeta-function on the critical line. Also, at the final workshop, K Soundararajan spoke on work carried out together with Z Rudnick in which they showed how one may, in

the case of discrete families, prove a lower bound of the correct order for the conjectured moments. The result was known for the Riemann zeta function (Titchmarsh) but the discrete family analogue was open until the conference on Matrix Ensembles and L-Functions.

The interactions during the programme also pressed home the importance of numerical data in this field where random matrix methods lead to conjectures about number theoretical quantities. Skills in computation will be of great importance in the future, especially to young researchers entering the field, as more and more data is needed to test the sophisticated conjectures generated by random matrix theory.

Several developments were made in various aspects of random matrix theory. J Keating and F Mezzadri discovered a new approach to computing the entropy of entanglement in quantum spin chains using random matrix theory. This established an unexpected connection with conformal field theory and statistical mechanics that led to discussions with the participants of the programme Statistical Mechanics of Molecular and Cellular Biological Systems which was running parallel to ours, and in particular with M Fisher. Other aspects of random matrix theory treated during the programme that ended up in ongoing research projects include: the generalization of the Fisher-Hartwig conjecture to all classical compact groups, on which E Basor and P Forrester are currently working; experts on integrable systems and the Riemann-Hilbert problem, among whom are A Its and J Harnad, attended the programme giving substantial contributions; and new approaches to spectral correlations at the edge of the spectra of non-Hermitian random matrices were developed by B Khoruzhenko, F Mezzadri and L Pastur.

In general this programme has been very successful in gaining more acceptance for random matrix theory in number theory, along with more willingness from proponents of the two fields to work together. It has sparked interest in this area amongst young researchers and those at the boundaries of the subject, and it has resulted in renewed vitality and a host of new problems to be worked on.