Programme "Spectral Theory and Partial Differential Equations"

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Spectral theory and partial differential equations stand at a meeting point of several different parts of mathematics and physics. Within mathematics it links spectral properties of elliptic and parabolic operators to the geometry and topology of the underlying manifold. Within physics it links, for example, the stability of matter to the properties of the potentials in the Schrödinger operators.

The programme concentrated on two global areas of Spectral Theory: Spectral Geometry and Spectral Theory of Schrödinger operators. The spectral geometric issues addressed during the programme were concerned with connection between the bottom of the spectrum, nodal lines, multiplicities of eigenvalues and the geometric properties of the region or manifold. Questions related to the Schrödinger operators covered the trace formulae(Lieb-Thirring inequalities, absolute continuity for multi-dimensional operators), theory of periodic operators (detailed properties of the band-gap spectrum, absolute continuity, the number of gaps) and scattering theory (scattering matrix properties, propagation of singularities). Many of these questions have important applications in physics (statistical physics, large particle systems, quantum mechanics, photonic crystals). This programme together with the one-week conference allowed one to reflect on major achievements, and to identify the key targets for the remainder of this decade. The programme emphasised the multidisciplinary character of the research in this area, and capitalised on the interaction of experts working on different aspects of Spectral Theory. It attracted a large number of young researchers from UK and other European countries, and we are confident that it will have a strong impact on the development of analysis and adjacent disciplines in the UK by reinforcing the interest in this vibrant area of mathematics.

The Programme focused on open problems and possible new developments in Spectral Geometry (week 1), Theory of Schrödinger Operators (week 3), and Scattering Theory and Heat Semigroups (week 4). The recent rapid developments in the field of Spectral Theory and its Applications were summarised during the Workshop in week 2. A total of 28 lectures surveyed these developments. In addition there were 11 contributions of young scientists to a poster session. The lecturers reported on recent results covering a wide range of topics. In particular, these included spectral gaps, nodal lines of eigenfunctions, large time heat kernel behaviour, scattering theory, universal eigenvalue bounds, periodic Schrödinger operators, spectral asymptotics, semilinear and quasilinear pde's.

In weeks 1,3, and 4 there were 2 seminar talks each, allowing for intense collaboration and discussion among the participants.

The Programme initiated many new directions of research in the field. These are for example illustrated by giving some details below:

(1) Robert Seeley , Peter Gilkey and Michiel van den Berg investigated the heat flow in compact Riemannian manifolds with singular initial data, and the question was raised whether an asymptotic series for the heat content for small time exists, similar to the case with smooth initial data. Seeley obtained in the mean time a suitable construction/representation of the parametrics for this problem, while van den Berg obtained possible extensions to non compact, geodesically complete Riemannian manifolds.The work is in progress, and further developments will be highlighted at a Conference in Blaubeuren later this year.

(2) Of particular interest was the lecture by L.Polterovich "Nodal inequalities on surfaces". Polterovich studied the distribution of the extrema of a Laplace eigenfunction on a closed , compact surface. Inequalities by Kronrod and Yomdin played a crucial role in the investigation. Several remarkable inequalities featured. For example prove that if M is a planar domain, f is a smooth function on M, 0 on the boundary of M then $\max |f| < (2\pi)^{-1} \int H_f$, where the integral is with respect to the surface measure on M ,and H_f is the operator norm of the Hessian of f. Rodrigo

Banuelos and others tried (so far in vain) to find a calculus proof of this fact. However, van den Berg and Banuelos obtained (independently) a weaker inequality which is nevertheless sharp in the case of a disc.

(3) Prof. V. Buslaev and Prof. A. Sobolev discussed a possibility of a joint work on the absolute continuity of periodic operators.
(4) Rodrigo Banuelos and Brian Davies discussed the problems related to the spectral gap for the Steklov operator, and Banuelos suggested the use of stochastic calculus in the computation of the expected lifetime of an alpha stable process to obtain estimates for the spectral gap in the Steklov operator.
(5) Michael Levitin and Leonid Parnovski collaborate to prove a class of new inequalities for the eigenvalues of the Dirichlet and Neumann Laplacians on regions in Euclidean space using commutator estimates. This work is in progress.
(6) Peter Gilkey gave a lecture on Complex Osserman algebraic curvature tensors and Clifford families , and completed a manuscript on this subject.
(7) Prof. V. Buslaev and Dr. A. Pushnitski started working on a joint project on the classical mechanics analog of the spectral shift function.
(8) Prof. G. Rozenblum and Prof. A. Sobolev began a joint project on the discrete spectrum asymptotics for the perturbed Landau operator.
(9) Prof. B. Helffer, Prof. T. Hoffman-Ostenhof and prof. M. van den Berg actively discussed spectral minimal partitions of open, bounded connected sets in Euclidean space.
(10) Prof. P. Kuchment and Prof. A. Sobolev began writing a joint paper on the properties of band function for periodic operators.

It is evident that the research in Spectral Geometry is shifting from semiclassical analysis, and large eigenvalue asymptotics, to the structure of nodal domains, bottom of the spectrum of the Laplacian, in particular spectral gaps for Laplacians and other operators, and that some of the problems in this area have become tractable using a variety of techniques and tools.

While Spectral Theory and Partial Differential Equations has been the centre of attention for more than a century by both the mathematics- and physics communities the Programme highlighted the rapid development across a broad range of subjects. Much recent progress has been made in the understanding of the asymptotic behaviour of spectral functions (spectral counting function, heat content,heat trace and heat kernels),and the general properties of periodic and magnetic Schrödinger operators.The Programme gave the opportunity to focus attention on a number of important questions related to the nodal lines of Laplace and Schrödinger eigenfunctions , spectral gaps of these and other operators and properties of the band gap spectrum, absolute continuity and number of gaps of periodic operators. Many of these questions have important applications in physics , and this was highlighted by lectures on e.g. the Casimir effect. Much progress and new developments were reported notably in new Cwickel - Lieb - Rozenbljum type inequalities. The Workshop emphasised the multidisciplinary character of the field and capitalised on the interaction of the more than 90 participants working on different aspects of Spectral Theory.