Singularity Theory

24 July to 22 December 2000

Report from the Organisers: VI Arnold (Moscow and Paris IX), JW Bruce (Liverpool),
V Goryunov (Liverpool),
D Siersma (Utrecht)

Scientific Background and Objectives
Organisation
Participation
Meetings and Workshops
Outcome and Achievements

Scientific Background and Objectives

Singularities arise naturally in a huge number of different areas of mathematics and science. As a consequence Singularity Theory lies at the crossroads of the paths connecting the most important areas of applications of mathematics with its most abstract parts. For example, it connects the investigation of optical caustics with simple Lie algebras and regular polyhedra theory, while also relating hyperbolic PDE wave fronts to knot theory and the theory of the shape of solids to commutative algebra.

The main goal in most problems of singularity theory is to understand the dependence of some objects of analysis and geometry, or phenomena from physics or some other science, on parameters. For generic points in the parameter space their exact values influence only quantitative aspects of the phenomena, their qualitative, topological features remaining stable under small changes of parameter values.

However, for certain exceptional values of the parameters these qualitative features may suddenly change under a small variation of the parameter. This change is called a perestroika, bifurcation or catastrophe in different branches of the sciences. A typical example is that of Morse surgery, describing the perestroika of the level variety of a function as the function crosses through a critical value. (This has an important complex counterpart: the Picard-Lefschetz theory concerning the branching of integrals.) Other familiar examples include caustics and outlines or profiles of surfaces obtained from viewing or projecting from a point, or in a given direction.

In spite of its fundamental character and the central position it now occupies in mathematics, singularity theory is a surprisingly young subject. So, for example, one can consider the singularities arising from the orthogonal projections of a generic surface in 3-space, a problem of surely classical interest. Their classification was completed as recently as 1979. In one sense singularity theory can be viewed as the modern equivalent of the differential calculus, and this explains its central position and wide applicability. In its current form the subject started with the fundamental discoveries of Whitney (1955), Thom (1958), Mather (1970) and Brieskorn (1971). Substantial results and exciting new developments within the subject have continued to flow in the intervening years, while the theory has embodied more and more applications.

The major idea of this programme was to bring together experts within the field and those from adjacent areas where singularity theory has existing or potential application, such as

wave propagation, dynamical systems, quantum field theory, differential and algebraic geometry, and others. It was the programme's aim both to foster exciting new developments within singularity theory, and also to build bridges to other subjects where its tools and philosophy prove useful.

Organisation

The programme was planned by Arnold, Bruce and Siersma. Unfortunately, Arnold was not able to attend the programme at all due to a cycling accident. The day-to-day organisation was carried out by Goryunov and Siersma. There was very valuable assistance from Zakalyukin, Wall, Giblin and Nikulin for specific workshops during the programme. In addition to the main meetings, the programme ran seminars on a regular weekly basis on Tuesday and Thursday afternoons, with at least 2 talks in each session. There were also extra, occasional seminars arranged by the organisers and other participants.

Two of the participants gave lecture series on special subjects. From September to October, a series of 9 lectures on Frobenius manifolds was given by B Dubrovin. Frobenius manifolds are now finding numerous applications in physics. However, they are a rather new (but highly natural) topic for Singularity Theory. In November, another series of lectures, on ramified integrals, was given by VA Vassiliev, devoted to generalisations of works by Newton and Atiyah. Both lecture series attracted excellent audiences.

In December the frequency of talks increased and the penultimate week was run on a full conference scale, with up to 5 talks daily.

N Nekrasov, AN Varchenko, VA Vassiliev and VV Goryunov spoke in a two-day Discussion Meeting on Topological Methods in the Physical Sciences held at the Royal Society in November. The meeting was organised together with the parallel programme on Geometry and Topology of Fluid Flows, and brought together leading pure and applied mathematicians from the UK and other countries. It was attended by many British specialists and was a great success.

Participation

The programme was attended by 53 long-term and 87 short-term visitors. Some of them came to the Institute several times. In addition, there was very strong participation in the conferences, workshops and other events during the programme. Dubrovin, as Rothschild Visiting Professor, made a remarkable contribution to the programme during his stay.

The majority of experts in the field attended the programme, and a large number of leading experts in adjacent areas visited the Institute. There was a strong presence from Europe and Russia (the latter allowed by the generous support from the Leverhulme Trust), as well as from North America.

Many of the meetings and individual talks, plus special lecture courses, attracted mathematicians from both Cambridge and other British universities. The EU support for two of the meetings and the Junior Membership scheme were highly useful for attendance by PhD students and young postdocs. A number of participants gave talks in other UK departments, such as Edinburgh, Hull, Leeds, Liverpool and Warwick.

Meetings and Workshops

NATO Advanced Study Institute / EC Summer School: *New Developments in Singularity Theory***, 31 July - 11 August 2000**

Organisers: D Siersma, VM Zakalyukin, JP Brasselet, VA Vassiliev and CTC Wall
This first meeting was designed to study developments in singularity theory, especially during the last 5 years. Its key topics included
• Singular complex varieties: global theory
• Singular complex varieties: local theory
• Singularities of holomorphic maps
• Singularities of real maps
• Study of discriminant spaces and Vassiliev- type invariants
The primary purpose of the ASI was the introduction of new developments in singularity theory to a broader audience, thus establishing new contacts and advancing a broad front of research. The importance of pedagogical skills had been borne in mind in choosing the ASI lecturers, as the instructional aspect was regarded as central to the success of the meeting.
GM Greuel, A Fruehbis-Krueger and C Lossen (all from Kaiserslautern) demonstrated the latest version of the computer package Singular which the group in Kaiserslautern has been developing over the last decade for the specific needs of singularity theory. The meeting finished with a problem session. The forum was remarkably large and served as an excellent start to the entire programme: the total number of participants exceeded 95, many of them younger mathematicians. This was made possible thanks to the combined generous support from NATO and the EU. A volume of proceedings is now being published by Kluwer.
Some of the ASI participants stayed on at the Institute for one week to attend the mini-workshop on polynomial and meromorphic functions conducted by A Dimca.

Workshop on *Applications to Wave Propagation Theory and Dynamical Systems*, **25 - 29 September 2000**
Organisers: RM Roberts and VM Zakalyukin
The topics of the second meeting included:
• Singularities in symplectic and contact spaces: caustics and wave fronts, shock waves, Hamilton-Jacobi equation, Lagrangian intersections, Legendre knots.
• Applications to control theory and differential equations, game theory. Singularities in subriemannian geometry, optimization, Pfaffian systems.
• Singularities of momentum maps, energy-momentum maps and integrable Hamiltonian systems. Monodromy in Hamiltonian systems, applications to physics.
• Bifurcations of (relative) equilibria of Hamiltonian systems and time-reversible equivariant dynamical systems. Singularity theory and KAM theory.
The meeting consisted of 35 lectures, with 53 official participants. It attracted many specialists
in Hamiltonian systems and this provided a considerable source of new questions (including those arising from physical experiments) to which Singularity Theory is very likely to find answers.

Euroworkshop on *Applications to Quantum Field Theory*, **23 - 27 October 2000**
Organisers: B Dubrovin, V Goryunov, N Nekrasov and V Nikulin
Over the last years, new and very interesting relations have been discovered between singularity theory and algebraic geometry on one hand, and quantum field theory and string theory on the other. For example, mirror symmetry for Calabi-Yau manifolds, Gromov-Witten invariants, Frobenius manifolds and integrable systems became important tools in string theory and quantum field theory. The October Euroworkshop was devoted to further interactions between all of these areas. Supported by the EU, it brought together leading specialists and young researchers in areas very rapidly developing nowadays. It was one of the most inspiring events in the entire programme bringing a large number of new problems

into the area of singularity theory. There were 24 lectures with good spacing between them allowing numerous discussions.

Informal Discussion Meeting on *Different Aspects of Singularity Theory*, **24 - 25 November 2000**

Organiser: D Siersma

The meeting concentrated on deformation theory, analysis and topology of singularities. This was a rather small scale session (21 participants), but intensive and productive. The discussions were very stimulating and useful for establishing new links with other branches of mathematics.

Satellite Workshop at the University of Liverpool on *Applications of Singularity Theory to Geometry*, **16 - 21 December 2001**

Organisers: JW Bruce, J Damon, PJ Giblin, VV Goryunov and CTC Wall

The final event of the programme was a Satellite Meeting in Liverpool. Its theme was one of the most traditional in the field: much of Singularity Theory was inspired by geometrical problems. Thom's early work on differential geometry via families of functions has borne enormous fruit in a richer understanding of the higher geometry of surfaces, and this in turn has found application in other fields such as computer vision. Projections of surfaces to planes, giving apparent contours, and the general theories of caustics and wave fronts are other examples where new techniques were motivated by geometrical problems. Remarkable duality connections have been found between some of these problems, and there are applications to algebraic geometry and other mathematical fields.

This workshop took as its theme interactions between singularity theory and geometry in its many modern guises. Besides the topics mentioned above one could mention Gauss mappings, the geometry of discriminants and bifurcation diagrams, coadjoint orbits, billiards, arrangements and the global geometry of singular waves and varieties.

The number of requests for talks in Liverpool was very high and it was decided to run a week-long full-scale pre-workshop at the INI, which included a most inspiring lecture series by A Losev exposing new ideas of applying singularity theory to the study of Frobenius structures.

There were 34 talks in Liverpool itself. The meeting was supported by the LMS conference grant and a special grant from the INI. It was attended by 63 participants.

Outcome and Achievements

The programme was highly successful. It did achieve its main goal to bring together the best researchers working on singularities and specialists in other branches of mathematics and physics where singularity theory either already is an efficient tool or can become such a tool in the future. The programme was an excellent school for its younger participants. There was a certain amount of interaction with the participants of the parallel programme on Geometry and Topology of Fluid Flows. Contacts with Pelz, Ricca, Michor, Khesin, Shnirelman and others helped our participants to realise the current needs of this branch of applied mathematics.

Over the programme, its participants worked both on their long-term projects and on new problems that came to their attention during their stay. The comments of participants were uniformly positive, and all reported productive discussions and new collaborations. A high number of papers were reported as either submitted or being in preparation during the programme.

Anisov worked on Matveev's complexity of three-manifolds. He obtained an upper bound (which is very likely to be precise) for the complexity of torus fibrations over the circle. Bogaevsky and Ishikawa studied singularities of Legendre mappings, in particular a

generalisation of Mather's theory to the Legendrian case.

Buchstaber and Rees continued their joint work on symmetric products of linear spaces which is a multi-dimensional version of the Vieta theorem.

Campillo and Olivares made considerable progress in describing singular one-dimensional foliations in terms of the associated polar map.

A lot of attention during the programme was devoted to the famous Jacobian Conjecture requiring deep understanding of singularities of plane algebraic curves at infinity. Campillo, Piltant and Reguera studied singularities of the characteristic cone of the surface obtained by blowing up the infinite points of such a curve. Chekanov worked on the Arnold-Givental conjecture claiming that a Lagrangian submanifold, which is a fixed-point set of an anti-symplectic involution, must have sufficiently many points of intersection with its image under a Hamiltonian symplectomorphism. He succeeded in constructing a counter-example to the degenerate part of this conjecture. The result can be seen as the first step to disproving the degenerate part of Arnold's famous symplectic fixed-point conjecture, whose non-degenerate part was proven recently by Fukaya-Ono and others. Davydov, with the involvement of Zhitomirskii, continued his study of singularities of implicit differential equations. He obtained new results in two directions: on generic singularities of higher order equations, and on normal forms of first order equations involving one dependent and many independent variables. The results are promising to be very important for applications. Dubrovin continued his research on applications of Frobenius manifolds to integrable hierarchies and to Gromov-Witten invariants of higher genera. He also started new investigations, together with Goryunov and Kazarian, on applications of Frobenius manifolds to singularities of functions on complete intersections and on singular space curves. Ebeling and Gusein-Zade made considerable progress in their work on an algebraic formula for the index of a differential form at an isolated complete intersection singularity. Gaffney worked on his long-term project to give an integral closure formulation to all equisingularity conditions controlled by analytic inequalities. Together with Houston, he started a project on finding necessary and sufficient conditions for families of finitely determined map-germs from 3-space to 3-space to be Whitney equisingular. Damon, Gaffney and Mond worked on a conjecture claiming that the image of a map-germ from n-space to (n+1)-space is a free star divisor in a sense of Damon. This would yield a formula for the image Milnor number. Gaffney, Trotman and Wilson finished their paper on equisingularity of sections. Denef, Melle and Veys worked on the Monodromy Conjecture on the topological zeta-function of a holomorphic function-germ.

Kazarian obtained very impressive results on classifying spaces of singularities and Thom polynomials, on characteristic classes related to singularities of Lagrange and Legendre mappings. Matveev obtained new strong results on the global theory of geodesically equivalent metrics. Merkov, helped by Vassiliev and Kazarian, made considerable progress in the study of Vassiliev-type invariants of ornaments (collections of plane curves no three of which have common points) and doodles (triple-point-free collections of plane curves). Polyak worked on Kontsevich's universal formula for a deformation quantisation of the algebra of functions on a real linear space. He succeeded in interpreting this formula as a degree of a map from a certain configuration to a multi-dimensional torus. Polyak also obtained a very elegant generalisation of the classical Crofton formula to the case of generic immersed plane curves instead of only convex ones. Pushkar continued working on his long-term and extremely promising project on Legendrian K-theory and relative Morse theory. Ruas, using the methods of Gaffney and Damon, made progress in the problem of finding sufficient conditions for topological triviality in families of function-germs on analytic varieties. Scherbak and Varchenko worked on the resonance case of the problem of behaviour of critical points of products of powers of linear functions. This question turned

out to be closely related to the representations of sl(2). Also Scherbak finished a paper on boundary singularities and non-crystallographic Coxeter groups. Sedykh studied admissible homotopies of space curves, that is those not involving in particular curves with self-intersections and inflexion points. This allowed him to solve Arnold's problem on the impossibility of deforming certain curves to curves without flattening points via such homotopies.

Shapiro, Kazarian and Goryunov studied Hurwitz spaces developing an approach to calculate Hurwitz numbers in the general case of a meromorphic function with multiple complicated branch points. Shapiro also worked on understanding relations between various Poisson structures on the space of unipotent upper triangular matrices.

Tibar wrote a paper on a problem of describing vanishing cycles in non-generic Lefschetz pencils on complex and symplectic spaces. Together with Siersma, he worked on understanding deformations of polynomials at infinity. Vassiliev finished the formulation of a purely combinatorial algorithm for calculating combinatorial formulae for knot invariants. This is an important part of his extensive programme on effective calculation of cohomology classes of spaces of non-singular geometrical objects. He wrote 4 papers and edited a translation of one of his books. Wall and du Plessis worked on their papers on theorems of Cayley-Baeharach type and on generic projections.

Zakalyukin and Goryunov made considerable progress in their study of the monodromy in vanishing homology of families of 2×2 matrices. Together with Bortakovsky, Zakalyukin proved that local singularities in generic completely non-integrable logical-dynamical systems with a low number of switches correspond to generic corner singularities. Together with Giblin, Zakalyukin worked on classification of generic singularities of envelopes of families of chords in affine geometry.

The programme was an international event of extremely high significance for Singularity Theory and related subjects. It provided a very valuable boost of new ideas and introduced many exciting problems to solve.

Back to Top