STOCHASTIC SYSTEMS FOR ANOMALOUS DIFFUSION: FINAL REPORT

1 July–18 December 2024

Organizers:

Codina Cotar (University College London), Aleksandar Mijatović (University of Warwick & The Alan Turing Institute), Anastasia Papavasiliou (University of Warwick), Ellen Powell (Durham University), Kavita Ramanan (Brown University), Kilian Raschel (CNRS & Université d'Angers), Perla Sousi (University of Cambridge), Andrew Wade (Durham University).

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7th May 2025

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Brief background/historical information

The programme "Stochastic systems for anomalous diffusion" was held from 1 July–18 December 2024, and brought together over 120 participants from across the world, with affiliations in Europe, the Americas, Asia, and Australasia, and with a range of scientific backgrounds from probability theory, stochastic analysis, mathematical physics, statistics and machine learning, and mathematical modelling. Specific thematic aspects were explored through four very active one-week workshops; two shorter workshops emphasized applications, industrial engagement, and diversity, equality and inclusion. An ongoing regular seminar programme, including a number of instructional mini-courses, ran throughout the course of the programme, in parallel to a full programme of social activities.

Diffusion refers to the movement of a particle or larger object through space subject to random effects. Mathematical models for diffusion phenomena give rise to stochastic processes, including classes of processes known collectively as diffusions or random walks. Such processes are ubiquitous in stochastic modelling single-particle or many-particle systems in physics (conformation of polymer molecules, mass transport, etc.), ecology (foraging, migration, dispersal, etc.), biology (microbe locomotion, etc.), chemistry (surface science, etc.) and many other areas. Diffusions and random walks are also primary ingredients of stochastic sampling and optimization algorithms in computational statistics and machine learning.

Anomalous diffusion describes processes that exhibit behaviour deviating fundamentally from the simplest diffusion models. Various physical, biological, or social mechanisms can produce anomalous diffusion. For example, processes may interact in complex ways with their past evolution, e.g., in systems with feedback, learning, or resource depletion, or with their spatial environment, e.g., in spaces with boundary reflections, anisotropy, or other non-trivial structure. Boundary reflections can drive a process to accelerate or confine it; learning or adaptive dynamics can force a process to dissipate faster; environmental inhomogeneities can trap or slow down diffusion; these are all "anomalous" behaviours.

In applications, to access fresh resources, roaming animals prefer not to retrace their steps; the excluded volume effect in polymers ensures that no two monomers can occupy the same physical space; topographical, botanical, or chemical gradients in an environment lend a preferred direction to an organism's motion. For statistical sampling and learning, processes that explore space faster than ordinary diffusions, or that adapt their behaviour according to previous learning, can lead to more efficient algorithms. Mathematical phenomena explored for processes exhibiting anomalous diffusion include limit behaviour, scaling limits, geometry and occupation statistics, mixing, and so on. Analysis of such processes can call on a range of tools from probability theory and stochastic analysis.

The programme was designed to explore a variety of novel phenomena and mechanisms for anomalous diffusion in mathematical models, and to bring together researchers in probability, stochastic analysis, and related areas, as well as (through applications of anomalous diffusion) mathematical physicists, ecologists, and materials scientists, and (through sampling algorithms) researchers in computational statistics and machine learning. In constructing the scientific programme, the organizers drew on the expertise of the Scientific advisory committee, consisting of Professor Jean Bertoin (Universität Zürich), Professor Gordon Slade (University of British Columbia), and Professor Ruth Williams (University of California, San Diego).

Programme timeliness, scope and outline

Overview of topics addressed and research areas involved. The main location of the scientific theme of the programme was the theory and applications of stochastic processes, one of the central areas of (pure and applied) probability theory. A common theme is 'anomalous diffusion', which runs through all of the topics, and several of the topics concern stochastic processes with constraints, either imposed by their own previous trajectory, or extrinsically, such as through domain boundaries.

The programme was built around 4 core thematically linked workshops, as well as two shorter workshops addressing specific applications. The workshops, ordered chronologically but labelled according to their designation in the programme, were:

(W01) Self-interacting processes (8–12 July).
(W02) Stochastic reflection (5–9 August), a Satellite Meeting held in Jock Colville Hall at Churchill College, Cambridge.
(W05) Modelling and applications of anomalous diffusions (4–6 September), including an Open for Business Day (OFBW69, 5 September), organized with the Newton Gateway team, and an EDI-themed panel discussion.
(W06) One-day workshop on Stochastic systems in active matter (4 October).
(W03) Geometry, occupation fields, and scaling limits (28 October–1 November).
(W04) Monte Carlo sampling: beyond the diffusive regime (25–29 November).

We give a short summary of the scientific scope of the workshops, the motivation for their thematic framing, and outline some of the ways in which the themes of the workshops interact, particularly with reference to anomalous diffusion.

(W01) Self-interacting processes. *Workshop organizers: Codina Cotar (University College London), Tyler Helmuth (Durham University), and Pierre Tarrès (NYU Shanghai).* Non-Markovian models exhibiting anomalous diffusion are of long-standing interest in physics, chemistry, ecology, and elsewhere [66, 69]. Examples include foraging animals that deplete resources and hence explore superdiffusively [13], and variations of the self-avoiding walk paradigm for models of macromolecules in solution. Mechanisms of self-interaction often are mediated by how much time the process spent in a given region in the past (its *occupation measure*); such non-Markov processes arise in numerous applications: roaming animals [83], investment strategies [5], adaptive sampling [4], queueing [32], and many more [11, 12, 28, 31, 68, 73, 86, 87].

In the case where the interaction is local, i.e., the dynamics depend on the occupation measure only in the neighbourhood of the current position, a rich class of models goes under the name of 'reinforced processes' [12, 73]. On the other hand are models with global interaction, where the dynamics are driven by a geometric functional of the occupation measure, such as its centre of mass, or the convex hull [5, 28]; here there can be significantly different behaviours, driven either by statistics of mean-field or extremal type. One specific model is the *elephant random walk* (ERW), introduced in physics [79], which has generated much recent interest in the mathematics literature [9, 17] due to its links to urns [9] and martingales [14, 15].

Another point of view considers static models chosen according to a given measure on paths. The most celebrated such example is the *self-avoiding walk* (SAW) [61], which models polymer chains in solution subject to excluded volume effects. Famous conjectures state that first-order scaling of n-step planar SAW is $n^{3/4}$ (the *Flory exponent* [36]) and the scaling limit is *Schramm–Loewner evolution* (SLE) with parameter $8/3$ [55]. There are recent results concerning SAW in presence of a reflecting wall [57], providing a modern link with the theme of (W02). Let us mention other recent trends in SAW: simulation (new estimates for critical temperatures and exponents) [10], combinatorial aspects via formal languages [59] for SAW on general graphs.

Avoidance of the past over short or longer time-scales can lead to anomalous exploration of space (explored in (W03)) and superdiffusive scaling, which can be beneficial for state-space exploration in sampling and optimization algorithms (explored in (W04)), just as it can be beneficial for animals roaming in search of fresh food or water (some of these ecological modelling questions were examined in (W05) and (W06)). Self-attraction can lead to radical collapse of spatial exploration, and trapping, which can accelerate convergence once one has identified a neighbourhood of the optimum, or represent consensus formation in social dynamics.

(W02) Stochastic reflection. *Workshop organizers: Aleksandar Mijatović (University of Warwick & The Alan Turing Institute), Kavita Ramanan (Brown University), Kilian Raschel (CNRS & Université d'Angers), and Ruth Williams (University of California, San Diego).* Just as self-interaction corresponds to intrinsic (self)-organization, reflection corresponds to extrinsic drivers of dynamics. Reflecting random walks, or diffusions, are a fundamental class of constrained (or controlled, confined) stochastic processes that have been studied over several decades. Motivation comes from queueing (heavy traffic [23, 40, 44, 45, 75, 76]), communication networks [37], gas dynamics [54], or financial models [43, 47], for example. Reflecting processes also play an important role in descriptions of systems of order-preserving diffusing particles, such as the Atlas model of mathematical finance and its relatives [8, 27, 46, 47, 52, 70, 78].

Stochastic reflections can lead to rich "anomalous diffusion" phenomena: frequent reflections in thin domains can drive processes to superdiffusive (even explosive) behaviour, and in non-smooth domains, analogous effects can drive processes into corners from which they cannot escape. These processes are locally diffusive, but can exhibit (very) anomalous behaviour on large scales. Before questions of dynamics can be studied, one must resolve existence and/or uniqueness of processes, which is highly non-trivial, particularly in the case of non-smooth or unbounded domains [51, 60, 88]. Also of ongoing interest is the regularity of solutions [3]. One of the most fundamental questions concerns the behaviour of the trajectory of the reflecting process. For example, in smooth, compact domains, a key theme is to understand conditions for convergence to stationarity, bounds on mixing, explicit solutions for stationary distributions and/or sharp asymptotics [38, 50]. In unbounded domains (orthants and cones being the most well studied), key questions include recurrence/transience [74, 89] and anomalous diffusion/explosion [65].

In non-smooth domains, if the reflections are arranged correctly, the diffusion may get 'trapped' at a corner (hence existence/uniqueness can be particularly subtle [51]). In unbounded domains, corner trappings in the quarter-plane are mostly understood [88], but recent models from queueing and financial mathematics have generated much interest on higher dimensional phenomena [77] and persistence problems [24].

In parallel to the above mentioned probabilistic analyses, reflected random (or deterministic) walks have been studied for their combinatorial properties, see [41] for an early paper, in connection with Weyl chambers. One key motivation is that many other combinatorial objects (maps, queues, Young tableaux, etc.) are encoded by lattice walks confined to some domains, for instance the quarter plane [22]. A variety of tools have been developed to study these combinatorial models (such as complex analysis [35], analytic combinatorics in several variables [64], Galois theory of difference equations [34], in addition to probabilistic methods). For random walks reflected in Weyl chambers associated to a root system of a Lie algebra, tools from algebraic and combinatorial representation theory become available [18, 21, 56].

Rough paths and signatures [20, 29, 39], explored in more detail in (W05), link to stochastic reflection through reflected rough differential equations, where it has been shown that the uniqueness or not of solutions is inherently depended on the roughness of the driving path [1]. Computational and statistical applications of reflecting processes include numerical solution of boundary-value problems [58] or set estimation [30], linking to the theme of (W04).

(W05) Modelling and applications of anomalous diffusions. *Workshop organizers: Luca Giuggioli (University of Bristol) and Anastasia Papavasiliou (University of Warwick).* The primary focus of (W05) was to explore anomalous diffusions as models of complex real-world systems, with one of the days designated as an "Open for Business" day, organized in conjunction with the Newton Gateway team, aiming to engage a broad audience, including industry and the public sector. Speakers came from varied backgrounds, bringing in diverse perspectives, problems, and ideas, to include theoretical developments in the areas of statistical physics, stochastic analysis and rough path theory, as well as application-driven methodological advances coming from biophysics, environmental statistics, and statistical ecology. To highlight EDI issues relevant for academia, there was a panel discussion on "Taking into account career breaks and caring responsibilities when assessing potential".

(W06) Stochastic systems in active matter. *Workshop organizer: Luke Davis (University of Cambridge & University College London).* Active matter is a class of physical systems that exhibit rich collective behaviours such as spontaneous phase separation, flocking and swarming, collective actuation in activated solids, and active turbulence [53, 63, 85]. In these systems constituents, which may include suspensions of biological cells and bacteria, chemically coated colloids and polymers, and animals and robots, constantly turnover energy to sustain stochastic and collective dynamical states and as such are inherently non-equilibrium, thus presenting many challenges to existing ideas and techniques in statistical mechanics. For example, activation of constituents typically leads to anomalous diffusion, in contrast to the majority of passive systems that can be described using simple "Fickian" diffusion models. A recent theoretical challenge is in deriving frameworks to predict the optimal (and efficient) control of active matter far from equilibrium. A straightforward departure from equilibrium statistical mechanics is the time irreversibility of the dynamics of the constituents that, in turn, leads to non-universality of the Boltzmann distribution and the absence of a state function. One approach to help overcome the theoretical challenges in describing and understanding active matter is to build and to strengthen bridges between communities in theoretical and computational condensed matter physics, biophysics, and stochastic processes. This one-day workshop aimed to facilitate the dissemination of the physics and mathematics of stochastic systems in the context of active matter.

(W03) Geometry, occupation fields, and scaling limits. *Workshop organizers: Amine Asselah (Université Paris 12 Val de Marne), Ellen Powell (Durham University), and Perla Sousi (University of Cambridge).* Scaling limit theory deals with the convergence of discrete probability models to continuum limits, as characteristic scale parameters tend to zero. Dramatic recent progress concerns statistical physics models and statistical sampling algorithms. The most classical scaling limit is the convergence of random walks to Brownian motion; this is *universal*, since a broad class of discrete models converge to a unique limit object. One viable scheme in which to link discrete-time and continuous-time systems with reflection is to use a form of strong approximation via parametrized penalization of discrete models. In a penalized model [82], the process is allowed to wander outside the domain to which it is confined, rather than reflecting at the boundary, but the farther it gets from the domain, the greater it drifts back.

The *range* of a discrete-space stochastic processes is the set of sites visited up until the present time. For example, the range of a lattice random walk is the set of the first n visited sites. There are different ways to measure the size of this set. Looking at the volume has been studied extensively starting with Jain, Orey [48], Spitzer and Le Gall. Another notion that captures the shape of a set is the capacity, which is a measure of how "visible" the set is for a random walk coming from infinity. When a random walk spends more time around its trajectory than what is typically expected, looking at the capacity of its range can describe effectively the homogeneity of this folding. Capacity is well studied in potential theory and in analysis and is given by a variational characterization. In probability it can also be related to intersections of random walks and this makes it more amenable to analysis using probabilistic tools [6, 7]. Of particular interest here is the phase transition between low and high dimensions, as the degree of transience of the walk influences the correlation-scale of trajectories.

Anomalous diffusion as discussed above is tied up with the notion of how a process explores its state-space: does it do so faster or slower than standard diffusion? Pertinent here is how much time the process spends in a given region, and this leads to questions involving occupation fields for stochastic processes. These questions are central when considering processes for statistical sampling algorithms, where the rate at which a process explores space is critical to performance of the algorithm, as explored in (W04). Anomalous diffusion is reflected in non-classical scaling exponents and non-classical scaling limits. For unconstrained, Markovian processes, the universality classes of anomalous diffusion are well-represented by stable processes, but as explained above, many anomalous diffusion mechanisms exist that do not require large-scale jumps. Most mechanisms of self-interaction (W01) can be described either in terms of geometry of the previous trajectory (avoidance of a certain set) or the occupation field (reinforcement according to past accumulated time). Understanding the evolution of the process together with these characteristics was a theme of (W03). In a rough-path context (a focus of (W05)) the signature of the path characterizes the geometry of a path up to tree-like equivalence [42].

(W04) Monte Carlo sampling: beyond the diffusive regime. *Workshop organizers: Joris Bierkens (Delft University of Technology), Michael Faulkner (University of Warwick), Krzysztof Łatuszyński (University of Warwick), Samuel Livingstone (University College London), and Gareth Roberts (University of Warwick).* The final workshop focused on computational statistics, where simulation, sampling and optimization algorithms that include elements of anomalous diffusion are an exciting subject of contemporary work. Markov chain algorithms for sampling from a target probability distribution are of great interest in statistics and machine learning [26, 75]. Efficiency of the algorithm is determined in part by how fast the underlying stochastic processes mixes, and by how fast it explores the state space; hence the relevance of anomalous diffusion. In another direction, a Bayesian approach to parameter inference for Markov (or more general) processes often leads, under the appropriate measure, to process that interact with their past trajectory, as previous observations lead to updates for the transition probabilities. Indeed, reinforced random walks appear in the context of Bayesian inference for ordinary random walks.

Euler discretizations of Langevin diffusion and related stochastic gradient methods are widely used in computational statistics and machine learning [26, 33, 62, 67, 80]. In practice the discretization in the Euler scheme is often implemented in a time-dependent way, via a deterministic 'annealing' schedule. The time-varying nature of the process is important as it eliminates asymptotically the bias of the method in applications in Bayesian statistics [84], and in machine learning this ensures that stochastic gradient descent convergences to the global minimum. More generally, it is natural to take the schedule chosen in an adapted way, based on the trajectory of the chain observed up to the present time. This framework is related both to adaptive MCMC and stochastic approximation [72].

The nature of traditional sampling schemes based on Euler/Langevin dynamics is well-suited to sampling from unbounded distributions. For the case of compactly-supported distributions, one needs some constraint; reflection is a natural way to achieve this. Recent work has looked at confined Langevin schemes [25], reflecting Euler schemes [58], reflecting Hamiltonian MCMC [81], and confined Milstein schemes [16], among others. As well as statistical sampling applications, Feynman–Kac representations lie behind use of such algorithms to provide numerical methods for simulation of solutions of PDEs in domains with Neumann, Robin, or generalized boundary conditions.

There is increasing evidence across several different settings that processes with some short memory, or momentum, which are discouraged from revisiting recently-visited sites, can have significantly improved mixing and coverage properties [2, 49]. Recently, there has been particular interest in the use of such non-reversible Markov dynamics, including piecewise-deterministic Markov processes, in sampling algorithms. The so-called 'Zig-zag' process is one important example [19]. The idea of random walks with restricted backtracking has earlier roots motivated by applications in polymer chemistry and the behaviour of foraging animals [71], for example; this class of processes includes the correlated or persistent random walks of Gillis–Domb–Fisher, and their continuum analogues such as the telegraph process.

The above lists just some of the ways in which anomalous diffusion is of fundamental interest in modern developments in sampling algorithms. The workshop considered applications of the processes, methods, and results discussed in the preceding workshops, to in computational statistics and machine learning. In particular, constrained sampling algorithms via reflection link back to (W02), adaptive and momentum-driven samplers link to (W01), and aspects of mixing (i.e., the way in which processes explore space) linked to (W03).

Summary of activities

Plenary lectures and public talks. The programme included a number of inspiring plenary lectures and public talks by distinguished speakers.

- Professor Krzysztof Burdzy, University of Washington, delivered the Rothschild Public Lecture "On Archimedes' Principle".
- Professor Gordon Slade, University of British Columbia, delivered a Clay Public Lecture on "Self-avoiding walk, spin systems, and renormalisation".
- Professor Bálint Tóth, Alfréd Rényi Institute for Mathematics, Hungarian Academy of Sciences, and University of Bristol, delivered a Clay Public Lecture on "Diffusion in the random Lorentz gas".
- Professor Ruth Williams, University of California, San Diego, delivered the Kirk Public Lecture on "Managing bottlenecks using diffusion and reflection".

Fellowships, bursaries, and visiting appointments.

Kirk Distinguished Visiting Fellow.

- Professor Ruth Williams, University of California, San Diego.

This fellowship supports mathematicians from under-represented groups in higher mathematical research.

Rothschild Distinguished Visiting Fellow.

- Professor Krzysztof Burdzy, University of Washington.

NM Rothschild & Sons have endowed the Institute with the funds to host visits from leading global mathematicians.

Clay Mathematics Institute Fellows.

- Professor Bálint Tóth, University of Bristol & Alfréd Rényi Institute for Mathematics, Hungarian Academy of Sciences.
- Professor Gordon Slade, University of British Columbia.

Organizers of conferences and similar events can nominate a highly distinguished mathematician to attend to deliver a Clay Lecture, which might be either a designated plenary lecture within the conference program or a special lecture delivered to a wider audience.

Heilbronn Distinguished Visiting Fellowship.

- Professor Amarjit Budhiraja, Department of Statistics and Operations Research at the University of North Carolina at Chapel Hill.

The Heilbronn Institute for Mathematical Research is a national centre supporting research across a range of areas of mathematics in the UK. As of 2018, the Institute has generously granted the Isaac Newton Institute an endowment to support visits from pre-eminent mathematicians around the world.

Simons Foundation Fellows.

- Professor Christophe Andrieu, University of Bristol.
- Professor Codina Cotar, University College London.
- Professor Sergei Fedotov, University of Manchester.
- Professor Luca Giuggioli, University of Bristol.
- Professor Andrey Pilipenko, Ukrainian National Academy of Sciences, Kiev.
- Professor Igor Podlubny, Technical University of Kosice.

The Simons Foundation have generously granted the Isaac Newton Institute an endowment to support visits from pre-eminent mathematicians around the world.

INI Postdoctoral Fellowships in Mathematics.

- Dr Luke Davis, University of Cambridge & University College London.

Established in 2021, the INI Postdoctoral Fellowships in Mathematics scheme was created to enable exceptional early career researchers in the mathematical sciences to gain experience, foster independence and forge new connections.

UCL Visiting Professorships. University College London awarded 3 Visiting Professorships to early-career researchers, who also gave talks in UCL:

- Dr Matthew Dickson, University of British Columbia.
- Dr Lucile Laulin, University Paris Nanterre.
- Dr Yucheng Liu, University of British Columbia.

Early-Career Researcher bursaries. Programme participants:

- Dr Martin Chak, Bocconi University.
- Dr Josiah Park, Berkeley Lab.
- Dr Jun Yang, University of Copenhagen.

Workshop participants:

- Dr Simon Buchholz, Max Planck Institute, Tübingen.
- Dr Alice Contat, Université Sorbonne Paris.
- Dr Daniel Han, University of New South Wales.
- Dr Alexandre Legrand, Institut Camille Jordan.
- Dr Larissa Richards, University of Toronto.
- Dr Florian Schweiger, Université du Genève.
- Dr Minwei Sun, Howard College of Arts and Sciences.

Anomalous diffusion seminar series.

1. Wednesday 17th July, 2–3pm. Marija Vucelja, University of Virginia. "Anomalous thermal relaxation of physical systems".

2. Wednesday 17th July, 3–4pm. Bernard Bercu, University of Bordeaux. "A martingale approach for the elephant random walk with stops and the Ewens-Pitman process".
3. Tuesday 23rd July, 2–3pm. Hélène Guérin, Université du Québec à Montréal. "Stochastic epidemic models with varying infectivity and susceptibility".
4. Tuesday 23rd July, 3–4pm. Amarjit Budhiraja, University of North Carolina. "Invariant measures of the infinite Atlas model: domains of attraction, extremality, and equilibrium fluctuations".
5. Wednesday 24th July, 4–5pm. Gordon Slade, University of British Columbia. Clay Math Public Lecture: "Self-avoiding walk, spin systems, and renormalisation".
6. Tuesday 30th July, 10–11am. Andrey Pilipenko, National Academy of Sciences of Ukraine. Mini-course: "Generalized Skorokhod's reflecting problem" (Lecture 1).
7. Thursday 1st August, 10–11am. Andrey Pilipenko, National Academy of Sciences of Ukraine. Mini-course: "Perturbed random walks and a skew Brownian motion" (Lecture 2).
8. Friday 2nd August, 10–11am. Andrey Pilipenko, National Academy of Sciences of Ukraine. Mini-course: "On a skew stable Lévy process" (Lecture 3).
9. Monday 12th August, 4–5pm. Krzysztof Burdzy, University of Washington. Rothschild Public Lecture: "On Archimedes' Principle".
10. Wednesday 14th August, 2–3pm. Isao Sauzedde, University of Warwick. "Central limit theorem for superdiffusive processes".
11. 3–4pm. Vladislav Vysotskiy, University of Sussex. "Stationary random walks with a switch".
12. Monday 19th August, 2–3pm. Alexander Milovanov, ENEA C.R. Frascati & Max-Planck-Institut für Physik komplexer Systeme, Dresden. "The nonlinear Anderson problem: A vital, if openly recognized challenge".
13. Wednesday 21st August, 4–5pm. Ruth Williams, University of California, San Diego. Kirk Public Lecture: "Managing bottlenecks using diffusion and reflection".
14. Tuesday 27th August, 10am–12pm. Franz Merkl, Ludwig-Maximilians-Universität München, and Silke Rolles Technische Universität München. Mini-course: "An introduction to supersymmetry and the vertex-reinforced jump process" (Lectures 1 & 2).
15. Wednesday 28th August, 2–3pm. Matthew Dickson, University of British Columbia. "Expanding the Critical Intensity of Random Connection Models".
16. Thursday 29th August, 10am–12pm. Franz Merkl, Ludwig-Maximilians Universität München, and Silke Rolles Technische Universität München. Mini-course: "An introduction to supersymmetry and the vertex-reinforced jump process" (Lectures 3 & 4).
17. Tuesday 3rd September, 2–3pm. EDI panel discussion. "Taking into account career breaks and caring responsibilities when assessing potential".
18. Wednesday 11th September, 2–3pm. Luca Giuggioli, University of Bristol. "On a collection of techniques to model first-passage processes with discrete space-time variables".
19. Wednesday 18th September, 2–3pm. Christopher Hoffman, University of Washington. "Universality for Self-organized Criticality".
20. Wednesday 18th September, 3–4pm. Nikolai Leonenko, Cardiff University. "Dickman-type stochastic processes and non-local operators".
21. Wednesday 25th September, 2–3pm. Stjepan Šebek, University of Zagreb. "Bounds on the size of the convex hull of planar Brownian motion and related inverse processes".
22. Wednesday 25th September, 3–4pm. Adam Bobrowski, Lublin University of Technology. "Two fairly recent approximations of Walsh's spider process".
23. Thursday 3rd October, 2–3pm. Luke Davis, University of Cambridge & University College London. "Controlling discrete and continuous state non-equilibrium systems".
24. Wednesday 9th October, 3–4pm. Julien Randon-Furling, ENS Paris-Saclay & UM6P College of Computing. "Convex Hulls of Higher-Dimensional Random Walks and First-Passage Resetting".

25. Wednesday 16th October, 2–3pm. Alex Watson, University College London. "A growth-fragmentation found in the cone excursions of Brownian motion (and in the quantum disc)".
26. Wednesday 23rd October, 3–4pm. Bálint Tóth, University of Bristol & Rényi Institute for Mathematics. "Anomalous super-diffusion from interactions".
27. Wednesday 6th November, 2–3pm. Leandro Chiarini, Durham University. "Non-monotone absorbing state phase transitions".
28. Wednesday 6th November, 3–4pm. Wojciech Cygan, University of Wroclaw. "Stable random walks in cones
29. Monday 11th November, 2–3pm. Giorgos Vasdekis, Newcastle University. "Skew-symmetric schemes for robust sampling from diffusions".
30. Monday 11th November, 3–4pm. Alexandros Beskos, University College London. "A Closed-Form Transition Density Expansion for Elliptic and Hypo-Elliptic SDEs".
31. Wednesday 13th November, 2–3pm. Nikola Sandrić, University of Zagreb. "On Convex Hulls of Stable Random Walks".
32. Wednesday 13th November, 3–4pm. Kayvan Sadeghi, University College London. "Axiomatization of Interventional Probability Distributions".
33. Wednesday 20th November, 4–5pm. Bálint Tóth, University of Bristol & Rényi Institute for Mathematics. Clay Public Lecture: "Diffusion in the random Lorentz gas".
34. Thursday 21st November, 2–3pm. Gareth Roberts, University of Warwick. "Diffusive behaviour of non-reversible MCMC with application to Simulated Tempering".
35. Thursday 21st November, 3–4pm. Błażej Miasojedow, University of Warsaw. "Langevin Monte Carlo Beyond Lipschitz Gradient Continuity".
36. Monday 2nd December, 2–3pm. Miha Brešar, University of Warwick. "Subexponential lower bounds for f-ergodic Markov processes".
37. Monday 2nd December, 3–4pm. Michael Tretyakov, University of Nottingham. "Geometric numerical methods for confined Langevin dynamics".
38. Tuesday 3rd December, 14:00 to 16:00. Sam Power, University of Bristol. "Mini-course on Geometric Functional Inequalities for Markov Chains".
39. Wednesday 4th December, 2–3pm. Joris Bierkens, Delft University of Technology. "Piecewise Deterministic Monte Carlo for latent variable models: a case study".
40. Wednesday 4th December, 3–4pm. Neha Spenta Wadia, Simons Foundation. "A high-probability mixing time bound for Gibbs sampling from log-smooth strongly log-concave distributions".
41. Monday 9th December, 2–3pm. Andrew Swan, Ecole Polytechnique Fédérale de Lausanne. "Random walk isomorphism theorems for a new type of spin system".
42. Monday 9th December, 3–4pm. Shuo Qin, Beijing Institute of Mathematical Sciences and Applications. "Recurrence and Transience of Multidimensional Elephant Random Walks".
43. Wednesday 11th December, 2–3pm. Igor Podlubny, Technical University of Kosice. "Fractional-order differentiation meets generalized probabilities".
44. Wednesday 11th December, 3–4pm. Luke Hardcastle, University College London. "Piecewise Deterministic Markov Processes for transdimensional sampling from flexible Bayesian survival models".

Scientific outcomes

The programme has generated considerable momentum (see Table 1 below for some summary data) and is already generating a range of scientific outcomes, over shorter and longer terms (see Table 2 for feedback from participants). One of our central aims, which we believe will be one of the most significant long-term contributions of the programme, was co-location and interaction of experts in a variety of fields where stochastic systems for anomalous diffusion have been of increasing significance in recent years, namely researchers from probability, analysis, mathematical physics, statistics and machine learning, and adjacent areas. The programme has helped to bridge gaps and stimulate what we hope will be lasting connections and interactions, at a stage in development of the theory of anomalous diffusion in these different communities that should prove particularly fruitful for cross-fertilization and sharing of ideas, intuitions, research goals, techniques, and potential applications.

Activity	Number
Programme participants	114 (of which ≥ 37 early-career)
Geographic distribution	41% UK, 59% international
Total scientific talks	$>$155
Programme seminars	44
(W01) *Self-interacting processes*	22 speakers Total 72 participants
(W02) *Stochastic reflection* Satellite event	18 speakers, plus "lightning" and open problem talks Total 49 participants
(W05) *Modelling and applications* Including OFBW69	27 speakers Total 54 participants
(W06) *Stochastic systems in active matter*	11 speakers Total 55 participants
(W03) *Geometry, occupation fields, and scaling limits*	13 speakers Total 68 participants
(W04) *Monte Carlo sampling: beyond the diffusive regime*	20 speakers Total 86 participants

Table 1. Quantitative summary of programme activity.

Advancing research in the field. In the end-of-programme survey, **11** participants (33% of responses) reported that the programme had a significant effect in opening new research directions, and **19** participants (58% of responses) reported a partial effect. Furthermore, **9** participants (27% of responses) reported new applications of work in this area of which they were previously unaware.

Among many highlights, we list the following selection to show some of the breadth of developments explored during the programme.

- One notable strand from (W01) consisted of striking developments in the theory of multidimensional random walks in random media, where, until recently, it seemed like the rate of advance had slowed due to major technical obstacles, and insufficiently developed mathematical tools. Talks by Bálint Tóth (University of Bristol & Alfréd Rényi Institute for Mathematics) on "H_{-1} reloaded", by Augusto Teixeira (IMPA) on "CLT for a class of random walks in dynamic random environments", and by Christophe Sabot (Université Claude Bernard Lyon) on "The point of view of the particle for 2D random walks in Dirichlet environment", among others, displayed some of the exciting developments in this area. Other remarkable advances in the area were presented in talks by Felix Otto (Max-Planck-Institut für Mathematik) on "A critical drift-diffusion equation: intermittent behavior", and by Gordon Slade (University of British Columbia) on "Torus plateaux in high-dimensional critical phenomena". The deeper understanding of multidimensional anomalous diffusion, including slow-down phenomena, that is being developed in this setting seems likely also to inform the other themes of the programme, particularly performance of sampling algorithms.

- A striking aspect of (W02) and related activity was the combination of sophisticated, deep developments for some of the fundamental classes of reflecting processes, novel approaches for degenerate classes that fall outside the standard settings, and continued relevance of reflected processes for applications. Particular themes involved *infinite-dimensional* problems, either formulated through diffusions with rank-driven interactions (cf. preprint [P2]) or reflected McKean-Vlasov SDEs which can model systems with constraints and mean-field interactions (preprint [P11]), problems with *multiple time-scales*, such as preprint [P13] and the talk by Rama Cont (University of Oxford) on "Stochastic dynamics of limit order books: a journey across time scales', and *non-smooth* domains, as in the talk by Cristina Costantini (Università degli Studi di Chieti-Pescara) on "Semimartingale obliquely reflecting diffusions in curved nonsmooth domains". All these aspects are relevant for particular applications, and present new challenges.
- (W03) explored exciting recent developments in random geometry, such as SLE, GFF, and associated critical structures that are central to modern scaling-limit theory, including talks by Nathanael Berestycki (Universität Wien) on "Weyl's law in Liouville quantum gravity" and by Léonie Papon (Durham University) on "A level line of the massive Gaussian free field". State-of-the art developments in occupation fields included those presented by Alberto Chiarini (Università degli Studi di Padova) in a talk on "Bulk deviations for the simple random walk".
- In (W04), a host of impressive recent developments in Monte Carlo algorithms which utilized some aspect of anomalous diffusion, either globally or locally, were presented. It was clear that this theory is undergoing rapid vigorous development, and there are many exciting possibilities ahead both for improvements to theoretical understanding of the performance of algorithms already being deployed, and for proposals of novel algorithms with potentially significant potential performance benefits. One notable example was presented by Andrea Bertazzi (École Polytechnique) in a talk on "Piecewise deterministic generative models", introducing a novel class of generative models based on piecewise deterministic Markov processes (as developed recently for sampling applications) with some desirable theoretical properties and promising numerical simulations on toy datasets.

Bringing communities together. In planning the programme, we had identified several areas where we felt the programme could make a significant contribution in bringing together traditionally separate communities of experts, who have been dealing with various aspects of anomalous diffusion with different motivations, stochastic models, and methodologies, but with considerable scope for cross-fertilization. Closer to applications are mathematical modellers in ecology, motive matter, fluids, and so on, as well as a large community of developing, studying, and deploying stochastic algorithms for optimization, simulation, and inference in machine learning and computational statistics. In both communities, the traditional diffusion framework is well-established, but developments using anomalous diffusion have been gaining significant momentum in the last few years, and the timing for sharing of ideas seemed ideal. On the more theoretical side, stochastic processes with interactions, constraints, reflections, and so on, is a topic that has held interest for several decades but which has gained renewed relevance in light of developments coming from stochastic algorithms on the one hand, and on significant advances in random geometry on the other hand.

Question	Response
Scientific quality	91% of responses "excellent"
Has your participation opened new research directions?	33% reported "yes" and 58% "partially"
Has your participation led to new collaborations?	47% reported "yes"
Did you visit other UK institutions as part of your trip?	26% reported "yes"
Did you attend seminars at the CMS?	41% reported "yes"

TABLE 2. Summary of programme impact and feedback (33 responses to the end of programme survey).

To give a few concrete examples of work emerging during the programme that, in our view, illustrate very exciting cross-fertilization of ideas, we mention:

- In preprint [P3] the authors study sampling algorithms using stochastic-momentum samplers (of the type discussed in (W04) e.g. [19]) subject to state-dependent *time-changes*, which is one of the most common mechanisms for anomalous diffusion studied in the mathematical physics literature (e.g. [66]). A similar mechanism is used for annealing schedules in machine learning algorithms. The preprint [P1] also uses time-changed Markov processes, now as part of a constructive solution theory for non-local equations, including an application to a pricing model from financial mathematics.
- In the context of modelling active matter (of the type discussed in (W06) e.g. [53]), a natural class of model has active agents moving with momentum, in *run-and-tumble* dynamics; a very similar approach has recently been shown to be effective in statistical samplers (of the type discussed in (W04) e.g. [19]). In statistical optimizers or samplers, one often uses multiple searching agents and *anticipated rejection* or *stochastic resetting* to abandon sampling runs where an agent appears to be trapped far from an optimum, or far from convergence. The preprint [P9] combines these dynamical features in a one-dimensional setting that is amenable to analysis.

Facilitating early-career researchers; EDI panel. A large number of PhD and early-career researchers engaged in the programme and workshops, supported by various funding sources. Local (Cambridge and nearby) early-career researchers also made good use of the programme events. The programme was integrated with the probability seminar in StatsLab (organized by Perla Sousi) enabling speakers to be shared between the programme seminar series and the StatsLab seminar, increasing the opportunity for networking among the programme participants and the local academics, envisioned to be a particular benefit to early-career researchers. The EDI-themed panel discussion hosted in (W05), "Taking into account career breaks and caring responsibilities when assessing potential", was also designed to highlight an issue which can particularly impact early-career researchers (due to the proportionately higher potential effect of a career break early in a career).

Collaborations. In the end-of-programme survey, **15** participants (47% of responses) reported that taking part in the programme led to new collaborations. The organizers have been informed of the following specific new collaborations that arose from activity during the programme.

- David Kramer-Bang (Aarhus University) and Stjepan Šebek (University of Zagreb).
- Adam Bobrowski (Lublin University of Technology) and Andrey Pilipenko (National Academy of Sciences of Ukraine): stochastic characterisation of all possible Brownian motions on star-like graphs and Portenko-type approximation of Walsh's process.
- Cristina Costantini (Università di Chieti-Pescara) and Andrey Pilipenko (National Academy of Sciences of Ukraine): reflecting Brownian motion in piecewise smooth cones with faces that meet in cusps, which is a process that arises in diffusion approximation of bandwidth sharing queues.
- Thomas Dreyfus (CNRS, University of Burgundy), Jules Flin (Télécom SudParis), Sandro Franceschi (Télécom SudParis): Degenerate systems of three Brownian particles with asymmetric collisions: invariant measure of gaps and differential properties.
- François Chapon (University of Toulouse), Sandro Franceschi (Télécom SudParis), Maxence Petit (Sorbonne Université), Kilian Raschel (CNRS, University of Angers): Distribution of the degenerate reflected Brownian motion in a wedge.
- Wojciech Cygan (University of Wrocław) and Sara Mazzonetto (Université de Lorraine).
- Alexander Milovanov (Frascati Research Centre), Alexander Iomin (Technion, Israel), and Jens Juul Rasmussen (Technical University of Denmark) formulated a new approach to characterize fluctuation-induced transport in magnetic confinement devices, an important milestone on the way to controlled fusion burn.
- Andrey Pilipenko (National Academy of Sciences of Ukraine), Aleksandar Mijatović, Isao Sauzedde (University of Warwick).
- Andrey Pilipenko (National Academy of Sciences of Ukraine) and Vladislav Vysotskiy (University of Sussex). The behaviour of perturbed random walks at crossing of a fixed level. VV reports: "Although this currently is at the state of discussion, it is likely that we will start a joint work within the next year. This would hardly be possible without the SSD problem."
- Bruno Toaldo (University of Turin) and Aleksandar Mijatović (University of Warwick).
- Frank van der Meulen (Vrije Universiteit Amsterdam) and Andi Wang (University of Warwick): Using the language of optics from category theory to study the compositional structure of a particular smoothing algorithm used in computational Bayesian inference.
- Vladislav Vysotskiy (University of Sussex) and Andrew Wade (Durham University) on convex hulls of planar random walks. VV reports: "We started a collaboration that aims to describe the structure of the convex hull of a planar random walk with a non-zero drift. We both work on convex hulls of random walks, and on a few occasions we tried to find a novel yet solvable problem in this topic, to work together at. This was not successful until the SSD program, which gave us a great opportunity to stay in a close contact for longer time. Finally, we found a very promising approach which will expectedly lead to a rather fine description of the structure of the convex hull. This is in turn key for further understanding of the hull in terms of the number of its vertices and its perimeter."

Public art contest. Associated to the programme is the "Anomalous Mathematical Patterns" Sci-Art Contest which ran between 4th December 2024–31st March 2025. We received in all **over 500 submissions** to the contest. The goals of the contest were to:

- showcase the interplay of mathematics (in particular of probability theory) and art in the modern world;

- explore the different ways in which artists and mathematicians approach mathematical concepts;
- find ways to communicate research in mathematics to as wide an audience as possible;
- inspire new generations towards mathematical education and training;
- potentially spark collaborations among and between mathematicians and artists.

Some potential mathematical themes that could be explored are: (ir)regularity and anomaly of mathematical patterns, interacting particle systems, fluid and gas dynamics, random networks, reflecting processes, interacting random walks and their geometry, diffusive processes, kinetic art. Have a look at this page for a more detailed description and for links to relevant articles that help to explain some of the central concepts connected to potential mathematical themes for the contest. There are three competition categories:

- Photograph, painting, film, print, animation;
- Textile, sculpture or other medium (such as 3D printing, laser cutting, CNC routing);
- AI and computer-generated art, which maybe also contains digitally or otherwise enhanced or altered photos.

There will be 11 cash awards, of which 9 will be picked by a panel of judges, 1 will be picked by an online vote on the competition website, and 1 will be picked by the participants at the "Stochastic systems for anomalous diffusion" research programme. The finalist entries may be exhibited in the UCL Cloisters, following the competition and throughout the Isaac Newton Institute building, Cambridge. Additionally, following the competition, the winners' work may be exhibited in various museums in the UK and elsewhere in Europe.

The contest has been advertised widely, including on many art/math pages. At the end of the contest, there will be a video with the finalists/winners.

Publications and/or grant applications

Publications and preprints. In the end-of-programme survey, **18** participants (54% of responses) reported that publications arising out of work initiated during the programme, or in which a substantial part was carried out during the programme, were either complete or in the pipeline. At the time of writing (March 2025) The organizers have been informed of the following publications and preprints that arose from the programme or where part of the work was undertaken during the course of the programme.

[P1] G. Ascione, E. Scalas, B. Toaldo, L. Torricelli: Time-changed Markov processes and coupled non-local equations. [https://arxiv.org/abs/2412.14956].

[P2] R. Atar, T. Ichiba: Rank-based stochastic differential inclusions and diffusion limits for a load balancing model. [https://arxiv.org/abs/2409.15121].

[P3] A. Bertazzi, G. Vasdekis: Sampling with time-changed Markov processes. [https://arxiv.org/abs/2501.15155].

[P4] I. Biočić, D.E. Cedeño-Girón, B. Toaldo: Sampling inverse subordinators and subdiffusions. [https://arxiv.org/abs/2412.15815].

[P5] M. Brešar, C. da Costa, A. Mijatović, A. Wade: Superdiffusive limits for Bessel-driven stochastic kinetics. [https://arxiv.org/abs/2401.11863].

[P6] C. da Costa, D. Thacker, A. Wade: Long-range one-dimensional internal diffusion-limited aggregation. [https://arxiv.org/abs/2411.10113].

[P7] M. Disertori, F. Merkl, S. Rolles: Restrictions of some reinforced processes to subgraphs. [https://arxiv.org/abs/2411.06195].

[P8] L. Facciaroni, C. Ricciuti, E. Scalas, B. Toaldo: Para-Markov chains and related non-local equations. [https://arxiv.org/abs/2412.14979].

[P9] M. Geueneau, S.N. Majumdar, G. Schehr: Run-and-tumble particle in one-dimensional potentials: mean first-passage time and applications. [https://doi.org/10.1103/PhysRevE.111.014144].

[P10] J. González Cázares, D. Kramer-Bang, A. Mijatović: Asymptotically optimal Wasserstein couplings for the small-time stable domain of attraction. [https://arxiv.org/abs/2411.03609].

[P11] P.D. Hinds, A. Sharma, M. Tretyakov: Well-posedness and approximation of reflected McKean-Vlasov SDEs with applications. [https://arxiv.org/abs/2412.20247].

[P12] A. Kundy., S.N. Majumdar, G. Schehr: The number of minima in random landscapes generated by constrained random walk and Lévy flights: universal properties. [https://arxiv.org/abs/2409.12906].

[P13] A. Mijatović, I. Sauzedde, A. Wade: Central limit theorem for superdiffusive reflected Brownian motion. [https://arxiv.org/abs/2412.14267].

[P14] A.V. Milovanov, A. Iomin, and J.J. Rasmussen, Turbulence spreading and anomalous diffusion on combs. Submitted.

[P15] M. Pricop-Jeckstadt, V. Patrangenaru, R.L. Paige: 2D Oriented Projective Shape Analysis. [https://www.newton.ac.uk/documents/preprints/NI24004].

Edited volume. A subset of the organizers (Cotar, Mijatović, Raschel, Wade) will edit a "lecture notes" volume titled *Stochastic Systems for Anomalous Diffusion* as a physical footprint for the programme to encapsulate key themes, present the state-of-the art, and stimulate further work. This will provide a rare opportunity to shape some of the territory that the programme has explored, at a formative stage for the field. Before the end of 2025 a proposal will be submitted to Cambridge University Press to publish the volume under the LMS Lecture Notes series. So far the following contributors have agreed to provide chapters, some of which will be based on material delivered as lecture courses during the programme:

- Pablo Alonso-Martin, Horatio Boedihardjo, Anastasia Papavasiliou: Statistical inference for stochastic differential equation driven by fractional Brownian motion.
- Joris Bierkens: Central limit theorems for piecewise deterministic Markov processes.
- Denis Denisov and Vitali Wachtel: Universality approach to fluctuations of random walks.
- Sandro Franceschi: Analytical, combinatorial and algebraic approaches to the study of reflected Brownian motion in a cone.
- Lionel Levine: Large language models and AI.
- Samuel Livingstone: Metropolis algorithm with heavy-tailed proposals.
- Andrey Pilipenko: Skorokhod reflection, skew reflection, perturbed random walks, scaling limits.
- Sam Power: Functional inequalities for Markov processes.
- Bruno Toaldo: Non-local approach to anomalous diffusion.
- Bálint Tóth: Diffusion in the random Lorentz gas.

Other books planned. The organizers have been informed of the following plans.

- Adam Bobrowski (Lublin University of Technology): *Generators of Feller processes*. AB reports "For a couple of years, I have been planning a book tentatively called 'Generators of Feller processes' (a follow-up to my *Generators of Markov chains*, Cambridge, 2020, and two other books), but the general idea I had about it was rather vague. After the month spend at INI the idea is much more clear, and definitely the research I am doing with Andrey [Pilipenko, mentioned in "Collaborations" above] will strongly influence the final shape of the book."

Grant applications. In the end-of-programme survey, **5** participants (15% of responses) reported that they will make applications for funding to support further research as a result of their participation in the programme. The organizers have been informed of the following specific grant applications that have been made arising from activity during the programme.

- Stjepan Šebek (University of Zagreb), with David Kramer-Bang (Aarhus University) and Mo Dick Wong (Durham University): distributional limits of geometric functionals of stable random walks.
- Bruno Toaldo (University of Turin) has submitted a proposal for an ERC Consolidator grant that was written during the programme, and another proposal for the FIS3 grant (Italian Science Fund) that was also written at the programme.

Multimedia items/links.

- Programme link at the Clay Mathematics Institute.
[https://www.claymath.org/events/stochastic-systems-for-anomalous-diffusion/]
- Living Proof Podcast "Exploring anomalous diffusion: An Interview with Aleks Mijatović and Codina Cotar".
[https://www.newton.ac.uk/media/podcasts/post/63-exploring-anomalous-diffusion-an-interview-with-aleks-mijatovic-and-codina-cotar/]
- "Anomalous Mathematical Patterns" Sci-Art contest.
[https://sites.google.com/view/anomalousmathematicalpatterns/home].
- Art contest link at the Heilbronn Institute.
[https://heilbronn.ac.uk/2024/10/10/anomalous-mathematical-patterns/].
- *Plus* dissemination article about the INI programme and about the Sci-Art contest
[https://plus.maths.org/content/calling-all-maths-friendly-artists].

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