Stochastic Partial Differential Equations

4 January – 2 July 2010

Final Report from the organisers: Z. Brzezniak (York), M. Hairer (Warwick), M. Röckner (Bielefeld), P. Souganidis (Chicago) and R. Tribe (Warwick).

Scientific background

Over the past two decades, the theory of Stochastic Partial Differential Equations (SPDEs) has emerged as a new area of mathematics that combines elements of both traditional PDE theory with tools from stochastic analysis and probability theory. Besides forming a branch of pure mathematics in its own right, the theory of SPDEs has been used in a number of applications. Often, SPDEs are used to model physical systems subject to the influence of internal, external or environmental noise, but they also arise when considering deterministic models with random initial conditions, or as tractable approximations to complex deterministic systems. In many cases, the presence of noise leads to new phenomena with many recent examples in the physical sciences, biology and financial modelling. Infinite dimensional equations also arise by filtering or sampling from finite stochastic dimensional systems.

The goal of the programme was to bring together the world leaders in SPDEs working on various aspects of the theory, but also on numerical approximations and on the application of stochastic PDEs to real-world problems. The programme also aimed at identifying future trends of the research in the field. >From the point of view of the fundamental theory (existence / uniqueness / regularity), it was clear there are interesting examples of stochastic PDEs for which some of the expected well-posedness results break down, either due to the roughness of the solutions, or due to the irregularity of their nonlinearity. Furthermore the current theory is largely for Gaussian noises, and one on-going emphasis was on systems driven by Lévy noise, for which the existing theory is still sparse. Another recent development that has demonstrated great potential during the programme is the use of the theory of rough paths to analyse solutions to SPDEs in various contexts. Finally, much work was on the study of qualitative features of specific models, for example the fractal dimension of hitting sets, the description of solutions by reduced models in certain regimes, the study of the appearance of intermittency, asymptotic behaviour e.t.c.

Organisation of the programme

We decided to organise the programme around a backbone of three week-long workshops. These were interwieved with two smaller focused 2-day mini-sessions, one satellite workshop, several focused mini-courses, and an extensive visitor's programme. Each of the main workshops consisted of about 30 talks and attracted about 90 participants. The mini-courses were given by Da Prato on Kolmogorov equations in infinite dimensions, by Ondrejat on the stochastic wave equations, and by Barbu on the differences between stochastic and deterministic nonlinear PDEs in the corresponding infinite-dimensional Cauchy problems.

Other prominent speakers giving exceptional lectures included those of Sinai on the application of the renormalisation group method in fluid dynamics and on the decay of Fourier modes of solutions

to the Navier-Stokes equations. Pierre-Louis Lions also gave two talks during one of the mini-sessions: a technical report on "Not so rough paths" and a talk intended for a general mathematical audience on "mean field games".

1. SPDEs opening workshop. 4 Jan – 8 Jan.

The opening conference concentrated on surveying the latest developments throughout the field. To this purpose, we invited many of the foremost experts on stochastic PDEs, so that a clear picture of the state of the art could emerge. Several lecturers discussed foundational issues on well-posedness of equations. We mention in particular Mytnik and Perkins's well-posedness result for equations with only Hölder-continuous coefficients, Flandoli and Priola's results on noise-induced uniqueness, Veraar, van Neerven and Weis work on maximal regularity for stochastic convolutions in Banach spaces and Da Prato and Zhang's results on models with reflection. The theory of large deviations for infinite-dimensional systems (addressed in particular by Cerrai, Millet, Sanz-Solé and Zabczyk) became a recurrent theme for the entire programme. Another recurrent theme that emerged during the workshop is that of stochastic fluid dynamics. Indeed, a significant group of participants (Kim, Hairer, Millet, Lototski, Barbu, Mattingly, Truman, Mohammed, and Stannat) reported on progress in the understanding of various aspects of the stochastic Navier-Stokes and Burgers models, an area of application which has seen intensive recent work.

2. SPDEs and their applications. 29 Mar – 1 Apr.

The aim of this workshop was to bring together researchers from a number of scientific areas that use stochastic PDEs as a mathematical tool and focused on the applications, rather than on the theory for its own sake. It included talks on dispersive models (stochastic Korteweg–de Vries and nonlinear Schrödinger equations) by Kuksin and by Gautier; on porous media equations by Röckner and Russo; on ferromagnetic modelling by Goldys; on surface growth models by Romito. The workshop invited several speakers from outside the stochastic community (Caffarelli, Otto, Luckhaus, Le Bris, Dirr) to help guide researchers towards potential areas of application for stochastic models. Outside the workshops there were further talks from physicists: Chantrell on stochastic ferromagnetism and Turitsyn on nonlinear photonics. Later in the programme Cutland gave an approach to the stochastic hydrodynamics via non-standard analysis, and R. Moser discussed the Ginzburg-Landau vortices driven by Landau-Lifshitz equation. Financial applications included Zabczyk on the Musiela SPDE with Levy noise and Techranchi on hedging in variance swap markets.

3. SPDEs: Approximation, asymptotics and computation. 28 Jun – 2 Jul.

The closing workshop for the programme focused on both theoretical and computational approximations to SPDEs. For numerical analysts, this is an area of great current interest, since many of the methods developed for the analysis of deterministic PDEs break down due to the irregularity of solutions. A number of recent developments in this direction, including for example stability results and error analysis, were presented by Faou, Hausenblas, Gyongy, Prohl, Voss, Larsson, and Buckwar. On a completely different note, several speakers (Stuart, Crisan, Mattingly) treated some of the challenges encountered when trying to implement Monte-Carlo methods in infinite-dimensional spaces. From a more theoretical point of view, an interesting class of problems was given by the asymptotic analysis of equations that include small parameters, especially in the realm of singular perturbation (Cerrai, Freidlin). Another type of asymptotic behaviour that was discussed is that of the large time description of stochastic reaction-diffusion equations (Nolen, Tribe, Lythe). Finally, we would like to mention the method of stochastic viscosity solutions that was presented in different contexts by Friz and Souganidis to treat fully non-linear problems.

4. Related smaller workshops.

- **Mini-session on Rough paths, SPDEs and related topics. 6-7 Apr.**
  This was a two-day mini-session concentrated on the theory of rough paths, with emphasis on the infinite-dimensional settings. It emerged as a powerful tool to tackle a number of problems related to the workshop, including the application of underlying ideas to numerical methods (Gyongy and Crisan), the analysis of Burgers-type equations (Hairer), the analysis of stochastic Hamilton-Jacobi equations (Souganidis, Friz), and a new functional analysis approach to (not so) rough paths (Lions).

- **Satellite meeting in York. 31 May – 4 Jun.**
  This satellite meeting combined the themes of this programme, together with certain specific topics of interest to some UK researchers who were unable to attend the programme in Cambridge. These topics included infinite-dimensional analysis (Li, Fang, Lévy, Raimond, Thalmaier), Lévy drivers, which are now required by modellers in financial mathematics as well as certain physical systems like climate models (Imkeller, Peszat, Hausenblas), Zakai equation (Krylov and Rozovskii) as well as homogenisation (Pardoux, Zygouras).

- **Mini-session on Filtering. 14-15 Jun.**
  With many of the world's leading filtering experts at the institute, we held a two-day mini-session on this theme. Weiss gave an account of his recent maximal regularity result for parabolic SPDEs. The session emphasised recent theoretical problems such as the smoothness, stability, long-time asymptotic of filtering equations, as well as the important numerical problem of implementation (grid methods, spectral methods and particle approximations).

Scientific Outcomes

The scientific reports and our feedback from the participants indicated that participants found the atmosphere for informal discussions at the INI extremely valuable and enjoyed the high level of scientific quality at the seminars and workshops. A number of common themes emerged in the work carried out during the programme.

Problems with a geometric slant arose in several contexts. Exploiting earlier results on the existence of solutions to the stochastic geometric wave equations (GWE) by Brzezniak and Ondrejat and on numerical approximations for the deterministic GWE by Prohl and Banas, these four authors began to investigate the numerical approximations to stochastic GWE. The numerical work focussed their attention on questions about ergodicity for such problems. Various aspects of other stochastic geometric equations like heat flows, stochastic mean curvature flows, and the Landau-Lifshitz-Gilbert (LLGE) equations were investigated by De Bouard, Goldys, Jegaraj, Millet, Yip, Weber and others. In particular, progress has been made towards proving non-uniqueness for 3D heat flow as well as large deviations principle for 1D stochastic LLGE. Related questions were discussed in relation to talks by R Moser (see below) and R Chantrell.

The recent theory of rough paths by Lyons and co-workers influenced many workers. Wong-Zakai approximations are well understood in the SDE case, and this theory clarifies the role played by the Levy area. In the recent years Gyongy has been working on the approximations of SPDEs in the language of Levy area. In joint work with Stinga, he has exploited area processes to obtain sharp almost sure rate of convergence results, and applied these results to the numerical approximation of non-linear filters. Souganidis worked on the generalisation of his earlier works with PL Lions on fully nonlinear SPDEs to other classes of equations, and had fruitful discussions on this and related

topics with Friz, Oberhauser and Lyons on various approaches to such equations and in particular on the notion of stochastic viscosity solutions. There was substantial discussion between Lyons and other participants (De Bouard, Friz, Hairer and others) on the applicability of rough paths theory in the theory of SPDEs, as well as between Lions and Lyons on the foundations of the theory of rough paths. There was a discussion about extending the viscosity solution approach to stochastic PDEs to other classes of equations driven by rough paths. From a different perspective, Hairer showed that certain SPDEs can be interpreted as processes taking values in the space of rough paths. This sheds light on the fact that some equations are classically ill-posed, but that natural approximations to them can still converge to a well-defined limit.

Malliavin calculus was exploited by many workers. Dalang and Sanz-Solé succeeded in the generalisations of previous works on the hitting probabilities for systems of stochastic wave equations (the new results are applicable to equations with multiplicative noise). Gauthier and E Nualart began to work on the applications of Malliavin calculus to nonlinear Schrödinger equations while Mueller and JL Wu got some interesting results on the hitting probabilities for SPDEs driven with Levy noise.

SPDEs driven by Lévy noise received considerable attention. For instance Flandoli drew Priola's attention to the problem of pathwise uniqueness for SDEs with jumps and irregular drift term. Priola, Zabczyk and Xu started to study the ergodic properties for semilinear SPDEs driven by cylindrical alpha-stable processes. Brzezniak, Hausenblas and Zhu proved a maximal inequality for stochastic convolutions in Banach spaces, while Hausenblas and Blömker worked on the amplitude equation for SPDEs with Levy noise.

There was a flurry of activity related to the question of ergodicity of Markov processes generated by SPDEs. Often progress on problems in stochastic fluid dynamics was a target. Several recent results on this topic, some of which were obtained or improved during the programme, were presented by Mattingly (on his work with Hairer on the exponential ergodicity for degenerate SPDEs), Bessaih (on the application of this work to shell models) and Szarek (on his work with Komorowski and Peszat on the implications of the e-property). Related results for the stochastic Burgers equations (SBE) were discussed by Sinai, Brzezniak, Peszat and Szarek, and the problem of the existence of solutions for multi-dimensional SBE was positively resolved by Goldys, Neklyudov and others. Flandoli, Brzezniak and Neklyudov worked on the stochastic Euler equation as a Brownian motion on the group of volume preserving transformations. Hairer and Maas completed work on the numerical approximation of stochastic Burgers-type equations, while Hairer and Weber started work on giving meaning to Burgers-type equations with multiplicative noise that are classically ill-posed. Thanks to a visit and an inspiring talk by Roger Moser, some participants started to work on the development of stochastic models of vortex motion for Ginzburg-Landau equations and stochastic equations of ferromagnetism. This is likely to lead to future formal collaboration between Moser and other participants. Connections between his results and more classical results about vortex models for Euler equations were discussed while Flandoli, Priola and others continued to work on the vortex models for stochastic the Euler equations.

Infinite dimensional analysis is becoming an established tool in this area. Bogachev, Da Prato and Röckner worked on uniqueness of Fokker-Planck equations in infinite dimensions, with applications to SPDEs with singular drift coefficients. Goldys and Da Prato and co-workers developed the theory of functions of bounded variation in Hilbert spaces. Their work is the first attempt to study BV functions in infinite dimensions in the case where the reference measure is not Gaussian. It was successfully applied to the existence and uniqueness of strong solutions for the stochastic reflection problem on a regular convex set in Hilbert space.

With a range of applications as wide as deterministic PDE, there was much other progress: Barbu

and coworkers proved existence and uniqueness of solutions for the stochastic porous media equations by employing the Doss-Sussman and Imkeller methods, thus generalising earlier results obtained for different types of problems; Röckner and co-workers established the existence or a random attractor for the stochastic porous media equation; F.T.Wang solved the long standing problem of dimension independent Harnack inequalities for multiplicative noise, and in particular for pure jump-type noise. Many collaborations were started during the programme: Buckdahn and Dorogevtsev on measure valued games; Crisan, Peszat and Otobe on inverse problems for SPDEs; Zaboronski and Tribe on coalescing particle systems; Chueshov and Schmalfuss on synchronisation of coupled systems; Russo and Girolami on stochastic calculus for non-semimartingales with values in Banach spaces; Dirr and Yip on the interface evolution in random media; Wanner and Blömker on the stochastic nucleation.

Several participants used the time to continue their work on research monographs or lecture notes. For instance, Kloeden worked on a joint book with Rasmussen on non-autonomous dynamical systems; Elworthy, Le Jan and Li worked on a book on a geometric approach to filtering; Hairer worked on a book giving an introduction to the theory of stochastic PDEs.

We anticipate a follow-up SPDE meeting in 2012.